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A new cubic B-spline method for linear fifth order boundary value problems

Feng-Gong Lang · Xiao-Ping Xu

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Abstract In this paper, we mainly study the numerical solution of linear fifth order boundary value problems by using cubic B-splines. Our algorithm develops not only the cubic spline approximation solution but also the approximation derivatives of first order to fourth order of the analytic solution at the same time. This new method has lower computational cost than many other methods and is second order convergent. Numerical examples are given to demonstrate the effectiveness of our method.

Keywords Cubic B-spline · Numerical solution · Fifth order boundary value problem

Mathematics Subject Classification (2000) 34K10 · 34K28 · 65D05 · 65D07

1 Introduction

In this paper, we study the numerical solutions of the following special linear fifth order boundary value problems (5BVP)

$$\begin{cases} y^{(5)}(x) + f(x)y(x) = g(x), & x \in [a,b], \\ y(a) = a_0, \quad y'(a) = a_1, \quad y''(a) = a_2, \\ y(b) = b_0, \quad y'(b) = b_1, \end{cases} \tag{1}$$

where a_0, a_1, a_2, and b_0, b_1 are given real constants, $f(x)$ and $g(x)$ are continuous on $[a,b]$. These problems generally arise in the mathematical modeling of viscoelastic flows [10, 15], and the conditions for existence and uniqueness of solutions of such

F.-G. Lang (✉) · X.-P. Xu
School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong 266071, People's Republic of China
e-mail: langfg@yahoo.cn

boundary value problems have been given in [1]. Currently, many numerical methods have been developed for these boundary value problems, for example, the decomposition method [24, 31], the homotopy perturbation method [20], the local polynomial regression method [8], and the variational iteration method [21–23, 32].

In the following, we mainly pay attention to the spline numerical solutions of such boundary value problems. Sextic spline approximate solution was first studied in [7], but it is only first order convergent. Later, Siddiqi et al. analyzed a method of second order convergent based on the consistency relations of sextic spline in [29], and Lamnii et al. discussed a different approach with second order convergence based on sextic spline interpolation and quasi-interpolation in [19]. Besides, Islam et al. also applied polynomial sextic spline to solve these special fifth order boundary value problems in [13]. At the same time, non-polynomial spline technique [26] and quartic spline technique [30] were also attempted by several scholars. However, we find the splines used in the above mentioned papers are all with higher degrees, which effect the computational efficiency in practical application. This motivates us to use cubic B-spline (a widely-used spline with lower degree) to solve these problems.

It is well known that cubic spline has been widely applied for the approximation solutions of boundary value problems. The use of cubic splines for the solution of linear second order boundary value problems was first suggested in [6], and was continued in [2–4, 9, 12, 14]. Besides, cubic spline was also used to solve linear third order boundary value problems [5], fourth order boundary value problems [25] and singular boundary value problems, see [17] and the references therein. Throughout this paper, we will use cubic spline to solve 5BVP (1). Our method is based on cubic spline interpolation. Comparisons with the other methods shows our results are well accepted. It is second order convergent and with lower computational cost. Moreover, as the byproducts of our methods, we also can get the approximate derivative values of $y'(x)$, $y''(x)$, $y^{(3)}(x)$ and $y^{(4)}(x)$ at the knots. This is another advantage of our method, because some methods can not obtain these values.

The remainder of this paper is organized as follows. In Sect. 2, we present some preliminary results of cubic B-splines for the sake of integrity; in Sect. 3, we mainly give the cubic B-spline solutions of the special linear fifth order boundary value problems based on the results in Sect. 2; Sect. 4 is devoted to the convergence analysis and the supporting examples are performed in Sect. 5; in Sect. 6, we extend our research and discuss the applicability of cubic spline to the general linear and nonlinear fifth order boundary value problems, two numerical examples are also given to show the efficiency of our method; finally, we conclude our paper in the last section.

2 Basics of cubic B-splines

2.1 Explicit representations of cubic B-splines

For an interval $I = [a, b]$, we divide it into n subintervals $I_i = [x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$) by the equidistant knots $x_i = a + ih$ ($i = 0, 1, \dots, n$), where $h = \frac{b-a}{n}$. The cubic spline space is defined as follows [11, 27, 28]:

$$S_3(I) = \{s(x) \in C^2(I) \mid s(x)_{|I_i} \in \mathbf{P}_3,\ i = 0, 1, \dots, n-1\},$$

where $s(x)_{|I_i}$ denotes the restriction of $s(x)$ over I_i, and $\mathbf{P}_3$ denotes the set of univariate cubic polynomials. The dimension of $S_3(I)$ is $n+3$, and the cubic B-splines are defined as

$$B_{-1}(x) = \frac{1}{6h^3}\begin{cases} (x_1 - x)^3, & \text{if } x \in [x_0, x_1] \\ 0, & \text{else} \end{cases}$$

$$B_0(x) = \frac{1}{6h^3}\begin{cases} h^3 + 3h^2(x_1 - x) + 3h(x_1 - x)^2 - 3(x_1 - x)^3, & \text{if } x \in [x_0, x_1], \\ (x_2 - x)^3, & \text{if } x \in [x_1, x_2], \\ 0, & \text{else}, \end{cases}$$

$$B_1(x) = \frac{1}{6h^3}\begin{cases} h^3 + 3h^2(x - x_0) + 3h(x - x_0)^2 - 3(x - x_0)^3, & \text{if } x \in [x_0, x_1], \\ h^3 + 3h^2(x_2 - x) + 3h(x_2 - x)^2 - 3(x_2 - x)^3, & \text{if } x \in [x_1, x_2], \\ (x_3 - x)^3, & \text{if } x \in [x_2, x_3], \\ 0, & \text{else}, \end{cases}$$

$$B_{n-1}(x) = \frac{1}{6h^3}\begin{cases} (x - x_{n-3})^3, & \text{if } x \in [x_{n-3}, x_{n-2}], \\ h^3 + 3h^2(x - x_{n-2}) + 3h(x - x_{n-2})^2 & \\ \quad -3(x - x_{n-2})^3, & \text{if } x \in [x_{n-2}, x_{n-1}], \\ h^3 + 3h^2(x_n - x) + 3h(x_n - x)^2 & \text{if } x \in [x_{n-1}, x_n], \\ \quad -3(x_n - x)^3, & \\ 0, & \text{else}, \end{cases}$$

$$B_n(x) = \frac{1}{6h^3}\begin{cases} (x - x_{n-2})^3, & \text{if } x \in [x_{n-2}, x_{n-1}], \\ h^3 + 3h^2(x - x_{n-1}) + 3h(x - x_{n-1})^2 & \\ \quad -3(x - x_{n-1})^3, & \text{if } x \in [x_{n-1}, x_n], \\ 0, & \text{else}, \end{cases}$$

$$B_{n+1}(x) = \frac{1}{6h^3}\begin{cases} (x - x_{n-1})^3, & \text{if } x \in [x_{n-1}, x_n], \\ 0, & \text{else}, \end{cases}$$

and for $i = 2, 3, \ldots, n-2$,

$$B_i(x) = \frac{1}{6h^3}\begin{cases} (x - x_{i-2})^3, & \text{if } x \in [x_{i-2}, x_{i-1}], \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 & \\ \quad -3(x - x_{i-1})^3, & \text{if } x \in [x_{i-1}, x_i], \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 & \\ \quad -3(x_{i+1} - x)^3, & \text{if } x \in [x_i, x_{i+1}], \\ (x_{i+2} - x)^3, & \text{if } x \in [x_{i+1}, x_{i+2}], \\ 0, & \text{else}. \end{cases}$$

We have $B_i(x) = B_{i+1}(x+h)$ (translation), and $\sum_{i=-1}^{n+1} B_i(x) \equiv 1$ (partition of unity). The values of $B_i(x)$, $B_i'(x)$ and $B_i''(x)$ at the knots are listed in Table 1.

Table 1 The values of $B_i(x)$, $B_i'(x)$ and $B_i''(x)$ at the knots

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	Else
$B_i(x)$	0	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	0	0
$B_i'(x)$	0	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0	0
$B_i''(x)$	0	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0	0

2.2 Cubic B-splines interpolation

For a given function $y(x)$ (assuming to be sufficiently smooth), there exists a unique cubic spline $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x) \in S_3(I)$ satisfying the interpolation conditions:

$$s(x_i) = y(x_i) \quad (i = 0, 1, \dots, n) \quad \text{and} \quad s'(a) = y'(a), \qquad s'(b) = y'(b).$$

For $j = 0, 1, \dots, n$, let $m_j = s'(x_j)$ and $M_j = s''(x_j)$, we have [12, 18]:

$$m_j = s'(x_j) = y'(x_j) - \frac{1}{180}h^4 y^{(5)}(x_j) + O(h^6); \tag{2}$$

$$M_j = s''(x_j) = y''(x_j) - \frac{1}{12}h^2 y^{(4)}(x_j) + \frac{1}{360}h^4 y^{(6)}(x_j) + O(h^6); \tag{3}$$

M_j can be applied to construct numerical difference formulae for $y^{(3)}(x_j)$, $y^{(4)}(x_j)$ $(j = 1, 2, \dots, n-1)$ and $y^{(5)}(x_j)$ $(j = 2, 3, \dots, n-2)$ as follows, where the errors are obtained by the Taylor series expansion.

$$\frac{M_{j+1} - M_{j-1}}{2h} = \frac{s^{(3)}(x_{j-}) + s^{(3)}(x_{j+})}{2} = y^{(3)}(x_j) + \frac{1}{12}h^2 y^{(5)}(x_j) + O(h^4); \tag{4}$$

$$\begin{aligned} \frac{M_{j+1} - 2M_j + M_{j-1}}{h^2} &= \frac{s^{(3)}(x_{j+}) - s^{(3)}(x_{j-})}{h} \\ &= y^{(4)}(x_j) - \frac{1}{720}h^4 y^{(8)}(x_j) + O(h^6); \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{M_{j+2} - 2M_{j+1} + 2M_{j-1} - M_{j-2}}{2h^3} &= \frac{\frac{M_{j+2}-M_j}{2h} - 2\frac{M_{j+1}-M_{j-1}}{2h} + \frac{M_j - M_{j-2}}{2h}}{h^2} \\ &= y^{(5)}(x_j) + O(h^2). \end{aligned} \tag{6}$$

Since $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$, by Table 1 and these equations, we get two tables.

Table 2 The approximate values of $y(x_j)$, $y'(x_j)$, $y''(x_j)$ and $y^{(3)}(x_j)$

	$y(x_j)$	$y'(x_j)$	$y''(x_j)$	$y^{(3)}(x_j)$
Approximate value	$s(x_j)$	$s'(x_j)=m_j$	$s''(x_j)=M_j$	$s^{(3)}(x_j)=\frac{M_{j+1}-M_{j-1}}{2h}$
Representation in c_j	$\frac{c_{j-1}+4c_j+c_{j+1}}{6}$	$\frac{c_{j+1}-c_{j-1}}{2h}$	$\frac{c_{j-1}-2c_j+c_{j+1}}{h^2}$	$\frac{c_{j+2}-2c_{j+1}+2c_{j-1}-c_{j-2}}{2h^3}$
Error order	$O(h^4)$	$O(h^4)$	$O(h^2)$	$O(h^2)$

3 Cubic B-spline solutions of 5BVP

Let $s(x)=\sum_{i=-1}^{n+1} c_i B_i(x)$ be the approximate solution of 5BVP (1) and $\widetilde{s(x)}=\sum_{i=-1}^{n+1} \widetilde{c_i} B_i(x)$ be the approximate spline of $s(x)$. Discretize 5BVP (1) at the knots we get ($i=2,3,\dots,n-2$):

$$y^{(5)}(x_i)+f(x_i)y(x_i)=g(x_i). \tag{7}$$

By Table 2 and Table 3, we turn (7) into:

$$\frac{c_{i+3}-4c_{i+2}+5c_{i+1}-5c_{i-1}+4c_{i-2}-c_{i-3}}{2h^5}+f_i\frac{c_{i-1}+4c_i+c_{i+1}}{6}=g_i+O(h^2), \tag{8}$$

where $f_i=f(x_i)$ and $g_i=g(x_i)$ be the values of $f(x)$ and $g(x)$ at the knots x_i ($i=0,1,\dots,n$) for short. Change (8) equivalently, we yield:

$$\begin{aligned}&3(c_{i+3}-4c_{i+2}+5c_{i+1}-5c_{i-1}+4c_{i-2}-c_{i-3})+f_ih^5(c_{i-1}+4c_i+c_{i+1})\\&\quad=6h^5g_i+O(h^7).\end{aligned} \tag{9}$$

Dropping the term $O(h^7)$ from (9), we get a linear system with $n-3$ linear equations ($i=2,3,\dots,n-2$) in $n+3$ unknowns c_i ($i=-1,0,1,\dots,n+1$), so six more equations are needed. By the three boundary conditions at $x=a$, we get:

$$\begin{aligned}&\begin{cases}y(a)=a_0\\y'(a)=a_1\\y''(a)=a_2\end{cases}\Rightarrow\begin{cases}c_{-1}+4c_0+c_1=6a_0\\-c_{-1}+c_1=2a_1h\\c_{-1}-2c_0+c_1=a_2h^2\end{cases}\\&\Rightarrow\begin{cases}c_{-1}=a_0-a_1h+\frac{1}{3}a_2h^2,\\c_0=a_0-\frac{1}{6}a_2h^2,\\c_1=a_0+a_1h+\frac{1}{3}a_2h^2.\end{cases}\end{aligned} \tag{10}$$

Similarly,

$$\begin{cases}y(b)=b_0\\y'(b)=b_1\end{cases}\Rightarrow\begin{cases}c_{n-1}+4c_n+c_{n+1}=6b_0,\\-c_{n-1}+c_{n+1}=2b_1h.\end{cases} \tag{11}$$

By (3), we can construct an approximate formula for $y^{(5)}(a)=g_0-f_0a_0$ as follows

Table 3 The approximate values of $y^{(4)}(x_j)$ and $y^{(5)}(x_j)$

	$y^{(4)}(x_j)$	$y^{(5)}(x_j)$
Approximate value	$s^{(4)}(x_j)=\frac{M_{j+1}-2M_j+M_{j-1}}{h^2}$	$s^{(5)}(x_j)=\frac{M_{j+2}-2M_{j+1}+2M_{j-1}-M_{j-2}}{2h^3}$
Representation in c_j	$\frac{c_{j+2}-4c_{j+1}+6c_j-4c_{j-1}+c_{j-2}}{h^4}$	$\frac{c_{j+3}-4c_{j+2}+5c_{j+1}-5c_{j-1}+4c_{j-2}-c_{j-3}}{2h^5}$
Error order	$O(h^4)$	$O(h^2)$

$$\frac{-5M_0+18M_1-24M_2+14M_3-3M_4}{2h^3}=y^{(5)}(a)+O(h^2), \tag{12}$$

where the coefficients are determined by maximizing the error order. By Table 2, we turn (12) into:

$$18c_0-60c_1+80c_2-55c_3+20c_4-3c_5=2h^5y^{(5)}(a)+5M_0h^2+O(h^7), \tag{13}$$

where $M_0=y''(a)=a_2$. Take (13), (9) and (11) together, we get n linear equations with c_i $(i=2,3,\dots,n+1)$ as unknowns since c_{-1}, c_0 and c_1 have been yielded from (10). Let $C=[c_2,c_3,\dots,c_{n+1}]^T$, $\widetilde{C}=[\widetilde{c_2},\widetilde{c_3},\dots,\widetilde{c_{n+1}}]^T$, $D=[d_1,d_2,\dots,d_n]^T$ and $E=[e_1,e_2,\dots,e_n]^T$, the linear system can be written in matrix notations

$$(A+h^5FB)C=D+E, \tag{14}$$

dropping the error vector E from (14), we get

$$(A+h^5FB)\widetilde{C}=D, \tag{15}$$

where

$$A=\begin{pmatrix} 80 & -55 & 20 & -3 & & & & & \\ 0 & 15 & -12 & 3 & & & & & \\ -15 & 0 & 15 & -12 & 3 & & & & \\ 12 & -15 & 0 & 15 & -12 & 3 & & & \\ -3 & 12 & -15 & 0 & 15 & -12 & 3 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -3 & 12 & -15 & 0 & 15 & -12 & 3 \\ & & & & & & 1 & 4 & 1 \\ & & & & & & -1 & 0 & 1 \end{pmatrix}, \tag{16}$$

$$F=\begin{pmatrix} 0 & & & & & & \\ & f_2 & & & & & \\ & & f_3 & & & & \\ & & & \ddots & & & \\ & & & & f_{n-2} & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & & & & & & \\ 4 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & 4 & 1 & & \\ & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

and

$$\begin{aligned} d_1 &= 2h^5 y^{(5)}(a) + 5M_0 h^2 - 18c_0 + 60c_1, \\ d_2 &= 6h^5 g_2 + 3c_{-1} - 12c_0 + 15c_1 - h^5 f_2 c_1, \\ d_3 &= 6h^5 g_3 + 3c_0 - 12c_1, \\ d_4 &= 6h^5 g_4 + 3c_1, \\ d_i &= 6h^5 g_i, \quad i = 5, \ldots, n-2, \\ d_{n-1} &= 6b_0, \\ d_n &= 2hb_1, \\ e_i &= O(h^7), \quad i = 1, 2, \ldots, n-2; \; e_{n-1} = e_n = 0. \end{aligned}$$

After solving the linear system (15), $\widetilde{c_i}$ $(i = 2, 3, \ldots, n+1)$ together with $\widetilde{c_{-1}} = c_{-1}$, $\widetilde{c_0} = c_0$ and $\widetilde{c_1} = c_1$ will be used to get the approximate spline solution $s(x) = \sum_{i=-1}^{n+1} \widetilde{c_i} B_i(x)$.

4 Convergence analysis

By (14) and (15), we have

$$(A + h^5 FB)(C - \widetilde{C}) = E. \tag{18}$$

A is invertible, and if we assume that

$$h^5 \|A^{-1}\|_\infty \|F\|_\infty \|B\|_\infty < 1, \tag{19}$$

then $(I + h^5 A^{-1} FB)$ is also nonsingular. Hence, we get

$$C - \widetilde{C} = (I + h^5 A^{-1} FB)^{-1} A^{-1} E. \tag{20}$$

Generally, for $n \geq 7$, we have

$$\|A^{-1}\|_\infty \leq 7.24 \times 10^{-5} n^5 \simeq O(n^5). \tag{21}$$

If $f(x)$ satisfies

$$\|F\|_\infty \leq \sup_{x \in [a,b]} |f(x)| \leq \frac{1}{h^5 \|A^{-1}\|_\infty \|B\|_\infty} \leq \frac{10^5}{7.24 \times 6 \times (b-a)^5}, \tag{22}$$

then (19) holds without fail ($\|B\|_\infty = 6$). By (19) and (20), and note $\|E\|_\infty \le Kh^7$, we get

$$\begin{aligned}\|C-\widetilde{C}\|_\infty &\le \frac{\|A^{-1}\|_\infty\|E\|_\infty}{1-h^5\|A^{-1}\|_\infty\|F\|_\infty\|B\|_\infty}\\ &\le \frac{7.24(b-a)^5K}{10^5-6\times 7.24(b-a)^5\|F\|_\infty}h^2 \simeq O(h^2).\end{aligned}$$

Hence,

$$|s(x)-\widetilde{s(x)}| \le \|C-\widetilde{C}\|_\infty \sum_{i=-1}^{n+1} B_i(x) = \|C-\widetilde{C}\|_\infty \simeq O(h^2). \tag{23}$$

Generally, we have

$$|y(x)-\widetilde{s(x)}| \le |y(x)-s(x)| + |s(x)-\widetilde{s(x)}| \le O(h^4)+O(h^2) \simeq O(h^2). \tag{24}$$

5 Numerical tests

In this section, we give two examples to show the effectiveness of our method. First, consider the following fifth order boundary value problem.

Example 1

$$\begin{cases} y^{(5)}(x)+xy(x) = 5(x-1)\sin x + 5(x-x^2-5)\cos x, & x\in[0,1],\\ y(0)=5, \quad y'(0)=-5, \quad y''(0)=-5,\\ y(1)=0, \quad y'(1)=-5\cos(1), \end{cases}$$

where the analytic solution is $y(x) = 5(1-x)\cos x$. See Table 4 for the maximum absolute errors $MAE[y^{(\mu)}(x_i)] = \max_i |y^{(\mu)}(x_i) - s^{(\mu)}(x_i)|$, where $\mu = 0,1,2,3,4$ and $i = 1,2,\ldots,n-1$. It is easy to observe that the errors decrease like $O(h^2)$ when the original interval is refined, without considering the computer round-off errors. It shows that our method is second order convergent. We know that the numerical difference formulae are very sensitive to the computer round-off errors, so we suggest not using bigger n when computing the approximate value of $y^{(3)}(x_i)$ and $y^{(4)}(x_i)$. For example, see ♠ in Table 4 and Table 6.

Next, we compare our method with the other spline methods. Consider the next fifth order boundary value problem.

Example 2

$$\begin{cases} y^{(5)}(x)-y(x) = -(15+10x)e^x, & x\in[0,1],\\ y(0)=0, \quad y'(0)=1, \quad y''(0)=0,\\ y(1)=0, \quad y'(1)=-e, \end{cases}$$

Table 4 The maximum absolute errors of $y^{(\mu)}(x_i)$ for Example 1

n	$MAE[y(x_i)]$	$MAE[y'(x_i)]$	$MAE[y''(x_i)]$	$MAE[y^{(3)}(x_i)]$	$MAE[y^{(4)}(x_i)]$
10	1.156×10^{-4}	3.836×10^{-4}	1.724×10^{-2}	4.217×10^{-2}	7.001×10^{-2}
20	2.897×10^{-5}	9.340×10^{-5}	4.326×10^{-3}	1.132×10^{-2}	1.782×10^{-2}
40	7.256×10^{-6}	2.321×10^{-5}	1.082×10^{-3}	2.933×10^{-3}	4.501×10^{-3}
80	1.816×10^{-6}	5.793×10^{-6}	2.707×10^{-4}	7.465×10^{-4}	2.062×10^{-3}
160	4.557×10^{-7}	1.452×10^{-6}	6.774×10^{-5}	4.605×10^{-4}	2.767×10^{-1} ♠

Table 5 The maximum absolute errors of $y(x_i)$ for Example 2

	$n = 10$	$n = 20$	$n = 40$
Cubic B-spline method (this paper)	1.84×10^{-4}	4.54×10^{-5}	1.14×10^{-5}
Sixth-degree B-spline method [7]	0.1570	0.0747	0.0208
Sextic spline method [13]	2.76×10^{-3}	2.45×10^{-4}	2.01×10^{-5}
Sextic spline method [19]	\	1.94×10^{-6}	4.84×10^{-7}
Sextic spline method [29]	2.26×10^{-4}	1.33×10^{-5}	5.28×10^{-7}
Nonpolynomial sextic spline method [26]	1.71×10^{-10}	1.86×10^{-13}	1.61×10^{-13}
Quartic spline method [30]	3.60×10^{-3}	5.55×10^{-4}	7.66×10^{-5}
Finite difference method [16]	4.02×10^{-3}	3.91×10^{-3}	1.15×10^{-2}

where $y(x) = x(1-x)e^x$ is the analytic solution. Example 2 has been solved by sixth degree B-spline method [7], sextic spline method [13, 19, 29], non-polynomial sextic spline technique [26] (associated with 1, x, x^2, x^3, x^4, and $\sin x$, $\cos x$, or $\sinh x$, $\cosh x$), quartic spline method [30] and finite difference method [16]. The respective maximum absolute errors of $y(x_i)$ are given in Table 5.

At first glance, our method seems to be inferior to the methods in [19, 26, 29]. However, considering our method is based on cubic B-spline (lower degree) while the methods in [19, 29] are based on sextic spline (higher degree) and the non-polynomial spline technique [26] is based on the bases 1, x, x^2, x^3, x^4, $\sin x$ and $\cos x$ or $\sinh x$, $\cosh x$, we remark that our method has lower computational complexity. At the same time, from Table 4 and Table 5, we find the approximation errors of our method can be well accepted in fact. From this point of view, our method has its superiorities over them. In addition, we point out that the result of our method is an approximate cubic spline function of the analytic solution on the global interval $[a, b]$, while the results of [7, 13, 26, 29, 30] are only the approximate values at the knots of the analytic solution. Furthermore, our method is also applicable to the general linear fifth order boundary value problems (see next section), while some of the others methods are only limited to the special cases (1). In a word, our new method is able to compete with the other methods in practical applications.

6 Further discussion

6.1 General linear fifth order boundary value problems

In this subsection, we aim to solve the linear fifth order boundary value problems in the form of

$$\begin{cases} y^{(5)}(x) + p(x)y^{(4)}(x) + q(x)y^{(3)}(x) + u(x)y''(x) + v(x)y'(x) \\ \quad + f(x)y(x) = g(x), \quad x \in [a,b], \\ y(a) = a_0, \qquad y'(a) = a_1, \qquad y''(a) = a_2, \qquad y(b) = b_0, \\ y'(b) = b_1, \end{cases} \tag{25}$$

where $p(x)$, $q(x)$, $u(x)$, $v(x)$, $f(x)$ and $g(x)$ are continuous on $[a,b]$, a_0, a_1, a_2, and b_0, b_1 are given real constants. Let $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ be the approximate solution of (25), discretize (25) at the inter knots ($i = 2, 3, \dots, n-2$), we get

$$y^{(5)}(x_i) + p_i y^{(4)}(x_i) + q_i y^{(3)}(x_i) + u_i y''(x_i) + v_i y'(x_i) + f_i y(x_i) = g_i, \tag{26}$$

where $p_i = p(x_i)$, $q_i = q(x_i)$, $u_i = u(x_i)$, $v_i = v(x_i)$, $f_i = f(x_i)$ and $g_i = g(x_i)$ for short. By using the formulae in Table 2 and Table 3, and dropping $O(h^2)$ directly, we have

$$\begin{aligned} &\frac{c_{i+3} - 4c_{i+2} + 5c_{i+1} - 5c_{i-1} + 4c_{i-2} - c_{i-3}}{2h^5} \\ &\quad + p_i \frac{c_{i+2} - 4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}}{h^4} \\ &\quad + q_i \frac{c_{i+2} - 2c_{i+1} + 2c_{i-1} - c_{i-2}}{2h^3} + u_i \frac{c_{i-1} - 2c_i + c_{i+1}}{h^2} \\ &\quad + v_i \frac{c_{i+1} - c_{i-1}}{2h} + f_i \frac{c_{i-1} + 4c_i + c_{i+1}}{6} = g_i. \end{aligned}$$

Change it equivalently, we yield

$$\begin{aligned} &3(c_{i+3} - 4c_{i+2} + 5c_{i+1} - 5c_{i-1} + 4c_{i-2} - c_{i-3}) \\ &\quad + 6hp_i(c_{i+2} - 4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}) \\ &\quad + 3h^2 q_i(c_{i+2} - 2c_{i+1} + 2c_{i-1} - c_{i-2}) \\ &\quad + 6h^3 u_i(c_{i-1} - 2c_i + c_{i+1}) + 3h^4 v_i(c_{i+1} - c_{i-1}) \\ &\quad + h^5 f_i(c_{i-1} + 4c_i + c_{i+1}) = 6g_i h^5. \end{aligned} \tag{27}$$

This is a linear system with $n-3$ linear equations ($i = 2, 3, \dots, n-2$) in $n+3$ unknowns c_i ($i = -1, 0, 1, \dots, n+1$), where c_{-1}, c_0 and c_1 can be obtained by (10). Another two equations are determined by the boundary conditions at $x = b$, i.e. (11). We still need one more equation. We drive it as follows. Similarly to (12), by (3), we also have

$$y^{(4)}(a) = \frac{2M_0 - 5M_1 + 4M_2 - M_3}{h^2} + O(h^2),$$

$$y^{(3)}(a) = \frac{-3M_0 + 4M_1 - M_2}{2h} + O(h^2). \tag{28}$$

Substitute (12) and (28) into the following equation

$$y^{(5)}(a) + p_0 y^{(4)}(a) + q_0 y^{(3)}(a) + u_0 y''(a) + v_0 y'(a) + f_0 y(a) = g_0, \tag{29}$$

we get ($O(h^2)$ is dropped)

$$\frac{-5M_0 + 18M_1 - 24M_2 + 14M_3 - 3M_4}{2h^3} + p_0 \frac{2M_0 - 5M_1 + 4M_2 - M_3}{h^2}$$
$$+ q_0 \frac{-3M_0 + 4M_1 - M_2}{2h} = g_0 - f_0 a_0 - v_0 a_1 - u_0 a_2,$$

as a result, we have

$$\begin{aligned}
&(18c_0 - 60c_1 + 80c_2 - 55c_3 + 20c_4 - 3c_5) \\
&\quad + 2hp_0(-5c_0 + 14c_1 - 14c_2 + 6c_3 - c_4) \\
&\quad + h^2 q_0(4c_0 - 9c_1 + 6c_2 - c_3) \\
&= 2h^5(g_0 - f_0 a_0 - v_0 a_1 - u_0 a_2) + a_2(5h^2 - 4h^3 p_0 + 3h^4 q_0).
\end{aligned} \tag{30}$$

Take (30), (27) and (11) together, we obtain the following linear system

$$(A + 6hPB_1 + 3h^2 QB_2 + 6h^3 UB_3 + 3h^4 VB_4 + h^5 FB)C = W, \tag{31}$$

where A, F and B have been given in (16) and (17), and

$$B_1 = \begin{pmatrix}
-\frac{14}{3} & 2 & -\frac{1}{3} & & & & & \\
6 & -4 & 1 & & & & & \\
-4 & 6 & -4 & 1 & & & & \\
1 & -4 & 6 & -4 & 1 & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & 1 & -4 & 6 & -4 & 1 & \\
 & & & & & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0
\end{pmatrix},$$

$$B_2 = \begin{pmatrix}
2 & -\frac{1}{3} & & & & & & \\
0 & -2 & 1 & & & & & \\
2 & 0 & -2 & 1 & & & & \\
-1 & 2 & 0 & -2 & 1 & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & -1 & 2 & 0 & -2 & 1 & \\
 & & & & & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0
\end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & & & & & & \\ -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & \\ & & & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & & & & & & \\ 0 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -1 & 0 & 1 & & \\ & & & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned}
P &= \mathrm{diag}(p_0, p_2, p_3, \ldots, p_{n-2}, 0, 0), \\
Q &= \mathrm{diag}(q_0, q_2, q_3, \ldots, q_{n-2}, 0, 0), \\
U &= \mathrm{diag}(0, u_2, u_3, \ldots, u_{n-2}, 0, 0), \\
V &= \mathrm{diag}(0, v_2, v_3, \ldots, v_{n-2}, 0, 0), \\
C &= [c_2, c_3, \ldots, c_{n+1}]^T, \\
W &= [w_1, w_2, \ldots, w_n]^T, \\
w_1 &= 2h^5(g_0 - f_0a_0 - v_0a_1 - u_0a_2) + a_2(5h^2 - 4h^3p_0 + 3h^4q_0) \\
&\quad - c_0(18 - 10hp_0 + 4h^2q_0) - c_1(-60 + 28hp_0 - 9h^2q_0), \\
w_2 &= 6h^5g_2 + (3c_{-1} - 12c_0 + 15c_1) \\
&\quad - c_0(6hp_2 - 3h^2q_2) - c_1(-24hp_2 + 6h^2q_2 + 6h^3u_2 - 3h^4v_2 + h^5f_2), \\
w_3 &= 6h^5g_3 + (3c_0 - 12c_1) - c_1(6hp_3 - 3h^2q_3), \\
w_4 &= 6h^5g_4 + 3c_1, \\
w_i &= 6h^5g_i, \quad i = 5, \ldots, n-2, \\
w_{n-1} &= 6b_0, \\
w_n &= 2hb_1.
\end{aligned}$$

By (10) and (31), we get the unknowns c_i $(i = -1, 0, 1, \ldots, n+1)$, so, the approximate spline $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ can be yielded.

In the following, we use this method to solve the following example.

Table 6 The maximum absolute errors of $y^{(\mu)}(x_i)$ for Example 3

n	$MAE[y(x_i)]$	$MAE[y'(x_i)]$	$MAE[y''(x_i)]$	$MAE[y^{(3)}(x_i)]$	$MAE[y^{(4)}(x_i)]$
10	1.388×10^{-5}	5.011×10^{-5}	1.596×10^{-2}	1.884×10^{-2}	3.037×10^{-2}
20	3.540×10^{-6}	1.360×10^{-5}	4.028×10^{-3}	4.650×10^{-3}	8.882×10^{-3}
40	8.879×10^{-7}	3.572×10^{-6}	1.009×10^{-3}	1.159×10^{-3}	2.361×10^{-3}
80	2.222×10^{-7}	9.039×10^{-7}	2.525×10^{-4}	2.903×10^{-4}	7.345×10^{-4}
160	5.525×10^{-8}	2.267×10^{-7}	6.317×10^{-5}	1.209×10^{-4}	5.359×10^{-2} ♠

Example 3

$$\begin{cases} y^{(5)}(x) + (x-2)y^{(4)}(x) + 2y^{(3)}(x) - (x^2+2x-1)y''(x) \\ \qquad + (2x^2+4x)y'(x) - 2x^2 y(x) \\ \quad = 4e^x \cos x - 2x^4 + 4x^3 + 6x^2 - 4x + 2, \quad x \in [0,1], \\ y(0) = 0, \quad y'(0) = 2, \quad y''(0) = 6, \\ y(1) = 1 + 2e \sin 1, \quad y'(1) = 2e(\sin 1 + \cos 1) + 2, \end{cases}$$

where the analytic solution is $y(x) = 2e^x \sin x + x^2$. The maximum absolute errors $MAE[y^{(\mu)}(x_i)]$ ($\mu = 0, 1, 2, 3, 4$, $i = 1, 2, \ldots, n-1$) are summarized in Table 6. It is easy to observe that the errors decrease by $\frac{1}{4}$ when the original interval is refined by $\frac{1}{2}$.

6.2 General nonlinear fifth order boundary value problems

For the sake of integrity, in this subsection, we briefly discuss the nonlinear fifth order boundary value problems in the following form

$$\begin{cases} y^{(5)}(x) = f(x, y(x), y'(x), y''(x), y^{(3)}(x), y^{(4)}(x)), \quad x \in [a,b], \\ y(a) = a_0, \quad y'(a) = a_1, \quad y''(a) = a_2, \quad y(b) = b_0, \quad y'(b) = b_1, \end{cases} \tag{32}$$

where a_0, a_1, a_2, and b_0, b_1 are given real constants, and $f(x, y(x), y'(x), y''(x), y^{(3)}(x), y^{(4)}(x))$ is continuous on $[a,b]$. Similarly, let $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ be the approximate solution of (32), discretize it at the inter knots ($i = 2, 3, \ldots, n-2$), we get

$$y^{(5)}(x_i) = f(x_i, y(x_i), y'(x_i), y''(x_i), y^{(3)}(x_i), y^{(4)}(x_i)).$$

By Table 2 and Table 3, we yield

$$\begin{aligned} &\frac{c_{i+3} - 4c_{i+2} + 5c_{i+1} - 5c_{i-1} + 4c_{i-2} - c_{i-3}}{2h^5} \\ &\quad = f\Bigg(x_i, \frac{c_{i-1} + 4c_i + c_{i+1}}{6}, \frac{c_{i+1} - c_{i-1}}{2h}, \frac{c_{i-1} - 2c_i + c_{i+1}}{h^2}, \\ &\frac{c_{i+2} - 2c_{i+1} + 2c_{i-1} - c_{i-2}}{2h^3}, \frac{c_{i+2} - 4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}}{h^4}\Bigg). \end{aligned}$$

Table 7 The absolute errors of $y(x)$ at knots for Example 4

x	Our errors	Errors of [22]	Errors of [23]	Errors of [32]	Errors of [7]
0.0	0.0	0.0	0.0	0.0	0.0
0.1	1.3×10^{-7}	1.0×10^{-9}	2.3×10^{-7}	0.0	7.0×10^{-4}
0.2	4.2×10^{-7}	2.0×10^{-9}	1.6×10^{-6}	1.0×10^{-5}	7.2×10^{-4}
0.3	7.2×10^{-7}	1.0×10^{-8}	4.6×10^{-6}	1.0×10^{-5}	4.1×10^{-4}
0.4	9.4×10^{-7}	2.0×10^{-8}	8.9×10^{-6}	1.0×10^{-4}	4.6×10^{-4}
0.5	1.0×10^{-6}	3.1×10^{-8}	1.3×10^{-5}	3.2×10^{-4}	4.7×10^{-4}
0.6	9.3×10^{-7}	3.7×10^{-8}	1.6×10^{-5}	3.6×10^{-4}	4.8×10^{-4}
0.7	7.1×10^{-7}	4.1×10^{-8}	1.6×10^{-5}	1.4×10^{-4}	3.9×10^{-4}
0.8	4.1×10^{-7}	3.1×10^{-8}	1.2×10^{-5}	3.1×10^{-4}	3.1×10^{-4}
0.9	1.3×10^{-7}	1.4×10^{-8}	5.1×10^{-6}	5.8×10^{-4}	1.6×10^{-4}
1.0	0.0	0.0	0.0	9.9×10^{-5}	0.0

Note these nonlinear equations ($i = 2, 3, \dots, n-2$) as

$$\frac{c_{i+3} - 4c_{i+2} + 5c_{i+1} - 5c_{i-1} + 4c_{i-2} - c_{i-3}}{2h^5} = \Phi(c_{i-2}, c_{i-1}, c_i, c_{i+1}, c_{i+2}). \tag{33}$$

Besides, we also have five linear equations obtained by the five boundary conditions at $x = a$ and $x = b$, see (10) and (11). Furthermore, by (12), (28) and $y^{(5)}(a) = f(a, a_0, a_1, a_2, y^{(3)}(a), y^{(4)}(a))$, we have another nonlinear equation as follows

$$\begin{aligned} &\frac{-5a_2h^2 + 18c_0 - 60c_1 + 80c_2 - 55c_3 + 20c_4 - 3c_5}{2h^5} \\ &\quad = f\Bigg(a, a_0, a_1, a_2, \frac{-3a_2h^2 + 4c_0 - 9c_1 + 6c_2 - c_3}{2h^3}, \\ &\qquad \frac{2a_2h^2 - 5c_0 + 14c_1 - 14c_2 + 6c_3 - c_4}{h^4}\Bigg). \end{aligned} \tag{34}$$

Take (10), (34), (33) and (11) together, we get a nonlinear system with c_i ($i = -1, 0, 1, \dots, n+1$) as unknowns. After solving this nonlinear system, we get $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$. For the sake of comparison, we give a same example studied by many authors.

Example 4

$$\begin{cases} y^{(5)}(x) = e^{-x} y^2(x), \quad x \in [0, 1], \\ y(0) = y'(0) = y''(0) = 1, \\ y(1) = y'(1) = e, \end{cases}$$

where the analytic solution is $y(x) = e^x$.

We list the respective absolute errors of $y(x)$ at different knots in Table 7. Our errors are obtained by Matlab with $n = 50$, and the else errors are cited directly from the corresponding papers, where the errors of [22] are obtained by a polynomial of

Table 8 The absolute errors of $y'(x)$ and $y''(x)$ at knots for Example 4

x	Our errors of $y'(x)$	Errors of $y'(x)$ by [22]	Our errors of $y''(x)$	Errors of $y''(x)$ by [22]
0.0	0.00	0.00	0.00	0.00
0.1	2.38×10^{-6}	1.37×10^{-8}	2.18×10^{-5}	2.50×10^{-7}
0.2	3.15×10^{-6}	4.40×10^{-8}	3.97×10^{-5}	3.09×10^{-7}
0.3	2.72×10^{-6}	5.06×10^{-8}	5.39×10^{-5}	4.67×10^{-7}
0.4	1.52×10^{-6}	1.90×10^{-7}	6.43×10^{-5}	5.73×10^{-6}
0.5	5.93×10^{-8}	1.65×10^{-6}	7.12×10^{-5}	2.82×10^{-5}
0.6	1.61×10^{-6}	7.47×10^{-6}	7.48×10^{-5}	1.01×10^{-4}
0.7	2.74×10^{-6}	2.59×10^{-5}	7.52×10^{-5}	2.97×10^{-4}
0.8	3.09×10^{-6}	7.56×10^{-5}	7.25×10^{-5}	7.57×10^{-4}
0.9	2.29×10^{-6}	1.94×10^{-4}	6.71×10^{-5}	1.73×10^{-3}
1.0	0.00	4.52×10^{-4}	5.91×10^{-5}	3.62×10^{-3}

degree 13 (see (5.9), (5.11) and (5.12) of [22] for details). Obviously, our errors are also well accepted. Hence, our method is also effective in solving nonlinear fifth order boundary value problems.

Besides, we also compare our method with [22] for the absolute errors of $y'(x)$ and $y''(x)$, see Table 8, where the errors of $y'(x)$ and $y''(x)$ by [22] are obtained by differentiating the polynomial of degree 13 in [22]. From this table, we find that the errors of $y'(x)$ and $y''(x)$ by [22] are better than ours when $x \in [0.0, 0.5]$; while our errors are better than [22] when $x \in [0.5, 1.0]$. Moreover, we also observe that our errors are well-balanced; while the errors by [22] are increasing.

7 Conclusions

In this paper, we develop a cubic B-spline method for solving fifth order boundary value problems. Our method is very encouraging with second order convergence. The given numerical results show cubic spline is effective in approximating the analytic solution and its derivatives. We believe that cubic spline can also be applied to study other higher order boundary value problems.

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