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Positive solutions for a system of generalized Lidstone problems

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Abstract In this paper, we study the existence and multiplicity of positive solutions for the system of the generalized Lidstone problems

$$\begin{cases} (-1)^m x^{(2m)} = f(t, x, -x'', \dots, (-1)^{m-1} x^{(2m-2)}, y, -y'', \dots, (-1)^{n-1} y^{(2n-2)}), \\ (-1)^n y^{(2n)} = g(t, x, -x'', \dots, (-1)^{m-1} x^{(2m-2)}, y, -y'', \dots, (-1)^{n-1} y^{(2n-2)}), \\ ax^{(2i)}(0) - bx^{(2i+1)}(0) = cx^{(2i)}(1) + dx^{(2i+1)}(1) = 0 \quad (i = 0, 1, \dots, m-1), \\ ay^{(2j)}(0) - by^{(2j+1)}(0) = cy^{(2j)}(1) + dy^{(2j+1)}(1) = 0 \quad (j = 0, 1, \dots, n-1). \end{cases}$$

We use fixed point index theory to establish our main results based on a priori estimates achieved by utilizing some properties of concave functions, so that the nonlinearities f and g are allowed to grow in distinct manners, with one of them growing superlinearly and the other growing sublinearly.

Keywords Generalized Lidstone problem · Positive solution · Cone · Fixed point index · Concave function

Mathematics Subject Classification (2000) 34B18 · 45G15 · 45M20 · 47H07 · 47H11

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1 Introduction

In this paper we study the existence and multiplicity of positive solutions for the system of the generalized Lidstone problems

$$\begin{cases} (-1)^m x^{(2m)} = f(t, x, -x'', \dots, (-1)^{m-1} x^{(2m-2)}, y, -y'', \dots, (-1)^{n-1} y^{(2n-2)}), \\ (-1)^n y^{(2n)} = g(t, x, -x'', \dots, (-1)^{m-1} x^{(2m-2)}, y, -y'', \dots, (-1)^{n-1} y^{(2n-2)}), \\ ax^{(2i)}(0) - bx^{(2i+1)}(0) = cx^{(2i)}(1) + dx^{(2i+1)}(1) = 0 \quad (i = 0, 1, \dots, m-1), \\ ay^{(2j)}(0) - by^{(2j+1)}(0) = cy^{(2j)}(1) + dy^{(2j+1)}(1) = 0 \quad (j = 0, 1, \dots, n-1), \end{cases} \tag{1.1}$$

where $m \geq 1, n \geq 1$, $f, g \in C([0,1] \times \mathbb{R}_+^{m+n}, \mathbb{R}_+)$ $(\mathbb{R}_+ := [0, +\infty))$, $a, b, c, d \in \mathbb{R}_+$ with $ac + ad + bc > 0$. Here, by a positive solution (1.1) we mean a pair of nonnegative functions $(u, v) \in C^{2m}[0,1] \times C^{2n}[0,1]$ that solve (1.1) and satisfy $(u, v) \not\equiv (0, 0)$.

The so-called Lidstone problem

$$\begin{cases} (-1)^n x^{(2n)} = f(t, x, -x'', \dots, (-1)^{n-1} x^{(2n-2)}), \\ x^{(2i)}(0) = x^{(2i)}(1) = 0 \quad (i = 0, 1, \dots, n-1), \end{cases} \tag{1.2}$$

arises in many different areas of applied mathematics and physics, and has been extensively studied in recent years; for more details, we refer the reader to [3–6, 8, 19, 23, 26] and references cited therein. In [26], Yang studied the existence and uniqueness of positive solutions for the generalized Lidstone problem

$$\begin{cases} (-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1} u^{(2n-2)}), \\ au^{(2i)}(0) - bu^{(2i+1)}(0) = 0 \quad (i = 0, 1, \dots, n-1), \\ cu^{(2i)}(1) + du^{(2i+1)}(1) = 0 \quad (i = 0, 1, \dots, n-1), \end{cases} \tag{1.3}$$

where a, b, c, d are as in (1.1) and $f \in C([0,1] \times \mathbb{R}_+^n, \mathbb{R}_+)$. The main results obtained in [26] are formulated in terms of spectral radii of some associated linear integral operators and thus can be viewed as extensions of corresponding *sharp* results for the case $n = 1$ due to Liu et al. [17].

The existence of positive solutions for systems of nonlinear differential equations have been studied by many authors; see, for instance, [1, 2, 7, 10–12, 15–18, 20–22, 24, 27], to cite a few. In [16], Li et al. discussed, using Krasnoselskii's fixed point theorem, the existence and multiplicity of positive solutions for the system of third-order three-point boundary value problems

$$\begin{cases} -u''' = a(t) f(t, v), \\ -v''' = b(t) h(t, u), \\ u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta), \\ v(0) = v'(0) = 0, \quad v'(1) = \alpha v'(\eta). \end{cases} \tag{1.4}$$

In recent years, the existence question of positive radial solutions to boundary value problems for nonlinear elliptic partial differential equations or nonlinear elliptic systems have also attracted considerable attention. In [7], D.R. Dunninger et al. considered the existence of positive radial solutions for the elliptic system

$$\begin{aligned} &\Delta u + \lambda k_1(|x|) f(u, v) = 0, \quad R_1 < |x| < R_2, \\ &\Delta v + \lambda k_2(|x|) g(u, v) = 0, \quad R_1 < |x| < R_2, \end{aligned} \tag{1.5}$$

with the boundary conditions

$$\begin{aligned} &\alpha_1 u + \beta_1 \frac{\partial u}{\partial n} = \alpha_2 v + \beta_2 \frac{\partial v}{\partial n} = 0, \quad |x| = R_1, \\ &\gamma_1 u + \delta_1 \frac{\partial u}{\partial n} = \gamma_2 v + \delta_2 \frac{\partial v}{\partial n} = 0, \quad |x| = R_2. \end{aligned} \tag{1.6}$$

Based on the strong maximum principle, the authors obtained, under some suitable conditions, that there exists a positive number λ^* such that (1.5) and (1.6) admits two positive radial solutions for $0 < \lambda < \lambda^*$, one positive radial solution for $\lambda = \lambda^*$, and no positive radial solution for $\lambda > \lambda^*$.

Note that the orders $2m$ and $2n$ in (1.1) may be different. Such problems can be encountered in applied sciences (see, for example, [15]). In [18], Lü et al. considered the existence of multiple positive solutions for coupled singular differential ordinary equations

$$u^{(4)} = f(t, v), \qquad -v'' = g(t, u), \quad t \in (0, 1) \tag{1.7}$$

subject to the following boundary value conditions:

$$u(0) = u(1) = u''(0) = u''(1) = v(0) = v(1) = 0,$$

where $f, g \in C((0, 1) \times \mathbb{R}_+, \mathbb{R}_+)$. Su et al. considered a system of multi-point singular boundary value problems with possibly different orders, see [21]. Wei et al. discussed the existence of positive solutions of coupled boundary value problems for nonlinear ordinary differential equations, with one being of fourth order and the other being of second order, see [24].

However, the existence problem of positive solutions for systems of generalized Lidstone problems has not been extensively studied yet. To the best of our knowledge, only [13] is devoted to this direction. Our main difficulty here comes from the presence of derivatives of all even orders in the nonlinearities f and g in (1.1). To overcome this difficulty, as in [26], we first use the method of order reduction to transform (1.1) into an equivalent system of integro-integral equations, then prove the existence and multiplicity of positive solutions for the resulting system under appropriate conditions, thereby establishing our main results for (1.1). It is of interest to note that our nonlinearities f and g may grow both superlinearly and both sublinearly, and, more importantly, f and g are also allowed to grow in distinct manners, that is, one of them grows superlinearly and the other grows sublinearly. The main tool used in the proofs is fixed point index theory, combined with the a priori estimates of positive solutions. Our main results are formulated in terms of spectral radii of some related linear integral operators, and our a priori estimates for positive solutions are derived by developing some properties of positive concave functions and using Jensen's inequality. This, together with the fact that our nonlinearities may be of distinct growth, means that our methodology and main results here are entirely different from those in [18, 21, 24].

This paper is organized as follows. Section contains some preliminary results, including some basic facts recalled from [26]. The main results, namely Theorems 3.1–3.3, are stated and proved in Sect. 3. Finally, in Sect. 4, some examples are offered to illustrate our main results.

2 Preliminaries

Let

$$E := C([0,1],\mathbb{R}),\qquad \|u\| := \max_{t\in[0,1]}|u(t)|,\qquad P := \{u\in E:\ u(t)\ge 0,\ \forall t\in[0,1]\}.$$

Clearly, $(E,\|\cdot\|)$ is a real Banach space and P is a solid cone (see [14, p. 193]) on E. Let $\rho := ac+ad+bc>0$. It is easy to see that for any $f\in E$, $u\in C^2[0,1]$ solves the boundary value problem

$$\begin{cases} -u'' = f(t),\\ au(0)-bu'(0)=0,\\ cu(1)+du'(1)=0, \end{cases}$$

if and only if $u\in E$ can be represented by

$$u(t)=\int_0^1 k_1(t,s)f(s)\,ds,$$

where

$$k_1(t,s) := \frac{1}{\rho}\begin{cases} (b+as)(c+d-ct), & 0\le s\le t\le 1,\\ (b+at)(c+d-cs), & 0\le t\le s\le 1. \end{cases}\tag{2.1}$$

Define the completely continuous linear operators $L: E\to E$ by

$$(Lu)(t) := \int_0^1 k_1(t,s)u(s)\,ds,\qquad u\in E.\tag{2.2}$$

Then L is also a positive operator, i.e. $L(P)\subset P$. Let

$$k_i(t,s) := \int_0^1 k_1(t,\tau)k_{i-1}(\tau,s)\,d\tau\qquad (i=2,\dots).\tag{2.3}$$

It is easy to see that

$$(L^i u)(t) := \int_0^1 k_i(t,s)u(s)\,ds,\qquad u\in E$$

for $i = 1, 2, \ldots, n$. Moreover, L has the spectral radius $r(L) > 0$, whence $r(L^i) = r^i(L) > 0$ for each positive integer i. Let $\lambda_1 > 0$ be the first eigenvalue and $\varphi \in C^2[0,1] \cap P$ the associated eigenfunction of

$$\begin{cases} -u'' = \lambda u, \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0, \end{cases}$$

with $\int_0^1 \varphi(t)\,dt = 1$, which can be written in the form

$$\varphi(t) = \lambda_1 \int_0^1 k_1(t,s)\varphi(s)\,ds = \lambda_1 (L\varphi)(t). \tag{2.4}$$

Therefore $\lambda_1 = 1/r(L)$. Moreover, the symmetry of $k_1(t,s)$ implies that

$$\varphi(s) = \lambda_1^i \int_0^1 k_i(t,s)\varphi(t)\,dt = \lambda_1^i (L^i\varphi)(s) \tag{2.5}$$

for $i = 1, 2, \ldots$.

Lemma 2.1 (See [25, Lemma 2]) *Let k_1 be defined by (2.1) and*

$$h(t) := \frac{1}{M} \min\{a + bt, c + d - ct\}, \tag{2.6}$$

where $M := \max\{a + b, c + d\}$. Then

$$k_1(t,s) \geq h(t)k_1(\tau,s), \quad \forall t, s, \tau \in [0,1]. \tag{2.7}$$

Let h be given by (2.6) and φ by (2.4). Put $\omega := \int_0^1 h(t)\varphi(t)\,dt > 0$ and

$$P_0 := \left\{ u \in P: \int_0^1 \varphi(t)u(t)\,dt \geq \omega\|u\| \right\}. \tag{2.8}$$

Clearly, P_0 is also a cone on E.

Lemma 2.2 (See [27]). $L(P) \subset P_0$.

Lemma 2.3 (See [9]) *Suppose $\Omega \subset E$ is a bounded open set, $A: \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{0\}$ such that $u - Au \neq \lambda u_0$ for all $u \in \partial\Omega \cap P$ and $\lambda \geq 0$, then $i(A, \Omega \cap P, P) = 0$ where i indicates the fixed point index on P.*

Lemma 2.4 (See [9]) *Let $\Omega \in E$ be a bounded open set with $0 \in \Omega$. Suppose $A: \overline{\Omega} \cap P \to P$ is a completely continuous operator. If $u \neq \lambda Au$ for all $u \in \partial\Omega \cap P$ and $0 \leq \lambda \leq 1$, then $i(A, \Omega \cap P, P) = 1$.*

Lemma 2.5 *If $p:\mathbb{R}_+\to\mathbb{R}_+$ is concave, then p is nondecreasing. In addition, if there exist $0\le x_1<x_2$ such that $p(x_1)=p(x_2)$, then*

$$p(x)\equiv p(x_1)=p(x_2),\quad \forall x\ge x_1. \tag{2.9}$$

Moreover, the following inequality holds:

$$p(a+b)\le p(a)+p(b),\quad \forall a,b\in\mathbb{R}_+. \tag{2.10}$$

Proof For any $x_2>x_1\ge 0$, the concavity of p implies

$$p(x)\le p(x_2)+\frac{p(x_2)-p(x_1)}{x_2-x_1}(x-x_2) \tag{2.11}$$

for all $x>x_2$ and thus $p(x_1)\le p(x_2)$ by nonnegativity of p. In addition, if $p(x_1)=p(x_2)$, then (2.9) holds, as is seen from (2.11). The proof of (2.10) can be found in [27, Lemma 5]. This completes the proof. □

3 Existence and multiplicity of positive solutions

Let $u(t)=(-1)^{m-1}x^{(2m-2)}$, $v(t)=(-1)^{n-1}y^{(2n-2)}$. It is easy to see that (1.1) is equivalent to the system of integro-ordinary differential equations

$$\begin{cases} -u''(t)=f\Big(t,\int_0^1 k_{m-1}(t,s)u(s)\,ds,\dots,\int_0^1 k_1(t,s)u(s)\,ds,u(t),\\ \qquad\qquad \int_0^1 k_{n-1}(t,s)v(s)\,ds,\dots,\int_0^1 k_1(t,s)v(s)\,ds,v(t)\Big),\\ -v''(t)=g\Big(t,\int_0^1 k_{m-1}(t,s)u(s)\,ds,\dots,\int_0^1 k_1(t,s)u(s)\,ds,u(t),\\ \qquad\qquad \int_0^1 k_{n-1}(t,s)v(s)\,ds,\dots,\int_0^1 k_1(t,s)v(s)\,ds,v(t)\Big), \end{cases} \tag{3.1}$$

subject to the boundary conditions

$$au(0)-bu'(0)=cu(1)+du'(1)=0,\qquad av(0)-bv'(0)=cv(1)+dv'(1)=0.$$

Furthermore, the above system can be written in the form

$$\begin{cases} u(t)=\int_0^1 k_1(t,s)f\Big(s,\int_0^1 k_{m-1}(s,\tau)u(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)u(\tau)\,d\tau,u(s),\\ \qquad\qquad \int_0^1 k_{n-1}(s,\tau)v(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)v(\tau)\,d\tau,v(s)\Big)\,ds,\\ v(t)=\int_0^1 k_1(t,s)g\Big(s,\int_0^1 k_{m-1}(s,\tau)u(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)u(\tau)\,d\tau,u(s),\\ \qquad\qquad \int_0^1 k_{n-1}(s,\tau)v(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)v(\tau)\,d\tau,v(s)\Big)\,ds. \end{cases} \tag{3.2}$$

Define the operators A_i $(i=1,2):P\to P$ and $A:P\times P\to P\times P$ by

$$\begin{aligned}
&A_1(u,v)(t)\\
&\quad=\int_0^1 k_1(t,s)f\bigg(s,\int_0^1 k_{m-1}(s,\tau)u(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)u(\tau)\,d\tau,u(s),\\
&\qquad\qquad\int_0^1 k_{n-1}(s,\tau)v(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)v(\tau)\,d\tau,v(s)\bigg)\,ds,\\
&A_2(u,v)(t)\\
&\quad=\int_0^1 k_1(t,s)g\bigg(s,\int_0^1 k_{m-1}(s,\tau)u(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)u(\tau)\,d\tau,u(s),\\
&\qquad\qquad\int_0^1 k_{n-1}(s,\tau)v(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)v(\tau)\,d\tau,v(s)\bigg)\,ds,\\
&A(u,v)(t)=(A_1(u,v)(t),A_2(u,v)(t)).
\end{aligned}$$

Now $f\in C([0,1]\times\mathbb{R}_+^{m+n},\mathbb{R}_+)$ and $g\in C([0,1]\times\mathbb{R}_+^{m+n},\mathbb{R}_+)$ imply that A_i and A are completely continuous operators, and the existence of positive solutions for (3.2) is equivalent to that of positive fixed points of A. Let

$$\begin{aligned}
F_1(u,v)(s)=f\bigg(&s,\int_0^1 k_{m-1}(s,\tau)u(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)u(\tau)\,d\tau,u(s),\\
&\int_0^1 k_{n-1}(s,\tau)v(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)v(\tau)\,d\tau,v(s)\bigg),\\
F_2(u,v)(s)=g\bigg(&s,\int_0^1 k_{m-1}(s,\tau)u(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)u(\tau)\,d\tau,u(s),\\
&\int_0^1 k_{n-1}(s,\tau)v(\tau)\,d\tau,\dots,\int_0^1 k_1(s,\tau)v(\tau)\,d\tau,v(s)\bigg).
\end{aligned}$$

Then $F_i: P\to P$ $(i=1,2)$ are continuous and bounded. Since

$$A_i(u,v)(t)=\int_0^1 k_1(t,s)(F_i(u,v))(s)\,ds=(LF_i(u,v))(t),\quad i=1,2,$$

it follows, by Lemma 2.2, that $A_i(P)\subset P_0$. Let $N:=\max\{k_1(t,s)\colon 0\le t,\ s\le 1\}>0$ and $K:=\max\{N^i\colon i=0,1,\dots,n-1\}>0$. Clearly, $k_i(t,s)\le N^i$ for any $0\le t$, $s\le 1$ and $i=1,2,\dots,n$.

For the reason of notational brevity, we denote by $x:=(x_1,x_2,\dots,x_m)\in\mathbb{R}_+^m$ and $y:=(y_1,y_2,\dots,y_n)\in\mathbb{R}_+^n$. We now list our hypotheses.

(H1) There exist $p_1\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $q_1\in C(\mathbb{R}_+,\mathbb{R}_+)$ such that

(1) p_1 is concave on $\mathbb{R}_+$,

(2) there exist $\alpha_{1i}\ge 0$, $\beta_{1j}\ge 0$ $(i=1,2,\dots,m,\ j=1,2,\dots,n)$ and $C>0$ such that

$$\sum_{i=1}^m\sum_{j=1}^n \alpha_{1i}\beta_{1j}\lambda_1^{m+n-i-j+2}>1,$$

$$f(t,x,y) \geq p_1\left(\sum_{j=1}^{n}\beta_{1j}y_j\right) - C, \quad \forall (t,x,y) \in [0,1]\times \mathbb{R}_+^{m+n},$$

$$g(t,x,y) \geq q_1\left(\sum_{i=1}^{m}\alpha_{1i}x_i\right) - C, \quad \forall (t,x,y) \in [0,1]\times \mathbb{R}_+^{m+n},$$

and

$$p_1\left(\sum_{j=1}^{n}\beta_{1j}N^{n-j+1}q_1(z)\right) \geq \sum_{j=1}^{n}\beta_{1j}N^{n-j+1}z - C, \quad \forall z \in \mathbb{R}_+.$$

(H2) There exist $\xi_{1i} \geq 0$, $\eta_{1j} \geq 0$, $\gamma_{1i} \geq 0$, $\delta_{1j} \geq 0$ $(i = 1,2,\ldots,m,\ j = 1,2,\ldots,n)$ and $r_1 > 0$ such that

$$f(t,x,y) \leq \sum_{i=1}^{m}\xi_{1i}x_i + \sum_{j=1}^{n}\eta_{1j}y_j, \quad \forall (t,x,y) \in [0,1]\times \underbrace{[0,r_1]\times\cdots\times[0,r_1]}_{n+m},$$

$$g(t,x,y) \leq \sum_{i=1}^{m}\gamma_{1i}x_i + \sum_{j=1}^{n}\delta_{1j}y_j, \quad \forall (t,x,y) \in [0,1]\times \underbrace{[0,r_1]\times\cdots\times[0,r_1]}_{n+m},$$

and

$$\kappa_1 > 0, \quad \kappa_4 > 0, \quad \kappa := \kappa_1\kappa_4 - \kappa_2\kappa_3 > 0,$$

where

$$\kappa_1 := 1 - \sum_{i=1}^{m}\xi_{1i}\lambda_1^{m-i+1}, \qquad \kappa_2 := \sum_{j=1}^{n}\eta_{1j}\lambda_1^{n-j+1},$$

$$\kappa_3 := \sum_{i=1}^{m}\gamma_{1i}\lambda_1^{m-i+1}, \qquad \kappa_4 := 1 - \sum_{j=1}^{n}\delta_{1j}\lambda_1^{n-j+1}.$$

(H3) There exist $p_2 \in C(\mathbb{R}_+,\mathbb{R}_+)$ and $q_2 \in C(\mathbb{R}_+,\mathbb{R}_+)$ such that
(1) p_2 is concave on $\mathbb{R}_+$,
(2) there exist $\alpha_{2i} \geq 0$, $\beta_{2j} \geq 0$ $(i = 1,2,\ldots,m,\ j = 1,2,\ldots,n)$ and $r_2 > 0$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{2i}\beta_{2j}\lambda_1^{m+n-i-j+2} > 1,$$

$$f(t,x,y) \geq p_2\left(\sum_{j=1}^{n}\beta_{2j}y_j\right), \quad \forall (t,x,y) \in [0,1]\times \underbrace{[0,r_2]\times\cdots\times[0,r_2]}_{n},$$

$$g(t,x,y) \geq q_2\left(\sum_{i=1}^{m}\alpha_{2i}x_i\right), \quad \forall (t,x,y) \in [0,1]\times \underbrace{[0,r_2]\times\cdots\times[0,r_2]}_{m},$$

and

$$p_2\left(\sum_{j=1}^{n}\beta_{2j}N^{n-j+1}q_2(z)\right)\geq\sum_{j=1}^{n}\beta_{2j}N^{n-j+1}z,\quad\forall z\in[0,r_2].$$

(H4) There exist $\xi_{2i}\geq 0$, $\eta_{2j}\geq 0$, $\gamma_{2i}\geq 0$, $\delta_{2j}\geq 0$ ($i=1,2,\ldots,m$, $j=1,2,\ldots,n$) and $C>0$ such that

$$f(t,x,y)\leq\sum_{i=1}^{m}\xi_{2i}x_i+\sum_{j=1}^{n}\eta_{2j}y_j+C,\quad\forall(t,x,y)\in[0,1]\times\mathbb{R}_+^{m+n},$$

$$g(t,x,y)\leq\sum_{i=1}^{m}\gamma_{2i}x_i+\sum_{j=1}^{n}\delta_{2j}y_j+C,\quad\forall(t,x,y)\in[0,1]\times\mathbb{R}_+^{m+n},$$

and

$$\tilde{\kappa}_1>0,\quad\tilde{\kappa}_4>0,\quad\tilde{\kappa}:=\tilde{\kappa}_1\tilde{\kappa}_4-\tilde{\kappa}_2\tilde{\kappa}_3>0,$$

where

$$\tilde{\kappa}_1:=1-\sum_{i=1}^{m}\xi_{2i}\lambda_1^{m-i+1},\qquad\tilde{\kappa}_2:=\sum_{j=1}^{n}\eta_{2j}\lambda_1^{n-j+1},$$

$$\tilde{\kappa}_3:=\sum_{i=1}^{m}\gamma_{2i}\lambda_1^{m-i+1},\qquad\tilde{\kappa}_4:=1-\sum_{j=1}^{n}\delta_{2j}\lambda_1^{n-j+1}.$$

(H5) $f(t,x,y)$ and $g(t,x,y)$ are nondecreasing in x_i ($i=1,2,\ldots,m$) and y_j ($j=1,2,\ldots,n$), and there exists $\Lambda>0$ such that

$$f(t,\Lambda,\Lambda,\ldots,\Lambda)<\frac{\Lambda}{\mu},\qquad g(t,\Lambda,\Lambda,\ldots,\Lambda)<\frac{\Lambda}{\mu},\quad\forall t\in[0,1],$$

where $\mu:=\max_{t\in[0,1]}\int_0^1 k_1(t,s)\,ds>0$.

As noted in Sect. 2, $\lambda_1=1/r(L)$. We adopt the convention in the sequel that $C_1,C_2,\ldots$ stand for different positive constants.

Theorem 3.1 *If* (H1) *and* (H2) *hold, then* (1.1) *has at least one positive solution* $(u,v)\in(C^{2n}[0,1]\times C^{2n}[0,1])\cap(P\times P)\setminus\{0\}$.

Proof It suffices to prove that (3.2) has at least one positive solution. Indeed, by (H1) we have for any $(u,v)\in P\times P$ and $t\in[0,1]$

$$A_1(u,v)(t)\geq\int_0^1 k_1(t,s)p_1\left(\sum_{j=1}^{n-1}\beta_{1j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\beta_{1n}v(s)\right)ds-C_1\tag{3.3}$$

and

$$A_2(u,v)(t) \geq \int_0^1 k_1(t,s)q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau+\alpha_{1m}u(s)\right)ds - C_1. \tag{3.4}$$

We claim that the set

$$\mathcal{M}_1 := \left\{(u,v)\in P\times P\colon\ (u,v)=A(u,v)+\lambda(\varphi,\varphi), \lambda\geq 0\right\}$$

is bounded in $P\times P$, where φ is determined by (2.4). Indeed, $(u,v)\in\mathcal{M}_1$ implies $u(t)\geq A_1(u,v)(t)$ and $v(t)\geq A_2(u,v)(t)$. By (3.3) and (3.4), we have

$$u(t)\geq \int_0^1 k_1(t,s)p_1\left(\sum_{j=1}^{n-1}\beta_{1j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\beta_{1n}v(s)\right)ds - C_1 \tag{3.5}$$

and

$$v(t)\geq \int_0^1 k_1(t,s)q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau+\alpha_{1m}u(s)\right)ds - C_1. \tag{3.6}$$

Note that $p_1:\mathbb{R}_+\to\mathbb{R}_+$ is concave and $\sum_{j=1}^n\beta_{1j}k_{n-j+1}(s,\tau)/(\sum_{j=1}^n\beta_{1j}N^{n-j+1})$ $\in[0,1]$ holds for every $(s,\tau)\in[0,1]\times[0,1]$. As a result, we have

$$p_1\left(\frac{\sum_{j=1}^n\beta_{1j}k_{n-j+1}(s,\tau)}{\sum_{j=1}^n\beta_{1j}N^{n-j+1}}z\right)\geq\frac{\sum_{j=1}^n\beta_{1j}k_{n-j+1}(s,\tau)}{\sum_{j=1}^n\beta_{1j}N^{n-j+1}}p_1(z) \tag{3.7}$$

for all $z\in\mathbb{R}_+$ and s,τ $[0,1]$. Now (3.5), (3.6) and (3.7), together with Jensen's inequality, imply

$$\begin{aligned}
&p_1\left(\sum_{j=1}^{n-1}\beta_{1j}\int_0^1 k_{n-j}(s,\tau)(v(\tau)+C_1)\,d\tau+\beta_{1n}(v(s)+C_1)\right)\\
&\quad\geq p_1\left(\sum_{j=1}^{n-1}\beta_{1j}\int_0^1 k_{n-j}(s,\tau)\left[\int_0^1 k_1(\tau,\varsigma)\right.\right.\\
&\qquad\left.\times q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(\varsigma,\zeta)u(\zeta)\,d\zeta+\alpha_{1m}u(\varsigma)\right)d\varsigma\right]d\tau\\
&\qquad\left.+\beta_{1n}\int_0^1 k_1(s,\tau)q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(\tau,\varsigma)u(\varsigma)\,d\varsigma+\alpha_{1m}u(\tau)\right)d\tau\right)\\
&\quad\geq p_1\left(\sum_{j=1}^{n}\beta_{1j}\int_0^1 k_{n-j+1}(s,\tau)\right.
\end{aligned}$$

$$\times q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(\tau,\varsigma)u(\varsigma)\,d\varsigma+\alpha_{1m}u(\tau)\right)d\tau\Bigg)$$

$$\geq \frac{1}{\sum_{j=1}^n\beta_{1j}N^{n-j+1}}\sum_{j=1}^n\beta_{1j}\int_0^1 k_{n-j+1}(s,\tau)p_1$$

$$\left(\sum_{j=1}^n\beta_{1j}N^{n-j+1}q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(\tau,\varsigma)u(\varsigma)\,d\varsigma\right.\right.$$

$$\left.\left.+\alpha_{1m}u(\tau)\right)\right)d\tau \tag{3.8}$$

for every $(u,v)\in\mathcal{M}_1$. So Lemma 2.5 implies

$$p_1\left(\sum_{j=1}^{n-1}\beta_{1j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\beta_{1n}v(s)\right)$$

$$\geq p_1\left(\sum_{j=1}^{n-1}\beta_{1j}\int_0^1 k_{n-j}(s,\tau)(v(\tau)+C_1)\,d\tau+\beta_{1n}(v(s)+C_1)\right)$$

$$-p_1\left(C_1\sum_{j=1}^{n-1}\beta_{1j}\int_0^1 k_{n-j}(s,\tau)\,d\tau+C_1\beta_{1n}\right)$$

$$\geq \frac{1}{\sum_{j=1}^n\beta_{1j}N^{n-j+1}}\sum_{j=1}^n\beta_{1j}\int_0^1 k_{n-j+1}(s,\tau)p_1\left(\sum_{j=1}^n\beta_{1j}N^{n-j+1}\right.$$

$$\left.\times q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(\tau,\varsigma)u(\varsigma)\,d\varsigma+\alpha_{1m}u(\tau)\right)\right)d\tau-C_2 \tag{3.9}$$

for every $(u,v)\in\mathcal{M}_1$. In view of (3.5), we have for every $(u,v)\in\mathcal{M}_1$

$$u(t)\geq\int_0^1 k_1(t,s)\Bigg[\frac{1}{\sum_{j=1}^n\beta_{1j}N^{n-j+1}}\sum_{j=1}^n\beta_{1j}\int_0^1 k_{n-j+1}(s,\tau)p_1\Bigg(\sum_{j=1}^n\beta_{1j}N^{n-j+1}$$

$$\times q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(\tau,\varsigma)u(\varsigma)\,d\varsigma+\alpha_{1m}u(\tau)\right)\Bigg)d\tau-C_2\Bigg]ds-C_1$$

$$\geq \frac{1}{\sum_{j=1}^n\beta_{1j}N^{n-j+1}}\sum_{j=1}^n\beta_{1j}\int_0^1 k_{n-j+2}(t,\tau)p_1\Bigg(\sum_{j=1}^n\beta_{1j}N^{n-j+1}$$

$$\times q_1\left(\sum_{i=1}^{m-1}\alpha_{1i}\int_0^1 k_{m-i}(\tau,\varsigma)u(\varsigma)\,d\varsigma+\alpha_{1m}u(\tau)\right)\Bigg)d\tau-C_3. \tag{3.10}$$

Now (H1) implies

$$u(t) \geq \frac{1}{\sum_{j=1}^{n} \beta_{1j} N^{n-j+1}} \sum_{j=1}^{n} \beta_{1j} \int_0^1 k_{n-j+2}(t,\tau) \Bigg(\sum_{j=1}^{n} \beta_{1j} N^{n-j+1} \\ \times \Bigg(\sum_{i=1}^{m-1} \alpha_{1i} \int_0^1 k_{m-i}(\tau,\varsigma) u(\varsigma)\, d\varsigma + \alpha_{1m} u(\tau) \Bigg) - C \Bigg) d\tau - C_3 \\ \geq \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{1i} \beta_{1j} \int_0^1 k_{m+n-i-j+2}(t,s) u(s)\, ds - C_4. \tag{3.11}$$

Multiply by $\varphi(t)$ on both sides of the above and integrate over [0, 1] and use (2.5) to obtain

$$\int_0^1 \varphi(t) u(t)\, dt \geq \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{1i} \beta_{1j} \lambda_1^{m+n-i-j+2} \int_0^1 \varphi(t) u(t)\, dt - C_4. \tag{3.12}$$

Consequently,

$$\int_0^1 \varphi(t) u(t)\, dt \leq \frac{C_4}{\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{1i} \beta_{1j} \lambda_1^{m+n-i-j+2} - 1} \tag{3.13}$$

for every $(u, v) \in \mathcal{M}_1$. Note that, by Lemma 2.1, $(u, v) \in \mathcal{M}_1$ implies $u \in P_0$ and $v \in P_0$. We then have for every $(u, v) \in \mathcal{M}_1$

$$\|u\| \leq \frac{C_4}{\sum_{i=1}^{m} \sum_{j=1}^{n} \omega \alpha_{1i} \beta_{1j} \lambda_1^{m+n-i-j+2} - \omega}. \tag{3.14}$$

Without loss of generality, we may assume $v \not\equiv 0$ for each $(u, v) \in \mathcal{M}_1$. Note $\int_0^1 \varphi(t)\, dt = 1$. By (3.5) and (2.5), we obtain

$$\|u\| \geq \int_0^1 \varphi(t) u(t)\, dt \\ \geq \int_0^1 \varphi(t) \Bigg(\int_0^1 k_1(t,s) \\ \times p_1 \Bigg(\sum_{j=1}^{n-1} \beta_{1j} \int_0^1 k_{n-j}(s,\tau) v(\tau)\, d\tau + \beta_{1n} v(s) \Bigg) ds - C_1 \Bigg) dt \\ \geq \int_0^1 \varphi(t) \\ \times \Bigg(\int_0^1 k_1(t,s) p_1 \Bigg(\sum_{j=1}^{n-1} \beta_{1j} \int_0^1 k_{n-j}(s,\tau) v(\tau)\, d\tau + \beta_{1n} v(s) \Bigg) ds \Bigg) dt - C_1$$

$$\geq \frac{p_1(\sum_{j=1}^n \beta_{1j} N^{n-j}\|v\|)}{\sum_{j=1}^n \beta_{1j} N^{n-j}\|v\|} \int_0^1 \varphi(t)\bigg(\int_0^1 k_1(t,s)$$

$$\times \bigg(\sum_{j=1}^{n-1} \beta_{1j} \int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau + \beta_{1n}v(s)\bigg)\,ds\bigg)\,dt - C_1$$

$$= \frac{p_1(\sum_{j=1}^n \beta_{1j} N^{n-j}\|v\|)}{\sum_{j=1}^n \beta_{1j} N^{n-j}\|v\|} \sum_{j=1}^n \beta_{1j}\lambda_1^{n-j+1} \int_0^1 \varphi(t)v(t)\,dt - C_1$$

$$\geq \frac{p_1(\sum_{j=1}^n \beta_{1j} N^{n-j}\|v\|)}{\sum_{j=1}^n \beta_{1j} N^{n-j}} \sum_{j=1}^n \omega\beta_{1j}\lambda_1^{n-j+1} - C_1. \tag{3.15}$$

Therefore,

$$p_1\bigg(\sum_{j=1}^n \beta_{1j} N^{n-j}\|v\|\bigg) \leq \frac{\sum_{j=1}^n \beta_{1j} N^{n-j}(\|u\| + C_1)}{\sum_{j=1}^n \omega\beta_{1j}\lambda_1^{n-j+1}},$$

for all $(u,v) \in \mathcal{M}_1$. By (H1) we have $\lim_{z\to+\infty} p(z) = +\infty$, and thus there exists $C_5 > 0$ such that

$$\|v\| \leq C_5. \tag{3.16}$$

We find from (3.14) and (3.16) that $\mathcal{M}_1$ is bounded. Taking $R > \sup\{\|(u,v)\|\colon (u,v) \in \mathcal{M}_1\}$, we have

$$(u,v) \neq A(u,v) + \lambda(\varphi,\varphi), \quad \forall (u,v) \in \partial\Omega_R \cap (P \times P),\ \lambda \geq 0.$$

Now Lemma 2.3 yields

$$i(A, \Omega_R \cap (P \times P), P \times P) = 0. \tag{3.17}$$

Let $\rho_1 := r_1/K$ and

$$\mathcal{M}_2 := \Big\{(u,v) = \lambda A(u,v)\colon (u,v) \in \partial\Omega_{\rho_1} \cap (P \times P),\ 0 \leq \lambda \leq 1\Big\}.$$

We claim $\mathcal{M}_2 = \{0\}$. Indeed, by (H2), we have for all $(u,v) \in \overline{\Omega}_{\rho_1} \cap (P \times P)$ and $t \in [0,1]$

$$A_1(u,v)(t) \leq \int_0^1 k_1(t,s)\bigg[\sum_{i=1}^{m-1} \xi_{1i} \int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau + \xi_{1m}u(s)$$

$$+ \sum_{j=1}^{n-1} \eta_{1j} \int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau + \eta_{1n}v(s)\bigg]\,ds$$

$$\leq \sum_{i=1}^m \xi_{1i} \int_0^1 k_{m-i+1}(t,s)u(s)\,ds$$

$$+\sum_{j=1}^{n}\eta_{1j}\int_0^1 k_{n-j+1}(t,s)v(s)\,ds, \tag{3.18}$$

and

$$\begin{aligned}A_2(u,v)(t) &\le \int_0^1 k_1(t,s)\Big[\sum_{i=1}^{m-1}\gamma_{1i}\int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau+\gamma_{1m}u(s)\\&\quad+\sum_{j=1}^{n-1}\delta_{1j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\delta_{1n}v(s)\Big]ds\\&\le \sum_{i=1}^{m}\gamma_{1i}\int_0^1 k_{m-i+1}(t,s)u(s)\,ds\\&\quad+\sum_{j=1}^{n}\delta_{1j}\int_0^1 k_{n-j+1}(t,s)v(s)\,ds.\end{aligned} \tag{3.19}$$

If $(u,v)\in\mathcal{M}_2$, then (3.18) and (3.19) imply

$$\begin{aligned}u(t) &\le \int_0^1 k_1(t,s)\Big[\sum_{i=1}^{m-1}\xi_{1i}\int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau+\xi_{1m}u(s)\\&\quad+\sum_{j=1}^{n-1}\eta_{1j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\eta_{1n}v(s)\Big]ds\\&\le \sum_{i=1}^{m}\xi_{1i}\int_0^1 k_{m-i+1}(t,s)u(s)\,ds\\&\quad+\sum_{j=1}^{n}\eta_{1j}\int_0^1 k_{n-j+1}(t,s)v(s)\,ds\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}v(t) &\le \int_0^1 k_1(t,s)\Big[\sum_{i=1}^{m-1}\gamma_{1i}\int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau+\gamma_{1m}u(s)\\&\quad+\sum_{j=1}^{n-1}\delta_{1j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\delta_{1n}v(s)\Big]ds\\&\le \sum_{i=1}^{m}\gamma_{1i}\int_0^1 k_{m-i+1}(t,s)u(s)\,ds\end{aligned}$$

$$+\sum_{j=1}^{n}\delta_{1j}\int_0^1 k_{n-j+1}(t,s)v(s)\,ds. \tag{3.21}$$

Multiply by $\varphi(t)$ on both sides of the preceding inequalities and integrate over [0, 1] and use (2.5) to obtain

$$\int_0^1 \varphi(t)u(t)\,dt \le \sum_{i=1}^{m}\xi_{1i}\lambda_1^{m-i+1}\int_0^1 \varphi(t)u(t)\,dt + \sum_{j=1}^{n}\eta_{1j}\lambda_1^{n-j+1}\int_0^1 \varphi(t)v(t)\,dt \tag{3.22}$$

and

$$\int_0^1 \varphi(t)v(t)\,dt \le \sum_{i=1}^{m}\gamma_{1i}\lambda_1^{m-i+1}\int_0^1 \varphi(t)u(t)\,dt + \sum_{j=1}^{n}\delta_{1j}\lambda_1^{n-j+1}\int_0^1 \varphi(t)v(t)\,dt, \tag{3.23}$$

which can be written in the form

$$\begin{bmatrix}\kappa_1 & -\kappa_2\\ -\kappa_3 & \kappa_4\end{bmatrix}\begin{bmatrix}\int_0^1 \varphi(t)u(t)\,dt\\ \int_0^1 \varphi(t)v(t)\,dt\end{bmatrix} \le \begin{bmatrix}0\\0\end{bmatrix}.$$

Now (H2) implies

$$\begin{bmatrix}\int_0^1 \varphi(t)u(t)\,dt\\ \int_0^1 \varphi(t)v(t)\,dt\end{bmatrix} \le \frac{1}{\kappa}\begin{bmatrix}\kappa_4 & \kappa_2\\ \kappa_3 & \kappa_1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

so that $\int_0^1 \varphi(t)u(t)\,dt = \int_0^1 \varphi(t)v(t)\,dt = 0$ and therefore $u \equiv 0$, $v \equiv 0$. This proves $\mathcal{M}_2 = \{0\}$, as claimed. As a result, we have

$$(u,v) \ne \lambda A(u,v), \quad (u,v) \in \partial\Omega_{\rho_1} \cap (P \times P), \quad 0 \le \lambda \le 1$$

and by Lemma 2.4

$$i(A, \partial\Omega_{\rho_1} \cap (P \times P), P \times P) = 1. \tag{3.24}$$

Combining (3.17) and (3.24) gives

$$i(A, (\Omega_R \backslash \overline{\Omega}_{\rho_1}) \cap (P \times P), P \times P) = 0 - 1 = -1.$$

Hence the operator A has at least one fixed point on $(\Omega_R \backslash \overline{\Omega}_{\rho_1}) \cap (P \times P)$ and (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.2 *If* (H3) *and* (H4) *hold*, *then* (1.1) *has at least one positive solution* $(u, v) \in (C^{2n}[0,1] \times C^{2n}[0,1]) \cap (P \times P) \setminus \{0\}$.

Proof It suffices to prove that (3.2) has at least one positive solution. Indeed, (H3) implies

$$A_1(u,v)(t) \geq \int_0^1 k_1(t,s) \times p_2\left(\sum_{j=1}^{n-1} \beta_{2j} \int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau + \beta_{2n}v(s)\right) ds \quad (3.25)$$

and

$$A_2(u,v)(t) \geq \int_0^1 k_1(t,s) q_2\left(\sum_{i=1}^{m-1} \alpha_{2i} \int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau + \alpha_{2m}u(s)\right) ds \quad (3.26)$$

for any $(u, v) \in P \times P$ and $t \in [0, 1]$. Let

$$\mathcal{M}_3 := \{(u,v) = A(u,v) + \lambda(\varphi,\varphi),\ (u,v) \in \partial\Omega_{\rho_2} \cap (P \times P),\ 0 \leq \lambda \leq 1\}, \quad (3.27)$$

where $\rho_2 = r_2/K$. We claim that $\mathcal{M}_3 \subset \{0\}$. Indeed, if $(u, v) \in \mathcal{M}_3$, then (3.25) and (3.26) imply

$$u(t) \geq \int_0^1 k_1(t,s) p_2\left(\sum_{j=1}^{n-1} \beta_{2j} \int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau + \beta_{2n}v(s)\right) ds \quad (3.28)$$

and

$$v(t) \geq \int_0^1 k_1(t,s) q_2\left(\sum_{i=1}^{m-1} \alpha_{2i} \int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau + \alpha_{2m}u(s)\right) ds. \quad (3.29)$$

Now the same argument as used in deriving (3.11) can be used to show

$$\begin{aligned} u(t) \geq \int_0^1 k_1(t,s) p_2\Bigg(&\sum_{j=1}^{n-1} \beta_{2j} \int_0^1 k_{n-j}(s,\tau)\Bigg[\int_0^1 k_1(\tau,\varsigma) \\ &\times q_2\left(\sum_{i=1}^{m-1} \alpha_{2i} \int_0^1 k_{m-i}(\varsigma,\zeta)u(\zeta)\,d\zeta + \alpha_{2m}u(\varsigma)\right) d\varsigma\Bigg] d\tau \\ &+ \beta_{2n}\Bigg[\int_0^1 k_1(s,\tau) q_2\left(\sum_{i=1}^{m-1} \alpha_{2j} \int_0^1 k_{m-i}(\tau,\varsigma)u(\varsigma)\,d\varsigma + \alpha_{2m}u(\tau)\right) d\tau\Bigg]\Bigg) ds \end{aligned}$$

$$\geq \int_0^1 k_1(t,s)p_2\bigg(\sum_{j=1}^{n}\beta_{2j}\int_0^1 k_{n-j+1}(s,\tau)$$

$$\times q_2\bigg(\sum_{i=1}^{m-1}\alpha_{2j}\int_0^1 k_{m-i}(\tau,\varsigma)u(\varsigma)\,d\varsigma+\alpha_{2m}u(\tau)\bigg)\bigg)\,d\tau$$

$$\geq \frac{1}{\sum_{j=1}^{n}\beta_{2j}N^{n-j+1}}\sum_{j=1}^{n}\beta_{2j}\int_0^1 k_{n-j+2}(t,s)p_2\bigg(\sum_{j=1}^{n}\beta_{2j}N^{n-j+1}$$

$$\times q_2\bigg(\sum_{i=1}^{m-1}\alpha_{2i}\int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau+\alpha_{2m}u(s)\bigg)\bigg)\,ds$$

$$\geq \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{2i}\beta_{2j}\int_0^1 k_{m+n-i-j+2}(t,s)u(s)\,ds \tag{3.30}$$

for every $(u,v)\in\mathcal{M}_3$. Multiply by $\varphi(t)$ on both sides of the above and integrate over $[0,1]$ and use (2.5) to obtain

$$\int_0^1\varphi(t)u(t)\,dt\geq\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{2i}\beta_{2j}\lambda_1^{m+n-i-j+2}\int_0^1\varphi(t)u(t)\,dt, \tag{3.31}$$

so that $\int_0^1\varphi(t)u(t)\,dt=0$, whence $u(t)\equiv 0$. By (3.28), we have

$$\int_0^1 k_1(t,s)p_2\bigg(\sum_{j=1}^{n-1}\beta_{2j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\beta_{2n}v(s)\bigg)\,ds\leq 0,$$

and hence $p_2(\sum_{j=1}^{n-1}\beta_{2j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\beta_{2n}v(s))\equiv 0$ and $v(t)\equiv 0$ by Lemma 2.5. This proves that $\mathcal{M}_3\subset\{0\}$, as claimed. As a result, we have

$$(u,v)\neq A(u,v)+\lambda(\varphi,\varphi),\quad (u,v)\in\partial\Omega_{\rho_2}\cap(P\times P),\quad 0\leq\lambda\leq 1$$

and by Lemma 2.3

$$i(A,\partial\Omega_{\rho_2}\cap(P\times P),P\times P)=0. \tag{3.32}$$

On the other hand, by (H4), we have for all $(u,v)\in P\times P$

$$A_1(u,v)(t)\leq\int_0^1 k_1(t,s)\bigg[\sum_{i=1}^{m-1}\xi_{2i}\int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau+\xi_{2m}u(s)$$

$$+\sum_{j=1}^{n-1}\eta_{2j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\eta_{2n}v(s)\bigg]\,ds+C_6$$

$$\leq \sum_{i=1}^{m} \xi_{2i} \int_0^1 k_{m-i+1}(t,s)u(s)\,ds$$

$$+ \sum_{j=1}^{n} \eta_{2j} \int_0^1 k_{n-j+1}(t,s)v(s)\,ds + C_6 \tag{3.33}$$

and

$$A_2(u,v)(t) \leq \int_0^1 k_1(t,s)\Bigg[\sum_{i=1}^{m-1} \gamma_{2i} \int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau + \gamma_{2m}u(s)$$

$$+ \sum_{j=1}^{n-1} \delta_{2j} \int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau + \delta_{2n}v(s)\Bigg]ds + C_6$$

$$\leq \sum_{i=1}^{m} \gamma_{2i} \int_0^1 k_{m-i+1}(t,s)u(s)\,ds$$

$$+ \sum_{j=1}^{n} \delta_{2j} \int_0^1 k_{n-j+1}(t,s)v(s)\,ds + C_6. \tag{3.34}$$

Let

$$\mathcal{M}_4 := \Big\{(u,v) \in P \times P\colon\ (u,v) = \lambda A(u,v),\ 0 \leq \lambda \leq 1\Big\}.$$

We claim that $\mathcal{M}_4$ is bounded. Indeed, if $(u,v) \in \mathcal{M}_4$, then (3.33) and (3.34) imply

$$u(t) \leq \int_0^1 k_1(t,s)\Bigg[\sum_{i=1}^{m-1} \xi_{2i} \int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau + \xi_{2m}u(s)$$

$$+ \sum_{j=1}^{n-1} \eta_{2j} \int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau + \eta_{2n}v(s)\Bigg]ds + C_6$$

$$\leq \sum_{i=1}^{m} \xi_{2i} \int_0^1 k_{m-i+1}(t,s)u(s)\,ds$$

$$+ \sum_{j=1}^{n} \eta_{2j} \int_0^1 k_{n-j+1}(t,s)v(s)\,ds + C_6 \tag{3.35}$$

and

$$v(t) \leq \int_0^1 k_1(t,s)\Bigg[\sum_{i=1}^{m-1} \gamma_{2i} \int_0^1 k_{m-i}(s,\tau)u(\tau)\,d\tau + \gamma_{2m}u(s)$$

$$+\sum_{j=1}^{n-1}\delta_{2j}\int_0^1 k_{n-j}(s,\tau)v(\tau)\,d\tau+\delta_{2n}v(s)\Bigg]\,ds+C_6$$

$$\le\sum_{i=1}^{m}\gamma_{2i}\int_0^1 k_{m-i+1}(t,s)u(s)\,ds$$

$$+\sum_{j=1}^{n}\delta_{2j}\int_0^1 k_{n-j+1}(t,s)v(s)\,ds+C_6. \tag{3.36}$$

Multiply by $\varphi(t)$ on both sides of the preceding inequalities and integrate over [0, 1] and use (2.5) to obtain

$$\int_0^1\varphi(t)u(t)\,dt\le\sum_{i=1}^{m}\xi_{2i}\lambda_1^{m-i+1}\int_0^1\varphi(t)u(t)\,dt$$

$$+\sum_{j=1}^{n}\eta_{2j}\lambda_1^{n-j+1}\int_0^1\varphi(t)v(t)\,dt+C_6 \tag{3.37}$$

and

$$\int_0^1\varphi(t)v(t)\,dt\le\sum_{i=1}^{m}\gamma_{2i}\lambda_1^{m-i+1}\int_0^1\varphi(t)u(t)\,dt$$

$$+\sum_{j=1}^{n}\delta_{2j}\lambda_1^{n-j+1}\int_0^1\varphi(t)v(t)\,dt+C_6, \tag{3.38}$$

which can be written in the form

$$\begin{bmatrix}\tilde\kappa_1 & -\tilde\kappa_2\\ -\tilde\kappa_3 & \tilde\kappa_4\end{bmatrix}\begin{bmatrix}\int_0^1\varphi(t)u(t)\,dt\\ \int_0^1\varphi(t)v(t)\,dt\end{bmatrix}\le\begin{bmatrix}C_6\\ C_6\end{bmatrix}.$$

Now (H4) implies

$$\begin{bmatrix}\int_0^1\varphi(t)u(t)\,dt\\ \int_0^1\varphi(t)v(t)\,dt\end{bmatrix}\le\frac{1}{\tilde\kappa}\begin{bmatrix}\tilde\kappa_4 & \tilde\kappa_2\\ \tilde\kappa_3 & \tilde\kappa_1\end{bmatrix}\begin{bmatrix}C_6\\ C_6\end{bmatrix}$$

and thus

$$\int_0^1\varphi(t)u(t)\,dt\le\frac{\tilde\kappa_2+\tilde\kappa_4}{\tilde\kappa}C_6,\qquad\int_0^1\varphi(t)v(t)\,dt\le\frac{\tilde\kappa_1+\tilde\kappa_3}{\tilde\kappa}C_6,\quad\forall(u,v)\in\mathcal{M}_4.$$

Note that, by Lemma 2.1, $(u,v)\in\mathcal{M}_4$ implies $u\in P_0$ and $v\in P_0$. Therefore

$$\|u\|\le\frac{\tilde\kappa_2+\tilde\kappa_4}{\omega\tilde\kappa}C_6,\qquad\|v\|\le\frac{\tilde\kappa_1+\tilde\kappa_3}{\omega\tilde\kappa}C_6$$

for every $(u, v) \in \mathcal{M}_4$. This proves the boundedness of $\mathcal{M}_4$. Taking $R > \sup\{\|(u, v)\| : (u, v) \in \mathcal{M}_4\}$ and $R > \rho_2$, we have

$$(u, v) \neq \lambda A(u, v), \qquad (u, v) \in \partial\Omega_R \cap (P \times P), \quad 0 \leq \lambda \leq 1$$

and by Lemma 2.4

$$i(A, \partial\Omega_R \cap (P \times P), P \times P) = 1. \tag{3.39}$$

We find from (3.32) and (3.39),

$$i(A, (\Omega_R \backslash \overline{\Omega}_{\rho_2}) \cap (P \times P), P \times P) = 1 - 0 = 1.$$

Hence the operator A has at least one fixed point on $(\Omega_R \backslash \overline{\Omega}_{\rho_2}) \cap (P \times P)$ and (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.3 *If* (H1), (H3) *and* (H5) *hold, then* (1.1) *has at least two positive solutions* $(u_1, v_1) \in (C^{2n}[0, 1] \times C^{2n}[0, 1]) \cap (P \times P) \setminus \{0\}$, $(u_2, v_2) \in (C^{2n}[0, 1] \times C^{2n}[0, 1]) \cap (P \times P) \setminus \{0\}$.

Proof By (H5), the following inequalities

$$f(t, x, y) \leq f(t, \Lambda, \Lambda, \dots, \Lambda) < \frac{\Lambda}{\mu}, \qquad g(t, x, y) \leq g(t, \Lambda, \Lambda, \dots, \Lambda) < \frac{\Lambda}{\mu},$$

hold for a.e. $t \in [0, 1]$ and all $(x, y) \in [0, \Lambda] \times \cdots \times [0, \Lambda]$. Consequently,

$$A_1(u, v)(t) < \int_0^1 \frac{\Lambda}{\mu} k_1(t, s)\, ds \leq \Lambda, \quad \forall (t, u, v) \in [0, 1] \times \partial\Omega_\Lambda,$$

and

$$A_2(u, v)(t) < \int_0^1 \frac{\Lambda}{\mu} k_1(t, s)\, ds \leq \Lambda, \quad \forall (t, u, v) \in [0, 1] \times \partial\Omega_\Lambda.$$

The preceding inequalities lead to

$$\|A(u, v)\| < \|(u, v)\|, \quad \forall ((u, v) \in \partial\Omega_\Lambda \cap (P \times P), \tag{3.40}$$

and thus

$$(u, v) \neq \lambda A(u, v), \quad \forall (u, v) \in \partial\Omega_\Lambda \cap (P \times P), \quad 0 \leq \lambda \leq 1.$$

Now Lemma 2.4 implies

$$i(A, \Omega_\Lambda \cap (P \times P), P \times P) = 1. \tag{3.41}$$

On the other hand, in view of (H1) and (H3), we may take a sufficiently large $R > \Lambda$ and a sufficiently small $r_2 \in (0, \Lambda)$ so that (3.17) and (3.32) hold (see the proofs of Theorems 3.1 and 3.2). The combination of (3.17), (3.32) and (3.41) then gives

$$i(A, (\Omega_R \backslash \overline{\Omega}_\Lambda) \cap (P \times P), P \times P) = 0 - 1 = -1,$$

and

$$i(A, (\Omega_\Lambda \backslash \overline{\Omega}_{r_2}) \cap (P \times P), P \times P) = 1 - 0 = 1.$$

Hence A has at least two fixed points, one on $\Omega_R \backslash \overline{\Omega}_\Lambda \cap (P \times P)$ and the other on $\Omega_\Lambda \backslash \overline{\Omega}_{r_2} \cap (P \times P)$. Therefore (1.1) has at least two positive solutions. This completes the proof. □

4 Examples

In this section we offer three examples to illustrate our main results. In what follows we assume that $\alpha_i \geq 0$, $\beta_j \geq 0$, $\xi_i \geq 0$, $\eta_j \geq 0$ and $\sum_{j=1}^n \beta_j > 0$, $\sum_{i=1}^m \xi_i > 0$.

Example 1 Let

$$f(t, x, y) := \left(\sum_{i=1}^m \alpha_i x_i + \sum_{j=1}^n \beta_j y_j \right)^\alpha, \qquad g(t, x, y) := \left(\sum_{i=1}^m \xi_i x_i + \sum_{j=1}^n \eta_j y_j \right)^\beta,$$

where $\alpha > 1$, $\beta > 1$. Obviously, both f and g grow superlinearly.

Claim (H1) *and* (H2) *hold with* $p_1(v) := v$ *and* $q_1(u) := u^{(\beta+1)/2}$.

Proof First, take $k > 0$ so that $k^2(\sum_{i=1}^m \alpha_i \lambda_1^{m-i+1})(\sum_{j=1}^n \beta_j \lambda_1^{n-j+1}) > 1$, and $\alpha_{1i} = k\alpha_i$, $\beta_{1j} = k\beta_j$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$. Then

$$\sum_{i=1}^m \sum_{j=1}^n \alpha_{1i} \beta_{1j} \lambda_1^{m+n-i-j+2} = k^2 \left(\sum_{i=1}^m \alpha_i \lambda_1^{m-i+1} \right) \left(\sum_{j=1}^n \beta_j \lambda_1^{n-j+1} \right) > 1.$$

This says the first inequality in (2) of (H1) holds. Next, notice

$$\liminf_{\sum_{j=1}^n \beta_{1j} y_j \to +\infty} \frac{f(t, x, y)}{p_1(\sum_{j=1}^n \beta_{1j} y_j)} = \liminf_{\sum_{j=1}^n k\beta_j y_j \to +\infty} \frac{f(t, x, y)}{\sum_{j=1}^n k\beta_j y_j} = +\infty$$

uniformly for $t \in [0, 1]$ and $(x_1, \ldots, x_m) \in \mathbb{R}_+^m$,

$$\liminf_{\sum_{i=1}^n \alpha_{1i} x_i \to +\infty} \frac{g(t, x, y)}{q_1(\sum_{i=1}^m \alpha_{1i} x_i)} = \liminf_{\sum_{i=1}^m k\alpha_i x_i \to +\infty} \frac{g(t, x, y)}{(\sum_{i=1}^m k\alpha_i x_i)^{(\beta+1)/2}} = +\infty$$

uniformly for $t \in [0, 1]$ and $(y_1, \dots, y_n) \in \mathbb{R}_+^n$, and

$$p_1\left(\sum_{j=1}^{n} \beta_{1j} N_{n-j+1} q_1(z)\right) = \sum_{j=1}^{n} k\beta_j N_{n-j+1} z^{(\beta+1)/2} \to +\infty$$

as $z \to +\infty$. The last three limits combined imply that there is a constant $C > 0$ such that the remaining three inequalities in (2) of (H1) hold. Consequently (H1) holds with $p_1(v) = v$ and $q_1(u) = u^{(\beta+1)/2}$, as claimed. It is easy to verify that (H2) holds. This completes the proof. □

With some necessary changes, Examples 2 and 3 below can be similarly verified to satisfy their respective conditions claimed for them.

Example 2 Let

$$f(t, x, y) := \left(\sum_{i=1}^{m} \alpha_i x_i + \sum_{j=1}^{n} \beta_j y_j\right)^{\alpha}, \qquad g(t, x, y) := \left(\sum_{i=1}^{m} \xi_i x_i + \sum_{j=1}^{n} \eta_j y_j\right)^{\beta},$$

where $0 < \alpha < 1$, $0 < \beta < 1$. Obviously, both f and g grow sublinearly. Now (H3) and (H4) hold with $p_1(v) := v$ and $q_1(u) := u^{(\beta+1)/2}$.

Example 3 Let

$$f(t, x, y) := \ln\left(\left(\sum_{i=1}^{m} \alpha_i x_i\right)^{\alpha} + \left(\sum_{j=1}^{n} \beta_j y_j\right)^{\beta} + 1\right),$$

$$g(t, x, y) := \exp\left(\left(\sum_{i=1}^{m} \xi_i x_i\right)^{2\alpha} + \left(\sum_{j=1}^{n} \eta_j y_j\right)^{2\beta}\right) - 1,$$

where $\alpha > 1$, $\beta > 1$. Obviously, f grows sublinearly at $+\infty$, whereas g grows superlinearly at $+\infty$. Now (H1) and (H2) hold with $p_1(v) = \ln(1 + v)$ and $q_1(u) = \exp(u^2) - 1$.

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