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Existence of multiple positive solutions for nth-order p-Laplacian m-point singular boundary value problems

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Abstract In this paper, by using fixed point theorem, we prove the existence of multiple positive solutions for a class of nth-order p-Laplacian m-point singular boundary value problem. The interesting point is that the nonlinear term f explicitly involves the each-order derivative of variable $u(t)$.

Keywords p-Laplacian operator · nth-order m-point singular boundary value problem · Positive solutions · Fixed points

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1 Introduction

The singular ordinary differential equations arise in the fields of gas dynamics, Newtonian fluid mechanics, the theory of boundary layer and so on. The theory of singular boundary value problems has become an important area of investigation in recent years (see [1–20] and the references therein), where the solvability and the existence of positive solutions for nonlinear m-point boundary value problem have been studied by many authors. For example, by using coincidence degree theory or Leray-Schauder continuation theorem, Feng et al. [4] and Ma [9] studied the existence of

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solutions for the following m-point boundary value problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), & 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{j=1}^{m-2} a_j x(\xi_j), \end{cases} \tag{1}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, a_j \in R$, with all of the a_j having the same sign. John R. Graef and Bo Yang [5] discussed the following boundary value problem

$$\begin{cases} u^{(n)}(t) + \lambda g(t) f(u(t)) = 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, \quad u^{(n-2)}(1) = \sum_{j=1}^{m} \alpha_j u^{(n-2)}(\eta_j), \end{cases} \tag{2}$$

where $\frac{1}{2} \leq \eta_1 < \eta_2 < \cdots < \eta_m < 1, \alpha_j > 0, \sum_{j=1}^{m} \alpha_j = 1, \lambda > 0, i = 0, 1, \ldots, n-2$. By using Krasnosel'skii fixed point theorem, they discussed the existence and nonexistence of positive solutions to the problem according to the value of λ. Jiang [8] studied the following m-point boundary value problem

$$\begin{cases} u^{(n)}(t) + f(t, u(t), \ldots, u^{(n-2)}(t), u^{(n-1)}(t)) = 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, \quad u^{(n-1)}(1) = \sum_{i=1}^{m-2} k_i u^{(n-1)}(\xi_i), \end{cases} \tag{3}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ k_i > 0,\ (i = 0, 1 \ldots, n-2), 0 < \sum_{i=1}^{m-2} k_i < 1$. By means of the five-functional fixed point theorem, he got the existence of multiple positive solutions for the problem (3) imposed some conditions on f.

For p-Laplacian m-point boundary value problems, Gupta [7] discussed the following m-point boundary value problem

$$\begin{cases} (\phi_p(u'(t)))' = f(t, x(t), x'(t)) + e(t), & 0 < t < 1, \\ u(0) = 0, \quad \phi_p(u'(1)) = \sum_{j=1}^{m-2} \alpha_j \phi_p(u'(\eta_j)), \end{cases} \tag{4}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$. By applying Leary-Schauder Continuation theorem, he obtained the existence of positive solutions to the problem (4). In recent papers [1, 2, 6, 14, 20] and [3], the authors studied p-Laplacian multipoint boundary value problem

$$(\phi_p(u'(t)))' + q(t) f(u(t), u'(t)) = 0, \quad 0 < t < 1, \tag{5}$$

subject to some boundary value conditions. They established the existence result of positive solutions by fixed point theorems, coincidence degree theory or monotone iterative method. Very recently, Su [13] studied the existence of positive solutions for the following nonlinear nth-order m-point singular boundary value problem with p-Laplacian operator,

$$\begin{cases} (\phi_p(u^{(n-1)}(t)))' + h(t) f(u(t)) = 0, & 0 < t < 1. \\ u^{(i)}(0) = 0, \quad \phi_p(u^{(n-1)}(1)) = \sum_{j=1}^{m-2} \alpha_j \phi_p(u^{(n-1)}(\eta_j)), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ h$ is singular at $t = 0$ and $t = 1$, and $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1,\ \alpha_j \geq 0, \sum_{j=1}^{m-2} \alpha_j < 1,\ i = 0, 1, \ldots, n-2$.

But with the author's acknowledge, there is no results on the existence of positive solutions for nth-order m-point p-Laplacian operator singular boundary value problem, in which the nonlinear term f depend on the each-order derivative of variable $u(t)$. In this paper, we will consider the following nth-order m-point singular boundary value problem

$$(\phi_p(u^{(n-1)}(t)))' + h(t)f(t, u(t), u(t)', \ldots, u^{(n-1)}(t)) = 0, \quad 0 < t < 1, \tag{6}$$

$$u^{(i)}(0) = 0, \qquad \phi_p(u^{(n-1)}(1)) = \sum_{j=1}^{m-2} \alpha_j \phi_p(u^{(n-1)}(\eta_j)), \tag{7}$$

where $\phi_p(s)$ is p-Laplacian operator, i.e. $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = (\phi_p)^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, h is singular at $t = 0$ and $t = 1$, and $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\alpha_j \geq 0$, $\sum_{j=1}^{m-2} \alpha_j < 1$, $i = 0, 1, \ldots, n-2$. We will mainly study the existence of positive solutions for nonlinear singular boundary value problem (6), (7). The interesting point is that the nonlinear term f explicitly involves the each-order derivative of variable $u(t)$.

This paper is organized as follows. In Sect. 2, we give some definitions and lemmas which are needed throughout this paper. In Sect. 3, we investigate the existence of multiple positive solutions for the boundary value problem (6), (7). At first, we construct a second order multi-point boundary value problem which is equivalent to (6), (7). Then, by imposing growth conditions on f we prove that problem (6), (7) has at least three positive solutions by means of the Leggett-Williams fixed point theorem. Our results extended the corresponding results in the references. In Sect. 4, we give an example to illustrate our main result.

For convenience, we list the following hypotheses.

(H_1) $f \in C([0, 1] \times [0, \infty)^n, [0, \infty))$;
(H_2) $h : (0, 1) \to [0, \infty)$, $0 < \int_0^1 h(t)dt < +\infty$, $\int_{\eta_{m-2}}^1 h(t)dt > 0$;
(H_3) $\alpha_j \geq 0$, $\sum_{j=1}^{m-2} \alpha_j < 1$.

2 Preliminaries

In this section, we give some definitions and lemmas which will be needed in the sequel.

Definition 2.1 ([21]) Assume that K is a cone in a real Banach space E, a map Ψ is said to be a nonnegative continuous concave functional on K, if $\Psi : K \to [0, \infty)$ is continuous, and

$$\Psi(\lambda x + (1-\lambda)y) \geq \lambda\Psi(x) + (1-\lambda)\Psi(y),$$

for all $x, y \in K$, $\lambda \in [0, 1]$.

Let a, b, r be positive constants, Ψ a nonnegative continuous concave functional on K, we define two convex sets as follows,

$$K_r = \{x \in K : \|x\| < r\}, \qquad K(\Psi, a, b) = \{y \in K : \Psi(y) \geq a, \|y\| \leq b\}.$$

The following Leggett-Willams fixed point theorem will be needed in our proof.

Theorem 2.1 ([21]) *Let K be a cone in a real Banach space E. $c > 0$, $A : \overline{K_c} \to \overline{K_c}$ a completely continuous map and Ψ be nonnegative continuous concave functional on K with $\Psi(y) \leq \|y\|$, for all $y \in \overline{K_c}$. Suppose that there exist positive constants r, a, b with $0 < r < a < b \leq c$, such that*

(i) $\{y \in K(\Psi, a, b) : \Psi(y) > a\} \neq \emptyset$, *and* $\Psi(Ay) > a$, *for all* $y \in K(\Psi, a, b)$,
(ii) $\|Ay\| < r$, *for all* $y \in \overline{K_r}$,
(iii) $\Psi(Ay) > a$, *for all* $y \in K(\Psi, a, c)$ *with* $\|Ay\| > b$.

Then A has at least three fixed points $y_1, y_2, y_3 \in K_c$ such that

$$\|y_1\| < r; \quad \Psi(y_2) > a; \quad \|y_3\| > r; \quad \Psi(y_3) < a.$$

Throughout this paper, we will assume that $E = \{u | u \in C^1[0, 1], u(0) = 0\}$.

Lemma 2.1 *Suppose that $u \in E$, then*

$$\|u\|_0 \leq \|u'\|_0,$$

where $\|u\|_0 = \max_{t\in[0,1]} |u(t)|$.

Proof From $u(0) = 0$, we have $u(t) - u(0) = \int_0^t u'(s)ds$, for $t \in [0, 1]$. i.e. $\|u\|_0 \leq \|u'\|_0$. The proof is completed. □

It is obvious that E is a Banach space with the norm $\|u\| = \|u'\|_0$, for $u \in E$. Let

$$K = \{u \in E : u \text{ is concave, nondecreasing and nonnegative on } [0, 1]\},$$

then K is a cone in E.

Lemma 2.2 *If $u \in K$, then*

$$\min_{t\in[\frac{\eta_{m-2}}{2}, \eta_{m-2}]} u(t) \geq \frac{\eta_{m-2}}{2} \min_{t\in[\eta_{m-2}, 1]} u(t).$$

Proof From the fact that u is concave and nonnegative on [0, 1], we have

$$\frac{u(\frac{\eta_{m-2}}{2}) - u(0)}{\frac{\eta_{m-2}}{2}} \geq \frac{u(\eta_{m-2}) - u(0)}{\eta_{m-2}}, \tag{8}$$

i.e.

$$u\left(\frac{\eta_{m-2}}{2}\right) \geq \frac{1}{2} u(\eta_{m-2}) \geq \frac{\eta_{m-2}}{2} u(\eta_{m-2}). \tag{9}$$

Since u is nondecreasing, we obtain that

$$\min_{t\in[\frac{\eta_{m-2}}{2},\eta_{m-2}]} u(t) \geq \frac{\eta_{m-2}}{2} \min_{t\in[\eta_{m-2},1]} u(t). \quad (10)$$

The proof is completed. □

3 Main results

In this section, we will discuss the existence of multiple positive solutions for the boundary value problem (6), (7).

Let $v(t) = u^{(n-2)}(t), t \in [0, 1]$. We define an operator $A_k : C^1[0, 1] \to C^1[0, 1]$ by

$$(A_k v)(t) = \int_0^t g_k(t, s)v(s)ds, \quad (11)$$

where $g_k(t, s) = \frac{(t-s)^{k-1}}{(k-1)!}$, $1 \leq k \leq n-2$. For any $1 \leq k \leq n-2$, we have

$$(A_k v)^{(k)}(t) = v(t), \qquad (A_k v)^{(i)}(0) = 0 \quad 0 \leq i \leq k-1.$$

Now, we consider boundary value problem

$$(\phi_p(v'(t)))' + h(t)f(t, (A_{n-2}v)(t), \ldots, (A_1 v)(t), v(t), v'(t)) = 0,$$
$$t \in (0, 1), \quad (12)$$

$$v(0) = 0, \qquad \phi_p(v'(1)) = \sum_{j=1}^{m-2} \alpha_j \phi_p(v'(\eta_j)). \quad (13)$$

Lemma 3.1 *Let $u = A_{n-2}v$, then u is a solution of boundary value problem* (6), (7) *if and only if v is a solution of boundary value problem* (12), (13).

Proof Let u be a solution of (6), (7), then from $u = A_{n-2}v$, we obtain that $v = u^{(n-2)}, v' = u^{(n-1)}$. It follows from $u^{(l)}(0) = 0$, $(l = n-k-2, \ldots, n-2.\ k = 1, 2, \ldots, n-2)$ and (11), we have that

$$(A_k v)(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} u^{(n-2)}(s)ds = u^{(n-k-2)}(t), \quad (k = 1, 2, \ldots, n-2), \quad (14)$$

$$v(0) = u^{(n-2)}(0) = 0,$$
$$\phi_p(v'(1)) = \sum_{j=1}^{m-2} \alpha_j \phi_p(u^{(n-1)}(\eta_j)) = \sum_{j=1}^{m-2} \alpha_j \phi_p(v'(\eta_j)). \quad (15)$$

From (14) and (15) we know that v is the solution of (12) and (13).

Conversely, if v is a solution of (12) and (13), then

$$u^{(n-2)}(t)=(A_{n-2}v)^{(n-2)}(t)=v(t),\qquad u^{(n-1)}(t)=v'(t), \tag{16}$$

$$\begin{aligned} u^{(k)}(t)=(A_{n-2}v)^{(k)}(t)&=\left(\int_0^t\frac{(t-s)^{n-3}}{(n-3)!}v(s)ds\right)^{(k)}\\ &=\int_0^t\frac{(t-s)^{n-3-k}}{(n-3-k)!}v(s)ds=(A_{n-2-k}v)(t), \end{aligned}\tag{17}$$

$$\phi_p(u^{(n-1)}(1))=\phi_p(v'(1))=\sum_{j=1}^{m-2}\alpha_j\phi_p(v'(\eta_j))=\sum_{j=1}^{m-2}\alpha_j\phi_p(u^{(n-1)}(\eta_j)),$$
$$k=0,1,\ldots,n-3. \tag{18}$$

From (16) and (17) we know that $u^{(k)}(0)=0$. $k=0,1,\ldots,n-2$. So by (16)–(18) we obtain that u is a solution of (6), (7). The proof is completed. □

Note that $g_{n-2}(t,s)\geq 0$, if $v(t)\geq 0(\not\equiv 0)$ is a solution of boundary value problem (12), (13), then $u=A_{n-2}v$ is a positive solution of (6), (7).

For convenience, we denote $M,\ N,\ L$ and l, by

$$M=\int_0^1\phi_q\left(\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\sum_{j=1}^{m-2}\alpha_j\int_{\eta_j}^1h(\tau)d\tau+\int_0^1h(\tau)d\tau\right)ds,$$

$$L=\max\left\{\int_0^1h(t)dt,\max_{t\in[\eta_1,\eta_{m-2}]}h(t)\right\},$$

$$N=\int_0^{\eta_{m-2}}\phi_q\left(\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\int_{\eta_{m-2}}^1h(\tau)d\tau\right)ds,$$

$$l=\min\left\{\int_0^{\eta_1}h(t)dt,\int_{\eta_{m-2}}^1h(t)dt\right\}.$$

Remark 3.1 By (H_2) and (H_3), it is easy to see that $0<N\leq M$.

We also note that $v(t)$ is a solution of (12), (13), if and only if

$$\begin{aligned} v(t)=\int_0^t\phi_q\Bigg(&\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\sum_{j=1}^{m-2}\alpha_j\\ &\times\int_{\eta_j}^1h(\tau)f(\tau,A_{n-2}v(\tau),\ldots,A_1v(\tau),v(\tau),v'(\tau))d\tau\\ &+\int_s^1h(\tau)f(\tau,A_{n-2}v(\tau),\ldots,A_1v(\tau),v(\tau),v'(\tau))d\tau\Bigg)ds. \end{aligned}$$

We define a nonnegative continuous concave functional $\Psi : K \to [0, \infty)$ by

$$\Psi(v) = \min_{t\in[\eta_{m-2},1]} v(t), \quad v \in K.$$

Obviously, $\Psi(v) \le \|v\|$, for $v \in K$.

Theorem 3.1 *Suppose that* (H_1), (H_2) *and* (H_3) *hold, and* r, a, b, c *are constants, such that* $0 < r < a < b \le c$ *and* $a < \min\{\frac{\eta_{m-2} lb}{L}, \frac{Nc}{M}, \frac{\eta_{m-2}}{2-\eta_{m-2}} b\}$. *If* f *satisfies the following conditions*

(H_4) $f(t, u_0, u_1, \dots, u_{n-1}) < \phi_p(\frac{r}{M}), (t, u_0, u_1, \dots, u_{n-1}) \in [0, 1] \times \prod_{k=n-2}^{1} [0, \frac{r}{k!}] \times [0, r] \times [0, r]$.

(H_5) $f(t, u_0, u_1, \dots, u_{n-1}) < \phi_p(\frac{c}{M}), (t, u_0, u_1, \dots, u_{n-1}) \in [0, 1] \times \prod_{k=n-2}^{1} [0, \frac{c}{k!}] \times [0, c] \times [0, c]$.

(H_6) $f(t, u_0, u_1, \dots, u_{n-1}) > \phi_p(\frac{a}{N}), (t, u_0, u_1, \dots, u_{n-1}) \in [\eta_{m-2}, 1] \times \prod_{k=n-2}^{1} [\frac{a\eta_{m-2}^{k+1}}{2^{k+1}k!}, \frac{b}{k!}] \times [a, b] \times [0, b],$.

(H_7) $f_0 \phi_p(\frac{L}{l}) \int_{\eta_{m-2}}^{1} h(s)ds \ge f_1 \int_0^1 h(s)ds$,

where

$$f_0 = \min_{(t,u_0,u_1,\dots,u_{n-1})\in[\eta_{m-2},1]\times\prod_{k=n-2}^{1}[\frac{a\eta_{m-2}^{k+1}}{2^{k+1}k!}, \frac{b}{k!}]\times[a,b]\times[0,b]} f(t, u_0, u_1, \dots, u_{n-1}),$$

$f_1 = \max_{(t,u_0,u_1,\dots,u_{n-1})\in[0,1]\times\prod_{k=n-2}^{1}[0,\frac{c}{k!}]\times[0,c]\times[0,c]} f(t, u_0, u_1, \dots, u_{n-1})$. *Then the problem* (6), (7) *has at least three positive solutions* u_1, u_2 *and* u_3, *such that*

$$\|u_1^{(n-2)}\| < r, \qquad \Psi(u_2^{(n-2)}) > a, \quad \textit{and} \quad \|u_3^{(n-2)}\| > r, \quad \textit{with } \Psi(u_3^{(n-2)}) < a.$$

Proof First, we define an operator $A : K \to E$ by

$$\begin{aligned}(Av)(t) = \int_0^t \phi_q\Bigg(&\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\sum_{j=1}^{m-2}\alpha_j \\ &\times \int_{\eta_j}^1 h(\tau) f(\tau, A_{n-2}v(\tau), \dots, A_1 v(\tau), v(\tau), v'(\tau))d\tau \\ &+ \int_s^1 h(\tau) f(\tau, A_{n-2}v(\tau), \dots, A_1 v(\tau), v(\tau), v'(\tau))d\tau\Bigg)ds.\end{aligned}$$

From (H_1) and (H_2), we can easily prove that A is a completely continuous operator. Let $v \in K$, then $(Av)(t) \ge 0$,

$$\begin{aligned}(Av)'(t) = \phi_q\Bigg(&\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\sum_{j=1}^{m-2}\alpha_j \\ &\times \int_{\eta_j}^1 h(s) f(s, A_{n-2}v(s), \dots, A_1 v(s), v(s), v'(s))ds\end{aligned}$$

$$+\int_t^1 h(s)f(s, A_{n-2}v(s), \ldots, A_1v(s), v(s), v'(s))ds\Big)$$

is decreasing, so $(Av)''(t) \leq 0$. Therefore, A is a map from K to K. It is obvious that v is a solution of (12), (13) if and only if v is a fixed point of A in K. Let $v \in \overline{K_c}$, then $\|v\| \leq c$, therefore, we have that

$$\begin{aligned} A_kv(t) &\leq \max_{t\in[0,1]}\left|\int_0^t g_k(t,s)v(s)ds\right| \leq \max_{t\in[0,1]}\int_0^t g_k(t,s)|v(s)|ds \\ &\leq c\max_{t\in[0,1]}\int_0^t \frac{(t-s)^{k-1}}{(k-1)!}ds = c\max_{t\in[0,1]}\frac{t^k}{k!} = \frac{c}{k!}, \quad 1\leq k\leq n-2. \end{aligned}$$

It follows from (H_5) that

$$\begin{aligned} (Av)'(t) &= \phi_q\Big(\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\sum_{j=1}^{m-2}\alpha_j \\ &\quad \times\int_{\eta_j}^1 h(s)f(s, A_{n-2}v(s), \ldots, A_1v(s), v(s), v'(s))ds \\ &\quad +\int_t^1 h(s)f(s, A_{n-2}v(s), A_{n-3}v(s), \ldots, A_1v(s), v(s), v'(s))ds\Big) \\ &\leq \phi_q\Big(\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\sum_{j=1}^{m-2}\alpha_j \\ &\quad \times\int_{\eta_j}^1 h(s)f(s, A_{n-2}v(s), \ldots, A_1v(s), v(s), v'(s))ds \\ &\quad +\int_0^1 h(s)f(s, A_{n-2}v(s), \ldots, A_1v(s), v(s), v'(s))ds\Big) \\ &< \frac{c}{M}\phi_q\Big(\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\sum_{j=1}^{m-2}\alpha_j\int_{\eta_j}^1 h(s)ds + \int_s^1 h(s)ds\Big) = c \end{aligned}$$

i.e. $\|Av\| \leq c$. This implies that $A: \overline{K_c} \to \overline{K_c}$ is a completely continuous operator. Similarly, (H_4) implies that $A(\overline{K_r}) \subset K_r$, this means that condition (ii) of Theorem 2.1 holds.

Secondly, we show that condition (i) of Theorem 2.1 holds.

If we let $v(t) = \frac{a+b}{2}t$, for $t \in [0,1]$, then $v \in \{y \in K(\Psi, a, b) : \Psi(y) > a\}$. Thus, $\{y \in K(\Psi, a, b) : \Psi(y) > a\}$ is not empty set. Now if $v \in K(\Psi, a, b)$, then $\Psi(v) \geq a$ and $\|v\| \leq b$. This implies that $a \leq v(t) \leq b, t \in [\eta_{m-2}, 1]$. Since $(A_kv)'(t) = \int_0^t \frac{(t-s)^{k-2}}{(k-2)!}v(s)ds \geq 0, (2 \leq k \leq n-2)$, and $(A_1v)'(t) = v(t) \geq 0$, we know that $(A_kv)(t)$ is increasing with respect to t, for $t \in [0,1]$, $1 \leq k \leq n-2$. By Lemma 2.2, we get that

$$\begin{aligned}
\min_{t\in[\eta_{m-2},1]}(A_k v)(t) &= \int_0^{\eta_{m-2}} \frac{(\eta_{m-2}-s)^{k-1}}{(k-1)!}v(s)ds \geq \int_{\frac{\eta_{m-2}}{2}}^{\eta_{m-2}} \frac{(\eta_{m-2}-s)^{k-1}}{(k-1)!}v(s)ds \\
&\geq \int_{\frac{\eta_{m-2}}{2}}^{\eta_{m-2}} \frac{(\eta_{m-2}-s)^{k-1}}{(k-1)!} \min_{s\in[\frac{\eta_{m-2}}{2},\eta_{m-2}]} v(s)ds \\
&= \min_{t\in[\frac{\eta_{m-2}}{2},\eta_{m-2}]} v(t) \int_{\frac{\eta_{m-2}}{2}}^{\eta_{m-2}} \frac{(\eta_{m-2}-s)^{k-1}}{(k-1)!}ds \\
&\geq \frac{\eta_{m-2}}{2} \min_{t\in[\eta_{m-2},1]} v(t) \int_{\frac{\eta_{m-2}}{2}}^{\eta_{m-2}} \frac{(\eta_{m-2}-s)^{k-1}}{(k-1)!}ds \geq \frac{\eta_{m-2}^{k+1}a}{2^{k+1}k!}. \\
\max_{t\in[\eta_{m-2},1]}(A_k v)(t) &= \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!}v(s)ds \leq \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!}\|v\|ds \\
&\leq b\int_0^1 \frac{(1-s)^{k-1}}{(k-1)!}ds = \frac{b}{k!}.
\end{aligned}$$

It follows from (H_6) that

$$\begin{aligned}
\Psi(Av) &= \min_{t\in[\eta_{m-2},1]}(Av)(t) \\
&= \int_0^{\eta_{m-2}} \phi_q\left(\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\sum_{j=1}^{m-2}\alpha_j\right. \\
&\quad \times \int_{\eta_j}^1 h(\tau)f(\tau, A_{n-2}v(\tau),\ldots,A_1v(\tau),v(\tau),v'(\tau))d\tau \\
&\quad \left.+\int_s^1 h(\tau)f(\tau, A_{n-2}v(\tau),\ldots,A_1v(\tau),v(\tau),v'(\tau))d\tau\right)ds \\
&\geq \int_0^{\eta_{m-2}} \phi_q\left(\frac{\sum_{j=1}^{m-2}\alpha_j}{1-\sum_{j=1}^{m-2}\alpha_j}\right. \\
&\quad \times \int_{\eta_{m-2}}^1 h(\tau)f(\tau, A_{n-2}v(\tau),\ldots,A_1v(\tau),v(\tau),v'(\tau))d\tau \\
&\quad \left.+\int_{\eta_{m-2}}^1 h(\tau)f(\tau, A_{n-2}v(\tau),\ldots,A_1v(\tau),v(\tau),v'(\tau))d\tau\right)ds \\
&= \int_0^{\eta_{m-2}} \phi_q\left(\frac{1}{1-\sum_{j=1}^{m-2}\alpha_j}\right.
\end{aligned}$$

$$\times \int_{\eta_{m-2}}^{1} h(\tau) f(\tau, A_{n-2}v(\tau), \ldots, A_1 v(\tau), v(\tau), v'(\tau)) d\tau \Bigg) ds$$

$$> \frac{a}{N} \int_0^{\eta_{m-2}} \phi_q \left(\frac{1}{1 - \sum_{j=1}^{m-2} \alpha_j} \int_{\eta_{m-2}}^{1} h(\tau) d\tau \right) ds = a.$$

Thus, condition (i) of Theorem 2.1 holds.

Finally, we show that (iii) of Theorem 2.1 holds. Let $v \in K(\Psi, a, c)$, and $\|Av\| > b$. Then $a \le v(t) \le c$, for $t \in [\eta_{m-2}, 1]$. It is easy to check that $\min_{t\in[0,1]} A_k v(t) = 0$, $\max_{t\in[0,1]} A_k v(t) \le \frac{c}{k!}$. By ($H_7$), we obtain that

$$\int_{\eta_{m-2}}^{1} h(s) f(s, A_{n-2}v(s), A_{n-3}v(s), \ldots, A_1 v(s), v(s), v'(s)) ds$$
$$\ge \frac{\int_0^1 h(s) f(s, A_{n-2}v(s), A_{n-3}v(s), \ldots, A_1 v(s), v(s), v'(s)) ds}{\phi_p(\frac{L}{l})}.$$

Therefore,

$$\Psi(Av) = \min_{t\in[\eta_{m-2},1]} (Av)(t)$$
$$= \int_0^{\eta_{m-2}} \phi_q \Bigg(\frac{1}{1 - \sum_{j=1}^{m-2} \alpha_j} \sum_{j=1}^{m-2} \alpha_j$$
$$\times \int_{\eta_j}^{1} h(\tau) f(\tau, A_{n-2}v(\tau), \ldots, A_1 v(\tau), v(\tau), v'(\tau)) d\tau$$
$$+ \int_s^1 h(\tau) f(\tau, A_{n-2}v(\tau), \ldots, A_1 v(\tau), v(\tau), v'(\tau)) d\tau \Bigg) ds$$
$$\ge \int_0^{\eta_{m-2}} \phi_q \Bigg(\frac{1}{1 - \sum_{j=1}^{m-2} \alpha_j}$$
$$\times \int_{\eta_{m-2}}^{1} h(\tau) f(\tau, A_{n-2}v(\tau), \ldots, A_1 v(\tau), v(\tau), v'(\tau)) d\tau \Bigg) ds$$
$$\ge \frac{\eta_{m-2} l}{L} \phi_q \Bigg(\frac{1}{1 - \sum_{j=1}^{m-2} \alpha_j}$$
$$\times \int_0^1 h(\tau) f(\tau, A_{n-2}v(\tau), \ldots, A_1 v(\tau), v(\tau), v'(\tau)) d\tau \Bigg)$$
$$\ge \frac{\eta_{m-2} l}{L} \phi_q \Bigg(\frac{1}{1 - \sum_{j=1}^{m-2} \alpha_j} \sum_{j=1}^{m-2} \alpha_j$$

$$\times \int_{\eta_j}^{1} h(\tau) f(\tau, A_{n-2}v(\tau), \dots, A_1 v(\tau), v(\tau), v'(\tau))d\tau$$
$$+ \int_{0}^{1} h(\tau) f(\tau, A_{n-2}v(\tau), A_{n-3}v(\tau), \dots, A_1 v(\tau), v(\tau), v'(\tau))d\tau\Bigg)$$
$$= \frac{\eta_{m-2} l}{L}(Av)'(0) \geq \frac{\eta_{m-2} l}{L}\|Av\| > a.$$

By Theorem 2.1, there exist at least three positive solutions v_1, v_2 and $v_3 \in K$ for boundary value problem (12) and (13). It follows from $u = A_{n-2}v$ and Lemma 3.1 that (6) and (7) there exist at least three positive solutions $u_1,\ u_2$ and u_3 satisfying

$$\|u_1^{(n-2)}\| < r, \qquad \Psi(u_2^{(n-2)}) > a, \quad \text{and} \quad \|u_3^{(n-2)}\| > r, \quad \text{with } \Psi(u_3^{(n-2)}) < a.$$

The proof is complete. □

4 Example

In this section, we give an example as an application of our results. We consider the following boundary value problem

$$\begin{cases} (\phi_p(u'''(t)))' + h(t) f(t, u(t), u'(t), u''(t), u'''(t)) = 0, & 0 < t < 1 \\ u(0) = u'(0) = u''(0) = 0, & \phi_p(u'''(1)) = \sum_{j=1}^{2} \alpha_j \phi_p(u'''(\eta_j)), \end{cases} \tag{19}$$

where

$$f(t,u,v,w,x) = \begin{cases} \frac{uv}{10^4} + \frac{1}{80}w^3 + \frac{\sin^2 x}{300}, & w \in [0, \frac{1}{2}] \\ \frac{uv}{10^4} + \frac{1}{2240}(121w - 57) + \frac{\sin^2 x}{300}, & w \in [\frac{1}{2}, 1] \\ \frac{uv}{10^4} - \frac{666}{175\times 49^2}(w - \frac{51}{2})^2 + \frac{49}{50} + \frac{\sin^2 x}{300}, & w \in [1, 50] \\ \frac{uv}{10^4} + \frac{10}{7w} + \frac{\sin^2 x}{300}, & w \in [50, +\infty], \end{cases}$$

and

$$\alpha_1 = \frac{3}{8}, \qquad \alpha_2 = \frac{1}{2}, \qquad \eta_1 = \frac{1}{9}, \qquad \eta_2 = \frac{15}{16},$$
$$p = 3, \qquad q = \frac{3}{2}, \qquad h(t) = 10(1-t)^{-\frac{1}{2}}.$$

By simple calculating, we can get that

$$M = (40 + 40\sqrt{2})^{\frac{1}{2}}, \qquad N = \frac{15\sqrt{10}}{8}, \qquad L = 40, \qquad l = \frac{20(3 - 2\sqrt{2})}{3}.$$

If we choose $r = \frac{1}{2}, a = 1, b = 43, c = 50,$ then we can get that $f(t,u,v,w,x)$ satisfies

$$f(t,u,v,w,x) < \phi_p\left(\frac{r}{M}\right) = \frac{1}{160(1+\sqrt{2})},$$

for $(t,u,v,w,x)\in[0,1]\times[0,\frac{1}{4}]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}]$;

$$f(t,u,v,w,x)<\phi_p\left(\frac{c}{M}\right)=\frac{250}{4(1+\sqrt{2})},$$

for $(t,u,v,w,x)\in[0,1]\times[0,25]\times[0,50]\times[0,50]\times[0,50]$;

$$f(t,u,v,w,x)>\phi_p\left(\frac{a}{N}\right)=\frac{32}{1125},$$

for $(t,u,v,w,x)\in[\frac{15}{16},1]\times[\frac{15^3}{16^4},\frac{43}{2}]\times[\frac{15^2}{16^2\times 4},43]\times[1,43]\times[0,43]$. Thus, (H_4), (H_5) and (H_6) hold. On the other hand, we have that

$$\min_{(u,v,w,x)\in[\frac{15^3}{16^4},\frac{43}{2}]\times[\frac{15^2}{16^2\times 4},43]\times[1,43]\times[0,43]} f(u,v,w,x)=\frac{15^5}{4^{13}\times 10^4}+\frac{1}{35},$$

$$t\in\left[\frac{15}{16},1\right],$$

$$\max_{(u,v,w,x)\in[0,25]\times[0,50]\times[0,50]\times[0,50]} f(u,v,w,x)=\frac{1250}{10^4}+\frac{49}{50}+\frac{1}{300},\quad t\in[0,1],$$

$$\int_{\frac{15}{16}}^1 h(t)dt=5,\qquad \int_0^1 h(t)dt=20,\qquad \phi_p\left(\frac{L}{l}\right)=\frac{36}{17-12\sqrt{2}}.$$

These imply that

$$\left(\frac{15^5}{4^{13}\times 10^4}+\frac{1}{35}\right)\times\frac{36}{17-12\sqrt{2}}\times 5>\left(\frac{1250}{10^4}+\frac{49}{50}+\frac{1}{300}\right)\times 20.$$

That is, (H_7) holds. Therefore, all conditions of the Theorem 3.1 are satisfied. It follows from Theorem 3.1 that boundary value problem (19) has at least three positive solutions u_1,u_2,u_3 such that

$$\|u_1''\|<\frac{1}{2},\qquad \min_{t\in[\frac{15}{16},1]} u_2''(t)>1,\qquad \|u_3''\|>\frac{1}{2},\quad \text{with}\ \min_{t\in[\frac{15}{16},1]} u_3''(t)<1.$$

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