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Methods for solving singular boundary value problems using splines: a review

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Abstract This paper surveys and reviews papers of spline solution of singular boundary value problems. Among a number of numerical methods used to solve two-point singular boundary value problems, spline methods provide an efficient tool. Techniques collected in this paper include cubic splines, non-polynomial splines, parametric splines, B-splines and TAGE method.

Keywords Singular boundary value problems · Quasi-linearization · Splines · B-spline · Finite difference method · TAGE method

Mathematics Subject Classification (2000) 65L10 · 65L12 · 34B16

1 Introduction

Consider a second-order linear differential equation

$$p_0(x)y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0 \tag{1}$$

where p_0, p_1 and p_2 are analytic at some point $x = a_0$. Such a point is said to be ordinary point if $p_0(a_0) \neq 0$ and that the solution of (1) can be represented by a power series in powers of $(x - a_0)$. If $p_0(a_0) = 0$, a_0 is called a singular point of (1). In such a case rewriting equation in the form

$$y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0 \tag{2}$$

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where $P_i(x) = p_i(x)/p_0(x)$, $i = 1, 2$. We see that the coefficients $P_1(x)$, $P_2(x)$ fail to be analytic at $x = a_0$. Singularities are divided into two kinds; regular singular points and irregular singular points. The point $x = a_0$ is said to be regular singular point, if $(x - a_0)P_1(x)$ and $(x - a_0)^2 P_2(x)$ are analytic at a_0; otherwise it is an irregular singularity.

Some classes of linear and non-linear singular boundary value problems (BVPs) considered by different authors are

$$y'' + f(x)y' + g(x) = h(x), \quad a < x \le b : y(a) = \alpha,\ y(b) = \beta, \tag{3}$$

$$x^{-\alpha}(x^{\alpha}u')' = f(x, u), \quad 0 < x \le 1 : u(0) = A,\ u(1) = B, \tag{4}$$

$$y''(x) + \frac{\alpha}{x}y'(x) = f(x, y), \quad y'(0) = 0,\ y(1) = \beta. \tag{5}$$

These types of singular boundary value problems for ordinary differential equations arise very frequently in several areas of sciences and engineering such as analysis of heat conduction through a solid with heat generation, Thomas–Fermi model in atomic physics and in the study of generalized axially symmetric potentials after separation of variables. These problems also occur very frequently in the study of electro hydrodynamics and the theory of thermal explosions. These also arise in physiology for the study of various tumor problems, the study of steady states oxygen diffusion in a spherical cell with Michaelis–Menten uptake kinetics and the study of heat sources distribution in human head.

Keeping in view such a wide scope of singular boundary value problems, these have been studied by several authors. Albasiny and Hoskins [5, 6] have obtained spline solutions by solving a set of equations with a tridiagonal matrix of coefficients. Bickley [8] has considered the use of cubic spline for solving linear two point boundary value problems. Fyfe [26] discussed the application of deferred corrections to the method suggested by Bickley, by considering linear boundary value problems. For $\alpha \in (0, 1)$, (4) has been extensively discussed. In the linear case, Jamet [35] considered a standard three-point finite difference scheme with uniform mesh and has shown that the error is $O(h^{1-\alpha})$ in maximum norm. Schreiber [65] considered the application of splines to the self-adjoint equation; again the function f was assumed to be linear. Ciarlet et al. [21] considered the application of Rayleigh–Ritz–Galerkin method and improved Jamet's result by showing that the error is $O(h^{2-\alpha})$ in uniform mesh for their Galerkin approximation. Gustafsson [32] gave a numerical method for the linear problem by representing the solution as a series expansion on a subinterval near the singularity and using the difference method for the remaining interval. Also, they constructed a Compact second and fourth-order scheme. Reddien [59], Reddien and Schumaker [60] considered the collocation method and the projection method for singular two-point boundary value problems and studied the existence, uniqueness and convergence rates of these methods. Doedel and Reddien [23] considered high-order finite difference methods for (4) where f is linear. Chawla and Katti [16, 17] constructed three kinds of finite difference methods and showed that these methods are $O(h^2)$ convergent. In [19], using cubic spline, a new numerical $O(h^4)$-convergent method was constructed. In [34] the construction of spline finite difference method is discussed and proved to be $O(h^2)$-convergent. Han [30, 31] considered Richardson's extrapolation of spline approximation, correction of finite difference solution and spline approximation solution. In [15] Chawla et al. obtained a method for the singular two-point boundary value problem (1) based on a uniform mesh and showed that under quite general condition, their method provided $O(h^2)$ convergent approximation for all $\alpha \geq 1$. El-Gebeily et al. [1, 2, 25] considered a finite difference method for the solution of a general class of singular two-point boundary value problems, and obtained the rate of convergence of the method in the uniform norm. Related literature can also be found in [10, 15, 18, 22, 24, 27, 33, 50, 53, 62, 63] and references therein.

In comparison with the finite difference methods, spline solution has its own advantages. For example, once the solution has been computed the information required for spline interpolation between mesh points is available. This is particularly important when the solution of boundary value problem is required at various locations in the interval $[a, b]$. Theory of splines can be found in detail in [4, 9] and [45].

The present paper contains a survey on spline solutions of singular boundary value problems. It is arranged as follows. In Sect. 2, a survey of papers using splines (mainly cubic spline) to solve singular boundary value problems is given. In Sect. 3, a brief review on B-spline papers is developed. The papers included in Sect. 4 use TAGE method. In each section papers are arranged chronologically so that one can understand step by step development.

2 Spline solution of singular boundary value problems

F. Natterer [49] defines the boundary value problem of a first order singular system in such a way that the system can be interpreted as a Fredholm operator in some Banach space, and a-priori-estimates for functions belonging to the domain of this operator are proved. In the paper, the space of generalized vector valued spline functions is also defined and its interpolation properties are investigated. The finite dimensional approximating problems are then introduced by a projection method. The paper also contains the convergence proof. It is shown that for suitable non-uniform partitions, the convergence is—up to logarithmic terms—as fast as in the regular case.

In [65] R. Schreiber, basically analyses several finite element methods for singular two-point boundary value problems

$$-(x^{\sigma} u')' + qu = f, \quad 0 < x < 1 \text{ where } \sigma \in [0, 1) \tag{6}$$

and appropriate boundary conditions are imposed. The solution can be approximated by splines on a non-uniform mesh. Error bounds of optimal order are proved, and upper and lower bounds on the extent to which the mesh must be graded are obtained. Author also considers approximating the solution by functions of the form $x^{-\sigma} s(x)$, where $s(x)$ is a spline. Error bounds and numerical results for these weighted splines indicate that they are very efficient. For a third space, known error bounds are improved by using a mildly graded mesh.

Kadalbajoo and Raman in [39] have considered a second order linear differential equation

$$Ly(x) \equiv y'' + f(x)y' + g(x) = h(x), \quad a < x \leq b \tag{7}$$

s.t. $y(a) = \alpha$ and $y(b) = \beta$; where the coefficient functions $f(x)$ and $g(x)$ may not be analytic at $x = a$. To remove the singularity at the point $x = a$, authors used series expansion in a small interval near $x = a$ so that (7) has a solution of the form

$$y(x) = (x-a)^m \sum_{n=0}^{\infty} b_n (x-a)^n, \quad b_0 \neq 0. \tag{8}$$

The authors solve a regular boundary value problem over a reduced interval excluding singular point and match the solution to the expansion. Next, cubic spline procedure is developed for discretizing the resulting regular problem with step size h. After straightforward but long calculations, a three term recurrence relation is obtained. In order to solve this resulting difference system method of Invariant Imbedding is then used. Stability of the recurrence relation is also being established. Lastly two numerical examples are solved.

Remark 1 We notice that the test examples considered in the paper have also been treated in an earlier paper [41]. While solving regular singular boundary value problems in that paper, a continuous form of Invariant Imbedding has been employed, after removing the singularity, to reduce the resulting boundary value problem to initial value problems in ordinary differential equations, which have been solved by the Runge–Kutta–Fehlberg scheme with step size control. In the present study, authors use cubic spline approximation, after treating the singularity separately, and develop discrete Invariant Imbedding algorithm for the solution of the resulting algebraic problem. Since it is sometimes misleading to compare the results obtained by fixed step size routines to that obtained by variable step size initial value software, authors have solved the initial value problems for ODEs obtained in an earlier paper [41] by a fixed step routine based on the Runge–Kutta–Gill method. It can be checked from the analytical solution that the results obtained by the present method are much more accurate than the results obtained by fixed step size routine (Runge–Kutta–Gill method) for the initial value ODEs in [40]. Also an advantage of this method is that the coefficient matrix of the system is tridigonal and the method has an order of convergence $O(h^2)$.

In [34] Iyengar and Jain consider the class of singular two point boundary value problem

$$x^{-\alpha}(x^{\alpha}u')' = f(x,u), \quad 0 < x \leq 1 \tag{9}$$

s.t. $u(0) = A$, $u(1) = B$ or $u'(0) = 0, u(1) = B$. Authors construct splines and the three point finite difference methods using these splines for the solution of (9) in different cases: $\alpha \in (0,1)$, $\alpha = 0$, $\alpha = 1$ (cylindrical case) and $\alpha = 2$ (spherical case). Convergence of the spline difference methods and numerical examples are given.

Remark 2 These schemes are of $O(h^2)$ under appropriate conditions. The advantage of the spline approximation is that (9) may be solved with a particular step length h and the intermediate values, if required, can be computed using the spline. It is shown through numerical computations that these spline solutions are of the same accuracy as the two neighbouring finite difference ones. The numerical results show that the methods are robust and the spline gives good approximation at the intermediate points.

M.M. Chawla and R. Subramanian, H.L. Sathi in [20] described two methods for the solution of (weakly) singular two-point boundary-value problems:

$$x^{-\alpha}(x^{\alpha}y')' = f(x, y), \quad 0 < x \le 1 : y(0) = A, \ y(1) = B. \tag{10}$$

Consider the uniform mesh $x_i = ih$, $h = 1/N$, $i = 0, \dots, N$. Define the linear functionals $L_i(y) = y(x_i)$ and $M_i(y) = (x^{-\alpha}(x^{\alpha}y')')_{x=x_i}$. In both these methods a piecewise 'spline' solution is obtained in the form $S(x) = S_i(x)$, $x \in [x_{i-1}, x_i]$, $i = 1, \dots, N$, where in each subinterval $S_i(x)$ is in the linear span of some set of (non-polynomial) basis functions in the representation of the solution $y(x)$ of the two-point boundary value problem and satisfies the interpolation conditions:

$$\begin{aligned} L_{i-1}(S) &= L_{i-1}(y), \qquad L_i(S) = L_i(y), \\ M_{i-1}(S) &= M_{i-1}(y), \qquad M_i(S) = M_i(y). \end{aligned} \tag{11}$$

By definition S and $x^{-\alpha}(x^{\alpha}s')' \in C[0, 1]$. Conditions of continuity are derived to ensure that $x^{\alpha}s' \in C[0, 1]$. It follows that the unknown parameters y_i and $M_i(y)$, $i = 1, \dots, N-1$, must satisfy conditions of the form:

$$-\frac{1}{J_i}y_{i-1} + \left(\frac{1}{J_i} + \frac{1}{J_{i+1}}\right)y_i - \frac{1}{J_{i+1}}y_{i+1} + k_{i,i-1}M_{i-1} + k_{i,i}M_i + k_{i,i+1}M_{i+1} = 0. \tag{12}$$

The first method consists in replacing $M_i(y)$ by $f(x, y)$ and solving (12) to obtain the values y_i; this method is a generalization of the idea of Bickley [8] for the case of (weakly) singular two-point boundary-value problems and provides order h^2 uniformly convergent approximations over [0, 1]. As a modification of the above method, in the second one authors generate the solution $\bar{y}_i$, at the nodal points by adapting the fourth-order method of Chawla [14] and then use the conditions of continuity (12) to obtain the corresponding smooth approximations for $M_i(y)$ needed for the construction of the spline solution. The second-order and the fourth-order methods are illustrated computationally.

In [28, 29] Han Guoqiang discussed the constructions of three-point finite-difference approximation and a spline approximation for a class of singular two-point boundary value problems:

$$x^{-\alpha}(x^{\alpha}u')' = f(x, u), \quad 0 < x \le 1 \tag{13}$$

s.t. $u(0^+) = 0$, $u(1) = A, \alpha \ge 1$. The asymptotic error expansions of the numerical solutions of these problems are obtained. From these asymptotic error expansions we find that the finite-difference solution and the spline approximation solution approximate the exact solution from two sides. So we can obtain correct solution of high-order accuracy. Richardson's extrapolation can also be done and the accuracy of numerical solution can be improved greatly. The numerical results show the fourth-order convergences of the correct solution, extrapolation solution of spline approximation and extrapolation solution of finite-difference.

In [51], R.K. Pandey considers a class of singular two-point boundary value problems arising in physiology. The convergence of a spline method for singular BVPs has been established. The results obtained using this method are better than using the usual finite difference method with the same number of knots. Also this method produces a spline function which may be used to obtain the solution at any point in the range, whereas the finite difference methods only give the solution at the chosen knots.

In [44], the class of two-point singular boundary value problems of the form

$$x^{-\alpha}(x^{\alpha}u')' = f(x,u), \quad 0 < x \le 1 \tag{14}$$

s.t. $u(0) = A$, $u(1) = B$, $\alpha \in (0,1)$ has been considered and a spline and the three point finite difference method based on non-uniform mesh for the solution of given problem is constructed. Method is illustrated by two numerical examples.

Remark 3 The given scheme is second order convergent under appropriate conditions. The numerical results show that the method is robust and the spline gives good approximation at the intermediate points. The author in his later paper [43] discusses the construction of five point finite difference method using the splines. The new method obtained is shown to be order h^4 convergent for all $\alpha \in (0,1)$. In [42] the author again explains higher order method for singular boundary value problems by using spline function and results obtained for non-uniform mesh.

Kadalbajoo and Vivek Agrawal in [36], have considered a second order linear differential equation

$$u'' + p(x)u' + q(x)u = r(x), \quad a < x \le b \tag{15}$$

subject to boundary conditions $u(a) = \alpha$ and $u(b) = \beta$ where the coefficient functions $p(x)$ and $q(x)$ fail to be analytic at $x = a$. In order to remove the singularity at the point $x = a$ for the given problem, authors use series expansion in a neighborhood of $x = a$; Then, cubic spline procedure is developed for discretizing the resulting regular problem. Authors described a procedure producing an equal interval spline for use as an interpolating spline over the whole range $\delta \le x \le b$. The spline will give results to a prescribed accuracy at any point in the range and will involve the minimum convenient number of knots consistent with such accuracy. The choice of interval is determined by two separate considerations. First, one must ensure that the values at the knots are determined to sufficient accuracy. Secondly, assuming that the values at the knots are correct, the interpolation error at interval points of any interval is sufficiently small. These two requirements are not necessarily related although it is often found that when the second condition is satisfied the first will also hold.

Remark 4 An advantage of the method is that the coefficient matrix of the system is of the Hessenberg form, and the method has an order of convergence $O(h^4)$, where

h is the step size. While solving regular singular BVPs in [40], a continuous form of invariant imbedding has been employed, after removing the singularity, to reduce the resulting boundary value problem to initial value problems in ordinary defferential equations, which have been solved by the Runge–Kutta–Fehlberg scheme with step size control. Authors also used cubic spline approximation for the given problem, after treating the singularity separately, without using invariant imbedding concept. It can be checked from the analytical solution that the results obtained by the method are much more accurate than the results obtained by fixed step size routine (Runge–Kutta–Gill method) for the initial value ODEs in [40]. The superiority of the solutions obtained by the method is again evident here as compared to the solutions obtained by Jamet [35] and Reddien [59].

A.S.V. Ravi Kanth and Y.N. Reddy [58], considered the class of singular two-point BVPs

$$\begin{aligned} &y''(x) + \frac{k}{x}y'(x) + b(x)y(x) = c(x), \quad 0 < x < 1, \\ &y'(0) = 0 \quad \text{and} \quad y(1) = \beta. \end{aligned} \tag{16}$$

Since $x = 0$ is singular point, the above equation is modified at it. By L'Hospital rule, the given BVP is transformed into

$$\begin{aligned} &y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \\ &y'(0) = 0 \quad \text{and} \quad y(1) = \beta, \end{aligned} \tag{17}$$

where $p(x), q(x)$and $r(x)$ are defined appropriately. Then authors give the cubic spline $S(x)$ interpolating the function $y(x)$ at the chosen grid points. Ahlberg et al. [4] have shown that if the function $y(x) \in C^4[0, 1]$, then the Spline function $S(x)$ approximates $y(x)$ at all points in [0, 1] to fourth order in h (Prenter [52]). By employing spline on modified BVP (17), we get $(n + 1)$ equations with $(n + 1)$ unknowns. The matrix problem associated here is a tridiagonal algebraic system whose solution can easily be determined by an efficient algorithm called Thomas algorithm. Authors have used the method with four examples; a homogeneous singular boundary value problem, and three non-homogeneous singular boundary value problems, and have tabulated the numerical results as well as the exact solutions. The tables show that the method approximates the exact solution very well.

Remark 5 Authors have described and demonstrated the applicability of the cubic spline method for solving singular boundary value problems. It is a direct, simple, accurate, and easy to implement on computer. It is a practical method and can easily be implemented on computer to solve the problems.

In [56] A.S.V. Ravi Kanth and Vishnu Bhattacharya have used cubic spline method to analyze a class of non-linear singular boundary value problems defined by

$$\begin{aligned} &y''(x) + \frac{\alpha}{x}y'(x) = f(x, y), \\ &y'(0) = 0 \quad \text{and} \quad \alpha y(1) + \beta y'(1) = \gamma. \end{aligned} \tag{18}$$

The quasilinearization technique is used to reduce the given non-linear problem to a sequence of linear problems. The resulting set of differential equations is modified at the singular point and is treated by using cubic spline for finding the numerical solution. The numerical method is tested for its efficiency by considering two examples from physiology.

In [54], J. Rashidinia, Z. Mahmoodi and M. Ghasemi present a three point finite difference method based on uniform mesh using parametric spline for the class of singular two-point BVPs

$$x^{-\alpha}(x^{\alpha}y')' = f(x, y), \quad 0 < x \leq 1;\ y(0) = A \text{ and } y(1) = B. \tag{19}$$

Firstly authors have derived the formulation of spline function approximations. Accordingly consider a uniform mesh with knots $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ where $x_i = ih$, $h = 1/N$. A function $S_{\Delta}(x, p)$ of class $C^2([a, b])$, interpolating $y(x)$ at the knots $\{x_i\}$ and depending on a parameter p is called a parametric spline function and reduces to a cubic spline function in the interval $[x_{i-1}, x_i]$ as $p \to 0$. Then given problem is treated with the parametric spline.

Remark 6 In this method, by choosing different values of parameters, we can obtain the classes of second order methods. Solution of Bessel's equation of order two by this method gives more accurate results as compared to that in [58].

J. Rashidinia, R. Mohammadi and R. Jalilian in [55] solve a class of non-linear singular ordinary differential equations arising in physiology by a new method based on non-polynomial cubic spline. Consider,

$$y''(x) + \left(a + \frac{m}{x}\right)y'(x) = f(x, y), \quad 0 \leq x \leq 1 \tag{20}$$

with

$$\eta_1 y(0) + \zeta_1 y'(0) = \gamma_1 \quad \text{and} \quad \eta_2 y(1) + \zeta_2 y'(1) = \gamma_2.$$

Authors use the quasilinearization technique to reduce the given non-linear problem to a sequence of linear problems, then modify the resulting set of differential equations at the singular point then treat them by using non-polynomial cubic spline approximation. For each segment $[x_i, x_{i+1}]$, the non-polynomial spline has the form

$$S_{\Delta}(x) = a_i + b_i(x - x_i) + c_i \sin\tau(x - x_i) + d_i \cos\tau(x - x_i), \quad i = 0, 1, 2, \ldots, N-1. \tag{21}$$

The resulting system of algebraic equations is solved by using a tri-diagonal solver.

Remark 7 For $\alpha = 1/6$ and $\beta = 1/3$, this method is the second-order method and reduces to the method by Ravi Kanth and Bhattacharya [56]. For $\alpha = 1/12$ and $\beta = 5/12$, this new method is the fourth-order method. Authors have illustrated numerical solution of three problems including an oxygen diffusion problem and a non-linear heat conduction model of the human head.

In [57], Ravikanth has considered a class of non-linear singular BVP

$$y''(x) + \frac{\alpha}{x}y'(x) = f(x, y), \quad y'(0) = 0,\ y(1) = \beta,\ \alpha \geq 1. \tag{22}$$

The author assumes that f is continuous, $\partial f/\partial y$ exists, is continuous and positive. Due to the singularity at $x = 0$ on the left side of the differential equation, direct numerical techniques face convergence difficulties. Attempts by many researchers for the removal of the singularity are based on series expansion procedures in the neighborhood $(0, \delta)$ of singularity and then solve the regular boundary value problem in the interval $(\delta, 1)$ using numerical methods. In this paper, authors discuss a direct method based on cubic spline approximation for the solution of nonlinear singular two-point boundary value problems. The advantage of this method is that the coefficient matrix of the system is of the system of Hessenberg form. First they use the quasi-linearization technique to reduce the given non-linear problem to a sequence of linear problems. The resulting sets of differential equations are modified at the singular point and are treated by using cubic spline for finding the numerical solution. The numerical method is tested for its efficiency by considering three physical model problems from the literature.

Remark 8 As it is evident from the computational results solved in this paper, the method gives $O(h^4)$ accuracy. The results obtained using this method are better than using the usual finite difference method with the same number of knots.

3 B-spline methods

Kadalbajoo and Vivek Kumar in [37] discuss a homogeneous second order linear differential equation having regular singularity given by

$$u''(x) + f(x)u'(x) + g(x)u(x) = 0, \quad 0 \leq x \leq 1 \tag{23}$$

s.t. $u(0) = \alpha$ and $u(1) = \beta$. Functions f and g are not analytic at $x = 0$. It gives singularity at $x = 0$. This type of problems arises when partial differential equation reduced to ODE equation by physical symmetry. To remove the singularity authors used Chebychev economization near the singularity and boundary at $x = \delta$. For finding the numerical solution, B-spline method is used on resulting regular BVP which gives $O(h^4)$ accuracy. The results obtained by this method are better than using the finite difference method with the same number of knots.

Existence and uniqueness of solutions for such type of problems has been discussed in [35] using finite difference method. Reddien [61] used collocation method for the solution of such problem. In [32] series solution is used in the neighborhood of the singularity. In general, series solution may not produce an effective approximation near the singularity. For reducing the necessary number of terms in the series without increasing the errors Chebychev economization can be applied.

Remark 9 Comparing B-spline method, with the one used in [22], we remark that it produces a spline function allowing the solution pointwise on the range. But by finite difference method used in [22] we can find solutions only on chosen Knots.

In [13] Nazan Caglar and Hikmet Caglar, have solved Homogeneous and non-homogenous singular boundary value problems using B-splines. In a series of papers by Caglar et al. [11, 12], BVPs of second, third, fourth and fifth order are solved using third, fourth and sixth-degree splines. In [13], a third-degree B-spline is used to solve singular boundary value problems as the following form which is assumed to have a unique solution in the interval of integration [3],

$$y''(x) + \frac{k}{x} y'(x) + b(x)y(x) = c(x), \quad 0 < x < 1; \; y'(0) = 0 \text{ and } y(1) = \beta, \tag{24}$$

which arise in the study of generalized axially symmetric potentials after separation of variables has been employed. These problems also occur very frequently in the study of electro hydrodynamics and the theory of thermal explosions. In [13], authors discuss a direct method based on B-splines for a class of singular two-point boundary value problems. The original differential equation is modified at singular point. The B-spline approximation is then employed to solve the boundary value problem.

In [38] M.K. Kadalbajoo and V. Kumar, consider a class of singular two-point boundary value problems

$$x^{-k}(x^k u')' = f(x, u), \quad 0 < x \le 1 \tag{25}$$

with $u'(0) = 0$ and $u(1) = B$. Here $k \in (0, 1)$ (weakly singular) or also $k = 1, 2$ (strongly singular). For $k = 1$, the problem becomes a cylindrical one and for $k = 2$, it becomes spherical. B is a finite constant. It is well known that the problem above has a unique solution, if f is continuous, $\frac{\partial f}{\partial u}$ exists and is continuous and $\frac{\partial f}{\partial u} \ge 0$. The aim of the paper is to present a modified fourth order B-spline method to solve a certain class of linear and non-linear singular boundary value problems such as

$$u''(x) + \frac{k}{x} u'(x) = f(x, u), \tag{26}$$

with $u'(0) = 0$ and $u(1) = B$, where $k = 1$ or 2. On physical ground we expect a smooth solution. Indeed, symmetry implies that this smooth solution has a derivative that vanishes at the origin. This physical condition is just what is needed for the existence of the limit $\lim_{x \to 0} \frac{u'(x)}{x} = u''(0)$. On the other hand we can say that as discussed in [32], it may sometimes be very difficult or even not possible to obtain the series solution in the neighborhood of the singularity. Recently Shampine et al. [66, 67] modified the MATLAB solver bvp4c and solved these kinds of problems. In the case of non-linear problems, quasilinearization technique, originally developed by Bellman and Kalaba [7], has been used to reduce the given non-linear problem to a sequence of linear problems. The linear problem is modified at the singular point. The numerical experiments for the model problems have been given to illustrate the method. One of the problems discussed in this paper has earlier been discussed by Russell and Shampine [64]. A second particular one deals with the oxygen diffusion into a cell, in which an enzyme-catalyzed reaction occurs. A third one arises in a study of heat and mass transfer in a porous spherical catalyst with a first order reaction. There is a singular coefficient arising from the reduction of a partial differential equation to an ODE by symmetry. A non-linear problem arising in the equilibrium of isothermal gas spheres in Astronomy is also being solved.

Remark 10 It was shown from numerical result that the method gives $O(h^4)$ accuracy. Using deferred correction this B-spline method will give results to a prescribed accuracy at any point in the range and will involve the minimum convenient number of knots consistent with such accuracy. This method gives comparable results and is easy to compute. It also produces a spline function which may be used to obtain the solution point wise on the range, whereas the finite-difference methods and the invariant imbedding methods [39] give the solution only at the chosen knots.

4 TAGE methods

In [46], R.K. Mohanty and D.J. Evans present a fourth-order method based on cubic spline approximations for the numerical solution of non-linear singular two point boundary value problems. The AGE (Alternating Group Explicit) and Newton-AGE iteration methods which are suitable both on sequential and parallel computers are discussed both for linear and nonlinear singular problems. The convergence theory is briefly discussed. The numerical results obtained confirm the viability of the proposed method.

In [48], Authors consider general non-linear second order ODE

$$-u'' + f(r, u, u') = 0, \quad 0 < r < 1 \tag{27}$$

with $u(0) = A$ and $u(1) = B$, where A and B are finite constants. By assuming that for $0 < r < 1$ and $-\infty < u, v < \infty$; $f(r, u, v)$ is continuous; $\partial f/\partial u$ and $\partial f/\partial v$ exist and are continuous; $\partial f/\partial v > 0$ and $|\partial f/\partial v| \le k$ for some positive constant k. These conditions guaranty that the boundary value problem has a unique solution [41]. In the paper, authors discuss a fourth order cubic spline two parameter alternating group explicit (TAGE) method for the numerical solution of both linear and nonlinear singular two point boundary value problems. The cubic spline interpolation process itself is fourth order. Therefore, it is natural to look for a method using cubic spline which provides fourth order approximations. In the paper, fourth order accurate cubic spline TAGE and Newton-TAGE iterative methods are discussed and proved to be suitable for computation both on sequential and parallel computers. The convergence of the method for the real unsymmetric coefficient matrix is also discussed.

Remark 11 Numerical experiments show that the TAGE method is accurate and convergent. The proposed TAGE and Newton-TAGE iteration methods show the superiority over the corresponding SOR iteration method. Although the TAGE method involves more work, the developing of the TAGE group implies that parallelism can be easily applied advantageously. Since both AGE and TAGE method requires the same number of sweep operations to solve the system of equations, the TAGE method requires less computation to obtain the final solution. Hence TAGE method is more efficient than the AGE method for the numerical solution of singular two point BVP.

In [47] R.K. Mohanty, D.J. Evans and Noopur Khosla report a non-uniform mesh cubic spline method of accuracy $O(h_k^3)$ for the solution of non-linear singular two-point boundary value problems. The application of two-parameter alternating group explicit (TAGE) and Newton-TAGE iteration methods which are suitable for use on parallel computers is discussed. The error analysis for TAGE iteration method is discussed in detail. The numerical results confirm the utility of proposed cubic spline TAGE iteration methods.

5 Discussion and conclusion

This paper contains sufficiently large amount of material concerned with spline solution of linear and non-linear second order two-point singular boundary value problems (BVPs) in ordinary differential equations. This may help substantially to the researchers. Spline functions give simple and practical methods to solve singular boundary value problems. It is more advantageous than other available computational techniques. In comparison with the finite difference methods, spline solution has its own advantages. For example, once the solution has been computed the information required for spline interpolation between mesh points is available. This is particularly important when the solution of boundary value problem is required at various locations in the given interval.

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