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On 2-factors with cycles containing specified vertices in a bipartite graph

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Abstract Let $k \geq 1$ be an integer and $G = (V_1, V_2; E)$ a bipartite graph with $|V_1| = |V_2| = n$ such that $n \geq 2k + 2$. Our result is as follows: If $d(x) + d(y) \geq \lceil \frac{4n+k}{3} \rceil$ for any nonadjacent vertices $x \in V_1$ and $y \in V_2$, then for any k distinct vertices $z_1, \dots, z_k$, G contains a 2-factor with $k + 1$ cycles $C_1, \dots, C_{k+1}$ such that $z_i \in V(C_i)$ and $l(C_i) = 4$ for each $i \in \{1, \dots, k\}$.

Keywords Bipartite graph · Vertex-disjoint · Quadrilateral · 2-factor

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1 Terminology and introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges and we use Bondy and Murty [3] for terminology and notation not defined here. Let G be a graph. A set of subgraphs is said to be vertex-disjoint if no two of them have any common vertex in G. Let G_1 and G_2 be two subgraphs of G. If G_1 and G_2 have no common vertex in G, we define $E(G_1, G_2)$ to be the set of edges of G between G_1 and G_2, and let $e(G_1, G_2) = |E(G_1, G_2)|$. Let H be a subgraph of G and $u \in V(G)$, $N(u, H) = N_H(u)$ is the set of neighbors of u contained in H. We let $d_H(u) = d(u, H) = |N(u, H)|$. Clearly, $d(u, G)$ is the degree of u in G, and

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we write $d(x)$ to replace $d(x, G)$. For a subset U of $V(G)$ and a subgraph H in G, we define $d_H(U) = \sum_{x \in U} d_H(x)$. A 2-factor of G is a 2-regular spanning subgraph of G. Clearly, each component of a 2-factor of G is a cycle. Let C and P be a cycle and a path, respectively, we use $l(C)$ and $l(P)$ to denote the length of C and P, respectively. That is, $l(C) = |C|$ and $l(P) = |P| - 1$. A Hamiltonian cycle of G is a cycle which contains all vertices of G, and a Hamiltonian path of G is a path of G which contains every vertex in G.

Let $v_1, \dots, v_k$ be k distinct vertices in G, and let $C_1, \dots, C_k$ be k disjoint cycles passing through $v_1, \dots, v_k$, respectively, in G. Then we say that G has k disjoint cycles $C_1, \dots, C_k$ with respect to $\{v_1, \dots, v_k\}$. We say that G has a 2-factor with k cycles $C_1, \dots, C_k$ with respect to $\{v_1, \dots, v_k\}$, if $V(G) = V(C_1 \cup \dots \cup C_k)$. A cycle of length 4 is called a quadrilateral. For a cycle C with $l(C) = k$, we call that C be a k-cycle. Let v be a vertex and H be a subgraph of G. We say H is a v-subgraph if $v \in V(H)$. In particular, a v-cycle or a v-path is a cycle or path that passes through v, respectively. For a bipartite graph $G = (V_1, V_2; E)$, if $|V_1| = |V_2|$, then G is called balanced. For a bipartite graph $G = (V_1, V_2; E)$, we define

$$\sigma_{1,1}(G) = \min\{d(x) + d(y) | x \in V_1, y \in V_2, xy \notin E(G)\}.$$

When G is a complete bipartite graph, we define $\sigma_{1,1}(G) = \infty$.

Let P be a v-path, we define $\lambda(v, P) = \min\{|V(P_1)|, |V(P_2)|\}$, where P_1 and P_2 is two sub-paths in $P - v$. Let r be a real number, we use $\lceil r \rceil$ for the smallest integer that is greater than or equal to r.

Two interesting questions have been in the forefront of the study of 2-factors in a graph. Under what conditions will a 2-factor with prescribed properties exist? Under what conditions does a graph contains k vertex-disjoint cycles? For example, Corrádi and Hajnal [5] investigated the maximum number of disjoint cycles in a graph. They proved that if G is a graph of order $n \geq 3k$ with $\delta(G) \geq 2k$, then G contains k disjoint cycles. In particular, when the order of G is exactly $3k$, then G contains k disjoint triangles. EI-Zahar [6] conjectured if a graph G of order $n = n_1 + \dots + n_k$ with $n_i \geq 3$ $(1 \leq i \leq k)$ has minimum degree $\delta(G) \geq \lceil n_1/2 \rceil + \dots + \lceil n_k/2 \rceil$, then G contains k disjoint cycles of length $n_1, \dots, n_k$, respectively. He proved it for $k = 2$. Alon and Yuster [1] showed that for any $\epsilon > 0$, there exists k_0 such that if G is a graph of order $4k$ and $\delta(G) \geq (2 + \epsilon)k$ with $k \geq k_0$, then G contains k disjoint quadrilaterals. Komlós, Sáközy and Szemerédi [8] showed that for any graph H of order r with chromatic number k, there exist constant c and n_0 such that if $n \geq n_0$, $r|n$, and G is a graph of order n with $\delta(G) \geq (1 - 1/k)n + c$, then G contains n/r disjoint copies of H. This result come to close to EI-Zahar's conjecture in the case when $n_1, \dots, n_k$ are all equal to a fixed even integer. However, when $n_1, n_2, \dots, n_k$ are all equal to a fixed odd integer, a similar application of the above mentioned result will require the minimum degree of G to be approximately $2n/3$ which is not close to the condition in the EI-Zahar's conjecture. Other results about disjoint cycles can be found in [4, 7, 10–12].

Clearly, for a bipartite graph, quadrilateral is the smallest cycle. H. Matsumura [9] investigated the degree conditions that G contains k vertex disjoint quadrilaterals each of them contains a previously specified edges. He proved the following two theorems.

Theorem 1.1 [9] *Suppose* $k \geq 1$, $1 \leq s \leq k$, $n \geq 2k$, *and*

$$\sigma_{1,1}(G) \geq \max\left\{\left\lceil \frac{4n+2s-1}{3} \right\rceil, \left\lceil \frac{2n-1}{3} \right\rceil + 2k\right\}.$$

Then for any independent edges $e_1, \dots, e_k$, G *contains* k *vertex disjoint cycles* $C_1, \dots, C_k$ *such that* $e_i \in E(C_i)$, $|C_i| \leq 6$, *and there are at least* s 4*-cycles in* $\{C_1, \dots, C_k\}$.

Theorem 1.2 [9] *Suppose* $k \geq 1$, $1 \leq s \leq k$, $n \geq 2k$, *and*

$$\delta(G) \geq \max\left\{\left\lceil \frac{2n+2k+s}{4} \right\rceil, \left\lceil \frac{2n+4k}{5} \right\rceil\right\}.$$

Then for any independent edges $e_1, \dots, e_k$, G *contains* k *vertex disjoint cycles* $C_1, \dots, C_k$ *such that* $e_i \in E(C_i)$, $|C_i| = 4$ *for* $1 \leq i \leq s$, *and* $|C_i| \leq 6$ *for* $s+1 \leq i \leq k$.

In the rest of this paper, $G = (V_1, V_2; E)$ denotes a bipartite graph with partite sets V_1 and V_2 satisfying $|V_1| = |V_2| = n$. In this paper, we consider a similar problem with Theorem 1.1, i.e., we replace k independent edges with k distinct vertices. We obtain the following result.

Theorem 1.3 *Suppose* s *and* k *be two integers with* $1 \leq s \leq k$ *and let* $G = (V_1, V_2; E)$ *be a bipartite graph with* $|V_1| = |V_2| = n \geq 2s + 3(k-s)$. *If* $\sigma_{1,1}(G) \geq \lceil \frac{4n+s}{3} \rceil$, *then for any* k *distinct vertices* $v_1, \dots, v_k$, G *contains* k *vertex-disjoint cycles* $C_1, \dots, C_k$ *such that* $v_i \in V(C_i)$, $|C_i| \leq 6$ *for each* $i \in \{1, \dots, k\}$, *and there are* s 4*-cycles in* $\{C_1, \dots, C_k\}$.

When $s = k$, the following result is obvious.

Corollary 1.1 *Let* $k \geq 1$ *be an integer and* $G = (V_1, V_2; E)$ *be a bipartite graph with* $|V_1| = |V_2| \geq 2k$. *If* $\sigma_{1,1}(G) \geq \lceil \frac{4n+k}{3} \rceil$, *then for any* k *distinct vertices* $v_1, \dots, v_k$, G *contains* k *vertex-disjoint quadrilaterals* $C_1, \dots, C_k$ *such that* $v_i \in V(C_i)$ *for each* $i \in \{1, \dots, k\}$.

Since we have solved the packing problem by Theorem 1.3, the next step is to show that this collection of cycles can be transformed into a collection of cycles that form a partition of G. Our main result is as follows.

Theorem 1.4 *Let* $k \geq 1$ *be an integer and* $G = (V_1, V_2; E)$ *a bipartite graph with* $|V_1| = |V_2| = n \geq 2k + 2$. *If* $\sigma_{1,1}(G) \geq \lceil \frac{4n+k}{3} \rceil$, *then for any* k *distinct vertices* $z_1, \dots, z_k$, G *contains a* 2*-factor with* $k+1$ *cycles* $C_1, \dots, C_{k+1}$ *such that* $z_i \in V(C_i)$ *and* $|C_i| = 4$ *for each* $i \in \{1, \dots, k\}$.

Notes The following example shows that the degree condition in Theorem 1.4 is sharp when $k = 1$ and $n = 4$. Let G be a balanced bipartite graph consisting of two disjoint subgraphs P and C, where $P = x_1x_2x_3x_4$ is a path of order 4 with $x_1x_4 \notin E(G)$ and $C = v_1v_2v_3v_4v_1$ is a quadrilateral. Clearly, $n = 4$ and $k = 1$. Suppose that $\{x_1, v_1\} \subseteq V_1$. We define the neighbor set of x_i in C as follows: $N(x_1, C) = \{v_2, v_4\}$ and $N(x_4, C) = \{v_1, v_3\}$. Then for any nonadjacent vertices $u \in V_1$ and $v \in V_2$, $d(u) + d(v) \geq 5 = \lceil \frac{17}{3} \rceil - 1$. It is easy to check that G does not contain two disjoint quadrilaterals such that one of them passing through v_1.

The degree conditions in Theorems 1.3 and 1.4 come from our proof. However, for more general case, we do not know whether the degree condition is sharp. We believe that this is true.

2 Lemmas

We will use the notation $C[u, v]$ to denote the segment of the cycle C from u to v (including u and v) under some orientation of C, and $C[u, v) = C[u, v] - \{v\}$ and $C(u, v) = C[u, v] - \{u, v\}$.

Lemma 2.1 [2] *If $d(x, G) + d(y, G) \geq n + 1$ for any two nonadjacent vertices x and y with $x \in V_1$ and $y \in V_2$, then G is Hamiltonian.*

Lemma 2.2 [11] *Let $P = x_1x_2 \cdots x_s$ be a path in G with $s = 2r + d$, where $d = 0$ or 1. Let $x_0 \in V(G - P)$ such that $\{x_s, x_0\} \nsubseteq V_i$ for every $i \in \{1, 2\}$.* (a) *If $d(x_0, P) + d(x_s, P) \geq r + 1$, then G has a path $P^\star$ such that $V(P^\star) = V(P) \cup \{x_0\}$.* (b) *If $d = 0$ and $d(x_0, P) + d(x_s, P) \geq r + 1$, then G has a path $P^\star$ from x_0 to x_1 such that $V(P^\star) = V(P) \cup \{x_0\}$.*

Lemma 2.3 *Let C be a cycle in G, $x \in V(C)$, $u \in V(G - C) \cap V_1$, $v \in V(G - C) \cap V_2$ and $d_C(u) + d_C(v) \geq |C|/2 + 2$. Then, either $G[V(C) \cup \{v\}]$ contains a shorter cycle than C passing through x, or there exists $w \in N_C(u)$ such that $G[V(C) \cup \{v\} - \{w\}]$ contains a cycle passing through x.*

Proof Clearly, $d(v, C) \leq 2$, otherwise, $G[V(C) \cup \{v\}]$ contains a cycle shorter than C and passing through x. Since $d_C(u) + d_C(v) \geq |C|/2 + 2$, which implies that $d_C(v) = 2$ and $d_C(u) = |C|/2$. This means that $N_C(u) = V(C) \cap V_2$. Without loss of generality, we assume that $N_C(v) = \{a, b\}$ with $x \in V(C[b, a])$. Take any $w \in N_C(u) \cap C(a, b)$. Then $G[V(C) \cup \{v\} - \{w\}]$ contains a cycle passing through x. □

Remark 2.1 In Lemma 2.3, if $|C| = 4$, it is easy to see that there exists $w \in N_C(u)$ such that $G[V(C) \cup \{v\} - \{w\}]$ contains a 4-cycle. If $|C| = 6$ and $G[V(C) \cup \{v\}]$ contains no cycle which is shorter than C and passing through v, then there exists $w \in N_C(u)$ such that $G[V(C) \cup \{v\} - \{w\}]$ contains a 6-cycle. Note that we can exchange the role of u and v when $|C| \leq 6$.

Lemma 2.4 [10] *Let $P = x_1y_1 \cdots x_ky_k$ be a path in G, $k \geq 2$. If $d(x_1, P) + d(y_k, P) \geq k + 1$, then G contains a cycle C such that $V(C) = V(P)$.*

Lemma 2.5 *Let Q be a* 4*-cycle passing through v, P be a u-path of length* 5 *such that $V(Q) \cap V(P) = \emptyset$. Suppose that $\lambda(u, P) \neq 0, 1$ and $e(Q, P) \geq 9$, then $G[V(Q \cup P)]$ contains two disjoint cycles Q_1 and Q_2 such that $l(Q_1) = 4$ or 6, $l(Q_2) = 4$, $v \in V(Q_2)$ and $u \in V(Q_1)$.*

Proof Let $Q = a_1a_2a_3a_4a_1$, $P = x_1x_2 \cdots x_6$. Suppose that $G[V(Q \cup P)]$ does not contain two disjoint cycles $\{Q_1, Q_2\}$ with respect to $\{u, v\}$ such that $l(Q_1) = 6$ and $l(Q_2) = 4$, we will prove that $G[V(Q \cup P)]$ contains two disjoint 4-cycles $\{Q_1, Q_2\}$ with respect to $\{u, v\}$.

Without loss of generality, suppose that $\{a_1, x_1\} \subseteq V_1$. Since $\lambda(u, P) \neq 0, 1$, $u = x_3$ or x_4. If $u = x_3$ (or x_4) and $\{u, v\} \subseteq V_i$ for some $i \in \{1, 2\}$, then we may assume that $v = a_1$. If $u = x_3$ (or x_4) and u, v belong to different partite sets, we let $v = a_2$. By symmetry, it suffices to consider the case that $v = a_1$ and $u = x_3$.

Let $P_1 = x_1x_2x_3x_4$. It is easy to see that if $e(P_1, Q) \geq 7$, then $G[V(P \cup Q)]$ contains two disjoint 4-cycles Q_1 and Q_2 such that $u \in V(Q_1)$ and $v \in V(Q_2)$. Hence, $e(P_1, Q) \leq 6$. As $e(P, Q) \geq 9$, then $5 \leq e(P_1, Q) \leq 6$, $e(x_5x_6, Q) \geq 3$.

Case 1: $e(P_1, Q) = 6$ and $d(x_4, Q) = 2$. If $d(a_2, P_1) = 2$, then let $Q_1 = a_2x_1x_2x_3a_2$ and $Q_2 = x_4a_1a_4a_3x_4$, we are done. Hence, $d(a_2, P_1) \leq 1$ and $d(a_4, P_1) \leq 1$ by symmetry. As $e(P_1, Q) = 6$, it follows that $\{a_1, a_3\} \subseteq N(x_2)$ and $d(a_2, P_1) = d(a_4, P_1) = 1$. If $N(a_2, P_1) \cap N(a_4, P_1) = \emptyset$, without loss of generality, assume that $a_2x_1 \in E(G)$ and $a_4x_3 \in E(G)$. Then, $G[V(Q \cup P_1)]$ contains two disjoint 4-cycles $Q_1 = x_3a_4a_3x_4x_3$ and $Q_2 = x_1a_2a_1x_2x_1$ with $x_3 \in V(Q_1)$ and $a_1 \in V(Q_2)$, we are done. Hence, $N(a_2, P_1) \cap N(a_4, P_1) \neq \emptyset$. If $x_1 \in N(a_2, P_1) \cap N(a_4, P_1)$, then $G[V(P_1 \cup Q)]$ contains two required quadrilaterals $Q_2 = a_3x_2x_3x_4a_3$ and $Q_1 = x_1a_2a_1a_4x_1$. Hence, it remains the case that $x_3 \in N(a_2, P_1) \cap N(a_4, P_1)$. If $x_6a_1 \in E(G)$, then $G[V(Q \cup P)]$ contains two required quadrilaterals $Q_1 = x_3a_2a_3a_4x_3$ and $Q_2 = a_1x_4x_5x_6a_1$. So, $x_6a_1 \notin E(G)$. As $e(x_5x_6, Q) \geq 3$, it follows that $d(x_5, Q) = 2$. We see that $G[V(Q \cup P)]$ contains two required quadrilaterals $Q_1 = a_2x_3x_4x_5a_2$ and $Q_2 = x_2a_1a_4a_3x_2$.

Case 2: $e(P_1, Q) = 6$ and $d(x_4, Q) \leq 1$. If $d(x_4, Q) = 0$, then $d(x_i, Q) = 2$ for each $i \in \{1, 2, 3\}$. As $e(P, Q) \geq 9$, so, $d(x_5, Q) \geq 1$. Without loss of generality, say $x_5a_2 \in E(G)$, then we can choose $Q_1 = x_3a_2x_5x_4x_3$ and $Q_2 = x_1a_4a_1x_2x_1$. Hence, it remains the case that $d(x_4, Q) = 1$ and so $e(\{a_2, a_4\}, P_1) \geq 3$. Without loss of generality, say $d(a_2, P_1) = 2$.

Suppose that $x_4a_3 \in E(G)$. We conclude $d(x_2, Q) = 1$. Otherwise, $d(x_2, Q) = 2$. If $a_4x_1 \in E(G)$, then choose $Q_1 = x_4a_3x_2x_3x_4$ and $Q_2 = x_1a_2a_1a_4x_1$. If $a_4x_3 \in E(G)$, then let $Q_1 = x_3a_4a_3x_4x_3$ and $Q_2 = x_1a_2a_1x_2x_1$. Since we have shown that $d(x_2, Q) = 1$, it follows that $d(a_4, P_1) = 2$. If $x_2a_1 \in E(G)$, then let $Q_1 = x_3a_2a_3x_4x_3$ and $Q_2 = x_1x_2a_1a_4x_1$. If $x_2a_3 \in E(G)$, then let $Q_1 = x_3x_2a_3x_4x_3$ and $Q_2 = x_1a_2a_1a_4x_1$.

Hence, $x_4a_3 \notin E(G)$ and so $x_4a_1 \in E(G)$. Let us assume that $x_2a_3 \in E(G)$. Then $d(x_6, Q) = 1$, otherwise, $d(x_6, Q) = 2$, we see that $G[V(Q \cup P)]$ contains two required cycles $Q_1 = a_2x_1x_2x_3a_2$ and $Q_2 = x_6a_1a_4a_3x_6$. Consequently, it follows that $d(x_5, Q) = 2$. Then $G[V(Q \cup P)]$ contains two required cycles $Q_1 = a_2x_1x_2x_3a_2$ and $Q_2 = x_5x_4a_1a_4x_5$. Therefore $x_2a_3 \notin E(G)$, and so $d(a_4, P_1) = 2$ and $x_2a_1 \in E(G)$. If $d(x_5, Q) = 2$, then choose $Q_1 = x_3a_4x_5x_4x_3$ and $Q_2 = a_1a_2x_1x_2a_1$. If $d(x_6, Q) = 2$, then chose $Q_1 = a_4x_1x_2x_3a_4$ and $Q_2 = x_6a_1a_2a_3x_6$.

Case 3: $e(P_1, Q) = 5$. Note that in this case, $e(x_5x_6, Q) = 4$. If $a_4x_1 \in E(G)$, then $G[V(Q \cup P)]$ contains two disjoint cycles $Q_1 = a_4x_1x_2x_3x_4x_5a_4$ and $Q_2 = x_6a_1a_2a_3x_6$ such that $u = x_3 \in V(Q_1)$ and $v = a_1 \in V(Q_2)$, a contradiction. Thus, $a_4x_1 \notin E(G)$ and $a_2x_1 \notin E(G)$ by symmetry. If $d(a_3, P_1) = 2$, then $G[V(Q \cup P)]$ contains two required disjoint quadrilaterals $Q_1 = a_3x_2x_3x_4a_3$ and $Q_2 = x_5x_6a_1a_2x_5$, we have nothing to prove. Therefore, we may assume that $d(a_3, P_1) \leq 1$. As $e(Q, P_1) = 5$, it follows that $d(a_1, P_1) = 2$, $d(a_3, P_1) = 1$ and $N(a_2, P_1) = N(a_4, P_1) = \{x_3\}$. Suppose $a_3x_2 \in E(G)$, then $G[V(Q \cup P)]$ contains two required cycles $Q_1 = a_1a_4a_3x_2a_1$ and $Q_2 = a_2x_3x_4x_5a_2$. So, $a_3x_2 \notin E(G)$ and $a_3x_4 \in E(G)$. However, we see that $G[V(Q \cup P)]$ contains two required disjoint quadrilaterals $Q_1 = a_3a_4x_3x_4a_3$ and $Q_2 = a_1a_2x_5x_6a_1$. This proves the lemma. □

The following lemma is obvious from the proof of Lemma 2.5.

Lemma 2.6 [12] *Let Q be a 4-cycle passing through v, P be a u-path of length 5 such that $V(Q) \cap V(P) = \emptyset$. Suppose that $\lambda(u, P) \neq 0, 1$ and $e(Q, P) \geq 10$, then $G[V(Q \cup P)]$ contains two disjoint 4-cycles Q_1 and Q_2 such that $v \in V(Q_2)$ and $u \in V(Q_1)$.*

Lemma 2.7 *Let $d \geq 2$ be an integer. Let $P = x_1 \cdots x_{2d}$ be a path, $C_1 = v_1v_2v_3v_4v_1$ and $C_2 = u_1u_2u_3u_4u_1$ be two disjoint quadrilaterals of G with $\{v_1, u_1, x_1\} \subseteq V_1$. Suppose that $e(\{x_1, x_{2d}\}, C_1) = 4$. If $e(\{v_3, v_4, x_{2d}, x_{2d-1}, x_1, x_2\}, C_2) \geq 10$, then $G[V(P \cup C_1 \cup C_2)]$ contains two quadrilaterals C_1' and C_2' passing through v_1 and u_1, respectively, and a cycle C_{k+1} of length $2d$ such that C_1', C_2' and C_{k+1} are disjoint.*

Proof Since $e(\{v_3, v_4, x_{2d}, x_{2d-1}, x_1, x_2\}, C_2) \geq 10$, without loss of generality, we may assume that $e(\{v_3, x_1, x_{2d-1}\}, C_2) \geq 5$.
Fact 1: $e(\{v_3, x_1, x_{2d-1}\}, C_2) = 5$. Otherwise, suppose that $e(\{v_3, x_1, x_{2d-1}\}, C_2) = 6$. Then we obtain that $N(v_3, C_2) = N(x_1, C_2) = N(x_{2d-1}, C_2) = \{u_2, u_4\}$. If $N(v_4, C_2) = \{u_1, u_3\}$, then we have two quadrilaterals $C_1' = x_{2d}v_1v_2v_3x_{2d}$ and $C_2' = v_4u_1u_4u_3v_4$ passing through v_1 and v_2, respectively, and a cycle $C_{k+1} = x_1u_2x_{2d-1} \cdots x_1$ such that they are all disjoint. So, $d(v_4, C_2) \leq 1$. Similarly, $d(x_{2d}, C_2) \leq 1$. As $e(\{v_4, x_2, x_{2d}\}, C_2) \geq 10 - 6 = 4$, this implies $d(x_2, C_2) = 2$ and $d(x_{2d}, C_2) = 1$. If $x_{2d}u_3 \in E(G)$, then $G[V(C_1 \cup C_2 \cup P)]$ contains three required cycles: $C_1' = x_1v_2v_1v_4x_1$, $C_2' = v_3u_2u_1u_4v_3$ and $C_{k+1} = u_3x_2 \cdots x_{2d}u_3$. If $x_{2d}u_1 \in E(G)$, then $G[V(C_1 \cup C_2 \cup P)]$ contains three required cycles: $C_1' = x_1v_2v_1v_4x_1$, $C_2' = x_{2d}u_1u_4v_3x_{2d}$ and $C_{k+1} = u_2u_3x_2 \cdots x_{2d-1}u_2$.
Fact 2: $d(v_4, C_2) \leq 1$ and $d(x_{2d}, C_2) \leq 1$. Otherwise, assume that $d(v_4, C_2) = 2$. By Fact 1, there exists $u_i \in \{u_2, u_4\}$, by symmetry, say u_2, such that $u_2 \in N(x_1, C_2) \cap N(x_{2d-1}, C_2)$. Then $G[V(C_1 \cup C_2 \cup P)]$ contains three required cycles: $C_1' = x_{2d}v_1v_2v_3x_{2d}$, $C_2' = v_4u_1u_4u_3v_4$ and $C_{k+1} = x_1u_2x_{2d-1} \cdots x_2x_1$. Similarly, $d(x_{2d}, C_2) \leq 1$.

By Fact 1 and Fact 2, we obtain $e(\{v_3, v_4, x_1, x_2, x_{2d-1}, x_{2d}\}, P) \leq 9$, a contradiction. This proves the lemma. □

Let $F = \{v_1, \dots, v_k\}$ be a set of distinct vertices. A cycle C is called *admissible* if $|V(C) \cap F| = 1$ and $|C| \le 6$, and a set of disjoint cycles $\{C_1, \dots, C_r\}$ is *admissible* for $r \le k$ if each C_i is admissible.

3 Proof of Theorem 1.3

Proof Otherwise, let G be an edge-maximal counterexample to Theorem 1.3. Clearly, since G is not a complete bipartite graph, there are nonadjacent vertices $x \in V_1$ and $y \in V_2$ in G. Let G' be the graph obtained from G by adding the new edge xy. For any k distinct vertices $\{v_1, \dots, v_k\}$, by the maximality of G, G' contains k admissible cycles $C_1, \dots, C_k$ with respect to $\{v_1, \dots, v_k\}$ and there are at least s quadrilaterals in $\{C_1, \dots, C_k\}$. Without loss of generality, we may assume $xy \in E(C_k)$.

Claim 3.1 $k \ge 2$.

Proof Otherwise, suppose $k = 1$. This implies $s = 1$. Since for each nonadjacent vertices $x \in V_1$ and $y \in V_2$, $d(x) + d(y) \ge \frac{4n+1}{3} \ge n + 1$, G contains a Hamiltonian cycle by Lemma 2.1. Let $C = x_1 y_1 \cdots x_n y_n x_1$ be a Hamiltonian cycle of G with $x_1 \in V_1$. Without loss of generality, assume $v_1 = x_2$ (otherwise, we can relabel the index). Furthermore, if $n = 2$, then G contains a quadrilateral, a contradiction. Thus, we may assume that $n \ge 3$. We consider the path $P = x_1 y_1 x_2 y_2 x_3 y_3$ in G. Since G contains no quadrilateral with respect to v_1, therefore,

$$N(x_2, G) \cap N(x_1, G) = y_1, \qquad N(y_1, G) \cap N(y_2, G) = x_2 \quad \text{and}$$
$$N(x_2, G) \cap N(x_3, G) = y_2.$$

Then we have

$$d(x_1) + d(x_2) = |N(x_1, G) \cap N(x_2, G)| + |N(x_1, G) \cup N(x_2, G)| \le n + 1.$$

Similarly, $d(x_2) + d(x_3) \le n + 1$ and $d(y_1) + d(y_2) \le n + 1$. Furthermore, we observe that $d(x_2) \ge 2$ and $d(y_3) \le n - 1$ as G contains no quadrilateral with respect to v_1. So, it follows that

$$\sum_{x \in V(P)} d(x) \le 3n + 3 - d(x_2) + d(y_3) \le 4n. \tag{1}$$

On the other hand, it is easy to see that G contains three pairs of nonadjacent vertex $\{x_1, y_2\}$, $\{y_1, x_3\}$ and $\{x_2, y_3\}$, thus, $\sum_{x \in V(P)} d(x) \ge 3 \times \frac{4n+1}{3} = 4n + 1$, contradicting (1). □

By the choice of G, there exists $v \in \{v_1, v_2, \dots, v_k\}$ such that G contains $k - 1$ admissible cycles $C_1, C_2, \dots, C_{k-1}$ with respect to $\{v_1, v_2, \dots, v_k\} - \{v\}$

and $v \notin V(\sum_{i=1}^{k-1} C_i)$. We choose $v \in \{v_1, v_2, \dots, v_k\}$ and $k-1$ admissible cycles $C_1, \dots, C_{k-1}$ with respect to $\{v_1, \dots, v_k\} - \{v\}$ such that

$$\sum_{i=1}^{k-1} |C_i| \text{ is as small as possible.} \tag{2}$$

Note that there are at least $s-1$ 4-cycles in $\{C_1, \dots, C_{k-1}\}$.

Subject to (2), we choose $v \in \{v_1, v_2, \dots, v_k\}$ and $k-1$ admissible cycles $C_1, C_2, \dots, C_{k-1}$ with respect to $\{v_1, \dots, v_k\} - \{v\}$ such that

$$\text{The length of the longest } v\text{-path in } M = G - V\left(\sum_{i=1}^{k-1} C_i\right) \text{ is maximum.} \tag{3}$$

Subject to (2) and (3), we choose $v \in \{v_1, v_2, \dots, v_k\}$, $k-1$ vertex disjoint $C_1, C_2, \dots, C_{k-1}$ with respect to $\{v_1, \dots, v_k\} - \{v\}$ and P such that

$$\lambda(v, P) \text{ is maximum.} \tag{4}$$

Let $P = x_1 x_2 \cdots x_t$ be a longest v-path in M with $x_1 \in V_1$. By the maximality of G, we see that M contains a v-path of length at least 3, so $t \geq 4$. Without loss of generality, suppose that $v = v_k$ and $v_i \in V(C_i)$ for each $i \in \{1, 2, \dots, k-1\}$. Let $H = \sum_{i=1}^{k-1} C_i$, then $M = G - V(H)$ and $|M| = 2m$. Clearly, $m \geq 2$.

For convenience, in the following proof, let $T_1, \dots, T_l$ denote l disjoint quadrilaterals and $Q_{l+1}, \dots, Q_{k-1}$ denote $k-l-1$ disjoint 6-cycles in $C_1, \dots, C_{k-1}$, let $H_T = \bigcup_{i=1}^{l} T_i$ and $H_Q = \bigcup_{i=l+1}^{k-1} Q_i$, where $s \leq l+1 \leq k$. As $n \geq 2s + 3(k-s)$, we obtain $n = 2l + 3(k-1-l) + m$ and $m \geq l - s + 3$. Let $t = 2r + q$, when $q = 0$ or 1. Our proof includes several claims.

Claim 3.2 *$t = 2m$, i.e., P is a Hamiltonian path of M.*

Proof Otherwise, suppose that $t < 2m$. Let $x_0 \in V(M-P)$ such that $\{x_t, x_0\} \nsubseteq V_i$ for each $i \in \{1, 2\}$. By Lemma 2.2 and (3), $d(x_0, P) + d(x_t, P) \leq r$. Clearly, $d(x_0, M - V(P)) \leq m - r$ and so $d(x_0, M) + d(x_t, M) \leq m$. For each 6-cycle $C_i \in H_Q$, we may assume that $d_{C_i}(x_0) + d_{C_i}(x_t) \leq 4$, otherwise, by Lemma 2.3, Remark 2.1 and the choice of (2), there exists $w \in N_{C_i}(x_t)$ such that $G[V(C_i) \cup \{x_0\} - \{w\}]$ contains a 6-cycle C' passing through v_i. If we replace C_i with C', we see that $P + w$ is a longer path than P, which contradicts (3) while (2) still maintain. Hence, $d(x_0, H_Q) + d(x_t, H_Q) \leq 4(k-1-l)$. As $m \geq l - s + 3$, it follows that

$$d(x_0, H_T) + d(x_t, H_T) \geq \frac{4n+s}{3} - 4(k-1-l) - m = \frac{8l+s+m}{3} \geq 3l+1.$$

This implies that there exists $C_i \in H_T$ such that $d(x_0, C_i) + d(x_t, C_i) = 4$. That is, $d(x_0, C_i) = d(x_t, C_i) = 2$. Let $z \in V(C_i)$ with $zx_t \in E(G)$ be such that $C_i - z + x_0$ is an quadrilateral passing through v_i, then $P + z$ is longer than P in M, contradicting (3) while (2) still holds. Thus, $t = 2m$. So the claim holds. □

Claim 3.3 *If* $\lambda(v_k, P) = 0$ *or* 1, *then* M *is Hamiltonian.*

Proof By Claim 3.2, M contains a Hamiltonian path $P = x_1x_2\cdots x_{2m}$ passing through v_k. If $x_1x_{2m} \in E(G)$, then we have nothing to prove. So, $x_1x_{2m} \notin E(G)$. By symmetry, if $\lambda(v_k, P) = 0$, we may assume that $v_k = x_1$. If $\lambda(v_k, P) = 1$, we assume that $v_k = x_2$.

If there exists $C_i \in H_T$ such that $d(x_1, C_i) + d(x_{2m}, C_i) = 4$, by Remark 2.1, there exits $w \in V(C_i)$ with $x_1w \in E(G)$ such that $C_i - w + x_{2m}$ contains a quadrilateral C_i' passing through v_i. If we replace C_i with C_i', we see that M contains a v_k-path $P' = P - x_{2m} + w$. However, $\lambda(v_k, P') = \lambda(v_k, P) + 1$, contradicting (4) while (2) and (3) still maintain. Hence, $d(x_1, C_i) + d(x_{2m}, C_i) \leq 3$ for each $C_i \in H_T$ and so $d(x_1, H_T) + d(x_{2m}, H_T) \leq 3l$. By a similar argument, we can show that $d(x_1, C_i) + d(x_{2m}, C_i) \leq 4$ for each $C_i \in H_Q$ and so $d(x_1, H_Q) + d(x_{2m}, H_Q) \leq 4(k-1-l)$. As $m \geq l - s + 3$, it follows that

$$d(x_1, M) + d(x_{2t}, M) \geq \frac{4n+s}{3} - 3l - 4(k-1-l) = \frac{8l+4m+s}{3} - 3l \geq m+1.$$

By Lemma 2.4, M contains a hamiltonian cycle. This proves the claim. □

We continue the proof. If $m = 2$, then by Claim 3.3, $G[V(P)]$ contains a quadrilateral passing through v_k, denoted by C_k, then G contains k admissible cycles $C_1, \ldots, C_k$ such that $v_i \in V(C_i)$ for each $i \in \{1, \ldots, k\}$, and there are at least s 4-cycles in $\{C_1, \ldots, C_k\}$, a contradiction. Hence, we may assume that $m \geq 3$ in the following. By Claims 3.2 and 3.3, we can choose a v_k-path P of length 5 in M such that $\lambda(v_k, P) = 2$. Let $P = y_1y_2y_3y_4y_5y_6$ with $y_1 \in V_1$, then $v_k = y_3$ or y_4. Since there is at least $s-1$ 4-cycles in $\{C_1, C_2, \ldots, C_{k-1}\}$, then M contains no 4-cycle passing through v_k. Consequently, we obtain

$$N(y_3, M) \cap N(y_5, M) = y_4 \quad \text{and} \quad N(y_2, M) \cap N(y_4, M) = y_3.$$

If $v_k = y_3$, then $N(y_1, M) \cap N(y_3, M) = y_2$. Otherwise, $v_k = y_4$, then $N(y_4, M) \cap N(y_6, M) = y_5$. It follows that

$$d(y_3, M) + d(y_5, M) = |N(y_3, M) \cup N(y_5, M)| + |N(y_3, M) \cap N(y_5, M)| \leq m+1.$$

Similarly, $d(y_2, M) + d(y_4, M) \leq m+1$, $d(y_1, M) + d(y_3, M) \leq m+1$ or $d(y_4, M) + d(y_6, M) \leq m+1$. Without loss of generality, we assume that $v_k = y_3$.

Since M contains no quadrilateral passing through $v_k = y_3$, then $d(y_6, M) \leq m-1$. Note that $d(y_3, M) \geq 2$, then

$$\sum_{x \in V(P)} d(x, M) \leq 3m + 3 - d(y_3, M) + d(y_6, M) \leq 4m.$$

On the other hand, it is easy to see that $V(P)$ contains three pairs of nonadjacent vertex $\{y_1, y_4\}$, $\{y_2, y_5\}$ and $\{y_3, y_6\}$. Therefore,

$$\sum_{x \in V(P)} d(x, H) \geq 3 \times \frac{4n+s}{3} - 4m = 8l + 12(k-1-l) + s. \tag{5}$$

Claim 3.4 *For each* 6*-cycle* $C_i \in H_Q$*, we may assume that* $e(P, C_i) \leq 12$.

Proof Otherwise, assume that there exists $C_i \in H$ such that $e(P, C_i) \geq 13$. Let $C_i = a_1a_2a_3a_4a_5a_6a_1$ with $a_1 \in V_1$ and $v_i = a_1$, $P_1 = y_2y_3y_4y_5$. Clearly, $d_{C_i}(P_1) \geq 13 - d_{C_i}(\{y_1, y_6\}) \geq 13 - 6 = 7$. If $y_2a_1 \in E(G)$, then $|E(G) \cap \{y_2a_3, y_2a_5\}| = 0$, otherwise, without loss of generality, say $y_2a_3 \in E(G)$, then $G[V(C_i \cup P)]$ contains a 4-cycle $y_2a_1a_2a_3y_2$ passing through $a_1 = v_i$, which contradicts (2). Similarly, if $y_4a_1 \in E(G)$, then $|E(G) \cap \{y_4a_3, y_4a_5\}| = 0$. On the other hand, if there exists $x \in \{a_3, a_5\}$ such that $\{y_2x, y_4x\} \subseteq E(G)$, then $y_2y_3y_4xy_2$ is a 4-cycle passing through $v_k = y_3$, contradicting (2) again. Consequently, $d_{C_i}(\{y_2, y_4\}) \leq 3$ and the equality holds when $\{y_2a_1, y_4a_3, y_4a_5\} \subseteq E(G)$ or $\{y_4a_1, y_2a_3, y_2a_5\} \subseteq E(G)$. It follows that $d_{C_i}(\{y_3, y_5\}) \geq 7 - 3 = 4$, which implies that there exists $z \in \{a_2, a_4, a_6\}$ such that $\{zy_3, zy_5\} \subseteq E(G)$. Then we see that $G[V(C_i \cup P)]$ contains a 4-cycle $y_3zy_5y_4y_3$ passing through $v_k = y_3$, which contradicts (2) again. This proves the claim. □

Now, we are in the position to complete the proof of Theorem 1.3. Note that there is at least $s - 1$ 4-cycles in H. By (5) and Claim 3.4, we obtain

$$\sum_{x \in V(P)} d(x, H_T) \geq 8l + 12(k - 1 - l) + s - 12(k - 1 - l) = 8l + s. \tag{6}$$

As $s \geq 1$, this implies there exists $T_j \in H_T$ such that $e(T_j, P) \geq 9$. By Lemma 2.5, $G[V(T_j \cup P)]$ contains two disjoint cycles Q_1 and Q_2 such that $l(Q_1) = 4$ or 6, $l(Q_2) = 4$, $v_j \in V(Q_2)$ and $v_k \in V(Q_1)$. Replacing T_j with Q_2. If $l(Q_1) = 4$, then G contains desired k admissible cycles $H \cup (Q_1 \cup Q_2) - T_j$ with at least s quadrilaterals, a contradiction. Hence, it remains the case that $l(Q_1) = 6$, then G contains k admissible cycles $H \cup (Q_1 \cup Q_2) - T_j$ with respect to $\{v_1, \ldots, v_k\}$, this implies that H contains exactly $s - 1$ 4-cycles, which yields to $l = s - 1$. Now we rewrite (6) as follows:

$$\sum_{x \in V(P)} d(x, H_T) \geq 8l + s = 9l + 1.$$

This implies that there exists $C_i \in H_T$ such that $e(C_i, P) \geq 10$. By Lemma 2.6, $G[V(C_i \cup P)]$ contains two disjoint 4-cycles Q_1 and Q_2 such that $v_k = y_3 \in V(Q_1)$ and $v_i \in V(Q_2)$. Replace C_i with Q_2, we see that G contains G contains k vertex disjoint admissible cycles $C_1, \ldots, C_{i-1}, Q_2, C_{i+1}, \ldots, C_{k-1}, Q_1$ with respect to $\{v_1, v_2, \ldots, v_k\}$, and there are s 4-cycles in $H \cup (Q_2 \cup Q_1) - C_i$, a final contradiction. □

4 Proof of Theorem 1.4

Proof Let $k \geq 1$ be an integer and $G = (V_1, V_2; E)$ a bipartite graph with $|V_1| = |V_2| = n \geq 2k + 2$ such that $\sigma_{1,1}(G) \geq \lceil \frac{4n+k}{3} \rceil$. Suppose to the contrary, Theorem 1.4 is false. By Corollary 1.1, for any k distinct vertices $z_1, \ldots, z_k$, G contains k disjoint quadrilaterals $C_1, \dots, C_k$ with respect to $\{z_1, \dots, z_k\}$. We choose k disjoint quadrilaterals $C_1, \dots, C_k$ with respect to $\{z_1, \dots, z_k\}$ such that

$$\text{The length of the longest path in } G - V\left(\sum_{i=1}^{k} C_i\right) \text{ is maximum.} \tag{7}$$

Let $P = x_1x_2\cdots x_t$ be a longest path in $G - V(\sum_{i=1}^{k} C_i)$ with $x_1 \in V_1$. Set $H = \sum_{i=1}^{k} C_i$, $D = G - V(H)$ and $|D| = 2d$. As $n = 2k + d \geq 2k + 2$, we obtain $d \geq 2$. Let $t = 2r + q$, when $q = 0$ or 1. Our proof includes several claims.

Claim 4.1 $t = 2d$.

Proof Otherwise, suppose $t < 2d$. Let $x_0 \in V(D - P)$ such that $\{x_0, x_t\} \not\subseteq V_i$ with $i \in \{1, 2\}$. By Lemma 2.2 and (7), $d(x_0, P) + d(x_t, P) \leq r$. Clearly, $d(x_0, D - V(P)) \leq d - r$ and so $d(x_0, D) + d(x_t, D) \leq d$. Then

$$d(x_0, H) + d(x_t, H) \geq \left\lceil \frac{4n + k}{3} \right\rceil - d = 3k + \frac{d}{3}.$$

Since $d \geq 2$, the above inequality implies that there exists $C_i \in H$ such that $d(x_0, C_i) + d(x_t, C_i) = 4$, then by Remark 2.1, there exists $z \in V(C_i)$ such that $x_t z \in E(G)$ and $C_i - z + x_0$ is a quadrilateral passing through v_i, if we replace C_i with $C_i - z + x_0$, then $P + z$ is longer than P, contradicting (7). This proves the claim. □

If $x_1x_{2d} \in E(G)$, by Claim 4.1, D contains a hamiltonian cycle, denoted by C_{k+1}, then G contains a 2-factor with $k + 1$ cycles $C_1, \dots, C_{k+1}$ such that $v_i \in V(C_i)$ and $|C_i| = 4$ for each $i \in \{1, \dots, k\}$, a contradiction. Hence, $x_1x_{2d} \notin E(G)$. As $d \geq 2$, if $d(x_1, P) + d(x_{2d}, P) \geq d + 1$, by Lemma 2.4, D is hamiltonian and we are done. So, we may assume that $d(x_1, P) + d(x_{2d}, P) \leq d$. Consequently, we have

$$d(x_1, H) + d(x_{2d}, H) \geq \left\lceil \frac{4n + k}{3} \right\rceil - d \geq 3l + \frac{d}{3}.$$

The above inequality implies that there exists $C_i \in H$ such that $d(x_1, C_i) + d(x_{2d}, C_i) = 4$. Without loss of generality, we may assume that $C_i = C_1$. Let $C_1 = v_1v_2v_3v_4v_1$ with $z_1 = v_1 \in V_1$. If $v_3x_2 \in E(G)$, then $G[V(C_1 \cup P)]$ contains two disjoint cycles $C_1' = x_1v_2v_1v_4x_1$ and $C_{k+1} = v_3x_2\cdots x_{2d}v_3$ such that $l(C_1') = 4$ and $l(C_{k+1}) = 2d$, so, G contains a desired 2-factor with $k + 1$ cycles: $C_1', C_2, \dots, C_k, C_{k+1}$, a contradiction. Therefore, $v_3x_2 \notin E(G)$. Similarly, $v_4x_{2d-1} \notin E(G)$. If there exists some $C_i, i \in \{2, \dots, k\}$ such that $e(\{x_1, x_2, x_{2d-1}, x_{2d}, v_3, v_4\}, C_i) \geq 10$, then by Lemma 2.7, $G[V(C_1 \cup C_i \cup P)]$ contains three disjoint cycles C_1', C_i' and C_{k+1} such that $z_j \in V(C_j')$ and $l(C_j') = 4$ for each $j \in \{1, i\}$ and $l(C_{k+1}) = 2d$, thus, G contains a desired 2-factor with $k + 1$ cycles, a contradiction. Hence, $e(\{x_1, x_2, x_{2d-1}, x_{2d}, v_3, v_4\}, C_i) \leq 9$ for each $C_i \in H - C_1$. It follows that

$$e(\{x_1, x_2, x_{2d-1}, x_{2d}, v_3, v_4\}, D \cup C_1) \geq 3 \times \frac{4n + k}{3} - 9(k - 1) = 4d + 9. \tag{8}$$

Let $P_1 = v_4x_1 \cdots x_{2d-1}$ and $P_2 = v_3x_{2d} \cdots x_2$. Since $x_{2d}v_1v_2v_3x_{2d}$ is a quadrilateral passing through v_1, we have $d(v_4, P_1) + d(x_{2d-1}, P_1) \le d$ by Lemma 2.4. Similarly, $d(x_2, P_2) + d(v_3, P_2) \le d$. Therefore,

$$e(\{x_1, x_2, x_{2d-1}, x_{2d}, v_3, v_4\}, D \cup C_1) \le 3d + 12. \tag{9}$$

Combine (8) with (9), we obtain $4d + 9 \le 3d + 12$, this gives $d \le 3$. If $d = 3$, it follows from (9) that $d(v_4, P_1) + d(x_{2d-1}, P_1) = 3$, $d(x_2, P_2) + d(v_3, P_2) = 3$, $d(x_1, P) + d(x_{2d}, P) = 3$, $e(\{v_4, x_{2d-1}\}, D \cup C_1) = 7$ and $e(\{v_3, x_2\}, D \cup C_1) = e(\{x_1, x_{2d}\}, D \cup C_1) = 7$. Thus, $x_5v_2 \in E(G)$ and $x_2v_1 \in E(G)$. Then $G[V(C_1 \cup D)]$ contains two disjoint cycles $C_1' = x_6v_1v_4v_3x_6$ and $C_{k+1} = x_1v_2x_5x_4x_3x_2x_1$ such that C_1' passing through $z_1 = v_1$, and so, G contains a desired 2-factor with $k + 1$ cycles: $C_1', C_2, \dots, C_k, C_{k+1}$, a contradiction.

Hence, it remains the case that $d = 2$. Clearly, $v_3x_2 \notin E(G)$ and $v_4x_3 \notin E(G)$. In this case, by (8) and (9), we obtain $17 \le e(\{v_3, v_4\} \cup P, D \cup C_1) \le 18$. If $e(P \cup \{v_3, v_4\}, D \cup C_1) = 18$, it follows from (9) that $e(\{x_1, x_4\}, D \cup C_1) = e(\{v_3, x_2\}, D \cup C_1) = e(\{v_4, x_3\}, D \cup C_1) = 6$. Consequently, $x_2v_1 \in E(G)$ and $v_2x_3 \in E(G)$, we see that $G[V(C_1 \cup P)]$ contains two disjoint quadrilaterals $C_1' = v_1x_2x_3x_4v_1$ and $C_{k+1} = v_3v_2x_1v_4v_3$ such that C_1' passing through $z_1 = v_1$, and so G contains a desired 2-factor with k quadrilaterals $C_1', C_2, \dots, C_k$ with respect to $\{z_1, z_2, \dots, z_k\}$ and a cycle C_{k+1}, a contradiction. Hence, $e(P \cup \{v_3, v_4\}, D \cup C_1) = 17$. From the above argument, we see that $v_1x_2 \notin E(G)$. As $e(\{v_3, v_4\} \cup P, D \cup C_1) = 17$, it is easy to check that $v_2x_3 \in E(G)$. Then $G[V(C_1 \cup P)]$ contains two disjoint quadrilaterals $C_1' = x_4v_1v_4v_3x_4$ and $C_{k+1} = v_2x_1x_2x_3v_2$ such that C_1' passing through $v_1 = z_1$, so, G contains a 2-factor with k quadrilaterals $C_1', C_2, \dots, C_k$ with respect to $\{z_1, \dots, z_k\}$ and a cycle C_{k+1}, a final contradiction. This proves Theorem 1.4. □

5 Concluding remark

We propose the following conjecture to specify the length of C_i for $s < i \le k$ in Theorem 1.3 and conclude this paper.

Conjecture *Suppose s and k be two integers with $1 \le s \le k$ and let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \ge 2s + 3(k - s)$. If $\sigma_{1,1}(G) \ge \lceil \frac{4n+s}{3} \rceil$, then for any k distinct vertices $v_1, \dots, v_k$, G contains k vertex disjoint cycles $C_1, \dots, C_k$ such that $v_i \in V(C_i)$ for each $i \in \{1, \dots, k\}$, $|C_i| = 4$ for $1 \le i \le s$ and $|C_i| = 6$ for $s < i \le k$.*

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References

1. Alon, N., Yuster, R.: H-factors in dense graphs. J. Comb. Theory Ser. B **66**, 269–282 (1996)
2. Bondy, J.A., Chvátal, V.: A method in graph theory. Discrete Math. **15**, 111–135 (1976)

3. Bondy, J.A., Murty, U.S.R.: Graph Theory with Applications. North-Holland, Amsterdam (1976)
4. Brandt, S., Chen, G., Faudree, R., Gould, R.J., Lesniak, L.: Degree conditions for 2-factors. J. Graph Theory **24**, 165–173 (1997)
5. Corrádi, K., Hajnal, A.: On the maximal number of independent circuits in a graph. Acta Math. Acad. Sci. Hung. **14**, 423–439 (1963)
6. EI-Zahar, M.: On circuits in graphs. Discrete Math. **50**, 227–230 (1984)
7. Johansson, R.: On the bipartite case of EI-Zahar's conjecture. Discrete Math. **219**, 123–134 (2000)
8. Komlós, J., Sáközy, G.N., Szemerédi, E.: Proof of the Alon-Yuster conjecture. Discrete Math. **235**, 255–269 (2001)
9. Matsumura, H.: Vertex-disjoint 4-cycles containing specified edges in a bipartite graph. Discrete Math. **297**, 78–90 (2005)
10. Wang, H.: On 2-factors of a bipartite graph. J. Graph Theory **31**, 101–106 (1999)
11. Yan, J., Liu, G.: On 2-factors with prescribed properties in a bipartite graph. Acta Math. Sin. Engl. Ser. **22**, 1115–1120 (2006)
12. Yan, J., Liu, G.: On 2-factors with cycles containing specified edges in a bipartite graph. Discrete Math. (2008). doi:10.1016/j.disc.2007.11.072