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$[r, s, t]$-Coloring of $K_{n,n}$

Changqing Xu · Xianli Ma · Shouliang Hua

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Abstract Let $G = (V(G), E(G))$ be a simple graph. Given non-negative integers r, s, and t, an $[r, s, t]$-coloring of G is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-chromatic number $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-coloring. We determine $\chi_{r,s,t}(K_{n,n})$ in all cases.

Keywords Coloring · $[r, s, t]$-Coloring · $[r, s, t]$-Chromatic number

Mathematics Subject Classification (2000) 05C15

1 Introduction

Throughout this paper, we consider finite and simple graphs. Vertex coloring, edge coloring, and total coloring are three fundamental colorings of graphs (see [1, 3]). A. Kemnitz and M. Marangio [2] generalized all these classical colorings to $[r, s, t]$-coloring.

Given non-negative integers r, s, and t, an $[r, s, t]$-coloring of a graph G with vertex set $V(G)$ and edge set $E(G)$ is a mapping c from $V(G) \cup E(G)$ to the color set

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C. Xu (✉) · X. Ma
Department of Applied Mathematics, Hebei University of Technology, Tianjin 300401, People's Republic of China
e-mail: chqxu@hebut.edu.cn

S. Hua
Department of Mathematics, Anyang University of Technology, Anyang, Henan 455100, People's Republic of China

$\{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-chromatic number $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-coloring.

In [2], A. Kemnitz and M. Marangio gave some general bounds of $\chi_{r,s,t}(G)$, and presented exact values of $\chi_{r,s,t}(G)$ for some values of r, s, t, and for some graphs.

Let $K_{n,n}$ denote the complete bipartite graph in which each vertex class contains n vertices. In this note, we give the exact value of $\chi_{r,s,t}(K_{n,n})$ for every positive integer n, and for each value of r, s, t.

The minimum number of colors such that a graph G admits a vertex coloring, an edge coloring, or a total coloring, respectively, is the chromatic number $\chi(G)$, the chromatic index $\chi'(G)$, or the total chromatic number $\chi''(G)$. It is well known that $\chi(K_{n,n}) = 2$, $\chi'(K_{n,n}) = n$, $\chi''(K_{n,n}) = n + 2$.

2 The main result

To get $\chi_{r,s,t}(K_{1,1})$, we need the following result in [2].

Lemma 1 [2] *If* $\chi(G) = 2$ *then*

$$\chi_{r,0,t}(G) = \begin{cases} r+1 & \text{if } r \geq 2t, \\ 2t+1 & \text{if } t \leq r < 2t, \\ r+t+1 & \text{if } r < t. \end{cases}$$

Since $K_{1,1}$ has only one edge, so $\chi_{r,s,t}(K_{1,1}) = \chi_{r,0,t}(K_{1,1})$ for any non-negative integer s. Noticing that $\chi(K_{1,1}) = 2$, by Lemma 1, we can get $\chi_{r,s,t}(K_{1,1})$.

Corollary 1

$$\chi_{r,s,t}(K_{1,1}) = \begin{cases} r+1 & \text{if } r \geq 2t, \\ 2t+1 & \text{if } t \leq r < 2t, \\ r+t+1 & \text{if } r < t. \end{cases}$$

Proposition 1 *If* $r \geq s(n-1) + 2t$, *then* $\chi_{r,s,t}(K_{n,n}) = r + 1$.

Proof Since $K_{n,n}$ has at least two adjacent vertices, $\chi_{r,s,t}(K_{n,n}) \geq r + 1$. If $r \geq s(n-1) + 2t$, color the vertices in different classes respectively with 0 and r. Noting that $\chi'(K_{n,n}) = n$, the edge set of $K_{n,n}$ can be divided into n classes, where each edge class is a matching. So we can color the edges in the edge classes respectively with $t, s+t, \ldots$ and $s(n-1)+t$. We obtain an $[r, s, t]$-coloring of $K_{n,n}$. □

Remark 1 As in the proof in Proposition 1, the edge set of $K_{n,n}$ can be divided into n classes, where each edge class is a matching. So we can use $s(n-1)+1$ or more different colors to color the edges of $K_{n,n}$, such that the color difference between each pair of adjacent edges is at least s. For the sake of simplicity, in the following, we will directly say that "color the edges with $t, s+t, \ldots$ and $s(n-1)+t$" and so on.

In the following, suppose that $r < s(n-1) + 2t$. For any $[r, s, t]$-coloring c of $K_{n,n}$ ($n \geq 2$), let the minimum and maximum colors of the vertices be m_v and M_v, the minimum and maximum colors of the edges be m_e and M_e. We say that c is of type (w, x, y, z) if $w < x < y < z$, where $\{w, x, y, z\} = \{m_v, M_v, m_e, M_e\}$. For example, c is of type (m_v, m_e, M_v, M_e) if $m_v < m_e < M_v < M_e$.

Lemma 2 *The minimum k such that $K_{n,n}$ admits an $[r, s, t]$-coloring of type (m_v, M_v, m_e, M_e) (or (m_e, M_e, m_v, M_v)) is $r + t + s(n-1) + 1$.*

Proof Clearly, any such coloring uses at least $r + t + s(n-1) + 1$ colors. We can get an $[r, s, t]$-coloring of $K_{n,n}$ with $r + t + s(n-1) + 1$ colors by coloring the vertices in different classes respectively with 0 and r, and the edges with $r + t, r + t + s, \ldots, r + t + s(n-1)$. Similarly, we can get such a coloring of type (m_e, M_e, m_v, M_v). □

Lemma 3 *The minimum k such that $K_{n,n}$ admits an $[r, s, t]$-coloring of type (m_v, m_e, M_v, M_e) (or type (m_e, m_v, M_e, M_v)) is*

$$k = \begin{cases} s(n-1) + t + 1 & \text{if } s \geq 2t, r < s(n-1), \\ r + t + 1 & \text{if } s \geq 2t, r \geq s(n-1), \\ s(n-2) + 3t + 1 & \text{if } s < 2t, r < s(n-2) + 2t, \\ r + t + 1 & \text{if } s < 2t, r \geq s(n-2) + 2t. \end{cases}$$

Proof If $s \geq 2t$, any such coloring uses at least $\max\{s(n-1) + t + 1, r + t + 1\}$ colors. If $r < s(n-1)$, we can get a coloring of type (m_v, m_e, M_v, M_e) by coloring the edges with $t, s + t, \ldots, s(n-1) + t$, and the vertices in different classes respectively with 0 and $s(n-1)$. That is $k = s(n-1) + t + 1$. And if $r \geq s(n-1)$, we obtain a coloring of type (m_v, m_e, M_v, M_e) by coloring the vertices in different classes respectively with 0 and r, and the edges with $t, s + t, \ldots, s(n-2) + t, r + t$. So in this case, $k = r + t + 1$.

If $s < 2t$, any such coloring uses at least $\max\{s(n-2) + 3t + 1, r + t + 1\}$ colors. If $r < s(n-2) + 2t$, we can get a coloring of type (m_v, m_e, M_v, M_e) by coloring the vertices in different classes respectively with 0 and $s(n-2) + 2t$, and the edges with $t, s + t, \ldots, s(n-2) + t, s(n-2) + 3t$. If $r \geq s(n-2) + 2t$, we can obtain such a coloring by coloring the vertices in different classes respectively with 0 and r, and the edges with $t, s + t, \ldots, s(n-2) + t, r + t$.

Similarly, we can get the result when the coloring is of type (m_e, m_v, M_e, M_v). □

Lemma 4 *Let $n \geq 3$. The minimum k such that $K_{n,n}$ admits an $[r, s, t]$-coloring of type (m_e, m_v, M_v, M_e) is*

$$k = \begin{cases} s(n-1) + 1 & \text{if } s \geq 2t, r < s(n-1) - 2t, \\ r + 2t + 1 & \text{if } s \geq 2t, r \geq s(n-1) - 2t, \\ s(n-2) + 2t + r + 1 & \text{if } s + r < 2t, \\ s(n-3) + 4t + 1 & \text{if } 2t \leq r + s, s < 2t, r < s(n-3) + 2t, \\ r + 2t + 1 & \text{if } 2t \leq r + s, s < 2t, r \geq s(n-3) + 2t. \end{cases}$$

Proof Any such coloring uses at least $s(n-1)+1$ colors. If $s \geq 2t, r < s(n-1)-2t$, there is such a coloring, such that the edges are with colors $0, s, \ldots, s(n-1)$, respectively, and the vertices in different classes are respectively with colors t and $s(n-1)-t$.

If $s \geq 2t, r \geq s(n-1)-2t$, any such coloring uses at least $r+2t+1$ colors. And there exists such a coloring: the vertices in different classes are respectively with colors t and $r+t$, the edges are respectively with colors $0, s, \ldots, s(n-2), r+2t$.

If $s < 2t$, any such coloring uses at least $\min\{s(n-3)+4t+1, s(n-2)+2t+r+1\}$ colors. If $r+s < 2t$, $\min\{s(n-3)+4t+1, s(n-2)+2t+r+1\} = s(n-2)+2t+r+1$. There exists such a coloring: the edges are respectively with colors $0, s, 2s, \ldots, s(n-2), s(n-2)+2t+r$, the vertices in different classes are respectively with colors $s(n-2)+t$ and $s(n-2)+t+r$. If $2t \leq r+s$, $\min\{s(n-3)+4t+1, s(n-2)+2t+r+1\} = s(n-3)+4t+1$.

If $s < 2t$, $2t \leq r+s, r < s(n-3)+2t$, clearly, any such coloring uses at least $s(n-3)+4t+1$ colors. And there exists such a coloring: the vertices in different classes are respectively with colors t and $s(n-3)+3t$, the edges are respectively with colors $0, 2t, s+2t, \ldots, s(n-3)+2t, s(n-3)+4t$.

If $s < 2t$, $2t \leq r+s, r \geq s(n-3)+2t$, any such coloring uses at least $\max\{r+2t+1, s(n-3)+4t+1\} = r+2t+1$ colors. And there exists such a coloring: the vertices in different classes are respectively with colors t and $r+t$, the edges are respectively with colors $0, 2t, s+2t, \ldots, s(n-3)+2t, r+2t$. □

Lemma 5 *The minimum k such that $K_{n,n}$ admits an $[r, s, t]$-coloring of type (m_v, m_e, M_e, M_v) is $s(n-1)+2t+1$.*

Proof Obviously, any such coloring uses at least $s(n-1)+2t+1$ colors. An we can get such a coloring by coloring the vertices in different classes respectively with 0 and $s(n-1)+2t$, and the edges with $t, s+t, \ldots, s(n-1)+t$. □

Remark 2 Let c be an $[r, s, t]$-coloring with k colors of $K_{n,n}$. If $a = b$ for some $a, b \in \{m_v, M_v, m_e, M_e\}$, then it is easy to see that there exists an $[r, s, t]$-coloring c' with k' colors of $K_{n,n}$, such that c' is of type (w, x, y, z) where $\{w, x, y, z\} = \{m_v, M_v, m_e, M_e\}$ and $k' \leq k$.

Theorem 1 *Let $n \geq 3$, $r < s(n-1)+2t$, then*:

(1) *If $s \geq 2t$, then*

$$\chi_{r,s,t}(K_{n,n}) = \begin{cases} s(n-1)+1 & \text{if } r < s(n-1)-2t, \\ r+2t+1 & \text{if } s(n-1)-2t \leq r < s(n-1)-t, \\ s(n-1)+t+1 & \text{if } s(n-1)-t \leq r < s(n-1), \\ r+t+1 & \text{if } s(n-1) \leq r < s(n-1)+t, \\ s(n-1)+2t+1 & \text{if } s(n-1)+t \leq r. \end{cases}$$

(2) *If $r+s < 2t$, then*

$$\chi_{r,s,t}(K_{n,n}) = \begin{cases} r+t+s(n-1)+1 & \text{if } s < t,\ r < t, \\ s(n-1)+2t+1 & \text{if } s < t,\ r \geq t, \\ s(n-2)+2t+r+1 & \text{if } s \geq t,\ r < t. \end{cases}$$

(3) *If* $r + s \geq 2t$, $s < 2t$, *then*

$$\chi_{r,s,t}(K_{n,n}) = \begin{cases} s(n-1) + 2t + 1 & \text{if } s < t, \\ r + t + s(n-1) + 1 & \text{if } t \leq s < 2t,\ r < 3t - 2s, \\ s(n-3) + 4t + 1 & \text{if } t \leq s < 2t,\ 3t - 2s \leq r < s(n-3) + 2t, \\ r + 2t + 1 & \text{if } t \leq s < 2t,\ s(n-3) + 2t \\ & \qquad \leq r < s(n-2) + t, \\ s(n-2) + 3t + 1 & \text{if } t \leq s < 2t,\ s(n-2) + t \\ & \qquad \leq r < s(n-2) + 2t, \\ r + t + 1 & \text{if } t \leq s < 2t,\ s(n-2) + 2t \\ & \qquad \leq r < s(n-1) + t, \\ s(n-1) + 2t + 1 & \text{if } t \leq s < 2t,\ s(n-1) + t \leq r. \end{cases}$$

Proof Lemma 2 through Lemma 5 and Remark 2 give the result after a quick analysis. □

If $n = 2$, Lemma 4 doesn't apply. For $n = 2$, we give the following result.

Lemma 6 *Let* $n = 2$. *The minimum* k *such that* $K_{2,2}$ *admits an* $[r, s, t]$-*coloring of type* (m_e, m_v, M_v, M_e) *is* $s + 1$, *if* $s \geq r + 2t$, $r + 2t + 1$, *if* $s < r + 2t$.

Proof Any such coloring uses at least $\max\{s + 1, r + 2t + 1\}$ colors. And it is easy to see that there is an $[r, s, t]$-coloring with $s + 1$ colors, if $s \geq r + 2t$; and there is an $[r, s, t]$-coloring with $r + 2t + 1$ colors, if $s < r + 2t$. □

We get the following result by combining Lemmas 2, 3, 5, 6 and Remark 2, and hence we have found $\chi_{r,s,t}(K_{n,n})$ for all n.

Proposition 2 *Let* $r < s + 2t$, *then*:

(1) $\chi_{r,s,t}(K_{2,2}) = s + 1$, *if* $s \geq r + 2t$,
(2) $\chi_{r,s,t}(K_{2,2}) = r + s + t + 1$, *if* $s < t, r < t$,
(3) $\chi_{r,s,t}(K_{2,2}) = s + 2t + 1$, *if* $s < t, t \leq r$,
(4) $\chi_{r,s,t}(K_{2,2}) = r + 2t + 1$, *if* $t \leq s < 2t, r < t$,
(5) $\chi_{r,s,t}(K_{2,2}) = 3t + 1$, *if* $t \leq s < 2t, t \leq r < 2t$,
(6) $\chi_{r,s,t}(K_{2,2}) = r + t + 1$, *if* $t \leq s < 2t, 2t \leq r < s + t$,
(7) $\chi_{r,s,t}(K_{2,2}) = s + 2t + 1$, *if* $t \leq s < 2t, s + t \leq r$,
(8) $\chi_{r,s,t}(K_{2,2}) = s + t + 1$, *if* $2t \leq s < r + t, r < s$,
(9) $\chi_{r,s,t}(K_{2,2}) = r + 2t + 1$, *if* $2t \leq r + t \leq s < r + 2t$,
(10) $\chi_{r,s,t}(K_{2,2}) = r + t + 1$, *if* $2t \leq s < r + 2t, s \leq r < s + t$,
(11) $\chi_{r,s,t}(K_{2,2}) = s + 2t + 1$, *if* $2t \leq s < r + 2t, s + t \leq r$.

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References

1. Jensen, T.R., Toft, B.: Graph Coloring Problems. Wiley-Interscience, New York (1995)
2. Kemnitz, A., Marangio, M.: [r, s, t]-Coloring of graphs. Discrete Math. **307**, 199–207 (2007)
3. Yap, H.P.: Total Colorings of Graphs. Lecture Notes in Mathematics, vol. 1623. Springer, Berlin (1996)