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A two-grid discretization scheme for the Steklov eigenvalue problem

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Abstract In the paper, a two-grid discretization scheme is discussed for the Steklov eigenvalue problem. With the scheme, the solution of the Steklov eigenvalue problem on a fine grid is reduced to the solution of the Steklov eigenvalue problem on a much coarser grid and the solution of a linear algebraic system on the fine grid. Using spectral approximation theory, it is shown theoretically that the two-scale scheme is efficient and the approximate solution obtained by the scheme maintains the asymptotically optimal accuracy. Finally, numerical experiments are carried out to confirm the considered theory.

Keywords Steklov eigenvalue problem · Finite element · Two-grid · Error estimates

Mathematics Subject Classification (2000) 65N25 · 65N30 · 65N15

1 Introduction

Steklov eigenvalue problems in which the eigenvalue parameter appears in the boundary condition have received increasing attention in physical and mathematical fields. They appear in a number of applications. For instance, they are found in the study of surface waves (see [7]), in the analysis of stability of mechanical oscillators immersed in a viscous fluid (see [14] and the references therein), and in the study of the vibration modes of a structure in contact with an incompressible fluid (see, for

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example, [8]). Numerical methods of Steklov eigenvalue problems have been developed, and the optimal error estimates have been obtained (see [2, 9, 19, 26] and the references therein). Huang and Lü [17] analyzed extrapolation for solving BIE of Steklov eigenvalue problems and Li, Lin and Zhang [18] analyzed the extrapolation and superconvergence of the Steklov eigenvalue problem in order to improve numerical approximation accuracy. Armentano and Padra [4] proposed and analyzed an a posteriori error estimator, of the residual type, for the linear finite element approximation of the Steklov eigenvalue problem. However, in our recent work (see, for instance, [26]), we realize well that there is no analysis on a two-grid discretization scheme for Steklov eigenvalue problems.

Actually, in practical applications, it is equally a challenging task to adopt efficient methods to reduce the computational costs such as the computational time and the storage requirement and still not decrease the optimally numerical approximation accuracy of a problem, so that larger-scale and much more complicated problems can be settled. A two-grid finite element discretization scheme is considered one of these efficient methods. The scheme was first introduced by Xu [21, 22] for nonsymmetric and nonlinear elliptic problems. To the best of our knowledge, the technique has been successfully applied and further investigated for Poisson eigenvalue equations and integral equations in [23], nonlinear eigenvalue problems in [11], semilinear elliptic eigenvalue problems in [12], Schrödinger equation in [10, 15], second order elliptic problems in [20], and nonselfadjoint elliptic problem in [25]. Based on the technique, local and parallel finite element algorithms have been introduced and analyzed (see [16, 24] and the references therein).

Our objective in the paper is to present a two-grid finite element discretization scheme for solving Steklov eigenvalue problems, that is, the solution of a Steklov eigenvalue problem on a fine grid is reduced to the solution of a Steklov eigenvalue problem on a much coarser grid and the solution of a linear algebraic system on the fine grid.

We now would like to give a kind of detailed description of the main ideas and results in this paper.

Consider the following model problem (see, for instance, [4, 6, 18]):

$$-\Delta u + u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \lambda u \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset R^2$ is a polygonal domain and $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$.

The variational problem associated with (1) is given by: Find $\lambda \in R$ and $u \in H^1(\Omega)$, $u \neq 0$, such that

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H^1(\Omega), \tag{2}$$

where $a(u, v) = \int_\Omega \nabla u \nabla v + uv dx$ and $b(u, v) = \int_{\partial\Omega} uv ds$, $\|u\|_b = b(u, u)^{\frac{1}{2}}$. It is clear that $a(\cdot, \cdot)$ is symmetric, continuous and uniformly $H^1(\Omega)$-elliptic.

It is known that (2) has a sequence of pairs (λ_j, u_j), with positive λ_j diverging to $+\infty$ (see [4]). We assume that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$ (here each eigenvalue occurs as many times as given by its multiplicity), and the corresponding eigenfunctions $u_j \in H^{r+1}(\Omega)$ $(j = 1, 2, \ldots)$, where $r = 1$ if Ω is convex and $r < \frac{\pi}{w}$ (with w being the largest inner angle of Ω) otherwise.

Let π_h be a regular triangulation of Ω (see [13], P131) and S^h be a piecewise linear polynomial space associated with π_h, that is,

$$S^h = \{v \in H^1(\Omega), v|_T \in P_1, \ \forall T \in \pi_h\},$$

where h is the mesh size. $S^h \subset H^1(\Omega)$. Then the corresponding finite element approximation of (2) is the following: Find $\lambda_h \in R$ and $u_h \in S^h$, $\|u_h\|_b = 1$, such that

$$a(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in S^h. \tag{3}$$

It is well-know that (2) has finite eigenvalues. We arrange them as $0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{N_h,h}$ ($N_h = \dim S^h$), and their corresponding eigenfunctions are $u_{j,h}$, $j = 1, 2, \ldots, N_h$.

According to minimum-maximum principle, we know $\lambda_j \le \lambda_{j,h}$ ($j = 1, 2, \ldots, N_h$). We also have the following a priori error estimates (see, for example, [3]):

$$\|u_j - u_{j,h}\|_1 \le Ch^r,$$

$$\|u_j - u_{j,h}\|_b \le Ch^{\frac{3r}{2}},$$

$$|\lambda_j - \lambda_{j,h}| \le Ch^{2r}.$$

Here and hereafter, C denotes a generic positive constant independent of h which may not be the same at each occurrence.

A two-grid discretization scheme for (1) is designed as follows:

Step 1 Solve (3) on a coarse grid π_H: Find $(\lambda_{j,H}, u_{j,H}) \in R \times S^H$ such that $\|u_{j,H}\|_b = 1$ and

$$a(u_{j,H}, v) = \lambda_{j,H} b(u_{j,H}, v), \quad \forall v \in S^H,$$

where $j \in \{1, 2, \ldots, N_H = \dim S^H\}$.

Step 2 Solve the boundary value problem associated with (2) on a fine grid π_h: Find $u^*_{j,h} \in S^h$ such that

$$a(u^*_{j,h}, v) = \lambda_{j,H} b(u_{j,H}, v), \quad \forall v \in S^h.$$

Step 3 Compute the Rayleigh quotient

$$\lambda_{j,s} = a(u^*_{j,h}, u^*_{j,h})/b(u^*_{j,h}, u^*_{j,h}).$$

Then $(\lambda_{j,s}, u^*_{j,h})$ is regarded as the approximate solution of (2) obtained by the scheme. Note that the convergent order of the error $\|u_j - u_{j,H}\|_b$ can only reach $O(H^{\frac{3r}{2}})$. It brings us great difficulties in analyzing the two-grid error estimates. However, through the introduction of the negative space $H^{-\frac{1}{2}}(\partial\Omega)$ and using spectral approximation theory, we elaborately analyze two-grid finite element approximation and we can establish the following satisfying results (see Sect. 3)

$$\|u^*_{j,h} - u_j\|_1 \le C(H^{2r} + h^r) \quad \text{and} \quad |\lambda_{j,s} - \lambda_j| \le C(H^{4r} + h^{2r}).$$

These estimates mean that we can obtain asymptotically optimal errors by taking $h = O(H^2)$.

The remainder of this paper is organized as follows: In Sect. 2, several preliminary results including the regular estimates are provided. The significant results including the optimal two-grid discretization error estimates and a superclose relationship between the Ritz-Galerkin projection of the eigenvector and the finite element approximation to the eigenvector are analyzed in Sect. 3. Finally, in Sect. 4, numerical examples are presented to confirm the reliability and efficiency of the two-grid scheme.

2 Preliminaries

Let $H^s(\Omega)$ denote the Sobolev space with real order s on Ω, $\|\cdot\|_s$ be the norm on $H^s(\Omega)$ and $H^0(\Omega) = L_2(\Omega)$. Consider the source problem (4) associated with (2) and the approximate source problem (5) associated with (3): Find $u \in H^1(\Omega)$ such that

$$a(u, v) = b(f, v), \quad \forall v \in H^1(\Omega). \tag{4}$$

Find $u_h \in S^h$ such that

$$a(u_h, v) = b(f, v), \quad \forall v \in S^h. \tag{5}$$

Then, several regular estimates of the Steklov eigenvalue problem are presented in the following lemma, which will be used in the sequel.

Lemma 1 *If $f \in H^{\frac{1}{2}}(\partial\Omega)$, then there exists a unique solution $u \in H^{1+r}(\Omega)$ to (4), and*

$$\|u\|_{1+r} \le C\|f\|_{\frac{1}{2},\partial\Omega}; \tag{6}$$

if $f \in H^{-\frac{1}{2}}(\partial\Omega)$, then there exists a unique solution $u \in H^1(\Omega)$ to (4), and

$$\|u\|_1 \le C\|f\|_{-\frac{1}{2},\partial\Omega}. \tag{7}$$

Proof As regards the proof of (6), see Proposition 4.1 in [1] for details.

We now prove (7). Consider the problem (4), where $f \in H^{-\frac{1}{2}}(\partial\Omega)$. Using the trace theorem,

$$|b(f, v)| = \left|\int_{\partial\Omega} f v ds\right| \le \|f\|_{-\frac{1}{2},\partial\Omega}\|v\|_{\frac{1}{2},\partial\Omega} \le \|f\|_{-\frac{1}{2},\partial\Omega}\|v\|_{1,\Omega}, \quad \forall v \in H^1(\Omega).$$

Therefore, from Lax-Milgram Theorem we can deduce that there exists a unique solution $u \in H^1(\Omega)$ to (4) and $\|u\|_1 \le C\|f\|_{-\frac{1}{2},\partial\Omega}$. □

Now, we define a norm over $H^{-\frac{1}{2}}(\partial\Omega)$:

$$\|u\|_{-\frac{1}{2},\partial\Omega} = \sup_{g\in H^{\frac{1}{2}}(\partial\Omega)} \frac{|b(u,g)|}{\|g\|_{\frac{1}{2},\partial\Omega}}, \quad \forall u \in H^{-\frac{1}{2}}(\partial\Omega). \tag{8}$$

According to the source problem (4) associated with (2), we define the operator $A : H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ by

$$a(Af, v) = b(f, v), \quad \forall v \in H^1(\Omega).$$

Define $T : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ by $Tf = (Af)'$, where the prime denotes the restriction to $\partial\Omega$. Similarly, from (5), we define the operator $A_h : H^{-\frac{1}{2}}(\partial\Omega) \to S^h \subset H^1(\Omega)$ by

$$a(A_h f, v) = b(f, v), \quad \forall v \in S^h.$$

Define $T_h : H^{-\frac{1}{2}}(\partial\Omega) \to \delta S^h \subset L_2(\partial\Omega)$ by $T_h f = (A_h f)'$, where δS^h denotes the space of restrictions to $\partial\Omega$ of functions in S^h.

We observe that Af and $A_h f$ are the exact solution and the approximate solution of (4) respectively, and $a(Af - A_h f, v) = 0$, $\forall v \in S^h \subset H^1(\Omega)$. Define a Ritz-Galerkin projection operator $P_h : H^1(\Omega) \longrightarrow S^h$ by

$$a(u - P_h u, v) = 0, \quad \forall u \in H^1(\Omega),\ \forall v \in S^h.$$

Thus, $a(Af - P_h(Af), v) = 0$, $\forall v \in S^h$, since $Af \in H^1(\Omega)$, $\forall f \in H^{-\frac{1}{2}}(\partial\Omega)$. Then, it is easy to see that $A_h = P_h A$. Let $\eta_1(h) = \sup_{g\in H^{\frac{1}{2}}(\partial\Omega), \|g\|_{\frac{1}{2},\partial\Omega}=1} \inf_{\chi\in S^h} \|Ag - \chi\|_1$. Then, from (6), the following result holds:

$$\eta_1(h) \le Ch^r. \tag{9}$$

From [9], we know (22) has the equivalent operator form:

$$Tw = \mu w. \tag{10}$$

Namely, if $(\mu, w) \in R \times H^{-\frac{1}{2}}(\partial\Omega)$ is an eigenpair of (10), then (λ, Aw) is an eigenpair of (2), $\lambda = \frac{1}{\mu}$; conversely, if (λ, u) is an eigenpair of (2), then (μ, u') is an eigenpair of (10), $\mu = \frac{1}{\lambda}$. Also, (3) has the equivalent operator form:

$$T_h w_h = \mu w_h. \tag{11}$$

Namely, if (μ_h, w_h) is an eigenpair of (11), then $(\lambda_h, A_h w_h)$ is an eigenpair of (3), $\mu_h = \frac{1}{\lambda_h}$; conversely, if (λ_h, u_h) is an eigenpair of (3), then (μ_h, u_h') is an eigenpair of (11), $\mu_h = \frac{1}{\lambda_h}$.

For any $f, g \in H^{-\frac{1}{2}}(\partial\Omega)$,

$$b(Tf, g) = b(g, Tf) = b(g, Af) = a(Ag, Af) = a(Af, Ag) = b(f, Ag) = b(f, Tg).$$

So, $T : L_2(\partial\Omega) \to L_2(\partial\Omega)$ is a selfadjoint operator. Analogously, $T_h : L_2(\partial\Omega) \to L_2(\partial\Omega)$ is a selfadjoint operator. Also, the following results are valid.

Lemma 2 $\|T - T_h\|_{H^{-\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}\partial\Omega)} \to 0$ *as* $h \to 0$ *and* T *is a compact operator.*

Proof From (8), Lemma 1 and the definitions of A and A_h, we have

$$\begin{aligned}
\|(A - A_h)g\|_{-\frac{1}{2},\partial\Omega} &= \sup_{\varphi\in H^{\frac{1}{2}}(\partial\Omega)} \frac{|b((A - A_h)g, \varphi)|}{\|\varphi\|_{\frac{1}{2},\partial\Omega}} \\
&= \sup_{\varphi\in H^{\frac{1}{2}}(\partial\Omega)} \frac{|a(A\varphi, (A - A_h)g)|}{\|\varphi\|_{\frac{1}{2},\partial\Omega}} \\
&= \sup_{\varphi\in H^{\frac{1}{2}}(\partial\Omega),\chi\in S^h} \frac{|a((A - A_h)g, A\varphi - \chi)|}{\|\varphi\|_{\frac{1}{2},\partial\Omega}} \\
&\le \sup_{\varphi\in H^{\frac{1}{2}}(\partial\Omega),\chi\in S^h} \frac{C\|A\varphi - \chi\|_1\|(A - A_h)g\|_1}{\|\varphi\|_{\frac{1}{2},\partial\Omega}} \\
&\le \sup_{\varphi\in H^{\frac{1}{2}}(\partial\Omega)} \frac{Ch^r\|A\varphi\|_{1+r}\|Ag\|_1}{\|\varphi\|_{\frac{1}{2},\partial\Omega}} \le Ch^r\|Ag\|_1,
\end{aligned}$$

where we use (6) in the last inequality.

Thus, from (7) and the definitions of T and T_h, we obtain

$$\begin{aligned}
\|T - T_h\|_{H^{-\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}(\partial\Omega)} &= \sup_{g\in H^{-\frac{1}{2}}(\partial\Omega)} \frac{\|(T - T_h)g\|_{-\frac{1}{2},\partial\Omega}}{\|g\|_{-\frac{1}{2},\partial\Omega}} \\
&= \sup_{g\in H^{-\frac{1}{2}}(\partial\Omega)} \frac{\|(A - A_h)g\|_{-\frac{1}{2},\partial\Omega}}{\|g\|_{-\frac{1}{2},\partial\Omega}} \\
&\le \sup_{g\in H^{-\frac{1}{2}}(\partial\Omega)} \frac{Ch^r\|Ag\|_1}{\|g\|_{-\frac{1}{2},\partial\Omega}} \le Ch^r \to 0 \quad (h \to 0).
\end{aligned}$$

Note that T_h is a finite rank operator. Thus, T is a compact operator. □

At the end of this section, we present the following crucial (but straightforward) property of eigenvalue and eigenfunction approximation which is the basis of the two-grid analysis for the eigenvalues.

Lemma 3 *Let* (λ, u) *be an eigenpair of* (2)*. Then, for any* $w \in H^1(\Omega)$, $\|w\|_b \neq 0$,

$$\frac{a(w,w)}{\|w\|_b^2} - \lambda = \frac{\|w-u\|_1^2}{\|w\|_b^2} - \lambda \frac{\|w-u\|_b^2}{\|w\|_b^2}.$$

Proof See, for instance, Lemma 9.1 in [6] for details. □

3 Two-grid finite element discretization

In this section, for any λ_j we let $M(\lambda_j) = \{u_j : u_j$ is an eigenfunction of (2) corresponding to $\lambda_j\}$. Let $\sigma(T)$ and $\rho(T)$ denote the spectrum and resolvent sets of T, respectively. Let Γ be a circle in the complex plane centered at $\mu_j = \frac{1}{\lambda_j}$ which lies in $\rho(T)$ and which encloses no other points of $\sigma(T)$. Define the spectral projection associated with μ_j and T by

$$E = \frac{1}{2\pi i} \int_\Gamma (z - T)^{-1} dz.$$

Then, the following result is presented.

Lemma 4 *For any eigenvector* $u_{j,h}$ *of* (3), $\|u_{j,h}\|_b = 1$, *there exists a unique* $u_j = Eu_{j,h} \in M(\lambda_j)$, *such that*

$$\|u_{j,h} - u_j\|_{-\frac{1}{2},\partial\Omega} \le C\|(I - P_h)u_j\|_{-\frac{1}{2},\partial\Omega}.$$

Proof From Lemma 2, $\|T - T_h\|_{H^{-\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}\partial\Omega)} \to 0$ $(h \to 0)$. Then, using the same argument as that of Lemma 3.7 in [5] and [6], P699, we can get the desired result. □

On the basis of the above lemma, we can get the following result which is crucial to analyze the two-grid error estimate for the eigenfunctions.

Lemma 5 *Let* $(\lambda_{j,h}, u_{j,h})$ *with* $\|u_{j,h}\|_b = 1$ *be the* j*-th eigenpair of* (3)*. Then there exists* $u_j = Eu_{j,h} \in M(\lambda_j)$, *such that*

$$\|u_j - u_{j,h}\|_{-\frac{1}{2},\partial\Omega} \le Ch^{2r}.$$

Proof From the definitions of P_h and A, we get

$$\begin{aligned} |b(u - P_h u, g)| &= |a(u - P_h u, Ag)| = |a(u - P_h u, Ag - \chi)| \\ &\le C\|(I - P_h)u\|_1 \inf_{\chi \in S^h} \|Ag - \chi\|_1, \quad \forall \chi \in S^h. \end{aligned}$$

Taking it in (8), we obtain

$$\|(I - P_h)u\|_{-\frac{1}{2},\partial\Omega} \le C\eta_1(h)\|(I - P_h)u\|_1,$$

which, together with (9), Lemma 4 and $u_j \in H^{1+r}(\Omega)$, yields that there exists $u_j = E_h u_{j,h} \in M(\lambda_j)$ such that

$$\begin{aligned}\|u_j - u_{j,h}\|_{-\frac{1}{2},\partial\Omega} &\le C\|(I-P_h)u_j\|_{-\frac{1}{2},\partial\Omega}\\ &\le C\eta_1(h)\|(I-P_h)u_j\|_1\\ &\le C\eta_1(h)h^r\|u_j\|_{1+r} \le Ch^{2r}.\end{aligned}$$

□

Now, we are in a position to prove the main results in the paper.

Theorem 1 *Let* $(\lambda_{j,H}, u_{j,H})$ *be the* j*-th approximate eigenpair of* (2), $\|u_{j,H}\|_b = 1$. *Assume that* $(\lambda_{j,s}, u^*_{j,h})$ *is the approximate solution of* (2) *obtained by the two-grid discretization scheme. Then there exists* $u_j = Eu_{j,H} \in M(\lambda_j)$ *such that*

$$\|u^*_{j,h} - u_j\|_1 \le C(\lambda_{j,H} - \lambda_j + \lambda_{j,H}\|u_j - u_{j,H}\|_{-\frac{1}{2},\partial\Omega} + h^r), \tag{12}$$

$$|\lambda_{j,s} - \lambda_j| \le C((\lambda_{j,H} - \lambda_j)^2 + \lambda^2_{j,H}\|u_j - u_{j,H}\|^2_{-\frac{1}{2},\partial\Omega} + h^{2r}). \tag{13}$$

Proof From the definitions of A and A_h we have

$$\begin{aligned}u^*_{j,h} - u_j &= A_h(\lambda_{j,H}u_{j,H}) - A(\lambda_j u_j)\\ &= A_h(\lambda_{j,H}u_{j,H}) - A(\lambda_{j,H}u_{j,H}) + A(\lambda_{j,H}u_{j,H}) - A(\lambda_j u_j).\end{aligned}$$

Note that $A(\lambda_{j,H}u_{j,H}) \in H^{1+r}$. Thus, from the a priori error estimates, (6) and (7), we have

$$\begin{aligned}\|u^*_{j,h} - u_j\|_1 &\le \|A_h(\lambda_{j,H}u_{j,H}) - A(\lambda_{j,H}u_{j,H})\|_1 + \|A(\lambda_{j,H}u_{j,H} - \lambda_j u_j)\|_1\\ &\le C(h^r\|A(\lambda_{j,H}u_{j,H})\|_{1+r} + \|\lambda_j u_j - \lambda_{j,H}u_{j,H}\|_{-\frac{1}{2},\partial\Omega})\\ &\le C(h^r\|\lambda_{j,H}u_{j,H}\|_{\frac{1}{2},\partial\Omega} + \|(\lambda_j - \lambda_{j,H})u_j + \lambda_{j,H}(u_j - u_{j,H})\|_{-\frac{1}{2},\partial\Omega})\\ &\le C(h^r + \lambda_{j,H} - \lambda_j + \lambda_{j,H}\|u_j - u_{j,H}\|_{-\frac{1}{2},\partial\Omega}),\end{aligned}$$

i.e. (12) is valid.

Equation (13) follows directly from Lemma 3 and (12). □

Corollary *Under the assumptions of Theorem* 1, *the following two-grid error estimates hold*:

$$\|u^*_{j,h} - u_j\|_1 \le C(H^{2r} + h^r),$$

$$|\lambda_{j,s} - \lambda_j| \le C(H^{4r} + h^{2r}).$$

Proof It follows directly from Lemma 5, Theorem 1 and the a priori estimates in Sect. 1. □

At the end of the section, we show a superclose relationship between the Ritz-Galerkin projection of the eigenvector and the finite element approximation to the eigenvector.

Theorem 2 *Let $(\lambda_{j,h}, u_{j,h})$ be the j-th approximate eigenpair of* (2), $\|u_{j,h}\|_b = 1$. *Then there exists $u_j = Eu_{j,h} \in M(\lambda_j)$ such that*

$$\|P_h u_j - u_{j,h}\|_1 \leq C(\lambda_{j,h} - \lambda_j + \lambda_{j,h}\|u_j - u_{j,h}\|_{-\frac{1}{2},\partial\Omega}).$$

Proof From the definitions of A and A_h and $A_h = P_h A$, we have

$$\begin{aligned} P_h u_j - u_{j,h} &= P_h A(\lambda_j u_j) - A_h(\lambda_{j,h} u_{j,h}) \\ &= P_h A(\lambda_j u_j - \lambda_{j,h} u_{j,h}). \end{aligned}$$

From the definition of P_h, we have that $a(u - P_h u, P_h u) = 0$ for any $u \in H^1(\Omega)$, and thus $\|P_h u\|_a \leq C\|u\|_a$, $\forall u \in H^1(\Omega)$, from the properties of $a(\cdot,\cdot)$.

Then, from (7), we get

$$\begin{aligned} \|P_h u_j - u_{j,h}\|_1 &= \|P_h A(\lambda_j u_j - \lambda_{j,h} u_{j,h})\|_1 \\ &\leq \|A(\lambda_j u_j - \lambda_{j,h} u_{j,h})\|_1 \\ &\leq C\|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_{-\frac{1}{2},\partial\Omega} \\ &\leq C\|(\lambda_j - \lambda_{j,h})u_j + \lambda_{j,h}(u_j - u_{j,h})\|_{-\frac{1}{2},\partial\Omega} \\ &\leq C(\lambda_{j,h} - \lambda_j + \lambda_{j,h}\|u_j - u_{j,h}\|_{-\frac{1}{2},\partial\Omega}). \end{aligned}$$ □

4 Numerical experiments

Consider the problem (1), where $\Omega \subset R^2$ is a unit square domain $[0, 1] \times [0, 1]$ or L-shaped domain $[0, 1] \times [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ (see Figs. 1 and 2, respectively).

Besides, Figs. 1 and 2 show the initial triangulations of the unit square domain and the L-shaped domain respectively. We refine the initial triangulations in a uniform way (each triangle is divided into four similar triangles). We use the Matlab order 'eigs' to compute the first four approximate eigenvalues. Note that when Ω is the unit square domain the first four exact eigenvalues $\lambda_1 \in [0.2400790855, 0.2400791144]$,

Fig. 1 The unit square domain Ω

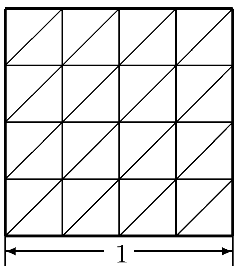

Fig. 2 The L-shaped domain Ω

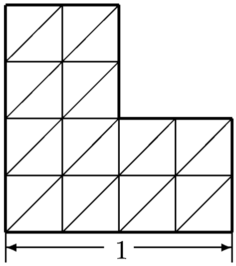

Table 1 The result for the unit square domain Ω

h	$\lvert\lambda_{1,h}-\lambda_{1,s}\rvert$	$\lvert\lambda_{2,h}-\lambda_{2,s}\rvert$	$\lvert\lambda_{3,h}-\lambda_{3,s}\rvert$	$\lvert\lambda_{4,h}-\lambda_{4,s}\rvert$
$\frac{\sqrt{2}}{64}$	3.100E–008	4.387E–005	5.557E–005	1.3926E–004
$\frac{\sqrt{2}}{144}$	6.500E–009	9.614E–006	1.231E–005	2.578E–005
$\frac{\sqrt{2}}{256}$	2.100E–009	3.264E–006	4.203E–006	7.871E–006
$\frac{\sqrt{2}}{576}$	5.000E–010	6.700E–007	8.670E–007	1.446E–006

Table 2 The result for the L-shaped domain Ω

h	$\lvert\lambda_{1,h}-\lambda_{1,s}\rvert$	$\lvert\lambda_{2,h}-\lambda_{2,s}\rvert$	$\lvert\lambda_{3,h}-\lambda_{3,s}\rvert$	$\lvert\lambda_{4,h}-\lambda_{4,s}\rvert$
$\frac{\sqrt{2}}{64}$	1.310E–008	6.151E–005	8.911E–005	1.376E–003
$\frac{\sqrt{2}}{144}$	2.900E–009	2.116E–005	1.962E–005	4.093E–004
$\frac{\sqrt{2}}{256}$	9.000E–010	1.068E–005	6.750E–006	1.564E–004
$\frac{\sqrt{2}}{576}$	2.000E–010	3.612E–006	1.412E–006	3.531E–005

$\lambda_2 \in [1.492302282, 1.492305003]$, $\lambda_3 \in [1.492302282, 1.492305388]$, and $\lambda_4 \in [2.082639338, 2.082659329]$; when Ω is the L-shaped domain the first four exact eigenvalues $\lambda_1 \in [0.1829642353, 0.1829642438]$, $\lambda_2 \in [0.8936027786, 0.8937363984]$, $\lambda_3 \in [1.688598128, 1.688606742]$, and $\lambda_4 \in [3.217841378, 3.217900202]$.

From Corollary, we have

$$|\lambda_{j,h} - \lambda_{j,s}| = O(H^{4r}) \approx Ch^{2r}.$$

Motivated by [23], we show the results in Tables 1–2 where $\lambda_{j,h}$, $j = 1, 2, 3, 4$, denote the approximate eigenvalues on the fine grid π_h and $\lambda_{j,s}$, $j = 1, 2, 3, 4$, are the approximate eigenvalues obtained by the two-grid discretization scheme. This results are consistent with the above estimate.

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