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A general iterative method for solving equilibrium problems, variational inequality problems and fixed point problems of an infinite family of nonexpansive mappings

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Abstract In this paper, we introduce and analyze a new general iterative scheme by the viscosity approximation method for finding the common element of the set of equilibrium problems, the set of fixed points of an infinite family of nonexpansive mappings and the set solutions of the variational inequality problems for an ξ-inverse-strongly monotone mapping in Hilbert spaces. We show that the sequence converge strongly to a common element of the above three sets under some parameters controlling conditions. The result extends and improves a recent result of Chang et al. (Nonlinear Anal. 70:3307–3319, 2009) and many others.

Keywords Nonexpansive mapping · ξ-inverse-strongly monotone mapping · Variational inequality problem · Equilibrium problem · Fixed points

Mathematics Subject Classification (2000) 46C05 · 47H09 · 47H10

1 Introduction

Equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization which has been extended and generalized in many directions using novel and innovative technique, see [1, 2]. Related

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to the equilibrium problems, we also have the problem of finding the fixed points of the nonexpansive mappings. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems and a set of the fixed points of finitely many nonexpansive mappings, see [4, 22, 28] and the references therein. In this paper, we suggest and analyze a new general iterative scheme by the viscosity approximation method for finding a common element of a set of the solutions of equilibrium problems, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of the variational inequality problems for an ξ-inverse-strongly monotone mappings in Hilbert spaces.

Let H be a real Hilbert space and let E be a nonempty closed convex subset of H. Let $S : E \longrightarrow E$ be a mapping. In the sequel, we will use $F(S)$ to denote the set of *fixed points* of S; that is, $F(S) = \{x \in E : Sx = x\}$. It is well known that if $E \subset H$ is nonempty, bounded, closed and convex and S is a nonexpansive self-mapping on E then $F(S)$ is nonempty; see, e.g., [20]. In addition, let a mapping $S : E \longrightarrow E$ is called *nonexpansive*, if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in E$. Recall also that a self-mapping $f : H \longrightarrow H$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in H$. In addition, let $B : E \longrightarrow H$ be a nonlinear mapping. The classical variational inequality which is denoted by $VI(E, B)$, is to find $x \in E$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in E. \tag{1.1}$$

The variational inequality problem has been extensively studied in literatures, see, e.g. [5, 27] and reference therein. Let F be a bifunction of $E \times E$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and let $F : E \times E \longrightarrow \mathbb{R}$ be an equilibrium function, that is,

$$F(x, x) = 0, \quad \forall x \in E.$$

The equilibrium problem is defined as follows:

$$\text{Find} \quad x \in E \quad \text{such that} \quad F(x, y) \geq 0, \quad \forall y \in E. \tag{1.2}$$

A solution of (1.2) is said to be an equilibrium point and the set of the equilibrium points is denoted by $EP(F)$, i.e.,

$$EP(F) = \big\{x \in E : F(x, y) \geq 0, \forall y \in E\big\}. \tag{1.3}$$

Given a mapping $T : E \longrightarrow H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in E$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in E$. Numerous problems in physics, saddle point problems, fixed point problems, variational inequality problems, optimization, and economics are reduce to find a solution of (1.2). Some methods have been proposed to solve the equilibrium problem; see, for instance [4, 10, 15]. Recently, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

We now recall some well-known concepts and results as follows.
Recall that the (nearest point) projection P_E from H onto E assigns to each $x \in H$

the unique point in $P_E x \in E$ satisfying the property

$$\|x - P_E x\| = \min_{y \in E} \|x - y\|.$$

The following characterizes the projection P_E.

Lemma 1.1 *For a given $x \in H$, $z \in E$ satisfies the inequality*

$$\langle z - x, y - z \rangle \geq 0, \quad \forall y \in E,$$

if and only if $z = P_E x$. It is well known that P_E is a firmly nonexpansive mapping of H onto E and satisfies

$$\|P_E x - P_E y\|^2 \leq \langle P_E x - P_E y, x - y \rangle, \quad \forall x, y \in H. \tag{1.4}$$

Moreover, $P_E x$ is characterized by the following properties: $P_E x \in E$ and for all $x \in H$, $y \in E$,

$$\langle x - P_E x, y - P_E x \rangle \leq 0. \tag{1.5}$$

Using Lemma 1.1, one can see that the variational inequality (1.1) is equivalent to a fixed point problem

Lemma 1.2 *The element $u \in E$ is a solution of the variational inequality (1.1) if and only if $u \in E$ satisfies the relation $u = P_E(u - \lambda Bu)$, where $\lambda > 0$ is a constant.*

It is clear from Lemma 1.2 that variational inequality and fixed point problem are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Definition 1.3 Let $B : E \longrightarrow H$ be a mapping. Then B is called

(1) *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in E.$$

(2) *ξ-strongly monotone* (see [3, 11]) if there exists a constant $\xi > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \xi \|x - y\|^2, \quad \forall x, y \in E.$$

(3) *k-Lipschitz continuous* if there exists a positive real number k such that

$$\|Bx - By\| \leq k\|x - y\|, \quad \forall x, y \in E.$$

(4) *ξ-inverse-strongly monotone* (see [3, 11]) if there exists a constant $\xi > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \xi \|Bx - By\|^2, \quad \forall x, y \in E.$$

Remark 1.4 It is obvious that any ξ-inverse-strongly monotone mapping B is monotone and $\frac{1}{\xi}$-Lipschitz continuous.

(5) A set-valued mapping $T : H \longrightarrow 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.
(6) A monotone mapping $T : H \longrightarrow 2^H$ is *maximal* if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of E into H and let $N_E v$ be the *normal cone* to E at $v \in E$, i.e., $N_E v = \{w \in H : \langle u - v, w\rangle \geq 0, \forall u \in E\}$ and define

$$Tv = \begin{cases} Bv + N_E v, & v \in E; \\ \emptyset, & v \notin E. \end{cases}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(E, B)$ (see, for example, [16]).

For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequalities for an ξ-inverse-strongly monotone, Takahashi and Toyoda [21] introduced the following iterative scheme:

$$\begin{cases} x_0 \in E \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_E(x_n - \lambda_n Bx_n), \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where B is an ξ-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\xi)$. They showed that if $F(S) \cap VI(E, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.6) converges weakly to some $z \in F(S) \cap VI(E, B)$. Recently, Iiduka and Takahashi [9] proposed a new iterative scheme as following

$$\begin{cases} x_0 = x \in E \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_E(x_n - \lambda_n Bx_n), \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where B is an ξ-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\xi)$. They showed that if $F(S) \cap VI(E, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to some $z \in F(S) \cap VI(E, B)$. Very recently, Su et al. [17] proposed the following new iterative scheme:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ z_n = \gamma_n x_n + (1 - \gamma_n) SP_E(x_n - \tau_n Bx_n), \\ y_n = \beta_n x_n + (1 - \beta_n) SP_E(z_n - \mu_n Bz_n), \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_E(y_n - \eta_n By_n), \quad \forall n \geq 1, \end{cases} \tag{1.8}$$

for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality for an ξ-inverse-strongly monotone mapping in a real Hilbert space, and a strong convergence theorem is established.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see e.g., [8, 23, 24, 26] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences.

A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F}\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle, \tag{1.9}$$

where F is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. Assume that A is *strongly positive* on H if there is a constant $\bar{\gamma}>0$ with property

$$\langle Ax,x\rangle\geq\bar{\gamma}\|x\|^2,\quad \forall x\in H. \tag{1.10}$$

Moreover, it is shown in [12] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1}=\epsilon_n\gamma f(x_n)+(1-\epsilon_n A)Sx_n, \tag{1.11}$$

converges strongly to $z=P_{F(S)}(I-A+\gamma f)(z)$. Recently, Plubtieng and Punpaeng [13] proposed the following iterative algorithm:

$$\begin{cases}F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, & \forall y\in H,\\ x_{n+1}=\epsilon_n\gamma f(x_n)+(I-\epsilon_n A)Su_n.\end{cases} \tag{1.12}$$

They prove that if the sequence $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle(A-\gamma f)z,x-z\rangle\geq 0,\quad z\in F(S)\cap EP(F), \tag{1.13}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap EP(F)}\frac{1}{2}\langle Ax,x\rangle-h(x), \tag{1.14}$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

Furthermore, for finding approximate common fixed points of an infinite countable family of nonexpansive mappings $\{T_n\}$ under very mild conditions on the parameters.

Let $T_1,T_2,T_3,\ldots$ be an infinite family of nonexpansive mappings of E into itself and let $\mu_1,\mu_2,\mu_3,\ldots$ be real numbers such that $0\leq\mu_i\leq 1$ for every $i\in\mathbb{N}$. For any $n\in\mathbb{N}$. Shimoji and Takahashi [19], define a mapping W_n of E into E as follows:

$$\begin{aligned}U_{n,n+1}&=I,\\ U_{n,n}&=\mu_nT_nU_{n,n+1}+(1-\mu_n)I,\\ U_{n,n-1}&=\mu_{n-1}T_{n-1}U_{n,n}+(1-\mu_{n-1})I,\\ &\vdots\\ U_{n,k}&=\mu_kT_kU_{n,k+1}+(1-\mu_k)I, \qquad (1.15)\\ U_{n,k-1}&=\mu_{k-1}T_{k-1}U_{n,k}+(1-\mu_{k-1})I,\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I.
\end{aligned}
$$

Such a mappings W_n is nonexpansive from E to E and it is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

On the other hand, Colao et al. [7] introduced and considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.2) and the set of common fixed points of a finite family of nonexpansive mappings on E. Starting with an arbitrary initial $x_0 \in E$, define a sequence $\{x_n\}$ recursively by

$$
\begin{cases}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\
x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)W_n u_n,
\end{cases}
\tag{1.16}
$$

where $\{\epsilon_n\}$ be a sequences in $(0,1)$. It is proved [7] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.16) converges strongly to $z \in \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F)$, where z is an equilibrium point for F and is the unique solution of the variational inequality (1.13), i.e., $z = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap EP(F)}(I - (A - \gamma f))z$. Very recently, Chang et al. [6] introduced an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.2) and the set of fixed points of an infinite family of nonexpansive mappings in a Hilbert space. Starting with an arbitrary initial $x_0 \in E$, define a sequence $\{x_n\}$,$\{k_n\}$,$\{y_n\}$ and $\{u_n\}$ recursively by

$$
\begin{cases}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E, \\
y_n = P_E(u_n - \lambda_n B u_n), \\
k_n = P_E(y_n - \lambda_n B y_n), \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, \quad \forall n \geq 1,
\end{cases}
\tag{1.17}
$$

where $\{W_n\}$ is the sequence generated by (1.15), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, $\{\lambda_n\}$ in a sequence in $[a,b] \subset (0, 2\xi)$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ and $\{u_n\}$ generated by (1.17) converge strongly to $z \in \bigcap_{n=1}^{\infty} F(T_n) \cap VI(E,B) \cap EP(F)$, where $z = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap VI(E,B)\cap EP(F)} f(z)$.

In this paper, motivated by iterative schemes considered in (1.8), (1.16) and (1.17) we will introduce a new iterative process (3.1) below for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of variational inequality (3.2) for an ξ-inverse-strongly monotone mapping in Hilbert spaces. Then, we prove a strong convergence theorem which is connected with Chang, Lee and Chan [6], Colao, Marino and Xu [7] and Su, Shang and Qin [17] and many others.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let E be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\longrightarrow$, respectively.

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (1.15). Then we can have the following crucial conclusions concerning W_n. You can find them in [19]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.1 [19] *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, for every $x \in E$ and $k \in \mathbb{N}$, the limit $\lim_{n\longrightarrow\infty} U_{n,k}x$ exists.*

Using Lemma 2.1, one can define a mapping W of E into itself as follows:

$$Wx = \lim_{n\longrightarrow\infty} W_n x = \lim_{n\longrightarrow\infty} U_{n,1}x, \tag{2.1}$$

for every $x \in E$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$. Throughout this paper, we will assume that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, we have the following results.

Lemma 2.2 [19] *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.3 [28] *If $\{x_n\}$ is a bounded sequence in E, then $\lim_{n\longrightarrow\infty} \|Wx_n - W_n x_n\| = 0$.*

Lemma 2.4 [14] *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$$

hold for each $y \in H$ with $y \ne x$.

Lemma 2.5 [12] *Let E be a nonempty closed convex subset of H and let f be a contraction of H into itself with $\alpha \in (0,1)$, and A be a strongly positive linear bounded self-adjoint on H with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \ge (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is a strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Lemma 2.6 [12] *Assume A be a strongly positive linear bounded self-adjoint on H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.*

Lemma 2.7 [18] *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\longrightarrow\infty}\beta_n \le \limsup_{n\longrightarrow\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\longrightarrow\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then*, $\lim_{n\longrightarrow\infty}\|z_n - x_n\| = 0$.

Lemma 2.8 [25] *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-l_n)a_n + \sigma_n, \quad n \ge 0,$$

where $\{l_n\}$ *is a sequence in* $(0,1)$ *and* $\{\sigma_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} l_n = \infty$,
(2) $\limsup_{n\longrightarrow\infty}\frac{\sigma_n}{l_n} \le 0$ *or* $\sum_{n=1}^{\infty}|\sigma_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.9 *Let* H *be a real Hilbert space, the following inequality hods*

(1) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$,
(2) $\|x+y\|^2 \ge \|x\|^2 + 2\langle y, x\rangle$,

for all $x, y \in H$.

For solving the equilibrium problem for a bifunction $F : E \times E \longrightarrow R$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in E$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \le 0$, $\forall x, y \in E$;
(A3) $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x,y)$, $\forall x, y, z \in E$;
(A4) for each $x \in E$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

Lemma 2.10 [2] *Let* E *be a nonempty closed convex subset of* H *and let* F *be a bifunction of* $E \times E$ *into* $\mathbb{R}$ *satisfying* (A1)–(A4). *Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in E$ *such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in E.$$

The following lemma was also given in [4].

Lemma 2.11 [4] *Assume that* $F : E \times E \longrightarrow R$ *satisfies* (A1)–(A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \longrightarrow E$ *as follows*:

$$T_r(x) = \left\{z \in E : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in E\right\}$$

for all $z \in H$. *Then, the following hold*:

(1) T_r *is single-valued*;

(2) T_r *is firmly nonexpansive, i.e.*,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H;$$

(3) $F(T_r) = EP(F)$; *and*
(4) $EP(F)$ *is closed and convex.*

3 Main results

In this section, we prove the strong convergence theorem for an infinite family of nonexpansive mappings in Hilbert spaces by using the viscosity approximation method.

Theorem 3.1 *Let E be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let $\{T_n\}$ be an infinite family of nonexpansive of E into itself. Let f be a contraction of H into itself with $\alpha \in (0,1)$, B be an ξ-inverse-strongly monotone mapping of E into H such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(E,B) \neq \emptyset$. Let A be a strongly positive linear bounded self-adjoint on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$,$\{y_n\}$,$\{k_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in E \text{ chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E, \\ y_n = \varphi_n u_n + (1-\varphi_n) P_E(u_n - \delta_n B u_n), \\ k_n = \alpha_n x_n + (1-\alpha_n) P_E(y_n - \lambda_n B y_n), \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big) W_n P_E(k_n - \tau_n B k_n), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{W_n\}$ is the sequence generated by (1.15) *and $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\varphi_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfy the following conditions*:

(i) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$, $\sum_{n=1}^{\infty} \epsilon_n = \infty$,
(ii) $\lim_{n\longrightarrow\infty} \alpha_n = \lim_{n\longrightarrow\infty} \varphi_n = 0$,
(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,
(iv) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$,
(v) $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\longrightarrow\infty} |\delta_{n+1} - \delta_n| = 0$ *and* $\lim_{n\longrightarrow\infty} |\tau_{n+1} - \tau_n| = 0$,
(vi) $\{\tau_n\}$, $\{\lambda_n\}$, $\{\delta_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\xi)$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Omega$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in \Omega. \tag{3.2}$$

Equivalently, we have $z = P_\Omega(I - A + \gamma f)(z)$.

Proof Since $\epsilon_n \longrightarrow 0$ by the condition (i), we may assume, without loss of generality, that $\epsilon_n \le (1-\beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.6, we know that if $0 \le \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$. We will assume that $\|I - A\| \le 1 - \bar{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I - \epsilon_n A)x, x\rangle &= 1 - \beta_n - \epsilon_n\langle Ax, x\rangle \\ &\ge 1 - \beta_n - \epsilon_n\|A\| \ge 0\end{aligned}$$

this show that $(1-\beta_n)I - \epsilon_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \epsilon_n A\| &= \sup\{|\langle((1-\beta_n)I - \epsilon_n A)x, x\rangle| : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \epsilon_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\le 1 - \beta_n - \epsilon_n\bar{\gamma}.\end{aligned}$$

Let $Q = P_\Omega$, where $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(E, B)$. Note that f be a contraction of H into itself with $\alpha \in (0, 1)$. Then, we have

$$\begin{aligned}&\|Q(I - A + \gamma f)(x) - Q(I - A + \gamma f)(y)\| \\ &\quad = \|P_\Omega(I - A + \gamma f)(x) - P_\Omega(I - A + \gamma f)(y)\| \\ &\quad \le \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\ &\quad \le \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\ &\quad \le (1 - \bar{\gamma})\|x - y\| + \gamma\alpha\|x - y\| \\ &\quad = (1 - \bar{\gamma} + \gamma\alpha)\|x - y\| \\ &\quad = (1 - (\bar{\gamma} - \alpha\gamma))\|x - y\|, \quad \forall x, y \in H.\end{aligned}$$

Since $0 < 1 - (\bar{\gamma} - \alpha\gamma) < 1$, it follows that $Q(I - A + \gamma f)$ is a contraction of H into itself. Therefore by the Banach Contraction Mapping Principle, which implies that there exists a unique element $z \in H$ such that $z = Q(I - A + \gamma f)(z) = P_\Omega(I - A + \gamma f)(z)$. First, we show that $I - \tau_n B$ is nonexpansive. Actually, for any $x, y \in E$, from the ξ-inverse-strongly monotone mapping definition on B and condition (vi), we have

$$\begin{aligned}&\|(I - \tau_n B)x - (I - \tau_n B)y\|^2 \\ &\quad = \|(x - y) - \tau_n(Bx - By)\|^2 \\ &\quad = \|x - y\|^2 - 2\tau_n\langle x - y, Bx - By\rangle + \tau_n^2\|Bx - By\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \|x-y\|^2 - 2\tau_n\xi\|Bx - By\| + \tau_n^2\|Bx - By\|^2 \\
&= \|x-y\|^2 + \tau_n(\tau_n - 2\xi)\|Bx - By\|^2 \\
&\leq \|x-y\|^2. \qquad (3.3)
\end{aligned}
$$

This implies that $I - \tau_n B$ is nonexpansive, and so are $I - \lambda_n B$ and $I - \delta_n B$.

We divide the proof of Theorem 3.1 into five steps.

Step 1. We claim that $\{x_n\}$ is bounded.

Indeed, let $p \in \Omega$ and let T_{r_n} be a sequence of mappings defined as in Lemma 2.11. Then $p = T_{r_n} p$. We note that $u_n = T_{r_n} x_n$. If follows that

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|.$$

Since $I - \lambda_n B$ and $I - \delta_n B$ are be nonexpansive and $p = P_E(p - \lambda_n Bp) = P_E(p - \delta_n Bp)$, we have

$$
\begin{aligned}
\|y_n - p\| &= \|\varphi_n(u_n - p) + (1-\varphi_n)(P_E(u_n - \delta_n Bu_n) - p)\| \\
&= \|\varphi_n(u_n - p) + (1-\varphi_n)(P_E(u_n - \delta_n Bu_n) - P_E(p - \delta_n Bp))\| \\
&\leq \varphi_n\|u_n - p\| + (1-\varphi_n)\|(u_n - \delta_n Bu_n) - (p - \delta_n Bp)\| \\
&= \varphi_n\|u_n - p\| + (1-\varphi_n)\|(I - \delta_n B)u_n - (I - \delta_n B)p)\| \\
&\leq \varphi_n\|u_n - p\| + (1-\varphi_n)\|u_n - p)\| \\
&= \|u_n - p\| \leq \|x_n - p\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|k_n - p\| &= \|\alpha_n(x_n - p) + (1-\alpha_n)(P_E(y_n - \lambda_n By_n) - p)\| \\
&= \|\alpha_n(x_n - p) + (1-\alpha_n)(P_E(y_n - \lambda_n By_n) - P_E(p - \lambda_n Bp))\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|(y_n - \lambda_n By_n) - (p - \lambda_n Bp)\| \\
&= \alpha_n\|x_n - p\| + (1-\alpha_n)\|(I - \lambda_n B)y_n - (I - \lambda_n B)p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| = \|x_n - p\|,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\epsilon_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) \\
&\quad + ((1-\beta_n)I - \epsilon_n A)(W_n P_E(k_n - \tau_n Bk_n) - p)\| \\
&\leq (1 - \beta_n - \epsilon_n\bar{\gamma})\|P_E(k_n - \tau_n Bk_n) - p\| \\
&\quad + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&= (1 - \beta_n - \epsilon_n\bar{\gamma})\|P_E(I - \tau_n B)k_n - p\|
\end{aligned}
$$

$$\begin{aligned}
&+ \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\leq (1 - \beta_n - \epsilon_n\bar{\gamma})\|k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\leq (1 - \beta_n - \epsilon_n\bar{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\leq (1 - \epsilon_n\bar{\gamma})\|x_n - p\| + \epsilon_n\gamma\|f(x_n) - f(p)\| + \epsilon_n\|\gamma f(p) - Ap\| \\
&\leq (1 - \epsilon_n\bar{\gamma})\|x_n - p\| + \epsilon_n\gamma\alpha\|x_n - p\| + \epsilon_n\|\gamma f(p) - Ap\| \\
&= (1 - (\bar{\gamma} - \alpha\gamma)\epsilon_n)\|x_n - p\| + (\bar{\gamma} - \alpha\gamma)\epsilon_n\frac{\|f(p) - Ap\|}{\bar{\gamma} - \alpha\gamma} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \alpha\gamma}\right\}.
\end{aligned}$$

By mathematical induction, we obtain

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \alpha\gamma}\right\}, \quad n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{k_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{Bu_n\}$, $\{Bk_n\}$ and $\{By_n\}$.

Step 2. We claim that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$.

On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E \tag{3.4}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in E. \tag{3.5}$$

Putting $y = u_{n+1}$ in (3.4) and $y = u_n$ in (3.5), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from the monotonicity of F, we get

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n|, \qquad (3.6)\end{aligned}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Put $\theta_n = P_E(k_n - \tau_n Bk_n)$, $\phi_n = P_E(y_n - \lambda_n By_n)$ and $\psi_n = P_E(u_n - \delta_n Bu_n)$. Since $I - \tau_n B$, $I - \lambda_n B$ and $I - \delta_n B$ are be nonexpansive, then we have the following some estimates:

$$\begin{aligned}\|\psi_{n+1} - \psi_n\| &= \|P_E(u_{n+1} - \delta_{n+1} Bu_{n+1}) - P_E(u_n - \delta_n Bu_n)\| \\ &\leq \|(u_{n+1} - \delta_{n+1} Bu_{n+1}) - (u_n - \delta_n Bu_n)\| \\ &= \|(u_{n+1} - \delta_{n+1} Bu_{n+1}) - (u_n - \delta_{n+1} Bu_n) + (\delta_n - \delta_{n+1}) Bu_n\| \\ &\leq \|(u_{n+1} - \delta_{n+1} Bu_{n+1}) - (u_n - \delta_{n+1} Bu_n)\| + (\delta_n - \delta_{n+1})\|Bu_n\| \\ &= \|(I - \delta_{n+1} B)u_{n+1} - (I - \delta_{n+1} B)u_n\| + (\delta_n - \delta_{n+1})\|Bu_n\| \\ &\leq \|u_{n+1} - u_n\| + (\delta_n - \delta_{n+1})\|Bu_n\|, \qquad (3.7) \\ \|\phi_{n+1} - \phi_n\| &= \|P_E(y_{n+1} - \lambda_{n+1} By_{n+1}) - P_E(y_n - \lambda_n By_n)\| \\ &\leq \|(y_{n+1} - \lambda_{n+1} By_{n+1}) - (y_n - \lambda_n By_n)\| \\ &= \|(y_{n+1} - \lambda_{n+1} By_{n+1}) - (y_n - \lambda_{n+1} By_n) + (\lambda_n - \lambda_{n+1}) By_n\| \\ &\leq \|(y_{n+1} - \lambda_{n+1} By_{n+1}) - (y_n - \lambda_{n+1} By_n)\| + (\lambda_n - \lambda_{n+1})\|By_n\| \\ &= \|(I - \lambda_{n+1} B)y_{n+1} - (I - \lambda_{n+1} B)y_n\| + (\lambda_n - \lambda_{n+1})\|By_n\| \\ &\leq \|y_{n+1} - y_n\| + (\lambda_n - \lambda_{n+1})\|By_n\| \qquad (3.8)\end{aligned}$$

and

$$\begin{aligned}\|\theta_{n+1} - \theta_n\| &= \|P_E(k_{n+1} - \tau_{n+1} Bk_{n+1}) - P_E(k_n - \tau_n Bk_n)\| \\ &\leq \|(k_{n+1} - \tau_{n+1} Bk_{n+1}) - (k_n - \tau_n Bk_n)\| \\ &= \|(k_{n+1} - \tau_{n+1} Bk_{n+1}) - (k_n - \tau_{n+1} Bk_n) + (\tau_n - \tau_{n+1}) Bk_n\| \\ &\leq \|(k_{n+1} - \tau_{n+1} Bk_{n+1}) - (k_n - \tau_{n+1} Bk_n)\| + (\tau_n - \tau_{n+1})\|Bk_n\| \\ &= \|(I - \tau_{n+1} B)k_{n+1} - (I - \tau_{n+1} B)k_n\| + (\tau_n - \tau_{n+1})\|Bk_n\|\end{aligned}$$

$$\leq \|k_{n+1} - k_n\| + (\tau_n - \tau_{n+1})\|Bk_n\|. \tag{3.9}$$

Observing that

$$\begin{cases} y_n = \varphi_n u_n + (1-\varphi_n)\psi_n \\ y_{n+1} = \varphi_{n+1} u_{n+1} + (1-\varphi_{n+1})\psi_{n+1}, \end{cases}$$

we obtain

$$y_n - y_{n+1} = \varphi_n(u_n - u_{n+1}) + (1-\varphi_n)(\psi_n - \psi_{n+1}) + (\psi_{n+1} - u_{n+1})(\varphi_{n+1} - \varphi_n),$$

which yields that

$$\|y_n - y_{n+1}\| \\ \leq \varphi_n\|u_n - u_{n+1}\| + (1-\varphi_n)\|\psi_n - \psi_{n+1}\| + \|\psi_{n+1} - u_{n+1}\||\varphi_{n+1} - \varphi_n|. \tag{3.10}$$

Substitution of (3.7) into (3.10) yields that

$$\begin{aligned} \|y_n - y_{n+1}\| &= \varphi_n\|u_n - u_{n+1}\| + (1-\varphi_n)\Big\{\|u_{n+1} - u_n\| + (\delta_n - \delta_{n+1})\|Bu_n\|\Big\} \\ &\quad + \|\psi_{n+1} - u_{n+1}\||\varphi_{n+1} - \varphi_n| \\ &= \|u_n - u_{n+1}\| + (1-\varphi_n)(\delta_n - \delta_{n+1})\|Bu_n\| \\ &\quad + \|\psi_{n+1} - u_{n+1}\||\varphi_{n+1} - \varphi_n| \\ &\leq \|u_n - u_{n+1}\| + M_2(|\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n|), \end{aligned} \tag{3.11}$$

where M_2 is a appropriate constant such that $M_2 = \max\{\sup_{n\geq 1}\|Bu_n\|, \sup_{n\geq 1}\|\psi_n - u_n\|\}$.

Observing that

$$\begin{cases} k_n = \alpha_n x_n + (1-\alpha_n)\phi_n \\ k_{n+1} = \alpha_{n+1} x_{n+1} + (1-\alpha_{n+1})\phi_{n+1}, \end{cases}$$

we obtain

$$k_n - k_{n+1} = \alpha_n(x_n - x_{n+1}) + (1-\alpha_n)(\phi_n - \phi_{n+1}) + (\phi_{n+1} - x_{n+1})(\alpha_{n+1} - \alpha_n),$$

which yields that

$$\|k_n - k_{n+1}\| \\ \leq \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|\phi_n - \phi_{n+1}\| + \|\phi_{n+1} - x_{n+1}\||\alpha_{n+1} - \alpha_n|. \tag{3.12}$$

Substitution of (3.8) into (3.12) yields that

$$\begin{aligned} \|k_n - k_{n+1}\| &= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\Big\{\|y_{n+1} - y_n\| + (\lambda_n - \lambda_{n+1})\|By_n\|\Big\} \\ &\quad + \|\phi_{n+1} - u_{n+1}\||\alpha_{n+1} - \alpha_n| \end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|y_{n+1} - y_n\| \\
&\quad + (1-\alpha_n)(\lambda_n - \lambda_{n+1})\|By_n\| + \|\phi_{n+1} - x_{n+1}\||\alpha_{n+1} - \alpha_n| \\
&\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|y_{n+1} - y_n\| \\
&\quad + M_3(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|), \qquad (3.13)
\end{aligned}
$$

where M_3 is a appropriate constant such that $M_3 = \max\{\sup_{n\ge 1}\|By_n\|, \sup_{n\ge 1}\|\phi_n - x_n\|\}$.

Substitution of (3.6) and (3.11) into (3.13), we obtain

$$
\begin{aligned}
\|k_n - k_{n+1}\| &\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|y_{n+1} - y_n\| \\
&\quad + M_3(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\Big\{\|u_n - u_{n+1}\| + M_2(|\delta_n - \delta_{n+1}| \\
&\quad + |\varphi_{n+1} - \varphi_n|)\Big\} + M_3(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|u_n - u_{n+1}\| + (1-\alpha_n)M_2(|\delta_n - \delta_{n+1}| \\
&\quad + |\varphi_{n+1} - \varphi_n|) + M_3(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\Big\{\|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n|\Big\} \\
&\quad + (1-\alpha_n)M_2(|\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n|) + M_3(|\lambda_n - \lambda_{n+1}| \\
&\quad + |\alpha_{n+1} - \alpha_n|) \\
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|x_{n+1} - x_n\| + (1-\alpha_n)\frac{M_1}{c}|r_{n+1} - r_n| \\
&\quad + (1-\alpha_n)M_2(|\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n|) + M_3(|\lambda_n - \lambda_{n+1}| \\
&\quad + |\alpha_{n+1} - \alpha_n|) \\
&\le \|x_n - x_{n+1}\| + \frac{M_1}{c}|r_{n+1} - r_n| \\
&\quad + M_2(|\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n|) + M_3(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\le \|x_n - x_{n+1}\| \\
&\quad + M_4(|r_{n+1} - r_n| + |\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n| + |\lambda_n - \lambda_{n+1}| \\
&\quad + |\alpha_{n+1} - \alpha_n|), \qquad (3.14)
\end{aligned}
$$

where M_4 is a appropriate constant such that $M_4 = \max\{\frac{M_1}{c}, M_2, M_3\}$. Substituting (3.14) into (3.9), we obtain

$$
\begin{aligned}
\|\theta_{n+1} - \theta_n\| &\le \|k_{n+1} - k_n\| + (\tau_n - \tau_{n+1})\|Bk_n\| \\
&\le \|x_n - x_{n+1}\| + M_4(|r_{n+1} - r_n| + |\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n| \\
&\quad + |\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) + (\tau_n - \tau_{n+1})\|Bk_n\|
\end{aligned}
$$

$$\leq \|x_n - x_{n+1}\| + M_5(|r_{n+1} - r_n| + |\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n| + |\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\tau_n - \tau_{n+1}|), \tag{3.15}$$

where M_5 is a appropriate constant such that $M_5 = \max\{\sup_{n\geq 1} \|Bk_n\|, M_4\}$.
Let $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, $n \geq 1$. Where

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\epsilon_n \gamma f(x_n) + ((1-\beta_n)I - \epsilon_n A)W_n\theta_n}{1-\beta_n}.$$

Then, we have

$$\begin{aligned} z_{n+1} - z_n &= \frac{\epsilon_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \epsilon_{n+1}A)W_{n+1}\theta_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\epsilon_n \gamma f(x_n) + ((1-\beta_n)I - \epsilon_n A)W_n\theta_n}{1-\beta_n} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \frac{\epsilon_n}{1-\beta_n}\gamma f(x_n) + W_{n+1}\theta_{n+1} - W_n\theta_n \\ &\quad + \frac{\epsilon_n}{1-\beta_n}AW_n\theta_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}AW_{n+1}\theta_{n+1} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - AW_{n+1}\theta_{n+1}) + \frac{\epsilon_n}{1-\beta_n}(AW_n\theta_n - \gamma f(x_n)) \\ &\quad + W_{n+1}\theta_{n+1} - W_{n+1}\theta_n + W_{n+1}\theta_n - W_n\theta_n. \end{aligned} \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\quad \leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AW_{n+1}\theta_{n+1}\|) \\ &\qquad + \frac{\epsilon_n}{1-\beta_n}(\|AW_n\theta_n\| + \|\gamma f(x_n)\|) + \|W_{n+1}\theta_{n+1} - W_{n+1}\theta_n\| \\ &\qquad + \|W_{n+1}\theta_n - W_n\theta_n\| - \|x_{n+1} - x_n\| \\ &\quad \leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AW_{n+1}\theta_{n+1}\|) \\ &\qquad + \frac{\epsilon_n}{1-\beta_n}(\|AW_n\theta_n\| + \|\gamma f(x_n)\|) + \|\theta_{n+1} - \theta_n\| \\ &\qquad + \|W_{n+1}\theta_n - W_n\theta_n\| - \|x_{n+1} - x_n\| \\ &\quad \leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AW_{n+1}\theta_{n+1}\|) \\ &\qquad + \frac{\epsilon_n}{1-\beta_n}(\|AW_n\theta_n\| + \|\gamma f(x_n)\|) + M_5(|r_{n+1} - r_n| + |\delta_n - \delta_{n+1}| \\ &\qquad + |\varphi_{n+1} - \varphi_n| + |\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\tau_n - \tau_{n+1}|) \\ &\qquad + \|W_{n+1}\theta_n - W_n\theta_n\|. \end{aligned} \tag{3.17}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}\|W_{n+1}\theta_n - W_n\theta_n\| &= \|\mu_1 T_1 U_{n+1,2}\theta_n - \mu_1 T_1 U_{n,2}\theta_n\| \\ &\le \mu_1 \|U_{n+1,2}\theta_n - U_{n,2}\theta_n\| \\ &= \mu_1 \|\mu_2 T_2 U_{n+1,3}\theta_n - \mu_2 T_2 U_{n,3}\theta_n\| \\ &\le \mu_1\mu_2 \|U_{n+1,3}\theta_n - U_{n,3}\theta_n\| \\ &\vdots \\ &\le \mu_1\mu_2\cdots\mu_n \|U_{n+1,n+1}\theta_n - U_{n,n+1}\theta_n\| \\ &\le M_6 \prod_{i=1}^{n} \mu_i, \end{aligned} \tag{3.18}$$

where $M_6 \ge 0$ is a constant such that $\|U_{n+1,n+1}\theta_n - U_{n,n+1}\theta_n\| \le M_6$ for all $n \ge 0$.

Combining (3.17) and (3.18), we deduce

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AW_{n+1}\theta_{n+1}\|) \\ &\quad + \frac{\epsilon_n}{1-\beta_n}(\|AW_n\theta_n\| + \|\gamma f(x_n)\|) + M_5(|r_{n+1} - r_n| + |\delta_n - \delta_{n+1}| \\ &\quad + |\varphi_{n+1} - \varphi_n| + |\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n| + |\tau_n - \tau_{n+1}|) \\ &\quad + M_6 \prod_{i=1}^{n} \mu_i.\end{aligned}$$

From the conditions (i), (iii), (iv), (v) and $0 < \mu_i \le b < 1, \forall i \ge 1$, we have

$$\limsup_{n\longrightarrow\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

It follows form Lemma 2.7 that

$$\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0.$$

It follows that

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.19}$$

Applying (iv), (v) and (3.19) to (3.6), (3.14) and (3.15), we obtain that

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n\| = \lim_{n\longrightarrow\infty} \|k_{n+1} - k_n\| = \lim_{n\longrightarrow\infty} \|\theta_{n+1} - \theta_n\| = 0. \tag{3.20}$$

Since $x_{n+1} = \epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_n\theta_n$, we have

$$\|x_n - W_n\theta_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n\theta_n\|$$

$$
\begin{aligned}
&= \|x_n - x_{n+1}\| + \|\epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_n\theta_n \\
&\quad - W_n\theta_n\| \\
&= \|x_n - x_{n+1}\| + \|\epsilon_n(\gamma f(x_n) - AW_n\theta_n) + \beta_n(x_n - W_n\theta_n)\| \\
&\le \|x_n - x_{n+1}\| + \epsilon_n(\|\gamma f(x_n)\| + \|AW_n\theta_n\|) + \beta_n\|x_n - W_n\theta_n\|,
\end{aligned}
$$

from which it follow that

$$
\|x_n - W_n\theta_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}(\|\gamma f(x_n)\| + \|AW_n\theta_n\|).
$$

By conditions (i), (iii) and (3.19), we obtain

$$
\lim_{n\longrightarrow\infty} \|W_n\theta_n - x_n\| = 0. \tag{3.21}
$$

Step 3. We claim that the following statements hold:

(s1) $\lim_{n\longrightarrow\infty}\|u_n - \theta_n\| = 0$;
(s2) $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$;
(s3) $\lim_{n\longrightarrow\infty}\|W_n u_n - u_n\| = 0$.

Since B is an ξ-inverse-strongly monotone, by the assumptions imposed on $\{\tau_n\}$ for any $p \in \Omega$, we have

$$
\begin{aligned}
\|\theta_n - p\|^2 &= \|P_E(k_n - \tau_n Bk_n) - P_E(p - \tau_n Bp)\|^2 \\
&\le \|(k_n - \tau_n Bk_n) - (p - \tau_n Bp)\|^2 \\
&= \|(k_n - p) - \tau_n(Bk_n - Bp)\|^2 \\
&\le \|k_n - p\|^2 - 2\tau_n\langle k_n - p, Bk_n - Bp\rangle + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - 2\tau_n\langle k_n - p, Bk_n - Bp\rangle + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 + \tau_n(\tau_n - 2\xi)\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 + a(b - 2\xi)\|Bk_n - Bp\|^2.
\end{aligned} \tag{3.22}
$$

Similarly, we have that

$$
\|\phi_n - p\|^2 \le \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2 \tag{3.23}
$$

and

$$
\|\psi_n - p\|^2 \le \|x_n - p\|^2 + \delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2. \tag{3.24}
$$

Observe that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p) + \beta_n(x_n - p) + \epsilon_n(\gamma f(x_n) - Ap)\|^2 \\
&\quad = \|((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p) + \beta_n(x_n - p)\|^2 + \epsilon_n^2\|\gamma f(x_n) - Ap\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(x_n) - Ap\rangle \\
&\le ((1-\beta_n-\epsilon_n\bar{\gamma})\|W_n\theta_n - p\| + \beta_n\|x_n - p\|)^2 + \epsilon_n^2\|\gamma f(x_n) - Ap\|^2 \\
&\quad + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(x_n) - Ap\rangle \\
&\le \Big((1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - p\| + \beta_n\|x_n - p\|\Big)^2 + c_n \\
&\le (1-\beta_n-\epsilon_n\bar{\gamma})^2\|\theta_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + 2(1-\beta_n-\epsilon_n\bar{\gamma})\beta_n\|\theta_n - p\|x_n - p\|| + c_n \\
&\le (1-\beta_n-\epsilon_n\bar{\gamma})^2\|\theta_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\epsilon_n\bar{\gamma})\beta_n\Big\{\|\theta_n - p\|^2 + \|x_n - p\|^2\Big\} + c_n \\
&= \Big[(1-\epsilon_n\bar{\gamma})^2 - 2(1-\epsilon_n\bar{\gamma})\beta_n + \beta_n^2\Big]\|\theta_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + ((1-\epsilon_n\bar{\gamma})\beta_n - \beta_n^2)(\|\theta_n - p\|^2 + \|x_n - p\|^2) + c_n \\
&= \Big[(1-\epsilon_n\bar{\gamma})^2 - (1-\epsilon_n\bar{\gamma})\beta_n\Big]\|\theta_n - p\|^2 + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - p\|^2 + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n, \qquad (3.25)
\end{aligned}
$$

where

$$
\begin{aligned}
c_n = {} & \epsilon_n^2\|\gamma f(x_n) - Ap\|^2 + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(x_n) - Ap\rangle \\
& + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(x_n) - Ap\rangle.
\end{aligned}
$$

It follows from condition (i) that

$$
\lim_{n\longrightarrow\infty} c_n = 0. \qquad (3.26)
$$

Substituting (3.22) into (3.25), and using condition (vi), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - p\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|x_n - p\|^2 + a(b-2\xi)\|Bk_n - Bp\|^2\Big\} \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})a(b-2\xi)\|Bk_n - Bp\|^2 + c_n \\
&\le \|x_n - p\|^2 + a(b-2\xi)\|Bk_n - Bp\|^2 + c_n.
\end{aligned}
$$

It follows that

$$\begin{aligned}
a(2\xi - b)\|Bk_n - Bp\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + c_n \\
&= (\|x_n - p\| - \|x_{n+1} - p\|)(\|x_n - p\| + \|x_{n+1} - p\|) + c_n \\
&\leq \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + c_n.
\end{aligned}$$

Since $c_n \longrightarrow 0$, $a, b \in (0, 2\xi)$ and from (3.19), we obtain

$$\lim_{n \longrightarrow \infty} \|Bk_n - Bp\| = 0. \tag{3.27}$$

Note that

$$\begin{aligned}
\|k_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(\phi_n - p)\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|\phi_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big\{\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2\Big\} \\
&= \|x_n - p\|^2 + (1 - \alpha_n)\lambda_n(\lambda_n - 2\xi)\|By_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 + (1 - \alpha_n)a(b - 2\xi)\|By_n - Bp\|^2
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\|y_n - p\|^2 &= \|\varphi_n(u_n - p) + (1 - \varphi_n)(\psi_n - p)\|^2 \\
&\leq \varphi_n\|u_n - p\|^2 + (1 - \varphi_n)\|\psi_n - p\|^2 \\
&\leq \varphi_n\|x_n - p\|^2 + (1 - \varphi_n)\Big\{\|x_n - p\|^2 + \delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2\Big\} \\
&= \|x_n - p\|^2 + (1 - \varphi_n)\delta_n(\delta_n - 2\xi)\|Bu_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 + (1 - \varphi_n)a(b - 2\xi)\|Bu_n - Bp\|^2.
\end{aligned}$$

Using (3.25) again, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|\theta_n - p\|^2 + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|P_E(k_n - \tau_n Bk_n) - P_E(p - \tau_n Bp)\|^2 \\
&\quad + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|(k_n - \tau_n Bk_n) - (p - \tau_n Bp)\|^2 \\
&\quad + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|(I - \tau_n B)k_n - (I - \tau_n B)p\|^2 \\
&\quad + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|k_n - p\|^2 \\
&\quad + (1 - \epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n.
\end{aligned} \tag{3.29}$$

Substituting (3.28) into (3.29), and using condition (vi), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|x_n-p\|^2 \\
&\quad + (1-\alpha_n)a(b-2\xi)\|By_n-Bp\|^2\Big\} \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&= (1-\epsilon_n\bar{\gamma})^2\|x_n-p\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)a(b-2\xi)\|By_n-Bp\|^2+c_n \\
&\leq \|x_n-p\|^2+(1-\alpha_n)a(b-2\xi)\|By_n-Bp\|^2+c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\alpha_n)a(2\xi-b)\|By_n-Bp\|^2 \\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n \\
&= (\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|)+c_n \\
&\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.
\end{aligned}$$

From $c_n \longrightarrow 0$, $a, b \in (0, 2\xi)$ and from (3.19), we arrive that

$$\lim_{n\longrightarrow\infty}\|By_n-Bp\|=0. \tag{3.30}$$

In a similar way, we can prove

$$\lim_{n\longrightarrow\infty}\|Bu_n-Bp\|=0. \tag{3.31}$$

On the other hand, we have

$$\begin{aligned}
\|\theta_n-p\|^2 &= \|P_E(k_n-\tau_nBk_n)-P_E(p-\tau_nBp)\|^2 \\
&= \|P_E(I-\tau_nB)k_n-P_E(I-\tau_nB)p\|^2 \\
&\leq \langle (I-\tau_nB)k_n-(I-\tau_nB)p, \theta_n-p\rangle \\
&= \frac{1}{2}\Big\{\|(I-\tau_nB)k_n-(I-\tau_nB)p\|^2+\|\theta_n-p\|^2 \\
&\quad - \|(I-\tau_nB)k_n-(I-\tau_nB)p-(\theta_n-p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|k_n-p\|^2+\|\theta_n-p\|^2-\|(k_n-\theta_n)-\tau_n(Bk_n-Bp)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n-p\|^2+\|\theta_n-p\|^2-\|k_n-\theta_n\|^2 \\
&\quad - \tau_n^2\|Bk_n-Bp\|^2+2\tau_n\langle k_n-\theta_n, Bk_n-Bp\rangle\Big\}
\end{aligned}$$

which yields that

$$\|\theta_n - p\|^2 \leq \|x_n - p\|^2 - \|k_n - \theta_n\|^2 + 2\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\|. \tag{3.32}$$

Similarly, we can prove

$$\|\phi_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - \phi_n\|^2 + 2\lambda_n\|y_n - \phi_n\|\|By_n - Bp\| \tag{3.33}$$

and

$$\|\psi_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - \psi_n\|^2 + 2\delta_n\|u_n - \psi_n\|\|Bu_n - Bp\|.$$

Substituting (3.32) into (3.25), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - p\|^2 + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|x_n - p\|^2 \\
&\quad - \|k_n - \theta_n\|^2 + 2\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\|\Big\} \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 - (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|k_n - \theta_n\|^2 \\
&\quad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\| + c_n \\
&\leq \|x_n - p\|^2 - (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|k_n - \theta_n\|^2 \\
&\quad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\| + c_n,
\end{aligned}$$

which yields that

$$\begin{aligned}
&(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|k_n - \theta_n\|^2 \\
&\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\| + c_n \\
&\quad \leq \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\qquad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\tau_n\|k_n - \theta_n\|\|Bk_n - Bp\| + c_n.
\end{aligned}$$

Applying (3.19), (3.27) and $c_n \longrightarrow 0$ as $n \longrightarrow \infty$ to the last inequality, we have

$$\lim_{n\longrightarrow\infty}\|k_n - \theta_n\| = 0. \tag{3.34}$$

Using (3.29) again, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|k_n - p\|^2 + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|\alpha_n(x_n - p) + (1-\alpha_n)(\phi_n - p)\|^2\Big\} \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\alpha_n\|x_n-p\|^2+(1-\alpha_n)\|\phi_n-p\|^2\Big\}\\
&\quad+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&=(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&\quad+(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\|\phi_n-p\|^2\\
&\quad+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\leq (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&\quad+(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\Big\{\|x_n-p\|^2-\|y_n-\phi_n\|^2\\
&\quad+2\lambda_n\|y_n-\phi_n\|\|By_n-Bp\|\Big\}+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&=(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\alpha_n\|x_n-p\|^2\\
&\quad+(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\|x_n-p\|^2\\
&\quad-(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\|y_n-\phi_n\|^2\\
&\quad+(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)2\lambda_n\|y_n-\phi_n\|\|By_n-Bp\|\\
&\quad+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&=(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n-p\|^2\\
&\quad-(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\|y_n-\phi_n\|^2\\
&\quad+(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)2\lambda_n\|y_n-\phi_n\|\|By_n-Bp\|\\
&\quad+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&=(1-\epsilon_n\bar{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\\
&\quad\times\|y_n-\phi_n\|^2+(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\\
&\quad\times 2\lambda_n\|y_n-\phi_n\|\|By_n-Bp\|+c_n\\
&\leq \|x_n-p\|^2-(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\|y_n-\phi_n\|^2\\
&\quad+(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\\
&\quad\times 2\lambda_n\|y_n-\phi_n\|\|By_n-Bp\|+c_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\|y_n-\phi_n\|^2\\
&\quad\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\\
&\qquad\times\lambda_n\|y_n-\phi_n\|\|By_n-Bp\|+c_n\\
&\quad\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\qquad+2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})(1-\alpha_n)\lambda_n\|y_n-\phi_n\|\|By_n-Bp\|+c_n.
\end{aligned}
$$

From (3.19) and (3.30), we obtain

$$\lim_{n\longrightarrow\infty}\|y_n-\phi_n\|=0. \tag{3.35}$$

By using the same argument, we can prove that

$$\lim_{n\longrightarrow\infty}\|u_n-\psi_n\|=0. \tag{3.36}$$

Notice that

$$k_n-\phi_n=\alpha_n(x_n-\phi_n)$$

and

$$y_n-\psi_n=\varphi_n(u_n-\psi_n).$$

Since $\alpha_n\longrightarrow 0$ and $\varphi_n\longrightarrow 0$ as $n\longrightarrow\infty$ respectively, we also have

$$\lim_{n\longrightarrow\infty}\|k_n-\phi_n\|=\lim_{n\longrightarrow\infty}\|y_n-\psi_n\|=0. \tag{3.37}$$

On the other hand, we observe

$$\|u_n-\theta_n\|\le\|u_n-\psi_n\|+\|\psi_n-y_n\|+\|y_n-\phi_n\|+\|\phi_n-k_n\|+\|k_n-\theta_n\|.$$

Applying (3.34), (3.35), (3.36) and (3.37), we have

$$\lim_{n\longrightarrow\infty}\|u_n-\theta_n\|=0. \tag{3.38}$$

For any $p\in\Omega$, note that T_r is firmly nonexpansive (Lemma 2.11); then we have

$$\begin{aligned}\|u_n-p\|^2&=\|T_{r_n}x_n-T_{r_n}p\|^2\le\langle T_{r_n}x_n-T_{r_n}p,x_n-p\rangle\\&=\langle u_n-p,x_n-p\rangle\\&=\frac{1}{2}\Big(\|u_n-p\|^2+\|x_n-p\|^2-\|x_n-u_n\|^2\Big)\end{aligned}$$

and hence

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|x_n-u_n\|^2.$$

Which together with (3.25) gives

$$\begin{aligned}\|x_{n+1}-p\|^2&\le((1-\epsilon_n\bar{\gamma})^2-\beta_n(1-\epsilon_n\bar{\gamma}))\|\theta_n-p\|^2\\&\quad+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\&=((1-\epsilon_n\bar{\gamma})^2-\beta_n(1-\epsilon_n\bar{\gamma}))\|(\theta_n-u_n)+(u_n-p)\|^2\\&\quad+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\&\le(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|\theta_n-u_n\|^2+\|u_n-p\|^2\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\langle \theta_n - u_n, u_n - p\rangle \Big\} + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|u_n - p\|^2 \\
&\quad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|\|u_n - p\| \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})((1-\beta_n-\epsilon_n\bar{\gamma})\Big\{\|x_n - p\|^2 - \|x_n - u_n\|^2\Big\} \\
&\quad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|\|u_n - p\| \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - p\|^2 \\
&\quad - (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - u_n\|^2 \\
&\quad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|\|u_n - p\| \\
&\quad + (1-\epsilon_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 - (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - u_n\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|^2 \\
&\quad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|\|u_n - p\| + c_n \\
&= (1-2\epsilon_n\bar{\gamma} + (\epsilon_n\bar{\gamma})^2)\|x_n - p\|^2 \\
&\quad - (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - u_n\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|^2 \\
&\quad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|\|u_n - p\| + c_n \\
&\le \|x_n - p\|^2 + (\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 \\
&\quad + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|^2 \\
&\quad - (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - u_n\|^2 \\
&\quad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|\|u_n - p\| + c_n,
\end{aligned}
$$

from which if follows that

$$
\begin{aligned}
&(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - u_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\epsilon_n\bar{\gamma})^2\|x_n - p\|^2 \\
&\qquad + (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|^2 \\
&\qquad + 2(1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n - u_n\|\|u_n - p\| + c_n
\end{aligned}
$$

$$\leq \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + (\epsilon_n\bar{\gamma})^2\|x_n - p\|^2$$
$$+ (1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|\theta_n - u_n\|^2$$
$$+ 2(1 - \epsilon_n\bar{\gamma})(1 - \beta_n - \epsilon_n\bar{\gamma})\|\theta_n - u_n\|\|u_n - p\| + c_n.$$

Using $\epsilon_n \longrightarrow 0$, $c_n \longrightarrow 0$, (3.19) and (3.38), we obtain

$$\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0. \tag{3.39}$$

Since $\liminf_{n\longrightarrow\infty} r_n > 0$, we obtain

$$\lim_{n\longrightarrow\infty} \left\| \frac{x_n - u_n}{r_n} \right\| = \lim_{n\longrightarrow\infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.40}$$

Furthermore, by the triangular inequality we also have

$$\|x_n - \theta_n\| \leq \|x_n - u_n\| + \|u_n - \theta_n\|,$$

thus form (3.38) and (3.39), we have

$$\lim_{n\longrightarrow\infty} \|x_n - \theta_n\| = 0. \tag{3.41}$$

Observe that

$$\|W_n\theta_n - \theta_n\| \leq \|W_n\theta_n - x_n\| + \|x_n - \theta_n\|.$$

Applying (3.21) and (3.41), we obtain

$$\lim_{n\longrightarrow\infty} \|W_n\theta_n - \theta_n\| = 0. \tag{3.42}$$

Let W be the mapping defined by (2.1). Since $\{\theta_n\}$ is bounded, applying Lemma 2.3 and (3.42), we have

$$\|W\theta_n - \theta_n\| \leq \|W\theta_n - W_n\theta_n\| + \|W_n\theta_n - \theta_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.43}$$

Step 4. We claim that $\limsup_{n\longrightarrow\infty}\langle(A - \gamma f)z, z - x_n\rangle \leq 0$, where z is the unique solution of the variational inequality $\langle(A - \gamma f)z, x - z\rangle \geq 0, \forall x \in \Omega$.

We know that $z = P_\Omega(I - A + \gamma f)(z)$ is a unique solution of the variational inequality (3.2). To show this inequality, we choose a subsequence $\{\theta_{n_i}\}$ of $\{\theta_n\}$ such that

$$\lim_{i\longrightarrow\infty} \langle(A - \gamma f)z, z - \theta_{n_i}\rangle = \limsup_{n\longrightarrow\infty}\langle(A - \gamma f)z, z - \theta_n\rangle.$$

Since $\{\theta_{n_i}\}$ is bounded, there exists a subsequence $\{\theta_{n_{i_j}}\}$ of $\{\theta_{n_i}\}$ which converges weakly to $w \in E$. Without loss of generality, we can assume that $\theta_{n_i} \rightharpoonup w$. From $\|W\theta_n - \theta_n\| \longrightarrow 0$, we obtain $W\theta_{n_i} \rightharpoonup w$. Next, We show that $w \in \Omega$, where $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(E, B)$.

First, we prove $w \in EP(F)$. Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E.$$

If follows from (A2) that,

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq -F(u_n, y) \geq F(y, u_n),$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$ and $u_{n_i} \rightharpoonup w$, it follows by (A4) that $F(y, w) \leq 0$ for all $y \in E$. For t with $0 < t \leq 1$ and $y \in E$, let $y_t = ty + (1-t)w$. Since $y \in E$ and $w \in E$, we have $y_t \in E$ and hence $F(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y)$$

and hence $F(y_t, y) \geq 0$. From (A3), we have $F(w, y) \geq 0$ for all $y \in E$ and hence $w \in EP(F)$.

Next, we show that $w \in \bigcap_{n=1}^{\infty} F(T_n)$. By Lemma 2.2, we have $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$. Assume $w \notin F(W)$. Since $\theta_{n_i} \rightharpoonup w$ and $w \neq Ww$, it follows by the Opial's condition (Lemma 2.4) that

$$\begin{aligned}
\liminf_{i \longrightarrow \infty} \|\theta_{n_i} - w\| &< \liminf_{i \longrightarrow \infty} \|\theta_{n_i} - Ww\| \\
&= \liminf_{i \longrightarrow \infty} (\|\theta_{n_i} - W\theta_{n_i} + W\theta_{n_i} - Ww\|) \\
&\leq \liminf_{i \longrightarrow \infty} (\|\theta_{n_i} - W\theta_{n_i}\| + \|W\theta_{n_i} - Ww\|) \\
&= \liminf_{i \longrightarrow \infty} \|W\theta_{n_i} - Ww\| \\
&\leq \liminf_{i \longrightarrow \infty} \|\theta_{n_i} - w\|.
\end{aligned}$$

This is a contradiction. Thus, we obtain $w \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.

Finally, Now we prove that $w \in VI(E, B)$.

Let

$$Tw_1 = \begin{cases} Aw_1 + N_E w_1, & w_1 \in E, \\ \emptyset, & w_1 \notin E. \end{cases}$$

Then, T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Aw_1 \in N_E w_1$ and $\theta_n \in E$, we have $\langle w_1 - \theta_n, w_2 - Bw_1\rangle \geq 0$. On the other hand, from $\theta_n = P_E(k_n - \tau_n Bk_n)$, we have

$$\langle w_1 - \theta_n, \theta_n - (k_n - \tau_n Bk_n)\rangle \geq 0,$$

and hence

$$\left\langle w_1 - \theta_n, \frac{\theta_n - k_n}{\tau_n} + Bk_n \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle w_1 - \theta_{n_i}, w\rangle &\geq \langle w_1 - \theta_{n_i}, Bw_1\rangle \\
&\geq \langle w_1 - \theta_{n_i}, Bw_1\rangle - \left\langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - k_{n_i}}{\tau_{n_i}} + Bk_{n_i} \right\rangle \\
&= \left\langle w_1 - \theta_{n_i}, Bw_1 - Ak_{n_i} - \frac{\theta_{n_i} - k_{n_i}}{\tau_{n_i}} \right\rangle \\
&= \langle w_1 - \theta_{n_i}, Bv - B\theta_{n_i}\rangle + \langle w_1 - \theta_{n_i}, B\theta_{n_i} - Bk_{n_i}\rangle \\
&\quad - \left\langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - k_{n_i}}{\tau_{n_i}} \right\rangle \\
&\geq \langle w_1 - \theta_{n_i}, B\theta_{n_i} - Bk_{n_i}\rangle - \left\langle w_1 - \theta_{n_i}, \frac{\theta_{n_i} - k_{n_i}}{\tau_{n_i}} \right\rangle. \qquad (3.44)
\end{aligned}$$

Noting that $\|\theta_{n_i} - k_{n_i}\| \longrightarrow 0$ as $n \longrightarrow \infty$ and B is Lipschitz continuous, hence from (3.44), we obtain

$$\langle w_1 - w, w_2\rangle \geq 0.$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(E, B)$.

That is $w \in \Omega$, where $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(E, B)$.

Since $z = P_\Omega(I - A + \gamma f)(z)$, it follows that

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle (A - \gamma f)z, z - x_n\rangle &= \limsup_{n\longrightarrow\infty}\langle (A - \gamma f)z, z - \theta_n\rangle \\
&= \lim_{i\longrightarrow\infty}\langle (A - \gamma f)z, z - \theta_{n_i}\rangle \\
&= \langle (A - \gamma f)z, z - w\rangle \leq 0. \qquad (3.45)
\end{aligned}$$

It follows from the last inequality, (3.21) and (3.41) that

$$\limsup_{n\longrightarrow\infty}\langle \gamma f(z) - Az, W_n\theta_n - z\rangle \leq 0. \qquad (3.46)$$

Step 5. Finally, we show that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Omega(I - A + \gamma f)(z)$.

Indeed, from (3.1) , we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\epsilon_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \epsilon_n A)W_n\theta_n - z\|^2 \\
&= \|((1 - \beta_n)I - \epsilon_n A)(W_n\theta_n - z) + \beta_n(x_n - z) + \epsilon_n(\gamma f(x_n) - Az)\|^2 \\
&= \|((1 - \beta_n)I - \epsilon_n A)(W_n\theta - z) + \beta_n(x_n - z)\|^2 + \epsilon_n^2\|\gamma f(x_n) - Az\|^2 \\
&\quad + 2\beta_n\epsilon_n\langle x_n - z, \gamma f(x_n) - Az\rangle
\end{aligned}$$

$$
\begin{aligned}
&+2\epsilon_n\langle((1-\beta_n)I-\epsilon_n A)(W_n\theta_n-z),\gamma f(x_n)-Az\rangle\\
&\leq\Big((1-\beta_n-\epsilon_n\bar{\gamma})\|W_n\theta_n-z\|+\beta_n\|x_n-z\|\Big)^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2\\
&+2\beta_n\epsilon_n\gamma\langle x_n-z,f(x_n)-f(z)\rangle+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&+2(1-\beta_n)\gamma\epsilon_n\langle W_n\theta_n-z,f(x_n)-f(z)\rangle\\
&+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle,\\
&\leq\Big((1-\beta_n-\epsilon_n\bar{\gamma})\|W_n\theta_n-z\|+\beta_n\|x_n-z\|\Big)^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2\\
&+2\beta_n\epsilon_n\gamma\|x_n-z\|\|f(x_n)-f(z)\|+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&+2(1-\beta_n)\gamma\epsilon_n\|W_n\theta_n-z\|\|f(x_n)-f(z)\|\\
&+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle,\\
&\leq\Big((1-\beta_n-\epsilon_n\bar{\gamma})\|\theta_n-z\|+\beta_n\|x_n-z\|\Big)^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2\\
&+2\beta_n\epsilon_n\gamma\|x_n-z\|\|f(x_n)-f(z)\|+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&+2(1-\beta_n)\gamma\epsilon_n\|\theta_n-z\|\|f(x_n)-f(z)\|\\
&+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle,\\
&\leq\Big((1-\beta_n-\epsilon_n\bar{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|\Big)^2+\epsilon_n^2\|\gamma f(x_n)-Az\|^2\\
&+2\beta_n\epsilon_n\gamma\alpha\|x_n-z\|^2+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&+2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n-z\|^2+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle\\
&=\Big[(1-\epsilon_n\bar{\gamma})^2+2\beta_n\epsilon_n\gamma\alpha+2(1-\beta_n)\gamma\epsilon_n\alpha\Big]\|x_n-z\|^2\\
&+\epsilon_n^2\|\gamma f(x_n)-Az\|^2+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle\\
&\leq\Big[1-2(\bar{\gamma}-\alpha\gamma)\epsilon_n\Big]\|x_n-z\|^2+\bar{\gamma}^2\epsilon_n^2\|x_n-z\|^2\\
&+\epsilon_n^2\|\gamma f(x_n)-Az\|^2+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&+2\epsilon_n^2\|A(W_n\theta_n-z)\|\|\gamma f(z)-Az\|
\end{aligned}
$$

$$= \Big[1 - 2(\bar{\gamma} - \alpha\gamma)\epsilon_n\Big]\|x_n - z\|^2 + \epsilon_n \Bigg\{ \epsilon_n\Big(\bar{\gamma}^2\|x_n - z\|^2 + \|\gamma f(x_n) - Az\|^2 + 2\|A(W_n\theta_n - z)\|\|\gamma f(z) - Az\|\Big) + 2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle W_n\theta_n - z, \gamma f(z) - Az\rangle \Bigg\}. \tag{3.47}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{W_n\theta_n\}$ are bounded, we can take a constant $M > 0$ such that

$$\bar{\gamma}^2\|x_n - z\|^2 + \|\gamma f(x_n) - Az\|^2 + 2\|A(W_n\theta_n - z)\|\|\gamma f(z) - Az\| \le M,$$

for all $n \ge 0$. It then follows that

$$\|x_{n+1} - z\|^2 \le (1 - l_n)\|x_n - z\|^2 + \epsilon_n\sigma_n, \tag{3.48}$$

where

$$l_n = 2(\bar{\gamma} - \alpha\gamma)\epsilon_n$$
$$\sigma_n = \epsilon_n M + 2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle W_n\theta_n - z, \gamma f(z) - Az\rangle.$$

Using condition (i), (3.45) and (3.46), we get $l_n \longrightarrow 0$, $\sum_{n=1}^{\infty} l_n = \infty$ and $\limsup_{n\longrightarrow\infty} \frac{\sigma_n}{l_n} \le 0$. Applying Lemma 2.8 to (3.48), we conclude that $x_n \longrightarrow z$ in norm. Finally, noticing

$$\|u_n - z\| = \|T_{r_n}x_n - T_{r_n}z\| \le \|x_n - z\|.$$

We also conclude that $u_n \longrightarrow z$ in norm. This completes the proof. □

Corollary 3.2 *Let E be a nonempty closed convex subset of a real Hilbert space H, let $\{T_n\}$ be an infinite family of nonexpansive mappings of E into itself and let B be an ξ-inverse-strongly monotone mapping of E into H such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(E, B) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let A be a strongly positive linear bounded self-adjoint on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$,$\{y_n\}$ and $\{k_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in E \text{ chosen arbitrary}, \\ y_n = \varphi_n x_n + (1-\varphi_n)P_E(x_n - \delta_n Bx_n), \\ k_n = \alpha_n x_n + (1-\alpha_n)P_E(y_n - \lambda_n By_n), \\ x_{n+1} = \epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_n P_E(k_n - \tau_n Bk_n), \quad \forall n \ge 1, \end{cases}$$

where $\{W_n\}$ is the sequence generated by (1.15) and $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\varphi_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfy the following conditions:

(i) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(ii) $\lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\varphi_n = 0$,
(iii) $0 < \liminf_{n\longrightarrow\infty}\beta_n \leq \limsup_{n\longrightarrow\infty}\beta_n < 1$,
(iv) $\lim_{n\longrightarrow\infty}|\lambda_{n+1}-\lambda_n| = \lim_{n\longrightarrow\infty}|\delta_{n+1}-\delta_n| = 0$ *and* $\lim_{n\longrightarrow\infty}|\tau_{n+1}-\tau_n| = 0$,
(v) $\{\tau_n\}, \{\lambda_n\}, \{\delta_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\xi)$.

Then, $\{x_n\}$ *converges strongly to a point* $z \in \Omega$ *which is the unique solution of the variational inequality*

$$\langle (A-\gamma f)z, x-z\rangle \geq 0, \quad \forall x \in \Omega.$$

Equivalently, we have $z = P_\Omega(I - A + \gamma f)(z)$.

Proof Put $F(x,y) = 0$ for all $x, y \in E$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1. Then, we have $u_n = P_C x_n = x_n$. So, by Theorem 3.1, we can conclude the desired conclusion easily. □

If $A = I, \gamma \equiv 1$ and $\gamma_n = 1 - \epsilon_n - \beta_n$ in Corollary 3.2, then we can obtain the following result immediately.

Corollary 3.3 *Let* E *be a nonempty closed convex subset of a real Hilbert space* H, *let* $\{T_n\}$ *be an infinite family of nonexpansive of* E *into itself and let* B *be an* ξ*-inverse-strongly monotone mapping of* E *into* H *such that* $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(E,B) \neq \emptyset$. *Let* f *be a contraction of* H *into itself with* $\alpha \in (0,1)$. *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{k_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in E \text{ chosen arbitrary}, \\ y_n = \varphi_n x_n + (1-\varphi_n) P_E(x_n - \delta_n B x_n), \\ k_n = \alpha_n x_n + (1-\alpha_n) P_E(y_n - \lambda_n B y_n), \\ x_{n+1} = \epsilon_n f(x_n) + \beta_n x_n + \gamma_n W_n P_E(k_n - \tau_n B k_n), \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ *is the sequence generated by* (1.15) *and* $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\varphi_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *satisfy the following conditions*:

(i) $\epsilon_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\longrightarrow\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$,
(iii) $\lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\varphi_n = 0$,
(iv) $0 < \liminf_{n\longrightarrow\infty}\beta_n \leq \limsup_{n\longrightarrow\infty}\beta_n < 1$,
(v) $\lim_{n\longrightarrow\infty}|\lambda_{n+1}-\lambda_n| = \lim_{n\longrightarrow\infty}|\delta_{n+1}-\delta_n| = 0$ *and* $\lim_{n\longrightarrow\infty}|\tau_{n+1}-\tau_n| = 0$,
(vi) $\{\tau_n\}, \{\lambda_n\}, \{\delta_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\xi)$.

Then, $\{x_n\}$ *converges strongly to a point* $z \in \Omega$ *which is the unique solution of the variational inequality*

$$\langle z - f(z), x-z\rangle \geq 0, \quad \forall x \in \Omega.$$

Equivalently, we have $z = P_\Omega f(z)$.

If $A = I, \gamma \equiv 1$, $\{\varphi_n\} = \{\alpha_n\} = 0$, $\{\delta_n\} = \{\lambda_n\} = \{\tau_n\}$ and $\gamma_n = 1 - \epsilon_n - \beta_n$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.4 ([6, Theorem 3.1]) *Let E be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (A1)–(A4), *let $\{T_n\}$ be an infinite family of nonexpansive of E into itself and let B be an ξ-inverse-strongly monotone mapping of E into H such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(E, B) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0, 1)$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in E \text{ chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E, \\ y_n = P_E(u_n - \lambda_n B u_n), \\ k_n = P_E(y_n - \lambda_n B y_n), \\ x_{n+1} = \epsilon_n f(x_n) + \beta_n x_n + \gamma_n W_n P_E(k_n - \lambda_n B k_n), \end{cases}$$

where $\{W_n\}$ is the sequence generated by (1.15) *and $\{\epsilon_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in* $(0, 1)$ *and $\{r_n\}$ is a real sequence in* $(0, \infty)$ *satisfy the following conditions*:

(i) $\epsilon_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\longrightarrow\infty} \epsilon_n = 0$, $\sum_{n=1}^{\infty} \epsilon_n = \infty$,
(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,
(iv) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$,
(v) $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$,
(vi) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\xi)$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Omega$ which is the unique solution of the variational inequality

$$\langle z - f(z), x - z\rangle \geq 0, \forall x \in \Omega.$$

Equivalently, we have $z = P_\Omega f(z)$.

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