J Appl Math Comput (2010) 34: 39–46
DOI 10.1007/s12190-009-0303-0

JAMC

A new semilocal convergence theorem for a fast iterative method with nondifferentiable operators

Hongmin Ren · Ioannis K. Argyros

Received: 24 June 2009 / Revised: 25 June 2009 / Published online: 9 July 2009

Abstract A new semilocal convergence theorem for a fast iterative method in Banach spaces is provided for approximating a solution of a nondifferentiable operator equation. A condition for divided differences of order one is considered in this paper, which generalizes the usual ones, i.e., Lipschitz continuous or Hölder continuous conditions. Note that no conditions of divided differences of order two are used. Therefore our results are of theoretical and practical interest. Finally, a numerical example is provided to show that the new iterative method compares favorably with earlier ones.

Keywords Semilocal convergence · Iterative method · Nondifferentiable operators · Banach space · Divided difference · Quadratic convergence

Mathematics Subject Classification (2000) 65G99 · 65H10 · 65K10 · 65B05 · 47H17 · 49M15

1 Introduction

In this study, we study the semilocal convergence of a fast iterative method in Banach spaces which is used to solve the nonlinear operator equation

$$F(x) = 0, \tag{1}$$

H. Ren (✉)
Department of Information and Electronics, Hangzhou Radio and TV University, Hangzhou 310012, Zhejiang, China
e-mail: rhm65@126.com

I.K. Argyros
Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
e-mail: iargyros@cameron.edu

where F is a continuous operator defined on an open convex domain Ω of a Banach space X with values in a Banach space Y. The differentiability of operator F is not assumed. That is, the well-known Newton's method cannot be applied. In this case an effective analysis method is to divide F into two parts: one is differentiable, say H; another is Lipschitz continuous, say G. Thus, (1) becomes

$$F(x) = H(x) + G(x) = 0. \tag{2}$$

About the solution of (2), Zabrejko and Nguen [12] used a modified Newton's method

$$x_{n+1} = x_n - H'(x_n)^{-1}(H(x_n) + G(x_n)) \quad (n \geq 0), \tag{3}$$

to obtain an approximate solution $x^\star$ of (2) and established a convergence theorem. Argyros [1] improved the result of Zabrejko and Nguen in some aspects. Han [5] proposed a general majorant method to analyze the convergence of (3) and gave a simpler and better convergence condition.

Another effective analysis method to find an approximate solution for a nondifferentiable (1) is to substitute the Fréchet derivative in Newton's method with a divided difference of order one. Let us denote by $\mathcal{L}(X, Y)$, the space of bounded linear operators from X to Y. An operator $[x, y; F] \in \mathcal{L}(X, Y)$ is called a divided difference of order one for the operator F on the points x and y $(x \neq y)$ if the following equality holds:

$$[x, y; F](x - y) = F(x) - F(y). \tag{4}$$

Then the Secant method in Banach spaces can be described by the following algorithm [10, 11]:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n) \quad (n \geq 0)\ (x_{-1},\ x_0 \in \Omega). \tag{5}$$

Hernandez and Rubio [6, 7] used the Secant method (5) to obtain an approximate solution $x^\star$ of the nondifferentiable equation (1), and some semilocal convergence theorems were established. Ren [9] improved the results of Hernandez and Rubio [6] in some aspects.

It is known that, under the Kantorovich conditions, Newton's method has quadratic convergence, whereas the Secant method has only a convergence order $1.618\ldots$ (see [3, 8]). Then, can we get an iterative method with quadratic convergence as Newton's method but using only a divided difference of order one instead of the Fréchet derivative in Newton's method? The answer is yes. Recently, Argyros [2, 4] study the convergence of the following iterative method:

$$x_{n+1} = x_n - [2x_n - x_{n-1}, x_{n-1}; F]^{-1} F(x_n) \quad (n \geq 0)\ (x_{-1},\ x_0 \in \Omega). \tag{6}$$

Under certain conditions, the quadratic convergence of the method (6) was proved. However, some conditions on the difference of order two (the definition of it, see [2, 4]) were used.

We should note that method (6) uses two function evaluations per step, and one inverse. From the efficiency point of view it is only fair to compare it with methods using the same information as the Secant method or Steffensen's method. Such a favorable comparison has already been made in [4]. Note also that if the analytic representation of F is not available method (6) is also an attractive replacement of Newton's method.

Here we study the method

$$x_{n+1} = x_n - L_n^{-1} F(x_n), \quad L_n = [2x_n - x_{n-1}, x_{n-1}; H] + [x_{n-1}, x_n; G]$$
$$(n \geq 0)\ (x_{-1},\ x_0 \in \Omega,\ 2x_0 - x_{-1} \in \Omega). \tag{7}$$

In this paper, we study the convergence of the method (7), but we only use the conditions on the difference of order one. In fact, we assume the following conditions:

$$\|[x, y; H] - [v, w; H]\| \leq \omega(\|x - v\|, \|y - w\|) \quad (\forall x, y, v, w \in \Omega), \tag{8}$$

$$\|[x, y; G] - [v, w; G]\| \leq \omega_1(\|x - v\|, \|y - w\|) \quad (\forall x, y, v, w \in \Omega), \tag{9}$$

where $\omega, \omega_1 : \mathbf{R}_+ \times \mathbf{R}_+$ are continuous nondecreasing functions in their two arguments. Clearly, this type of condition generalizes the classical Lipschitz continuous or Hölder continuous conditions. Moreover, in general, this condition does not require F to be differentiable. Then, we provide a semilocal convergence result for nondifferentiable operators in general. Finally, we provide a numerical example to show that method (7) is faster than (3) and (5).

2 Convergence study

Let $x \in X$ and $r > 0$. Denote $B(x, r)$ as an open ball around x with radius r and $\overline{B(x, r)}$ its closure. We have:

Theorem 1 *Assume that, for every pair of distinct points $x, y \in \Omega$, $2y - x \in \Omega$ and there exists a divided difference of order one $[2y - x, x; F] \in \mathcal{L}(X, Y)$. Let x_{-1}, $x_0 \in \Omega$, and further assume*:

(a) *the linear operator $L_0 = [2x_0 - x_{-1}, x_{-1}; H] + [x_{-1}, x_0; G]$ is invertible and there exist constants $\beta > 0$, $\alpha \geq 0$, $\eta > 0$ so that*:

$$\|L_0^{-1}\| \leq \beta, \qquad \|x_{-1} - x_0\| \leq \alpha, \qquad \|L_0^{-1} F(x_0)\| \leq \eta; \tag{10}$$

(b) *conditions (8) and (9) on divided differences of order one hold*;

(c) *we denote $m = \beta \max(\omega(\eta + \alpha, \alpha) + \omega_1(\alpha + \eta, 0), \omega(2\eta, \eta) + \omega_1(2\eta, 0))$, and assume that the equation*

$$u\left(1 - \frac{m}{1 - \beta[\omega(3u + \alpha, u + \alpha) + \omega_1(\alpha + u, u)]}\right) - \eta = 0 \tag{11}$$

has at least one positive zero, let R be the minimal positive one.

If $\beta[\omega(3R + \alpha, R + \alpha) + \omega_1(\alpha + R, R)] < 1$, $M = \frac{m}{1 - \beta[\omega(3R+\alpha, R+\alpha) + \omega_1(\alpha+R, R)]} < 1$, and $\overline{B(x_0, R)} \subset \Omega$, then the sequence $\{x_n\}$ given by the method (7) is well defined, remains in $B(x_0, R) \subset \Omega$, and converges to a unique solution $x^\star$ of (1) in $\overline{B(x_0, R)} \subset \Omega$.

Proof We prove, by induction, that the sequence $\{x_n\}$ given by (7) is well defined, and $x_n \in B(x_0, R)$ for any $n \geq 1$. By the definition of R, we have $R = \frac{\eta}{1-M} > \eta$. By the condition (a), x_1 is well defined, and $x_1 \in B(x_0, R)$. Using (8) and (9), we obtain

$$\begin{aligned}
\|I - L_0^{-1}L_1\| &= \|L_0^{-1}([2x_1 - x_0, x_0; H] - [2x_0 - x_{-1}, x_{-1}; H]) \\
&\quad + L_0^{-1}([x_0, x_1; G] - [x_{-1}, x_0; G])\| \\
&\leq \beta[\omega(\|2x_1 + x_{-1} - 3x_0\|, \|x_0 - x_{-1}\|) \\
&\quad + \omega_1(\|x_0 - x_{-1}\|, \|x_1 - x_0\|)] \\
&\leq \beta[\omega(2\eta + \alpha, \alpha) + \omega_1(\alpha, \eta)] \\
&\leq \beta[\omega(3R + \alpha, R + \alpha) + \omega_1(\alpha + R, R)] < 1. \qquad (12)
\end{aligned}$$

It follows from (12), and the Banach lemma on invertible operators [3, 8] that L_1^{-1} exists, so that

$$\|L_1^{-1}\| \leq \frac{\beta}{1 - \beta[\omega(3R + \alpha, R + \alpha) + \omega_1(\alpha + R, R)]}. \qquad (13)$$

Consequently, the iterate x_2 is well defined. We can also obtain from (4), (7), (8) and (9) that

$$\begin{aligned}
\|F(x_1)\| &= \|F(x_1) - F(x_0) + F(x_0)\| \\
&= \|H(x_1) - H(x_0) + G(x_1) - G(x_0) - L_0(x_1 - x_0)\| \\
&= \|([x_1, x_0; H] - [2x_0 - x_{-1}, x_{-1}; H] \\
&\quad + [x_1, x_0; G] - [x_{-1}, x_0; G])(x_1 - x_0)\| \\
&\leq [\omega(\|x_1 + x_{-1} - 2x_0\|, \|x_0 - x_{-1}\|) \\
&\quad + \omega_1(\|x_1 - x_{-1}\|, 0)]\|x_1 - x_0\| \\
&\leq [\omega(\eta + \alpha, \alpha) + \omega_1(\alpha + \eta, 0)]\|x_1 - x_0\|. \qquad (14)
\end{aligned}$$

Then, we get

$$\begin{aligned}
\|x_2 - x_1\| &\leq \|L_1^{-1}\|\|F(x_1)\| \\
&\leq \frac{m}{1 - \beta[\omega(3R + \alpha, R + \alpha) + \omega_1(\alpha + R, R)]}\|x_1 - x_0\| \\
&= M\|x_1 - x_0\|. \qquad (15)
\end{aligned}$$

Furthermore, we have

$$\|x_2 - x_0\| \leq (M + 1)\|x_1 - x_0\| \leq (M + 1)\eta < \frac{\eta}{1 - M} = R, \qquad (16)$$

and $x_2 \in B(x_0, R)$.

Now we will prove by induction that the following items are true for $j \geq 1$

(I_j) L_j^{-1} exists, so that $\|L_j^{-1}\| \leq \frac{\beta}{1-\beta[\omega(3R+\alpha, R+\alpha)+\omega_1(\alpha+R, R)]}$,

(II_j) $\|x_{j+1} - x_j\| \leq M\|x_j - x_{j-1}\| \leq M^j\|x_1 - x_0\| < \eta$.

Assuming that the linear operators L_j are invertible and $x_{j+1} \in B(x_0, R)$ for $1 \leq j \leq n-1$, and $n \geq 2$ is a fixed integer, we obtain

$$\begin{aligned}\|I - L_0^{-1}L_n\| &= \|L_0^{-1}([2x_n - x_{n-1}, x_{n-1}; H] - [2x_0 - x_{-1}, x_{-1}; H])\\ &\quad + L_0^{-1}([x_{n-1}, x_n; G] - [x_{-1}, x_0; G])\|\\ &\leq \beta[\omega(\|2(x_n - x_0) - (x_{n-1} - x_{-1})\|, \|x_{n-1} - x_{-1}\|)\\ &\quad + \omega_1(\|x_{n-1} - x_{-1}\|, \|x_n - x_0\|)]\\ &\leq \beta[\omega(3R + \alpha, R + \alpha) + \omega_1(\alpha + R, R)] < 1. \end{aligned} \tag{17}$$

It follows from (17), and the Banach lemma on invertible operators that L_n^{-1} exists, so that

$$\|L_n^{-1}\| \leq \frac{\beta}{1 - \beta[\omega(3R + \alpha, R + \alpha) + \omega_1(\alpha + R, R)]}. \tag{18}$$

Consequently, the iterate x_{n+1} is well defined. Similarly to (14), we can get

$$\begin{aligned}\|F(x_n)\| &= \|F(x_n) - F(x_{n-1}) + F(x_{n-1})\|\\ &= \|H(x_n) - H(x_{n-1}) + G(x_n) - G(x_{n-1}) - L_{n-1}(x_n - x_{n-1})\|\\ &= \|([x_n, x_{n-1}; H] - [2x_{n-1} - x_{n-2}, x_{n-2}; H]\\ &\quad + [x_n, x_{n-1}; G] - [x_{n-2}, x_{n-1}; G])(x_n - x_{n-1})\|\\ &\leq [\omega(\|x_n + x_{n-2} - 2x_{n-1}\|, \|x_{n-1} - x_{n-2}\|)\\ &\quad + \omega_1(\|x_n - x_{n-2}\|, 0)]\|x_n - x_{n-1}\|\\ &\leq [\omega((M^{n-1} + M^{n-2})\eta, M^{n-2}\eta)\\ &\quad + \omega_1((M^{n-1} + M^{n-2})\eta, 0)]\|x_n - x_{n-1}\|\\ &\leq [\omega(2\eta, \eta) + \omega_1(2\eta, 0)]\|x_n - x_{n-1}\| \leq \frac{m}{\beta}\|x_n - x_{n-1}\|. \end{aligned} \tag{19}$$

Thus, we obtain

$$\|x_{n+1} - x_n\| \leq \|L_n^{-1}\|\|F(x_n)\| \leq M\|x_n - x_{n-1}\| \leq M^n\|x_1 - x_0\| < \eta. \tag{20}$$

Consequently, from (10) and (II_j), it follows:

$$\begin{aligned}\|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\|\\ &\leq (M^n + M^{n-1} + \cdots + 1)\|x_1 - x_0\|\\ &\leq \frac{1 - M^{n+1}}{1 - M}\eta < \frac{\eta}{1 - M} = R. \end{aligned} \tag{21}$$

So, $x_{n+1} \in B(x_0, R)$ and the induction is complete.

Next, we prove that $\{x_n\}$ is a Cauchy sequence. For $k \geq 1$, we obtain

$$\begin{aligned}\|x_{n+k} - x_n\| &\leq \|x_{n+k} - x_{n+k-1}\| + \|x_{n+k-1} - x_{n+k-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq (M^{k-1} + M^{k-2} + \cdots + 1)\|x_{n+1} - x_n\| \\ &\leq \frac{1 - M^k}{1 - M} M^n \|x_1 - x_0\| < \frac{M^n}{1 - M}\|x_1 - x_0\|. \end{aligned} \tag{22}$$

Therefore, $\{x_n\}$ is a Cauchy sequence on a Banach space X as such it converges to $x^\star \in \overline{B(x_0, R)}$ (since $\overline{B(x_0, R)}$ is a closed set). Setting $n \to \infty$ in (19), we obtain $F(x^\star) = 0$.

To show the uniqueness, we assume that there exists another solution $y^\star \in \overline{B(x_0, R)}$, and consider the operator $A = [y^\star, x^\star; H] + [y^\star, x^\star; G]$. Since $A(y^\star - x^\star) = F(y^\star) - F(x^\star)$, if the operator A is invertible, then $x^\star = y^\star$. Indeed, we get in turn:

$$\begin{aligned}\|I - L_0^{-1}A\| &= \|L_0^{-1}(A - L_0)\| \\ &\leq \|L_0^{-1}\|\big(\|[y^\star, x^\star; H] - [2x_0 - x_{-1}, x_{-1}; H]\| \\ &\quad + \|[y^\star, x^\star; G] - [x_{-1}, x_0; G]\|\big) \\ &\leq \beta[\omega(\|y^\star + x_{-1} - 2x_0\|, \|x^\star - x_{-1}\|) \\ &\quad + \omega_1(\|y^\star - x_{-1}\|, \|x^\star - x_0\|)] \\ &\leq \beta[\omega(R + \alpha, R + \alpha) + \omega_1(R + \alpha, R)] < 1. \end{aligned} \tag{23}$$

and the operator A^{-1} exists.

That completes the proof of Theorem 1. □

3 Numerical examples

We complete this study with a numerical example where we compare methods (3), (5) and (7).

Example 1 Let $X = Y = (\mathbf{R}^2, \|.\|_\infty)$. Consider the system

$$\begin{aligned} 3x^2y + y^2 - 1 + |x - 1| &= 0, \\ x^4 + xy^3 - 1 + |y| &= 0. \end{aligned}$$

Set

$$\begin{aligned} &\|x\|_\infty = \|(x', x'')\|_\infty = \max\{|x'|, |x''|\}, \\ &H = (H_1, H_2), \quad G = (G_1, G_2), \quad \text{and} \quad F = H + G. \end{aligned}$$

For $x = (x', x'') \in \mathbf{R}^2$ we take

$$\begin{aligned} &H_1(x', x'') = 3(x')^2x'' + (x'')^2 - 1, \qquad H_2(x', x'') = (x')^4 + x'(x'')^3 - 1, \\ &G_1(x', x'') = |x' - 1|, \qquad G_2(x', x'') = |x''|. \end{aligned}$$

We shall take $[x, y; G] \in M_{2\times 2}(\mathbf{R})$ as

$$[x, y; G]_{i,1} = \frac{G_i(y', y'') - G_i(x', y'')}{y' - x'},$$
$$[x, y; G]_{i,2} = \frac{G_i(x', y'') - G_i(x', x'')}{y'' - x''}, \quad i = 1, 2,$$

provided that $y' \neq x'$ and $y'' \neq x''$. Otherwise define $[x, y; G]$ to be the zero matrix in $M_{2\times 2}(\mathbf{R})$. Similarly we define divided difference $[2y - x, x; H]$.

Using method (3) with $x_0 = (1, 0)$ we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	1	0	
1	1	0.333333333333333	3.333E-1
2	0.906550218340611	0.354002911208151	9.344E-2
3	0.885328400663412	0.338027276361322	2.122E-2
4	0.891329556832800	0.326613976593566	1.141E-2
5	0.895238815463844	0.326406852843625	3.909E-3
6	0.895154671372635	0.327730334045043	1.323E-3
7	0.894673743471137	0.327979154372032	4.809E-4
8	0.894598908977448	0.327865059348755	1.140E-4
9	0.894643228355865	0.327815039208286	5.002E-5
10	0.894659993615645	0.327819889264891	1.676E-5
11	0.894657640195329	0.327826728208560	6.838E-6
12	0.894655219565091	0.327827351826856	2.420E-6
13	0.894655074977661	0.327826643198819	7.086E-7
⋮			
39	0.894655373334687	0.327826521746298	5.149E-19

Using the method of chord or Secant method (5) with $x_{-1} = (5, 5)$, and $x_0 = (1, 0)$, we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
−1	5	5	
0	1	0	5.000E+00
1	0.989800874210782	0.012627489072365	1.262E-02
2	0.921814765493287	0.307939916152262	2.953E-01
3	0.900073765669214	0.325927010697792	2.174E-02
4	0.894939851625105	0.327725437396226	5.133E-03
5	0.894658420586013	0.327825363500783	2.814E-04
6	0.894655375077418	0.327826521051833	3.045E-04
7	0.894655373334698	0.327826521746293	1.742E-09
8	0.894655373334687	0.327826521746298	1.076E-14
9	0.894655373334687	0.327826521746298	5.421E-20

Using our method (7) with $x_{-1} = (5, 5)$, and $x_0 = (1, 0)$, we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
−1	5	5	
0	1	0	5.000E+00
1	0.909090909090909	0.363636363636364	3.636E-01
2	0.894886945874111	0.329098638203090	3.453E-02
3	0.894655531991499	0.327827544745569	1.271E-03
4	0.894655373334793	0.327826521746906	1.022E-06
5	0.894655373334687	0.327826521746298	6.089E-13
6	0.894655373334687	0.327826521746298	2.710E-20

We did not verify the hypotheses of Theorem 1 for the above starting points. However, it is clear that the hypotheses of Theorem 1 are satisfied for all three methods for starting points closer to the solution

$$x^\star = (0.894655373334687, 0.327826521746298)$$

chosen from the lists of the tables displayed above.

Hence method (7) converges faster than (3) suggested by Zabrejko and Nguen in [12] and the method of chord [2–4, 6–11].

Acknowledgements This work is supported by National Natural Science Foundation of China (No. 10871178), Natural Science Foundation of Zhejiang Province of China (No. Y606154), and Scientific Research Fund of Zhejiang Provincial Education Department of China (No. 20071362).

References

1. Argyros, I.K.: On the solution of equations with nondifferentiable and Ptak error estimates. BIT **30**, 752–754 (1990)
2. Argyros, I.K.: On a two-point Newton-like method of convergent order two. Int. J. Comput. Math. **82**, 219–233 (2005)
3. Argyros, I.K.: Computational theory of iterative methods. In: Chui, C.K., Wuytack, L. (eds.) Studies in Computational Mathematics, vol. 15. Elsevier, New York (2007)
4. Argyros, I.K.: A Kantorovich-type analysis for a fast iterative method for solving nonlinear equations. J. Math. Anal. Appl. **332**, 97–108 (2007)
5. Han, D.F.: The majorant method and convergence for solving nondifferentiable equations in Banach space. Appl. Math. Comput. **118**, 73–82 (2001)
6. Hernandez, M.A., Rubio, M.J.: The Secant method for nondifferentiable operators. Appl. Math. Lett. **15**, 395–399 (2002)
7. Hernandez, M.A., Rubio, M.J.: Semilocal convergence of the Secant method under mild convergence conditions of differentiability. Comput. Math. Appl. **44**, 277–285 (2002)
8. Ortega, J.M., Rheinbolt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York (1970)
9. Ren, H.M.: New sufficient convergence conditions of the Secant method for nondifferentiable operators. Appl. Math. Comput. **182**, 1255–1259 (2006)
10. Schmidt, J.W.: Regula-falsi Verfahren mit konsistenter Steigung und majoranten Prinzip. Period. Math. Hung. **5**, 187–193 (1974)
11. Sergeev, A.: On the method of chords. Sib. Mat. Z. **2**, 282–289 (1961)
12. Zabrejko, P.P., Nguen, D.F.: The majorant method in the theory of Newton-Kantorovich approximations and the Ptak error estimates. Numer. Funct. Anal. Optim. **9**, 671–684 (1987)