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On a class of nonlinear inhomogeneous Schrödinger equation

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Abstract In this paper, we study the inhomogeneous Schrödinger equation

$$i\varphi_t = -\triangle\varphi - |x|^b|\varphi|^{p-1}\varphi, \quad x \in \mathbb{R}^N.$$

By using variational methods and a refined interpolation inequality, we establish some simple but sharp conditions on the solutions which exist globally or blow up in a finite time. An interesting result is that we obtain the existence of global solution for arbitrarily large data.

Keywords Inhomogeneous Schrödinger equation · Unbounded coefficient · Global solutions · Blow-up solutions

Mathematics Subject Classification (2000) 35J20 · 35Q53

1 Introduction

Considered here is the following inhomogeneous Schrödinger equation

$$i\varphi_t = -\triangle\varphi - |x|^b|\varphi|^{p-1}\varphi, \quad b > 0,\ t > 0,\ x \in \mathbb{R}^N \tag{1.1}$$

with initial condition

$$\varphi(x, 0) = \varphi_0(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

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We are mainly concerned with the stability of standing waves of (1.1) and the existence of blow-up or global solutions of (1.1)–(1.2). The main results are contained in Theorems 3.3, 3.4 and 5.3.

Nonlinear Schrödinger equations arise naturally from various physical contexts. The homogeneous Schrödinger equation

$$i\varphi_t = -\triangle\varphi - |\varphi|^{p-1}\varphi \tag{1.3}$$

has been derived in the description of nonlinear waves such as propagation of a laser beam and plasma waves. Moreover, it was suggested that stable high power propagation can be achieved in plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduced nonlinearity inside the channel [11, 12]. Under these conditions, beam propagation can be modeled by the following inhomogeneous nonlinear Schrödinger equation

$$i\varphi_t = -\triangle\varphi - K(x)|\varphi|^{p-1}\varphi. \tag{1.4}$$

The solution φ denotes the electric field in laser and optics and $K(x)$ is proportional to the electric density. Further background on the nonlinear Schrödinger equations can be found in Berge [2].

Equation (1.3) has been studied extensively, see [1, 3, 5, 16–18] and the references therein. Observed the results in these references and their proofs, the following standard Gagliardo-Nirenberg [4] inequality

$$\int |u|^{p+1} \le C_{GN}\left(\int |\nabla u|^2\right)^{N(p-1)/4}\left(\int |u|^2\right)^{(2(p+1)-N(p-1))/4} \tag{1.5}$$

and the best constant C_{GN} played an essential role. For (1.4) in the case of $K(x)$ is bounded in $\mathbb{R}^N$, one can also use the inequality (1.5) to study the existence of local and global solutions of (1.4) as well as the stability of the standing-wave solutions, see [9, 13, 14]. It is also noted that Fukuizumi and Ohta [10] use Hardy inequality to obtain the instability of standing waves of (1.4) when the inhomogeneity K of nonlinearity behaves like $|x|^{-b}$ at infinity with $0 < b < 2$. But for (1.1) in the case of $b > 0$, none of the above mentioned methods can be applied. To explain that (1.1) is quite different from (1.4) with bounded $K(x)$, we point out that (1.1) has at least two conservation laws defined in the following

$$E(\varphi) = \frac{1}{2}\int |\nabla\varphi|^2 dx - \frac{1}{p+1}\int |x|^b|\varphi|^{p+1}dx \tag{1.6}$$

and

$$V(u) = \frac{1}{2}\int |\varphi|^2 dx. \tag{1.7}$$

In the case of $b = 0$ (or $|x|^b$ is replaced by a bounded $K(x)$), one can use (1.5) to estimate the nonlinear term $\int |\varphi|^{p+1}$ (or $\int K(x)|\varphi|^{p+1}$) and use the conservation laws (1.6) and (1.7) to get local and global solutions of Cauchy problem of (1.3) (or (1.4)).

While (1.5) can not be used to estimate $\int |x|^b|\varphi|^{p+1}dx$ due to the unbounded coefficient $|x|^b$. Thus the issue of whether or not particular of initial data generate a blow-up solution of (1.1) is more subtle.

To overcome this difficulty, we have established an improved inequality of Gagliardo-Nirenberg type interpolation [8] (see also Proposition 2.2 of the present paper) and used this inequality to study the existence of local solutions of (1.1)–(1.2). The main purpose of the present paper is to use the best constant $C(N, p, b)$ (see Proposition 2.4 of the present paper) to solve the following problems: Under what (optimal) conditions, the solutions of (1.1)–(1.2) exist globally in time and under what (optimal) conditions the solutions of (1.1)–(1.2) blow up in a finite time?

This paper is organized as follows. In Sect. 2, we recall some results on the improved inequality of Gagliardo-Nirenberg type interpolation and give a slight refined version of $C(N, p, b)$. In Sect. 3, we use the best constant $C(N, p, b)$ to study (1.1)–(1.2) in the critical nonlinearity, i.e., $p = 1 + \frac{4+2b}{N}$. We present a simple but sharp condition on the solutions of (1.1) which exist globally in time or blow up in a finite time. The main results are contained in Theorems 3.3 and 3.4. In Sect. 4, we study (1.1)–(1.2) in the supercritical nonlinearity case, i.e., $p > 1 + \frac{4+2b}{N}$. By establishing several invariant sets $\mathcal{S}_\pm$, we prove that for $\varphi_0 \in \mathcal{S}_+$ (resp. $\varphi_0 \in \mathcal{S}_-$) the solution of (1.1) exists globally in time (resp. blow up in a finite time), see Theorems 4.5 and 4.6. In Sect. 5, we prove that the solutions of (1.1)–(1.2) exist globally in time for one class of initial data whose norm can be as large as you want. The main result is contained in Theorem 5.3. In order to get these results, we use the ideas originated in [1, 3, 17], but we emphasize that throughout this paper, the uses of the improved Gagliardo-Nirenberg inequality and the relate best constant [7, 8] are essential.

Notations As above and henceforth, we denote the norm of the space $L^q(\mathbb{R}^n)$ by $\|\cdot\|_q$, $1 \le q \le \infty$ and denote the integral $\int_{\mathbb{R}^N} dx$ simply by $\int$ unless stated otherwise. We also denote various positive constants by C or C_j. For any t, the function $x \mapsto \varphi(x, t)$ is simply denoted by $\varphi(t)$ if no confusion occurs.

2 Preliminaries

In this section, we give some results on the improved inequality of Gagliardo-Nirenberg type interpolation [8]. Let $H^1(\mathbb{R}^N)$ be the standard Sobolev space with the standard norm. $H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N);\ u(x) = u(|x|)\}$. For $b \ge 0$ and $0 < p < +\infty$, we set

$$Y = \left\{ u \text{ is measurable in } \mathbb{R}^N;\ \int |x|^b |u|^{p+1} < +\infty \right\}$$

with the norm $\|u\|_Y^{p+1} := \int |x|^b|u|^{p+1}$. We denote by Y_r the space of radially symmetric functions in Y. By [7, 8], $H_r^1(\mathbb{R}^N) \hookrightarrow Y_r$ is continuous for $1 < p < \tilde{p}$, where

$$\tilde{p} = \begin{cases} \frac{N+2}{N-2} + \frac{2b}{N-1}, & \text{if } N \ge 3, \\ +\infty, & \text{if } N = 2. \end{cases}$$

Proposition 2.1 *Let $N \geq 2$, $b \geq 0$ and $1 + \frac{2b}{N-1} < p < \tilde{p}$. Then $H_r^1(\mathbb{R}^N) \hookrightarrow Y_r$ is compact.*

Remark 2.1 (i) When $b = 0$, this lemma is due to Strauss [16].

(ii) When $N \geq 3$, this lemma has been proved by Sintzoff [15] based on Rother's inequality. It seems that the proof used in [15] can not be used to the case of $N = 2$ since Rother's inequality holds only in the case of $N \geq 3$.

Proof of Proposition 2.1 By the previous remark, it suffices to prove Proposition 2.1 in the case of $N = 2$ and $b > 0$. Without loss of generality, assuming that $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ satisfying $u_n \rightharpoonup 0$ weakly in $H_r^1(\mathbb{R}^N)$, then we have that $\int |x|^b |u_n|^{p+1} < +\infty$ uniformly for n. For any $\varepsilon > 0$, we write

$$\int |x|^b |u_n|^{p+1} = \left(\int_{|x| \leq \varepsilon} + \int_{\varepsilon \leq |x| \leq \frac{1}{\varepsilon}} + \int_{|x| \geq \frac{1}{\varepsilon}} \right) |x|^b |u_n|^{p+1}. \tag{2.1}$$

We firstly consider the term $\int_{|x| \leq \varepsilon} |x|^b |u_n|^{p+1}$. Since $b > 0$, we obtain from the standard Gagliardo-Nirenberg inequality that

$$\int_{|x| \leq \varepsilon} |x|^b |u_n|^{p+1} \leq \varepsilon^b \int_{|x| \leq \varepsilon} |u_n|^{p+1} \leq C\varepsilon^b \left(\int |\nabla u_n|^2 \right)^{\frac{p-1}{2}} \int |u_n|^2 \leq C_1 \varepsilon^b,$$

where use has been made of the fact that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$. It follows that $\int_{|x| \leq \varepsilon} |x|^b |u_n|^{p+1}$ tends to 0 uniformly in n as $\varepsilon \to 0$. On the other hand, as $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$ and $N = 2$, we deduce from Strauss' inequality that

$$\begin{aligned} \int_{|x| \geq \frac{1}{\varepsilon}} |x|^b |u_n|^{p+1} &= \int_{|x| \geq \frac{1}{\varepsilon}} |x|^{b - \frac{p-1}{2}} |x|^{\frac{p-1}{2}} |u_n|^{p-1} |u_n|^2 \\ &\leq C \left(\int |\nabla u_n|^2 \right)^{\frac{p-1}{4}} \left(\int |u_n|^2 \right)^{\frac{p-1}{4}} \int_{|x| \geq \frac{1}{\varepsilon}} |x|^{b - \frac{p-2}{2}} |u_n|^2 \\ &\leq C_1 \left(\frac{1}{\varepsilon} \right)^{b - \frac{p-1}{2}}. \end{aligned} \tag{2.2}$$

It follows from $b - \frac{p-1}{2} < 0$ and $\{u_n\}$ being bounded in $H_r^1(\mathbb{R}^N)$ that the above integral tends to 0 uniformly in n as $\varepsilon \to 0$.

For the term $\int_{\varepsilon \leq |x| \leq \frac{1}{\varepsilon}} |x|^b |u_n|^{p+1}$, because $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$ we have first from the Rellich compact embedding theorem that

$$\int_{\varepsilon \leq |x| \leq \frac{1}{\varepsilon}} |u_n|^2 \to 0 \quad \text{as } n \to +\infty.$$

Using Strauss' inequality we get that

$$\int_{\varepsilon \leq |x| \leq \frac{1}{\varepsilon}} |x|^b |u_n|^{p+1} \leq C \int_{\varepsilon \leq |x| \leq \frac{1}{\varepsilon}} |u_n|^2 \to 0 \quad \text{as } n \to +\infty. \tag{2.3}$$

Since ε is arbitrary, we obtain from (2.1), (2.2) and (2.3) that $\int |x|^b|u_n|^{p+1} \to 0$ as $n \to +\infty$. The proof is complete. □

Proposition 2.2 [6, Theorem 1] *Let $N \geq 2$, $b \geq 0$ and $1 + \frac{2b}{N-1} < p < \tilde{p}$. Then there is a positive constant $C(N, p, b)$ depending only on N, p and b such that for any $u \in H_r^1(\mathbb{R}^N)$,*

$$\int |x|^b|u|^{p+1} \leq C(N, p, b)\Big(\int |\nabla u|^2\Big)^{\frac{N(p-1)-2b}{4}} \Big(\int |u|^2\Big)^{\frac{2(p+1)-(N(p-1)-2b)}{4}}. \tag{2.4}$$

Define

$$J(u) = \int \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}|\phi|^2 - \frac{1}{p+1}|x|^b|\phi|^{p+1}\right), \quad u \in H_r^1(\mathbb{R}^N)$$

and

$$d_0 = \inf\{J(\phi);\ \phi \neq 0,\ -\Delta\phi + \phi = |x|^b|\phi|^{p-1}\phi\}.$$

Proposition 2.3 [6] *Let $N \geq 2$, $b \geq 0$ and $1 + \frac{2b}{N-1} < p < \tilde{p}$. Then $d_0 > 0$ is achieved by some $Q \in H_r^1(\mathbb{R}^N)$ and Q is a ground state solution of*

$$-\Delta\phi + \phi - |x|^b|\phi|^{p-1}\phi = 0, \quad \phi \in H_r^1(\mathbb{R}^N) \tag{2.5}$$

Remark 2.2 Since $b > 0$, the uniqueness of the ground state solution of (2.5) is still **open**. We emphasize that d_0 is independent of the choice of the ground state solution $Q(x)$.

Proposition 2.4 *Let $N \geq 2$, $b \geq 0$ and $1 + \frac{2b}{N-1} < p < \tilde{p}$. Then the smallest positive constant $C(N, p, b)$ satisfying (2.4) is exactly as*

$$\begin{aligned} C(N, p, b) &= \frac{2(p+1)}{N(p-1)-2b}\left(\frac{2(p+1)-(N(p-1)-2b)}{N(p-1)-2b}\right)^{\frac{N(p-1)-2b-4}{4}} \|Q\|_2^{-(p-1)} \\ &= \frac{2(p+1)}{N(p-1)-2b}\left(\frac{2(p+1)-(N(p-1)-2b)}{N(p-1)-2b}\right)^{\frac{N(p-1)-2b-4}{4}} \\ &\quad \times \left(\frac{2(p+1)-(N(p-1)-2b)}{p-1} d_0\right)^{\frac{1-p}{2}}, \end{aligned}$$

where $Q \in H_r^1(\mathbb{R}^N)$ is the ground state solution of (2.5).

Proof The first equality of $C(N, p, b)$ has been proved in [7]. We only prove the second equality of $C(N, p, b)$. Note that for the ground state solution Q, we have

$$\begin{aligned} &\int |\nabla Q|^2 + \int |Q|^2 = \int |x|^b|Q|^{p+1} \quad \text{and} \\ &d_0 = \frac{1}{2}\int \left(|\nabla Q|^2 + |Q|^2\right) = \frac{p-1}{2(p+1)}\int |x|^b|Q|^{p+1} > 0. \end{aligned} \tag{2.6}$$

Moreover, for $Q^{\lambda}(x)=\lambda^{\frac{N}{2}}Q(x)$, we have that

$$\left(\frac{\partial}{\partial\lambda}J(Q^{\lambda})\right)\Big|_{\lambda=1}=\left\langle J'(Q),\frac{\partial Q^{\lambda}}{\partial\lambda}\Big|_{\lambda=1}\right\rangle=0. \tag{2.7}$$

On the other hand

$$\left(\frac{\partial}{\partial\lambda}J(Q^{\lambda})\right)\Big|_{\lambda=1}=\int|\nabla Q|^2-\frac{N(p-1)-2b}{2(p+1)}\int|x|^b|Q|^{p+1}. \tag{2.8}$$

It is deduced from (2.7) and (2.8) that

$$\int|\nabla Q|^2=\frac{N(p-1)-2b}{2(p+1)}\int|x|^b|Q|^{p+1}. \tag{2.9}$$

Equations (2.6) and (2.9) implies that

$$\int|x|^b|Q|^{p+1}=\frac{2(p+1)}{2(p+1)-(N(p-1)-2b)}\int|Q|^2.$$

Thus

$$\|Q\|_2^2=\frac{2(p+1)-(N(p-1)-2b)}{p-1}d_0. \tag{2.10}$$

Substituting this to the first equality of $C(N,p,b)$, we get the required result. □

Remark 2.3 We emphasize that $C(N,p,b)$ is independent of the choice of the ground state solution $Q(x)$ as stated in the second equality of $C(N,p,b)$.

3 Critical nonlinearity $p=1+(4+2b)/N$

In this section, we study (1.1)–(1.2) in the critical nonlinearity $p=1+(4+2b)/N$. Recalling the local existence results from [8], we have

Proposition 3.1 *Let $N\geq 2$, $b\geq 0$ and $1+2b/(N-1)<p<\tilde{p}$. For any $\varphi_0\in H_r^1(\mathbb{R}^N)$, there is a $T=T(\|\varphi_0\|_{H_r^1(\mathbb{R}^N)})>0$ and a unique solution φ of (1.1) with $\varphi\in C([0,T),H_r^1(\mathbb{R}^N))$ and $\varphi(0)=\varphi_0$. Moreover, we have the conserved particle number*

$$\int|\varphi(t)|^2=\int|\varphi_0|^2,\quad t\in[0,T) \tag{3.1}$$

and the conserved energy

$$E(\varphi(t))=\frac{1}{2}\int|\nabla\varphi|^2-\frac{1}{p+1}\int|x|^b|\varphi|^{p+1}=E(\varphi_0),\quad t\in[0,T), \tag{3.2}$$

where either $T=+\infty$ or $T<+\infty$ and $\lim_{t\to T^-}\|\varphi\|_{H_r^1(\mathbb{R}^N)}=+\infty$.

Proposition 3.2 *Let $N \geq 2$, $b \geq 0$, $1 + 2b/(N-1) < p < \tilde{p}$ and $\varphi_0 \in H_r^1(\mathbb{R}^N)$.*

- *If $N > 1 + \frac{b}{2}$ and $1 + 2b/(N-1) < p < 1 + (4+2b)/N$, then the existence time T obtained in Proposition 3.1 must be infinite.*
- *If $N > 1 + \frac{b}{2}$ and $p = 1 + (4+2b)/N$, then the existence time $T = +\infty$ for $\|\varphi_0\|_{H_r^1(\mathbb{R}^N)}$ small enough.*

Note that the "small enough" appeared in the above is "vague", the first result of the present paper is to answer the question: "How small?" The answer is simple.

Theorem 3.3 *Suppose $b > 0$, $N > 1 + b/2$ and $p = 1 + (4+2b)/N$ and Q is the ground state solution of (2.5). If $\varphi_0 \in H_r^1(\mathbb{R}^N)$ and*

$$\|\varphi_0\|_2 < \|Q\|_2, \tag{3.3}$$

then (1.1)–(1.2) has a global solution $\varphi(x,t) \in C(\mathbb{R}_+, H_r^1(\mathbb{R}^N))$.

Remark 3.1 $b = 0$, $N = 2$ and $p = 3$, $\|Q\|_2^2 \cong 11.70086$, see e.g. [17].

Proof of Theorem 3.3 Let $\varphi(x,t) \in C([0,T), H_r^1(\mathbb{R}^N))$ be a solution of (1.1)–(1.2) in the case of $p = 1 + (4+2b)/N$. Using Propositions 2.1 and 2.2, one has

$$\int |x|^b |\varphi|^{\frac{2(N+2+b)}{N}} \leq \frac{N+2+b}{N} \left(\int |Q|^2 \right)^{-\frac{2+b}{N}} \int |\nabla\varphi|^2 \left(\int |\varphi|^2 \right)^{\frac{2+b}{N}}. \tag{3.4}$$

It follows from (3.4) and the conserved energy that

$$\begin{aligned} E(\varphi_0) &= \frac{1}{2} \int |\nabla\varphi|^2 - \frac{N}{2(N+2+b)} \int |x|^b |\varphi|^{\frac{2(N+2+b)}{N}} \\ &\geq \frac{1}{2} \left(1 - \left(\frac{\int |\varphi|^2}{\int |Q|^2} \right)^{\frac{2+b}{N}} \right) \int |\nabla\varphi|^2. \end{aligned} \tag{3.5}$$

As

$$\int |\varphi|^2 \equiv \int |\varphi_0|^2 < \int |Q|^2,$$

$\int |\nabla\varphi|^2$ is bounded uniformly for $t \in [0,T)$. It is deduced from Proposition 3.1 that $\varphi(x,t)$ exists globally in $t \in [0, +\infty)$. □

Remark 3.2 According to (2.10), condition (3.3) is independent of the choice of the ground state solution Q. The condition (3.3) is sharp in the sense of the following Theorem 3.4.

Theorem 3.4 *Let $b > 0$, $N > 1 + b/2$ and $p = 1 + (4+2b)/N$. If $\varphi_0 = (1+\varepsilon)Q(x)$, where $Q(x)$ is the ground state solution of $-\Delta u + u = |x|^b |u|^{p-1} u$, then the solution $\varphi(x,t)$ of (1.1) blows up in a finite time.*

Proof For $\varphi_0 = (1+\varepsilon)Q(x)$ and $\varepsilon > 0$, we have that $\|\varphi_0\|_{L^2}^2 = (1+\varepsilon)^2\|Q\|_{L^2}^2 > \|Q\|_{L^2}^2$ since $Q \not\equiv 0$. Moreover we have $E(\varphi_0) = -2\varepsilon\|Q\|_{L^2}^2 + O(\varepsilon^2) < 0$. Since $Q(x)$ decays rapidly at infinity, $\int |x|^2|Q(x)|^2 < +\infty$. The result now follows from the following Lemma 3.5. □

Lemma 3.5 *Let* $b > 0$, $N > 1 + b/2$ *and* $p = 1 + (4+2b)/N$. *If* $E(\varphi_0) < 0$ *and* $\int |x|^2|\varphi_0|^2 < +\infty$, *then there exists* $0 < T < +\infty$ *such that*

$$\lim_{t\to T^-} \|\nabla\varphi\|_{L^2} = +\infty.$$

Proof Noting that for $h(t) = \frac{1}{2}\int |x|^2|\varphi(x,t)|^2$, we have from [6] that

$$h''(t) = 8E(\varphi_0).$$

If $E(\varphi_0) < 0$, then there is $T_0 > 0$ such that $\lim_{t\to T_0^-} h(t) = 0$. From

$$\int |\varphi|^2 \le C\left(\int |\nabla\varphi|^2\right)^{\frac{1}{2}}\left(\int |x|^2|\varphi|^2\right)^{\frac{1}{2}},$$

we know that there exists $0 < T < +\infty$ such that

$$\lim_{t\to T^-} \|\nabla\varphi\|_{L^2} = +\infty.$$ □

4 Supercritical nonlinearity $1 + (4+2b)/N < p < \tilde{p}$

After developing the existence of global solutions and the blow-up solutions of (1.1)–(1.2) in the critical nonlinearity $p = 1 + (4+2b)/N$, attentions are now focused on the existence of solutions of (1.1)–(1.2) in the supercritical nonlinearity $1 + (4+2b)/N < p < \tilde{p}$. The idea is to construct two sets which are invariant under the flow generated by Cauchy problem (1.1)–(1.2). We emphasize that the uses of Propositions 2.1 and 2.2 are essential. Firstly, we define

$$R(u) = \int \left(|\nabla u|^2 - \frac{N(p-1)-2b}{2(p+1)}|x|^b|u|^{p+1}\right), \quad u \in H_r^1(\mathbb{R}^N).$$

Proposition 2.2 implies that R is well defined and C^1 on $H_r^1(\mathbb{R}^N)$. Set

$$\mathcal{M} = \{u \in H_r^1(\mathbb{R}^N);\ u \neq 0,\ R(u) = 0\} \quad \text{and} \quad d_1 = \inf_{u\in\mathcal{M}} J(u).$$

We have that

Lemma 4.1 $0 < d_1 < +\infty$.

Proof For any $u \in \mathcal{M}$, we have that

$$J(u) = \left(\frac{1}{2} - \frac{2}{N(p-1)-2b}\right)\int |\nabla u|^2 + \frac{1}{2}\int |u|^2 > 0.$$

By (2.9), $Q \in \mathcal{M}$ and we have that $d_1 < \infty$. Therefore $0 \leq d_1 < +\infty$.

To see that $d_1 > 0$, we consider a minimizing sequence $\{u_n\} \subset \mathcal{M}$ such that

$$d_1 + o(1) = J(u_n) \quad \text{for } n \text{ large.}$$

Using the fact that $R(u_n) = 0$, we get that

$$d_1 + o(1) = \left(\frac{1}{2} - \frac{2}{N(p-1)-2b}\right)\int |\nabla u|^2 + \frac{1}{2}\int |u|^2.$$

Therefore $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$. Again using the fact that $\{u_n\} \subset \mathcal{M}$, we get from Proposition 2.2 that

$$\begin{aligned}\int |\nabla u_n|^2 &= \frac{N(p-1)-2b}{2(p+1)}\int |x|^b |u_n|^{p+1}\\ &\leq \frac{N(p-1)-2b}{2(p+1)} C(N,p,b)\|u_n\|_2^{\frac{2(p+1)-(N(p-1)-2b)}{2}}\|\nabla u_n\|_2^{\frac{N(p-1)-2b}{2}}\\ &\leq C\left(\int |\nabla u_n|^2\right)^{\frac{N(p-1)-2b}{4}}.\end{aligned}$$

It follows from the assumption of p that $\int |\nabla u_n|^2$ is bounded away from zero. Therefore $d_1 > 0$. The proof is complete. □

Lemma 4.2 *Let $b > 0$, $N > 1 + b/2$ and $1 + (4+2b)/N < p < \tilde{p}$. For $u \in H_r^1(\mathbb{R}^N)$ and $\lambda > 0$, let $u^\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x)$. Then there is unique $\mu > 0$ such that $R(u^\mu) = 0$ and $R(u^\lambda) > 0$ for $\lambda \in (0, \mu)$ and $R(u^\lambda) < 0$ for $\lambda \in (\mu, +\infty)$. Furthermore,*

$$J(u^\mu) \geq J(u^\lambda), \quad \forall \lambda > 0.$$

Proof By direct computations, we have that

$$R(u^\lambda) = \lambda^2 \int |\nabla u|^2 - \frac{N(p-1)-2b}{2(p+1)}\lambda^{\frac{N(p-1)-2b}{2}}\int |x|^b |u|^{p+1} \quad \text{and}$$

$$J(u^\lambda) = \frac{\lambda^2}{2}\int |\nabla u|^2 + \frac{1}{2}\int |u|^2 - \frac{1}{p+1}\lambda^{\frac{N(p-1)-2b}{2}}\int |x|^b |u|^{p+1}.$$

The assumption of p implies that there is $\mu > 0$ such that $R(u^\mu) = 0$ and $R(u^\lambda) > 0$ for $\lambda \in (0, \mu)$ and $R(u^\lambda) < 0$ for $\lambda \in (\mu, +\infty)$. Since

$$\frac{\partial}{\partial \lambda} J(u^\lambda) = \lambda^{-1} R(u^\lambda)$$

and $R(u^\mu) = 0$, we have that $J(u^\mu) \geq J(u^\lambda)$ for all $\lambda > 0$. □

Lemma 4.3 *d_1 is achieved by some $w(x) \in \mathcal{M}$.*

Proof Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence of d_1. We know from the proof of Lemma 4.1 that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$ and $\int |\nabla u_n|^2$ is bounded away from zero. Going if necessary to a subsequence, still denoted by $\{u_n\}$, we assume that $u_n \rightharpoonup u_0$ weakly in $H_r^1(\mathbb{R}^N)$. By Proposition 2.1 we have that $u_n \to u_0$ strongly in Y_r. We claim that $u_0 \neq 0$ since if otherwise from $R(u_n) = 0$, we get that

$$\int |\nabla u_n|^2 = \frac{N(p-1)-2b}{2(p+1)} \int |x|^b |u_n|^{p+1} \to \frac{N(p-1)-2b}{2(p+1)} \int |x|^b |u_0|^{p+1} = 0,$$

which contradicts to the fact that $\int |\nabla u_n|^2$ is bounded away from zero. So $u_0 \neq 0$ and Lemma 4.2 implies that there is a unique $\mu > 0$ such that $R(u_0^\mu) = 0$, where $u_0^\mu(x) = \mu^{\frac{N}{2}} u_0(\mu x)$. Moreover $u_n^\mu \rightharpoonup u_0^\mu$ weakly in $H_r^1(\mathbb{R}^N)$ and $u_n^\mu \to u_0^\mu$ strongly in Y_r. It is now deduced from Lemma 4.2 that

$$J(u_0^\mu) \le \liminf_{n\to\infty} J(u_n^\mu) \le \lim_{n\to\infty} J(u_n^\mu) = \inf_{u\in\mathcal{M}} J(u).$$

On the other hand $R(u_0^\mu) = 0$ and $u_0^\mu \neq 0$ imply that

$$J(u_0^\mu) \ge d_1 = \inf_{u\in\mathcal{M}} J(u).$$

Therefore taking $w = u_0^\mu(x) \in \mathcal{M}$, we have that $J(u_0^\mu) = d_1$. The proof is complete. □

Next, define

$$\mathcal{S}_+ = \{u \in H_r^1(\mathbb{R}^N);\ R(u) > 0,\ J(u) < d_1\} \quad \text{and}$$
$$\mathcal{S}_- = \{u \in H_r^1(\mathbb{R}^N);\ R(u) < 0,\ J(u) < d_1\}.$$

We have that

Lemma 4.4 *Both $\mathcal{S}_+$ and $\mathcal{S}_-$ are invariant sets of (1.1) in the sense that if $\varphi_0 \in \mathcal{S}_+$ (resp. $\mathcal{S}_-$), then the solution $\varphi(x,t)$ of (1.1)–(1.2) belongs to $\mathcal{S}_+$ (resp. $\mathcal{S}_-$) for all $t \in [0,T)$.*

Proof We only prove that $\mathcal{S}_-$ is an invariant set since the proof of the other is similar. Let $\varphi(x,t) := \varphi(t)$ be the solution of (1.1) with initial value $\varphi_0 \in \mathcal{S}_-$. In the first place, from the conserved identities $\int |\varphi(t)|^2 \equiv \int |\varphi_0|^2$ and

$$E(\varphi(t)) = \int \left(\frac{1}{2}|\nabla\varphi|^2 - \frac{1}{p+1}|x|^b|\varphi|^{p+1}\right) \equiv E(\varphi_0),$$

we get immediately that $J(\varphi) \equiv J(\varphi_0)$. Thus $J(\varphi) < d_1$ since $J(\varphi_0) < d_1$.

In the second place, we show $R(\varphi(t)) < 0$ for $t \in [0,T)$. If otherwise, from the continuity, there would be a $t_1 \in (0,T)$ such that $R(\varphi(t_1)) = 0$. Since $\varphi(t_1) \neq 0$ we have that $J(\varphi(t_1)) \ge d_1$, which contradicts to $J(\varphi(t)) < d_1$ for all $t \in (0,T)$. Therefore $R(\varphi(t)) < 0$ for $t \in [0,T)$. □

Remark 4.1 From Lemma 4.3, we know that $\mathcal{S}_+ \neq \emptyset$ and $\mathcal{S}_- \neq \emptyset$. Indeed, for $w(x)$ obtained in Lemma 4.3, we have that $w \neq 0$, $R(w) = 0$ and

$$d = \frac{1}{2}\int |w|^2 + \left(\frac{1}{2} - \frac{2}{N(p-1)-2b}\right)\int |\nabla w|^2.$$

Denote $w^\lambda(x) = \lambda^{\frac{N}{2}} w(\lambda x)$. We have from direct computations that $R(w^\lambda) > 0$ for $\lambda > 0$ but λ is small. Moreover, $J(w^\lambda) < J(w) = d_1$ for $\lambda > 0$ but λ is small enough. Therefore there is $\lambda_0 \in (0, 1)$ such that $w^{\lambda_0} \in \mathcal{S}_+$. Similarly, there is $\lambda_1 > 1$ such that $w^{\lambda_1} \in \mathcal{S}_-$.

Theorem 4.5 *Let* $N \geq 2$ *and* $b > 0$. *If* $\max\{1 + (4+2b)/N, 1 + 2b/(N-1)\} < p < \tilde{p}$, $\varphi_0 \in \mathcal{S}_-$ *and* $\int |x|^2 |\varphi_0|^2 < +\infty$, *then the solution* $\varphi(x,t)$ *of* (1.1) *blows up in a finite time*, *i.e.*, *there is* $0 < T_2 < +\infty$ *such that*

$$\lim_{t \to T_2^-} \|\nabla \varphi\|_{L^2} = +\infty.$$

Proof Let $\varphi(x,t)$ be the solution of (1.1) with initial data φ_0. For any fixed t, we simply denote $\varphi(x,t)$ by $\varphi(x)$. Using the notation $\varphi^\lambda(x) = \lambda^{\frac{N}{2}} \varphi(\lambda x)$, we have that

$$R(\varphi^\lambda) = \lambda^2 \int |\nabla \varphi|^2 - \frac{N(p-1)-2b}{2(p+1)} \lambda^{\frac{N(p-1)-2b}{2}} \int |x|^b |\varphi|^{p+1} \quad \text{and}$$

$$J(\varphi^\lambda) = \frac{\lambda^2}{2}\int |\nabla \varphi|^2 + \frac{1}{2}\int |\varphi|^2 - \frac{1}{p+1}\lambda^{\frac{N(p-1)-2b}{2}} \int |x|^b |\varphi|^{p+1}.$$

Note that $R(\varphi^\lambda) \to R(\varphi) < 0$ as $\lambda \to 1$ and $R(\varphi^\lambda) > 0$ for $\lambda > 0$ and λ sufficiently small. Since $R(\varphi^\lambda)$ is continuous on λ, we have a $\lambda_0 \in (0, 1)$ such that $R(\varphi^{\lambda_0}) = 0$, i.e. $\varphi^{\lambda_0} \in \mathcal{M}$. So $J(\varphi^{\lambda_0}) \geq d_1$. On the other hand,

$$\begin{aligned} J(\varphi) - J(\varphi^{\lambda_0}) &= \frac{1-\lambda_0^2}{2}\int |\nabla\varphi|^2 - \frac{1}{p+1}\left(1 - \lambda_0^{\frac{N(p-1)-2b}{2}}\right)\int |x|^b|\varphi|^{p+1} \\ &\geq \frac{1}{2}\left((1-\lambda_0^2)\int |\nabla\varphi|^2 - \left(1 - \lambda_0^{\frac{N(p-1)-2b}{2}}\right)\int |x|^b|\varphi|^{p+1}\right) \\ &= \frac{1}{2}\big(R(\varphi) - R(\varphi^{\lambda_0})\big). \end{aligned}$$

It follows that

$$R(\varphi) \leq 2(J(\varphi) - d_1) := \delta_0 < 0.$$

Noting that for $h(t) = \frac{1}{2}\int |x|^2 |\varphi(x,t)|^2$, we have that

$$h''(t) = 4R(\varphi).$$

So there is $T_1 > 0$ such that $\lim_{t\to T_1^-} h(t) = 0$. From

$$\int |\varphi|^2 \le C\left(\int |\nabla\varphi|^2\right)^{\frac{1}{2}}\left(\int |x|^2|\varphi|^2\right)^{\frac{1}{2}},$$

we know that there exists $0 < T_2 < +\infty$ such that

$$\lim_{t\to T_2^-} \|\nabla\varphi\|_{L^2} = +\infty.$$

The proof is complete. □

Theorem 4.6 *Let $N \ge 2$ and $b > 0$. If $\max\{1+(4+2b)/N, 1+2b/(N-1)\} < p < \tilde{p}$ and $\varphi_0 \in \mathcal{S}_+$, then the solution $\varphi(x,t)$ of (1.1) exists globally in time.*

Proof For $\varphi_0 \in \mathcal{S}_+$, we have from Lemma 4.4 that $\varphi(\cdot,t) \in \mathcal{S}_+$. $R(\varphi) > 0$ and $J(\varphi) < d_1$ implies that

$$\begin{aligned} d_1 &> \frac{1}{2}\int (|\nabla\varphi|^2 + |\varphi|^2) - \frac{1}{p+1}\int |x|^b|\varphi|^{p+1} \\ &> \frac{N(p-1)-2b-4}{N(p-1)-2b}\int |\nabla\varphi|^2 + \frac{1}{2}\int |\varphi|^2. \end{aligned}$$

It now follows from the assumption of p that $\int |\nabla\varphi(t)|^2$ is bounded with respect to $t \in [0,T)$. Therefore φ is bounded in $H_r^1(\mathbb{R}^N)$ and $\varphi(x,t)$ exists globally in time. □

Remark 4.2 We remark that the set $\mathcal{S}_+$ is bounded in $H_r^1(\mathbb{R}^N)$. Indeed for any $\phi \in \mathcal{S}_+$, we obtain from $J(\phi) < d_1$ and $R(\phi) > 0$ that

$$d_1 > \frac{N(p-1)-2b-4}{N(p-1)-2b}\int |\nabla\phi|^2 + \frac{1}{2}\int |\phi|^2.$$

Corollary 4.3 *Let $N \ge 2$, $b > 0$ and $\max\{1+(4+2b)/N, 1+2b/(N-1)\} < p < \tilde{p}$. If $\varphi_0 \in H_r^1(\mathbb{R}^N)$ and $\int (|\nabla\varphi_0|^2 + |\varphi_0|^2) < 2d_1$, then the solution $\varphi(x,t)$ of (1.1) exists globally in time.*

Proof According to the expression of J, we have that $J(\varphi_0) < d_1$. In addition, we claim that $R(\varphi_0) > 0$. If otherwise, from the expression of R, there were a $0 < \mu \le 1$ such that $R(\mu\varphi_0) = 0$. Thus the definition of d_1 implies that $J(\mu\varphi_0) \ge d_1$. On the other hand,

$$\int (|\nabla(\mu\varphi_0)|^2 + |\mu\varphi_0|^2) = \mu^2\int (|\nabla\varphi_0|^2 + |\varphi_0|^2) < 2\mu^2 d_1 \le 2d_1$$

implies that $J(\mu\varphi_0) < d_1$, which is a contradiction. So $R(\varphi_0) > 0$ holds. From the above argument, we get $\varphi_0 \in \mathcal{S}_+$, hence Theorem 4.6 yields the result of Corollary 4.3. □

5 Global solutions for large data

From Theorems 3.3 and 3.4, we see that when $p = 1 + (4 + 2b)/N$, $\|Q\|_2$ is the critical mass for the solutions of (1.1)–(1.2) which exist globally in time. When $\max\{1+(4+2b)/N, 1+2b/(N-1)\} < p < \tilde{p}$, Theorem 4.6 and Remark 4.2 imply the existence of global solutions for initial data contained in a suitable bounded subset of $H_r^1(\mathbb{R}^N)$. On the other hand, the prescribed initial data in Theorem 3.4 also seems to imply that the existence of blow-up solutions of (1.1)–(1.2) not only depends on the mass of the initial data but also on the profile of the initial data. So it is very reasonable to **conjecture** that for some class of initial data φ_0 with $\|\varphi_0\|_2 > \|Q\|_2$, but solutions of (1.1)–(1.2) exist globally in time. The main purpose of this section is to construct an unbounded subset $\mathcal{O}$ of $H_r^1(\mathbb{R}^N)$ such that for $\varphi_0 \in \mathcal{O}$ and $\max\{1 + (4 + 2b)/N, 1 + 2b/(N - 1)\} < p < \tilde{p}$, the solutions of (1.1)–(1.2) exist globally in time. Firstly, we need the following lemma from Bégout [3].

Lemma 5.1 *Let $I \subset \mathbb{R}$ be an open interval, $s_0 \in I$, $\theta > 1$, $a > 0$, $b > 0$ and $\Phi(s) \in C(I, \mathbb{R}_+)$. Set $f(y) = a - y + by^\theta$ for any $y \geq 0$. $y_* = (b\theta)^{-\frac{1}{\theta-1}}$ and $b_* = \frac{\theta-1}{\theta} y_*$. Assume that $\Phi(s_0) < y_*$, $a \leq b_*$ and $f \circ \Phi > 0$. Then $\Phi(s) < y_*$ for any $s \in I$.*

Proof Since $\Phi(s_0) < y_*$ and Φ is a continuous function, there exists a $\delta > 0$ such that $\Phi(s) < y_*$ for any $s \in (s_0 - \delta, s_0 + \delta) \subset I$. If $\Phi(s) < y_*$ were not true for any $s \in I$, by continuity, there would exist a $s_* \in I$ satisfying $\Phi(s_*) = y_*$. Then $f \circ \Phi(s_*) = f(y_*) = a - b_* \leq 0$. However, this is impossible from $f \circ \Phi > 0$. Therefore $\Phi(s) < y_*$ for any $s \in I$. The proof is complete. □

Next for $\lambda > 0$, we define a real function $V(\lambda)$ as follows:

$$\begin{aligned} V(\lambda) &= \left(\frac{N(p-1)-2b-4}{2(p+1)-(N(p-1)-2b)}\right)^{\frac{N(p-1)-2b-4}{2(2(p+1)-(N(p-1)-2b))}} \|Q\|_2^{\frac{2(p-1)}{2(p+1)-(N(p-1)-2b)}} \\ &\quad \times \lambda^{-\frac{N(p-1)-2b-4}{2(2(p+1)-(N(p-1)-2b))}} \\ &= \left(\frac{A-2}{B}\right)^{\frac{A-2}{2B}} \|Q\|_2^{\frac{p-1}{B}} \lambda^{-\frac{A-2}{2B}}, \quad \lambda > 0, \end{aligned} \tag{5.1}$$

where $A = \frac{N(p-1)-2b}{2}$ and $B = \frac{2(p+1)-(N(p-1)-2b)}{2}$. Set

$$\mathcal{O} = \{\phi \in H_r^1(\mathbb{R}^N);\ \|\phi\|_2 \leq V(\|\nabla\phi\|_2^2)\}.$$

Lemma 5.2 *If* $\max\{1 + (4 + 2b)/N, 1 + 2b/(N - 1)\} < p < \tilde{p}$, *then $\mathcal{O}$ is an unbounded subset of $H_r^1(\mathbb{R}^N)$.*

Proof For any $M > 0$, take $\phi \in H_r^1(\mathbb{R}^N)$ such that $\int |\phi(x)|^2 dx > M$. Denote $\phi^\mu(x) = \mu^{\frac{N}{2}} \phi(\mu x)$, then

$$\int |\phi^\mu(x)|^2 dx = \int |\phi(x)|^2 dx > M \quad \text{and} \quad \int |\nabla\phi^\mu(x)|^2 dx = \mu^2 \int |\nabla\phi(x)|^2 dx.$$

Therefore for μ small enough, we have that

$$\|\phi^\mu\|_2\left(\|\nabla\phi^\mu\|_2^2\right)^{\frac{A-2}{2B}} = \mu^{\frac{A-2}{B}}\|\phi\|_2\left(\|\nabla\phi\|_2^2\right)^{\frac{A-2}{2B}} \le \left(\frac{A-2}{B}\right)^{\frac{A-2}{2B}}\|Q\|_2^{\frac{p-1}{B}}.$$

Hence $\phi^\mu \in \mathcal{O}$ and $\|\phi^\mu\|^2_{H^1_r(\mathbb{R}^N)} = \mu^2\|\nabla\phi\|_2^2 + \|\phi\|_2^2 > M$. The proof is complete. □

Now we are in a position to give a global existence result for initial data in $\mathcal{O}$.

Theorem 5.3 *Let* $b > 0$, $N \ge 2$ *and* $\max\{1+(4+2b)/N, 1+2b/(N-1)\} < p < \tilde{p}$. *If* $\varphi_0 \in H^1_r(\mathbb{R}^N)$ *and* $\varphi_0 \in \mathcal{O}$, *then the solution* $\varphi(x,t)$ *of* (1.1) *exits globally in time. Moreover we have*

$$\int |\nabla\varphi(t)|^2 \le \frac{2N(p-1)-4b}{N(p-1)-2b-4}E(\varphi_0)$$

uniformly with respect to $t \in [0,+\infty)$.

Proof For any $t \in [0,T)$, applying Propositions 2.2 and 2.4 to $\varphi(t)$ and using the choice of A and B, we get that

$$\int |x|^b|\varphi|^{p+1} \le C(N,p,b)\|\varphi\|_2^B\|\nabla\varphi\|_2^A. \tag{5.2}$$

Denote $a = \int|\nabla\varphi_0|^2 > 0$. It is deduced from the energy identity and (5.1) that

$$\begin{aligned}\int |\nabla\varphi|^2 &= 2E(\varphi) + \frac{2}{p+1}\int |x|^b|\varphi|^{p+1}\\ &= 2E(\varphi_0) + \frac{2}{p+1}\int |x|^b|\varphi|^{p+1}\\ &< a + \frac{2}{p+1}\int |x|^b|\varphi|^{p+1}\\ &\le a + \frac{2C(N,p,b)}{p+1}\|\varphi_0\|_2^B\left(\int|\nabla\varphi|^2\right)^{\frac{A}{2}}.\end{aligned} \tag{5.3}$$

Let

$$\begin{aligned} b &:= \frac{2C(N,p,b)}{p+1}\|\varphi_0\|_2^B = \frac{2}{B}\left(\frac{B}{A}\right)^{\frac{A}{2}}\|Q\|_2^{-(p-1)}\|\varphi_0\|_2^B,\\ \theta &= \frac{N(p-1)-2b}{4} > 1,\end{aligned}$$

and

$$\Phi(t) = \int |\nabla\varphi(t)|^2.$$

Obviously $\Phi(0)=a$. At the same time, we define $f(y)=a-y+by^{\theta}$. Then (5.3) implies that

$$0<a-y+by^{\theta}, \quad \text{where } y=\Phi(t).$$

Denote

$$y_*=(b\theta)^{-\frac{1}{\theta-1}}, \qquad b_*=\frac{\theta-1}{\theta}y_*,$$

then $b_*<y_*$. By direct computation and the exact value of $C(N,p,b)$ we have that

$$y_*=\frac{A}{B}\|Q\|_2^{\frac{2(p-1)}{A-2}}\|\varphi_0\|_2^{-\frac{2B}{A-2}}$$

and

$$b_*=\frac{A-2}{B}\|Q\|_2^{\frac{2(p-1)}{A-2}}\|\varphi_0\|_2^{-\frac{2B}{A-2}}. \tag{5.4}$$

Moreover, using $\|\varphi_0\|_2\le V(\|\nabla\varphi_0\|_2^2)=V(a)$, we know that

$$\|\varphi_0\|_2\le\left(\frac{A-2}{B}\right)^{\frac{A-2}{2B}}\|Q\|_2^{\frac{p-1}{B}}a^{-\frac{A-2}{2B}},$$

which implies that

$$a\le\frac{A-2}{B}\|Q\|_2^{\frac{2(p-1)}{A-2}}\|\varphi_0\|_2^{-\frac{2B}{A-2}}=b_*<y_* \tag{5.5}$$

because of $\max\{1+(4+2b)/N, 1+2b/(N-1)\}<p<\tilde{p}$.

Now using (5.4), (5.5) and Lemma 5.1 (taking $s_0=0$), we get that $\Phi(t)<y_*$ for any $t\in[0,T)$. It follows from $\int|\varphi|^2\equiv\int|\varphi_0|^2$ that $\|\varphi(t)\|^2_{H_r^1(\mathbb{R}^N)}$ is bounded from above. In other words, the solutions of (1.1)–(1.2) with φ_0 satisfying $\|\varphi_0\|_2\le V(\|\nabla\varphi_0\|_2^2)$ exist globally in $t\in[0,+\infty)$.

Next, for the solutions obtained in the above, we give an explicit upper bound on the $\int|\nabla\varphi(t)|^2$. Note that

$$\begin{aligned}
E(\varphi_0)=E(\varphi)&=\frac{1}{2}\int|\nabla\varphi|^2-\frac{1}{p+1}\int|x|^b|\varphi|^{p+1}\\
&\ge\frac{1}{2}\int|\nabla\varphi|^2-\frac{C(N,p,b)}{p+1}\|\varphi\|_2^B\|\nabla\varphi\|_2^A\\
&\ge\frac{1}{2}\int|\nabla\varphi|^2-\frac{C(N,p,b)}{p+1}\|\varphi\|_2^B\left(\int|\nabla\varphi|^2\right)^{\frac{A}{2}}\\
&=\frac{1}{2}\int|\nabla\varphi|^2\left(1-\frac{2C(N,p,b)}{p+1}\|\varphi\|_2^B\left(\int|\nabla\varphi|^2\right)^{\frac{A}{2}-1}\right)\\
&=\frac{1}{2}\int|\nabla\varphi|^2\left(1-\frac{2}{A}\left[\left(\frac{AC(N,p,b)}{p+1}\|\varphi_0\|_2^B\right)^{-\frac{2}{A-2}}\left(\int|\nabla\varphi|^2\right)^{-1}\right]^{-\frac{A-2}{2}}\right).
\end{aligned}$$

Since $\Phi(t) < y_*$, there holds

$$\left(\frac{AC(N,p,b)}{p+1}\|\varphi_0\|_2^B\right)^{-\frac{2}{A-2}}\left(\int|\nabla\varphi|^2\right)^{-1} > 1$$

by using the exact value of $C(N,p,b)$ obtained in Proposition 2.4. Therefore

$$\left[\left(\frac{AC(N,p,b)}{p+1}\|\varphi_0\|_2^B\right)^{-\frac{2}{A-2}}\left(\int|\nabla\varphi|^2\right)^{-1}\right]^{-\frac{A-2}{2}} < 1$$

because of $\max\{1+(4+2b)/N, 1+2b/(N-1)\} < p < \tilde{p}$. It follows that

$$E(\varphi_0) \geq \frac{1}{2}\int|\nabla\varphi|^2\left(1-\frac{2}{A}\right),$$

which yields that

$$\int|\nabla\varphi(t)|^2 \leq \frac{2N(p-1)-4b}{N(p-1)-2b-4}E(\varphi_0).$$

The proof is complete. □

Remark Note that the set $\mathcal{O} = \{\phi \in H_r^1(\mathbb{R}^N);\ \|\phi\|_2 \leq V(\|\nabla\phi\|_2^2)\}$ is unbounded in $H_r^1(\mathbb{R}^N)$. So we get the existence of global solutions of (1.1) which allow the initial data as large as you want. On the other hand, from the definition of $V(\lambda)$ and Theorem 5.3, we know that $V(\lambda) \to \|Q\|_2$ as $p \to 1+(4+2b)/N$. So we obtain the sharp condition for global existence in the case of initial data $\|\varphi_0\|_2 < \|Q\|_2$, which coincides with Theorem 3.3. In the case of critical nonlinearity $p = 1+(4+2b)/N$, the condition (3.3) is sharp. However, we do not know whether or not the condition $\varphi_0 \in \mathcal{O}$ is sharp in the case of super critical nonlinearity $1+(4+2b)/N < p < \tilde{p}$.

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