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Approximation of derivative in a system of singularly perturbed convection-diffusion equations

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Abstract In this paper, a numerical method for a weakly coupled system of two singularly perturbed convection-diffusion second order ordinary differential equations with a small parameter multiplying the highest derivative is presented. Parameter-uniform error bounds for the numerical solution and also to numerical derivative are established. Numerical results are provided to illustrate the theoretical results.

Keywords System · Singular perturbation problems · Piecewise uniform meshes · Scaled derivative · Scaled discrete derivative

Mathematics Subject Classification (2000) 65L10 · G1.7

1 Introduction

A weakly coupled system of two singularly perturbed linear convection-diffusion two-point boundary value problem for second order ordinary differential equations with a small parameter multiplying the highest derivative is examined. Typically, in the narrow layer region of the domain the solution of the equations has steep gradients

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as the small parameter tends to zero. Robust parameter-uniform numerical methods for a system of singularly perturbed ordinary differential equations have been examined by a few authors [1–4, 6–12, 14–19]. In [2], a parameter-uniform finite difference method for a system of coupled singularly perturbed convection-diffusion equations is presented. It is proved that the scheme converges almost first-order uniformly with respect to the small parameter. In [10], a system of reaction-convection-diffusion type problems are discussed. It summaries the stability and convergence results for (upwind) finite difference discretizations. In [9], a system of coupled convection diffusion equations having diffusion parameters of different magnitudes is discussed. A robust convergence with respect to the perturbation parameters is obtained. In [17], the author present a computational method for a weakly coupled system of singularly perturbed reaction-diffusion equation with discontinuous source term.

While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite-difference approximation of their derivatives, even though such approximations are desirable in certain applications (flux or drag). It should be noted that for convection-diffusion problems, the attainment of high accuracy in a computed solution does not automatically lead to good approximation of derivatives of the true solution. In [5], for singularly perturbed convection-diffusion problems with continuous convection coefficient and source term for single differential equation estimates for numerical derivatives have been derived. Here the scaled derivative is taken on whole domain where as Natalia Kopteva and Martin Stynes [8] have obtained approximation of derivatives with scaling in the boundary layer region and without scaling in the outer region. It may be noted that the source term and convection coefficient are smooth for the problem considered in [8]. In [6], the authors have obtained bounds on the errors in approximations to the scaled derivative in the whole domain in the case of discontinuous source term. R. Mythili Priyadharshini and N. Ramanujam [15], have determined estimate for the scaled derivative in the boundary layer region and non-scaled derivative in the outer region for the boundary value problems with Robin type boundary conditions and discontinuous convection coefficient and source term. In [14], the authors have estimated the scaled derivative for a singularly perturbed second -order ordinary differential equation with discontinuous convection coefficient using hybrid difference scheme.

The present paper extends the results available in [5, pp. 51, 66] for a single one-dimensional singularly perturbed convection-diffusion equation to a weakly coupled system of two singularly perturbed convection-diffusion equations. In this paper, we obtain parameter-uniform approximations not only to the solution but also to its derivatives. Thus in this paper, motivated by the works of [1] and [8], bounds on the errors in approximating the first derivative of the solution with a weight in the fine mesh and without a weight in the coarse mesh are obtained.

Note: Through out this paper, C denotes a generic constant (sometimes subscripted) is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Let $y : D \longrightarrow \Re$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the maximum norm $\|y\|_D = \sup_{x\in D} |y(x)|$. In case of vectors $\bar{y} = (y_1, y_2)^T$, we define $|\bar{y}(x)| = (|y_1(x)|, |y_2(x)|)^T$ and $\|\bar{y}\|_D = \max\{\|y_1\|_D, \|y_2\|_D\}$.

2 Continuous problem

2.1 Statement of the problem

Find $u_1, u_2 \in Y \equiv C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$
\begin{aligned}
&L_1\bar{u}(x) \equiv -\varepsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x),\\
&\quad x \in \Omega,\\
&L_2\bar{u}(x) \equiv -\varepsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \quad (1)\\
&\quad x \in \Omega,\\
&u_1(0) = A_1, \qquad u_2(0) = A_2, \qquad u_1(1) = B_1, \qquad u_2(1) = B_2,
\end{aligned}
$$

$$b_{12}(x) \le 0, \qquad b_{21}(x) \le 0, \tag{2}$$

$$b_{11}(x) + b_{12}(x) \ge 0, \qquad b_{22}(x) + b_{21}(x) \ge 0, \tag{3}$$

where the functions $a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x), f_1(x)$ and $f_2(x)$ are sufficiently smooth on $\overline{\Omega}$, $\Omega = (0,1)$, $0 < \varepsilon \le 1$, $a_1(x) \ge \alpha_1 > 0$ and $a_2(x) \ge \alpha_2 > 0$. Let $\alpha = \min\{\alpha_1, \alpha_2\}$. The above system can be written in the vector form as

$$
\begin{aligned}
\mathbf{L}\bar{u}(x) &\equiv \begin{pmatrix} L_1\bar{u}(x) \\ L_2\bar{u}(x) \end{pmatrix}\\
&\equiv \begin{pmatrix} -\varepsilon\frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon\frac{d^2}{dx^2} \end{pmatrix} \bar{u}(x) + \begin{pmatrix} a_1(x)\frac{d}{dx} & 0 \\ 0 & a_2(x)\frac{d}{dx} \end{pmatrix} \bar{u}(x)\\
&\quad + \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix} \bar{u}(x) = \bar{f}(x), \quad x \in \Omega,\\
\bar{u}(0) &= (A_1, A_2)^T, \qquad \bar{u}(1) = (B_1, B_2)^T,
\end{aligned}
$$

where $\bar{u}(x) = (u_1(x), u_2(x))^T$, $\bar{f}(x) = (f_1(x), f_2(x))^T$.

2.2 Analytical results

In the following, the maximum principle for (1) is given. Then using this principle, a stability result is stated.

Theorem 1 *Suppose that a function $\bar{u}(x) = (u_1(x), u_2(x))^T$, $u_1(x), u_2(x) \in Y$ satisfies $\bar{u}(0) \ge \bar{0}$, $\bar{u}(1) \ge \bar{0}$ and $\mathrm{L}\bar{u}(x) \ge \bar{0}$, for all $x \in \Omega$. Then $\bar{u}(x) \ge \bar{0}$, for all $x \in \overline{\Omega}$.*

Proof Define $\bar{s}(x) = (s_1(x), s_2(x))^T$ as

$$s_1(x) = s_2(x) = 1 + x.$$

Then $s_1, s_2 \in Y$, $\bar{s}(x) > \bar{0}$, for all $x \in \overline{\Omega}$ and $\mathbf{L}\bar{s}(x) > \bar{0}$, $x \in \Omega$. So, we further define

$$\mu = \max\left\{\max_{x\in\overline{\Omega}}\left(\frac{-u_1}{s_1}\right)(x), \max_{x\in\overline{\Omega}}\left(\frac{-u_2}{s_2}\right)(x)\right\}.$$

Assume that the theorem is not true. Then $\mu > 0$ and there exists a point $x_0 \in \Omega$, such that either $(\frac{-u_1}{s_1})(x_0) = \mu$ or $(\frac{-u_2}{s_2})(x_0) = \mu$ or both. Also $(\bar{u} + \mu\bar{s})(x) \geq \bar{0}$, $\forall x \in \overline{\Omega}$.

Case (i): $(\frac{-u_1}{s_1})(x_0) = \mu$. That is, $(u_1 + \mu s_1)(x_0) = 0$. Therefore $(u_1 + \mu s_1)$ attains its minimum at $x = x_0$. Then,

$$\begin{aligned} 0 <& L_1(\bar{u} + \mu\bar{s})(x_0) \\ =& -\varepsilon(u_1 + \mu s_1)''(x_0) + a_1(x_0)(u_1 + \mu s_1)'(x_0) \\ &+ b_{11}(x_0)(u_1 + \mu s_1)(x_0) + b_{12}(x_0)(u_2 + \mu s_2)(x_0) \leq 0, \end{aligned}$$

which is a contradiction.

Case (ii): $(\frac{-u_2}{s_2})(x_0) = \mu$. Similar to Case (i), it leads to a contradiction.

Hence $\bar{u}(x) \geq \bar{0}$, $\forall x \in \overline{\Omega}$. □

In the rest of this $\|.\|$ means $\|.\|_{\overline{\Omega}}$.

Lemma 1 *If $u_1, u_2 \in Y$ then for $j = 1, 2$*

$$|u_j(x)| \leq C\{\max\{|u_1(0)|, |u_2(0)|\}, \max\{|u_1(1)|, |u_2(1)|\}, \|L_1\bar{u}\|_\Omega, \|L_2\bar{u}\|_\Omega\}, x \in \bar{\Omega}.$$

Proof Using appropriate barrier functions and applying Theorem 1, the present lemma can be proved. □

Lemma 2 *Let $\bar{u} = (u_1, u_2)^T$ be the solution of (1). Then, for $j = 1, 2$*

$$\begin{aligned} \|u_j^{(k)}\| &\leq C\varepsilon^{-k}\max\{\|f_j\|, \|\bar{u}\|\}, \quad \text{for } k = 1, 2 \\ \|u_j^{(3)}\| &\leq C\varepsilon^{-3}\max\{\|f_j\|, \|f_j'\|, \|\bar{u}\|\}, \end{aligned}$$

where C depends on $\|a_1\|$, $\|b_{11}\|$, $\|b_{12}\|$, $\|a_1'\|$, $\|a_2\|$, $\|b_{21}\|$, $\|b_{22}\|$ and $\|a_2'\|$.

Proof Using the technique adopted in [5, pp. 44, 45] and [12, p. 4], the present theorem can be proved. □

The sharper bounds on the derivatives of the solution are obtained by decomposing the solution $\bar{u}$ into regular and singular components as, $\bar{u} = \bar{v} + \bar{w}$, where $\bar{v} = (v_1, v_2)^T$ and $\bar{w} = (w_1, w_2)^T$. The regular component $\bar{v}$ can be written in the form $\bar{v} = \bar{v}_0 + \varepsilon\bar{v}_1 + \varepsilon^2\bar{v}_2$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$, $\bar{v}_2 = (v_{21}, v_{22})^T$ are defined respectively to be the solutions of the problems

$$\begin{pmatrix} a_1\frac{d}{dx} & 0 \\ 0 & a_2\frac{d}{dx} \end{pmatrix}\bar{v}_0 + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\bar{v}_0 = \bar{f}, \qquad \bar{v}_0(0) = \bar{u}(0),$$

$$\begin{pmatrix} a_1\frac{d}{dx} & 0 \\ 0 & a_2\frac{d}{dx} \end{pmatrix}\bar{v}_1 + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\bar{v}_1 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}\bar{v}_0, \qquad \bar{v}_1(0) = \bar{0} \quad \text{and}$$

$$\mathbf{L}\bar{v}_2 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}\bar{v}_1, \qquad \bar{v}_2(0) = \bar{0}, \qquad \bar{v}_2(1) = \bar{0}.$$

Thus the regular component $\bar{v}$ is the solution of

$$\mathbf{L}\bar{v} = \bar{f}, \qquad \bar{v}(0) = \bar{u}(0), \qquad \bar{v}(1) = \bar{v}_0(1) + \varepsilon\bar{v}_1(1) + \varepsilon^2\bar{v}_2(1).$$

Then the singular component $\bar{w}$ is the solution of $\mathbf{L}\bar{w} = \bar{0}$, $\bar{w}(0) = \bar{0}$, $\bar{w}(1) = \bar{u}(1) - \bar{v}(1)$. The following lemma provides the bound on the derivatives of the regular and singular components of the solution $\bar{u}$.

Lemma 3 *The solution $\bar{u}$ can be decomposed into the sum $\bar{u} = \bar{v} + \bar{w}$, where, $\bar{v}$ and $\bar{w}$ are regular and singular components respectively. Further, these components and their derivatives satisfy the bounds*

$$\|v_j^{(k)}\| \le C(1 + \varepsilon^{2-k}), \quad k = 0, 1, 2, 3, \ j = 1, 2 \quad \textit{and}$$
$$|w_j^{(k)}(x)| \le C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon}, \quad k = 0, 1, 2, 3, \ \forall x \in \overline{\Omega}, \ j = 1, 2.$$

Proof Using appropriate barrier functions, applying Theorem 1 and adopting the method of proof used in [5, p. 46] and [12, p. 6], the present lemma can be proved. □

3 Discrete problem

3.1 Statement of the problem

The system (1) is now discretised using a fitted mesh method composed of a standard finite difference operator on a fitted piecewise uniform mesh

$$\overline{\Omega}^N = \{x_i \mid x_i = 2i(1-\sigma)/N, 0 \le i \le N/2; \ x_i = x_{i-1} + 2\sigma/N, N/2 < i \le N\}$$

condensing at the boundary point $x_N = 1$, $h = \frac{2(1-\sigma)}{N}$ and $H = \frac{2\sigma}{N}$. The transition parameter σ is chosen to satisfy $\sigma = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\}$. The resulting fitted mesh finite difference method is to find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for $i = 0, 1, 2, \ldots, N$ such that for $x_i \in \Omega^N$,

$$\begin{aligned}
L_1^N\bar{U}(x_i) \equiv &- \varepsilon\delta^2 U_1(x_i) + a_1(x_i)D^- U_1(x_i)\\
&+ b_{11}(x_i)U_1(x_i) + b_{12}(x_i)U_2(x_i) = f_1(x_i),\\
L_2^N\bar{U}(x_i) \equiv &- \varepsilon\delta^2 U_2(x_i) + a_2(x_i)D^- U_2(x_i)\\
&+ b_{21}(x_i)U_1(x_i) + b_{22}(x_i)U_2(x_i) = f_2(x_i),\\
U_1(0) = u_1(0), \qquad &U_2(0) = u_2(0), \qquad U_1(1) = u_1(1), \qquad U_2(1) = u_2(1).
\end{aligned} \tag{4}$$

The finite difference operator δ^2 is the central difference operator defined as $\delta^2 U_j(x_i) = \frac{(D^+ - D^-)U_j(x_i)}{(x_{i+1} - x_{i-1})/2}$, for $j = 1, 2$, where, $D^+U_j(x_i) = \frac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i}$ and

$D^- U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}}$. The difference operator $\mathbf{L}^N$ can be defined as

$$\mathbf{L}^N \bar{U}(x_i) \equiv \begin{pmatrix} L_1^N \bar{U}(x_i) \\ L_2^N \bar{U}(x_i) \end{pmatrix}$$
$$\equiv \begin{pmatrix} -\varepsilon\delta^2 & 0 \\ 0 & -\varepsilon\delta^2 \end{pmatrix} \bar{U}(x_i) + \begin{pmatrix} a_1(x_i)D^- & 0 \\ 0 & a_2(x_i)D^- \end{pmatrix} \bar{U}(x_i)$$
$$+ \begin{pmatrix} b_{11}(x_i) & b_{12}(x_i) \\ b_{21}(x_i) & b_{22}(x_i) \end{pmatrix} \bar{U}(x_i) = \bar{f}(x_i).$$

3.2 Numerical solution estimates

Analogous to the continuous results stated in Theorem 1 and Lemma 1 one can prove the following results.

Theorem 2 *For any mesh function $\bar{\Psi}(x_i)$ assume that $\bar{\Psi}(x_0) \geq \bar{0}$, $\bar{\Psi}(x_N) \geq \bar{0}$ and $\boldsymbol{L}^N \bar{\Psi}(x_i) \geq \bar{0}$, for all $i = 1, \ldots, N-1$. Then $\bar{\Psi}(x_i) \geq \bar{0}$, for all $i = 0, \ldots, N$.*

Proof Define $\bar{S}(x_i) = (S_1(x_i), S_2(x_i))^T$ as $S_1(x_i) = S_2(x_i) = 1 + x_i$. Then $\bar{S}(x_i) > \bar{0}$ for all $x_i \in \overline{\Omega}^N$. So, we further define

$$\xi = \max\left\{ \max_{x_i \in \overline{\Omega}^N} \left(\frac{-\Psi_1}{S_1}\right)(x_i),\ \max_{x_i \in \overline{\Omega}^N} \left(\frac{-\Psi_2}{S_2}\right)(x_i) \right\}$$

Assume that the theorem is not true, then $\xi > 0$ and we have $(\bar{\Psi} + \xi\bar{S})(x_i) \geq 0$, for $x_i \in \overline{\Omega}^N$. For some $i = k$, we may have either $(\Psi_1 + \xi S_1)(x_k) = 0$ or $(\Psi_2 + \xi S_2)(x_k) = 0$ or both.

Case (i): $(\Psi_1 + \xi S_1)(x_k) = 0$. Then

$$\begin{aligned} 0 <& L_1^N (\bar{\Psi} + \xi\bar{S})(x_k) \\ &= -\varepsilon\delta^2(\Psi_1 + \xi S_1)(x_k) + a_1(x_k)D^-(\Psi_1 + \xi S_1)(x_k) \\ &+ b_{11}(x_k)(\Psi_1 + \xi S_1)(x_k) + b_{12}(x_k)(\Psi_2 + \xi S_2)(x_k) \leq 0, \end{aligned}$$

which is a contradiction.
Case (ii): $(\Psi_2 + \xi S_2)(x_k) = 0$. Similar to Case (i) it leads to a contradiction.
Hence $\bar{\Psi}(x_i) \geq \bar{0}$, $\forall x_i \in \overline{\Omega}^N$. □

Lemma 4 *Consider the scheme (4). If $\bar{Z}(x_i) = (Z_1(x_i), Z_2(x_i))^T$ is any mesh function then, for all $x_i \in \overline{\Omega}^N$,*

$$|Z_j(x_i)| \leq C \max\Big\{ \max\{|Z_1(x_0)|,\ |Z_1(x_N)|\}, \max\{|Z_2(x_0)|,\ |Z_2(x_N)|\},$$
$$\max_{1\leq i\leq N-1} |L_1^N \bar{Z}(x_i)|,\ \max_{1\leq i\leq N-1} |L_2^N \bar{Z}(x_i)| \Big\}, \quad j = 1, 2.$$

Proof Let

$$C_1 = C \max\Big\{ \max(|Z_1(0)|, |Z_1(1)|), \max(|Z_2(0)|, |Z_2(1)|), \max_{1\le i\le N-1} |L_1^N \bar{Z}(x_i)|, \max_{1\le i\le N-1} |L_2^N \bar{Z}(x_i)| \Big\}.$$

Define the mesh functions

$$\bar{W}^{\pm}(x_i) = \begin{pmatrix} C_1(1+x_i) \\ C_1(1+x_i) \end{pmatrix} \pm \bar{Z}(x_i).$$

Then we have $\bar{W}^{\pm}(x_0) \ge \bar{0}$, $\bar{W}^{\pm}(x_N) \ge \bar{0}$ and $L_\varepsilon^N \bar{W}^{\pm}(x_i) \ge \bar{0}$. By Theorem 2 we get the required result. □

The discrete solution $\bar{U}(x_i)$ can be decomposed into the sum $\bar{U}(x_i) = \bar{V}(x_i) + \bar{W}(x_i)$ where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ are regular and singular components respectively defined as

$$\mathbf{L}^N \bar{V}(x_i) = \bar{f}(x_i), \quad i = 1, \dots, N-1, \qquad \bar{V}(0) = \bar{v}(0), \qquad \bar{V}(1) = \bar{v}(1) \quad \text{and}$$
$$\mathbf{L}^N \bar{W}(x_i) = \bar{0}, \quad i = 1, \dots, N-1, \qquad \bar{W}(0) = \bar{0}, \qquad \bar{W}(1) = \bar{w}(1).$$

The error in the numerical solution can be written in the form $(\bar{U} - \bar{u})(x_i) = (\bar{V} - \bar{v})(x_i) + (\bar{W} - \bar{w})(x_i)$.

Lemma 5 *At each mesh point $x_i \in \overline{\Omega}^N$, the error of the regular component satisfies the estimate*

$$|(\bar{V} - \bar{v})(x_i)| \le \begin{pmatrix} CN^{-1}x_i \\ CN^{-1}x_i \end{pmatrix}.$$

Proof Using the barrier functions $\bar{\Psi}^{\pm}(x_i) = \binom{CN^{-1}x_i}{CN^{-1}x_i} \pm (\bar{V} - \bar{v})(x_i)$ and adopting the method of proof used in [5, Lemma 3.4], the present lemma can be proved. □

Lemma 6 *At each mesh point $x_i \in \Omega^N$, the error of the singular component satisfies the estimate*

$$|(\bar{W} - \bar{w})(x_i)| \le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

Proof We consider first the case $\sigma = \frac{1}{2}$ and so $\varepsilon^{-1} \le C \ln N$ and $h = N^{-1}$. By classical argument and using Lemma 3, we obtain $|L_1^N(\bar{W} - \bar{w})(x_i)| \le C\varepsilon^{-2}N^{-1}e^{-\alpha(1-x_i)/\varepsilon}$. We introduce the mesh functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)} \varepsilon^{-1} N^{-1} Y_1(x_i) \\ \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)} \varepsilon^{-1} N^{-1} Y_2(x_i) \end{pmatrix} \pm (\bar{W} - \bar{w})(x_i),$$

where γ is any constant satisfying $0 < \gamma < \alpha$ and

$$Y_1(x_i) = \frac{\lambda^i - 1}{\lambda^N - 1}, \quad \text{where } \lambda = 1 + \frac{\gamma h}{\varepsilon}.$$

It is easy to see that $(-\varepsilon\delta^2 + \gamma D^-)Y_1(x_i) = 0$, for $i = 1, \ldots, N-1$, $D^- Y_1(x_i) \geq \frac{\gamma}{\varepsilon} e^{-\gamma(1-x_i)/\varepsilon}$, for $1 \leq i \leq N$ and so $Y_1(x_i)$ increases monotonically with $0 \leq Y_1(x_i) \leq 1$. Further let $Y_2(x_i) \equiv Y_1(x_i)$. It is easy to show that $\bar{\Psi}^{\pm}(x_0) = \bar{0}$, $\bar{\Psi}^{\pm}(x_N) > \bar{0}$ and $\mathbf{L}^N \bar{\Psi}^{\pm}(x_i) > 0$, for $1 \leq i \leq N-1$. Hence by Theorem 2, we get $\bar{\Psi}^{\pm}(x_i) \geq \bar{0}$, which leads to the required result.

Now consider the case $\sigma = \frac{2\varepsilon}{\alpha} \ln N$. Suppose that $x_i \in [0, 1-\sigma]$. Using the triangle inequality we have $|(\bar{W} - \bar{w})(x_i)| \leq |\bar{W}(x_i)| + |\bar{w}(x_i)|$. Using Lemma 3, we have $|\bar{w}(x_i)| \leq \binom{CN^{-2}}{CN^{-2}}$. To obtain a similar bound for $|\bar{W}(x_i)|$, consider the mesh functions $Y_1(x_i)$, which is the solution of the constant coefficient problem

$$-\varepsilon\delta^2 Y_1(x_i) + \alpha D^- Y_1(x_i) = 0, \quad i = 1, \ldots, N-1,$$
$$Y_1(x_0) = e^{-\gamma/\varepsilon}, \qquad Y_1(x_N) = 1.$$

Further let $Y_2(x_i) \equiv Y_1(x_i)$. Let $\beta = \max\{|W_1(1)|, |W_2(1)|\}$ and introduce the mesh functions $\bar{\Psi}^{\pm}(x_i) = \binom{\beta Y_1(x_i)}{\beta Y_2(x_i)} \pm \bar{W}(x_i)$. It is easy to show that $\bar{\Psi}^{\pm}(x_0) > \bar{0}$, $\bar{\Psi}^{\pm}(x_N) \geq \bar{0}$ and $\mathbf{L}^N \bar{\Psi}^{\pm}(x_i) \geq \bar{0}$, for $1 \leq i \leq N-1$. Using the procedure adopted in [13, Chap. 7, Lemmas 1–3] and hence by Theorem 2, we get

$$|\bar{W}(x_i)| \leq \begin{pmatrix} \beta Y_1(x_i) \\ \beta Y_2(x_i) \end{pmatrix} \leq \begin{pmatrix} CN^{-2} \\ CN^{-2} \end{pmatrix}, \quad \text{for } x_i \in [0, 1-\sigma]. \tag{5}$$

Now for $x_i \in (1-\sigma, 1]$, the proof follows the same lines as for the case $\sigma = 1/2$, except that we use the discrete maximum principle on $[1-\sigma, 1]$ and the already established bound $|\bar{W}(x_{N/2})| \leq \binom{CN^{-2}}{CN^{-2}}$. For all i, $N/2 \leq i \leq N$, we introduce the mesh functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} \frac{Ce^{2\gamma H/\varepsilon}}{\gamma(\alpha-\gamma)} \sigma\varepsilon^{-1} N^{-1} Y_1(x_i) + C_1 N^{-2} \\ \frac{Ce^{2\gamma H/\varepsilon}}{\gamma(\alpha-\gamma)} \sigma\varepsilon^{-1} N^{-1} Y_2(x_i) + C_1 N^{-2} \end{pmatrix} \pm (\bar{W} - \bar{w})(x_i),$$

where γ is defined as before and for all i, $N/2 \leq i \leq N$, $Y_1(x_i) = \frac{\lambda^{i-N/2} - 1}{\lambda^{N/2} - 1}$, where, $\lambda = 1 + \frac{\gamma H}{\varepsilon}$, $D^- Y_1(x_i) \geq \frac{\gamma}{\varepsilon} e^{-(1-x_i)/\varepsilon}$ and $0 \leq Y_1(x_i) \leq 1$. Let $Y_2(x_i) \equiv Y_1(x_i)$. It is easy to show that $\bar{\Psi}^{\pm}(x_{N/2}) \geq 0$, $\bar{\Psi}^{\pm}(x_N) > 0$ and $\mathbf{L}^N \bar{\Psi}^{\pm}(x_i) \geq \bar{0}$ for $N/2 - 1 \leq i \leq N-1$. Then by Theorem 2 we get $\bar{\Psi}^{\pm}(x_i) \geq \bar{0}$. Thus we get the required result. □

Theorem 3 *Let $\bar{u}(x) = (u_1(x), u_2(x))^T$, for $x \in \bar{\Omega}$ be the solution of (1) and let $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$, for $x_i \in \bar{\Omega}^N$ be the numerical solution of problem (4). Then we have*

$$\sup_{0<\varepsilon\leq 1} \|U_1 - u_1\|_{\Omega^N} \leq CN^{-1} \ln N \quad \textit{and} \quad \sup_{0<\varepsilon\leq 1} \|U_2 - u_2\|_{\Omega^N} \leq CN^{-1} \ln N.$$

Proof Proof follows immediately, if one applies the above Lemmas 5 and 6 to $\bar{U} - \bar{u} = \bar{V} - \bar{v} + \bar{W} - \bar{w}$. □

4 Analysis on derivative approximation

In this section, we give the ε-uniform error estimate between the scaled derivative of the continuous solution and the corresponding numerical solution in the fine mesh region. Further, in the coarse mesh, an estimate is obtained without scaling the derivative.

We note that the errors

$$e_k(x_i) \equiv U_k(x_i) - u_k(x_i),$$

satisfy the equations for $k = 1, 2$

$$[-\varepsilon\delta^2 + a_k(x_i)D^-]e_k(x_i) = -[b_{k1}(x_i)e_1(x_i) + b_{k2}(x_i)e_2(x_i)] + \text{truncation error},$$

where, by Theorem 3, $[b_{k1}(x_i)e_1(x_i) + b_{k2}(x_i)e_2(x_i)] = O(N^{-1}\ln N)$. In the proofs of the following lemmas and theorems, we use the above equations, they are necessary single equation. Hence the analysis carried out in [5, §3.5] for single equation can applied immediately with a slight modifications where ever necessary. Therefore, proofs for some lemmas are omitted; for some of the theorems short proves are given.

4.1 Numerical derivative estimates

Lemma 7 *At each mesh point $x_i \in \Omega^N \cup \{1\}$ and all $x \in \bar{\Omega}_i = [x_{i-1}, x_i]$, we have*

$$|D^- u_j(x_i) - u_j'(x)| \le CN^{-1}, \quad \text{for } x_i \le 1 - \sigma, \ j = 1, 2 \quad \text{and}$$

$$|\varepsilon(D^- u_j(x_i) - u_j'(x))| \le CN^{-1}\ln N, \quad \text{for } x_i > 1 - \sigma, \ j = 1, 2,$$

where $(u_1(x), u_2(x))^T$ is the solution of (1).

Lemma 8 *Let $\bar{v}$ and $\bar{V}$ be the exact and discrete regular components of the solutions of 1 and 4 respectively. Then for all $x_i \in \Omega^N \cup \{1\}$, we have*

$$\max_{0<i\le N/2} |D^-(V_j - v_j)(x_i)| \le CN^{-1}, \quad \text{for } j = 1, 2 \quad \text{and}$$

$$\max_{N/2<i\le N} |\varepsilon D^-(V_j - v_j)(x_i)| \le CN^{-1}, \quad \text{for } j = 1, 2.$$

Proof We denote the error and the local truncation error, respectively at each mesh point by $e_j(x_i) = V_j(x_i) - v_j(x_i)$ and $\tau_j(x_i) = L_j^N e_j(x_i)$, for $j = 1, 2$. Since $e_j(x_N) = 0$, we have

$$|\varepsilon D^- e_j(x_N)| = \left| \frac{\varepsilon(e_j(x_N) - e_j(x_{N-1}))}{x_N - x_{N-1}} \right| \le C\varepsilon N^{-1}. \tag{6}$$

Now we write $\tau_j(x_i) = L_j^N e_j(x_i)$ in the form,

$$\begin{aligned}\varepsilon D^- e_j(x_k) - \varepsilon D^- e_j(x_{k+1}) + \frac{1}{2}(x_{k+1} - x_{k-1}) a_j(x_k) D^- e_j(x_k) \\ = \frac{1}{2}(x_{k+1} - x_{k-1})(\tau_j(x_k) - [b_{j1}(x_k)e_1(x_k) + b_{j2}(x_k)e_2(x_k)]).\end{aligned} \tag{7}$$

Summing and rearranging for each $i,\ N/2 < i < N$, we get

$$\begin{aligned}|\varepsilon D^- e_j(x_i)| \le |\varepsilon D^- e_j(x_N)| + \frac{1}{2}\sum_{k=i}^{N-1}(x_{k+1} - x_{k-1})(|\tau_j(x_k)| + |[b_{j1}(x_k)e_1(x_k) \\ + b_{j2}(x_k)e_2(x_k)]|) + \left|\frac{1}{2}\sum_{k=i}^{N-1}(x_{k+1} - x_{k-1}) a_j(x_k) D^- e_j(x_k)\right|.\end{aligned}$$

Using the telescoping effect for the last term, $|e_j(x_i)| \le CN^{-1}$ and $\|a_j'\| \le C$, we get for all $i,\ N/2 < i \le N$,

$$|\varepsilon D^- e_j(x_i)| \le CN^{-1}.$$

For $i \le N/2$, we rewrite (7) in the form

$$\begin{aligned}(1 + \rho_k) D^- e_j(x_k) = D^- e_j(x_{k+1}) \\ + \frac{\rho_k}{a_j(x_k)}\big(\tau_j(x_k) - b_{j1}(x_k)e_1(x_k) + b_{j2}(x_k)e_2(x_k)\big),\end{aligned} \tag{8}$$

where $\rho_k = \frac{a_j(x_k)(x_{k+1} - x_{k-1})}{\varepsilon}$. Summing the equations in (8) from $k = 1$ to $k = i = N/2$, we have

$$|D^- e_j(x_i)| = |D^- e_j(x_1)| \frac{(1 + \rho_i)^{-(i - \frac{N}{2} - 1)}}{1 + \rho_i} + CN^{-1} \le CN^{-1},$$

which completes the proof. □

Lemma 9 *For $\sigma = 1/2$, we have for all $x_i \in \overline{\Omega}^N$,*

$$|\varepsilon(W_j - w_j)(x_i)| \le CN^{-1} x_i \ln N, \quad \textit{for } j = 1, 2.$$

Proof Use the barrier functions $\bar{\Psi}^{\pm}(x_i) = \binom{\frac{C}{\alpha}\varepsilon^{-2}N^{-1}x_i}{\frac{C}{\alpha}\varepsilon^{-2}N^{-1}x_i} \pm (\bar{W} - \bar{w})(x_i)$ to establish the required bound. □

Lemma 10 *Let $\bar{w}$ and $\bar{W}$ be the exact and discrete singular components of the solutions of* (1) *and* (4) *respectively. Then for all $x_i \in \Omega^N \cup \{1\}$, we have*

$$\max_{0 < i \le N/2} |D^-(W_j - w_j)(x_i)| \le CN^{-1}, \quad \textit{for } j = 1, 2 \quad \textit{and}$$

$$\max_{N/2<i\le N} |\varepsilon D^-(W_j - w_j)(x_i)| \le CN^{-1}\ln N, \quad \textit{for } j = 1, 2.$$

Proof Consider the case $x_i \le 1-\sigma$. Using the procedure given in [1, Lemma 8], we have for $j = 1, 2$, $|W_j(x_i)| \le CN^{-2}$ and $|w_j(x_i)| \le CN^{-2}$, for all $x_i \le 1-\sigma$. Hence for $j = 1, 2$

$$|D^-(W_j - w_j)(x_i)| \le CN^{-1}, \quad x_i \in [0, 1-\sigma].$$

For $x_i = 1-\sigma$, we write $L_1^N \bar{W}(1-\sigma) = 0$ in the form

$$\begin{aligned}\varepsilon D^- W_1(x_{N/2+1}) =& \varepsilon D^- W_1(1-\sigma) + \frac{h+H}{2} a_1(1-\sigma) D^- W_1(1-\sigma)\\ &+ \frac{1}{2} b_{11}(1-\sigma)(h+H) W_1(1-\sigma)\\ &+ \frac{1}{2} b_{12}(1-\sigma)(h+H) W_2(1-\sigma).\end{aligned}$$

Using the estimate obtained at the point $(1-\sigma)$ and from the proof of Lemma 6, we obtain $|\varepsilon D^- W_1(x_{N/2+1})| \le CN^{-1}$. Now consider $x_i \in (1-\sigma, 1]$. For convenience we introduce the notation $\hat{e}_j(x_i) = (\hat{W}_j - \hat{w}_j)(x_i)$ and $\hat{\tau}_j(x_i) = L_j^N \hat{e}(x_i)$, $j = 1, 2$. We have already established that

$$|\hat{e}_j(x_i)| \le CN^{-1}\ln N \quad \text{and} \quad |\hat{\tau}_j(x_i)| \le C\sigma\varepsilon^{-2} N^{-1} e^{-\alpha(1-x_i)/\varepsilon}, \; j = 1, 2. \tag{9}$$

We write the equation $\hat{\tau}_j(x_i) = L_j^N \hat{e}(x_i)$ in the form

$$\begin{aligned}&\varepsilon D^-(\hat{e}_j(x_k) - \hat{e}_j(x_{k+1})) + \frac{1}{2} a_j(x_k)(x_{k+1} - x_{k-1}) D^- \hat{e}_j(x_k)\\ &\quad = \frac{1}{2}(x_{k+1} - x_{k-1})\hat{\tau}_j(x_k) - [b_{j1}(x_k)\hat{e}_1(x_k) + b_{j2}(x_k)\hat{e}_2(x_k)].\end{aligned}$$

Summing for $k = \frac{N}{2}$ to $i-1$ and rearranging we get

$$\begin{aligned}\varepsilon D^- \hat{e}_j(x_i) &= \varepsilon D^- \hat{e}_j(x_{N/2}) + \sum_{k=N/2}^{i-1} a_j(x_k)(\hat{e}_j(x_k) - \hat{e}_j(x_{k-1})) - \sum_{k=N/2}^{i-1} H\hat{\tau}_j(x_k)\\ &\quad + \sum_{k=N/2}^{i-1} [b_{j1}(x_k)\hat{e}_1(x_k) + b_{j2}(x_k)\hat{e}_2(x_k)], \; j = 1, 2.\end{aligned}$$

Hence using the result at the point $x_{N/2}$ and (9), we have $|\varepsilon D^- \hat{e}_j(x_i)| \le CN^{-1}(\ln N + \frac{\sigma}{\varepsilon}\frac{\alpha H/\varepsilon}{1-e^{-\alpha h/\varepsilon}})$. Hence $|\varepsilon D^- \hat{e}_j(x_i)| \le CN^{-1}\ln N$, $j = 1, 2$ as required.

When $\sigma = 1/2$, using the above arguments and Lemma 9, we get

$$|\varepsilon D^- \hat{e}_j(x_i)| \le CN^{-1}\ln N, \quad j = 1, 2.$$

which is the required result. □

Theorem 4 *Let $\bar{u}$ be the solution of* (1) *and $\bar{U}$ be the numerical solution of* (4). *Then, we have for $j = 1, 2$,*

$$\sup_{0<\varepsilon\leq 1} \|D^- U_j(x_i) - u'_j\|_{\overline{\Omega}_i} \leq CN^{-1}, \quad \text{for } 1 \leq i \leq N/2 \quad \text{and}$$

$$\sup_{0<\varepsilon\leq 1} \|\varepsilon(D^- U_j(x_i) - u'_j)\|_{\overline{\Omega}_i} \leq CN^{-1} \ln N, \quad \text{for } N/2+1 \leq i \leq N.$$

Proof Following the method of proof adopted in [5, Theorem 3.17], using the Lemmas 8 and 10 we get the required result. □

Remark 1 Let $\widetilde{U}_j$, $j = 1, 2$, denote the piecewise linear interpolant of the finite difference solution $\{U_j(x_i)\}_{i=0}^N$. As done in [5, p. 66], we get for $j = 1, 2$

$$\sup_{0<\varepsilon\leq 1} \|\widetilde{D}^- U_j - u'_j\|_{\overline{\Omega}_i} \leq CN^{-1}, \quad i = 1, \ldots, N/2, \quad \text{and}$$

$$\sup_{0<\varepsilon\leq 1} \|\varepsilon(\widetilde{D}^- U_j - u'_j)\|_{\overline{\Omega}_i} \leq CN^{-1} \ln N, \ i = N/2+1, \ldots, N,$$

where $\widetilde{D}^- U_j(x) = D^- U_j(x_i)$, for $x \in (x_{i-1}, x_i]$, $i = 1, \ldots, N$.

5 Numerical results

In this section, we present an example to illustrate the results obtained in the paper.

Example 1 Consider the singularly perturbed boundary value problem

$$\begin{gathered}
-\varepsilon u''_1(x) + 7u'_1(x) + (9+x)u_1(x) - 8u_2(x) = 2 + e^{-x}, \\
-\varepsilon u''_2(x) + 7u'_2(x) - 4u_1(x) + (5+x)u_2(x) = 1 + e^{-x}, \\
u_1(0) = 1, \qquad u_1(1) = 0, \qquad u_2(0) = 1, \qquad u_2(1) = 0.
\end{gathered}$$

Let U_j^N be a numerical approximation for the exact solution u_j on the mesh Ω^N and N is the number of mesh points. The exact solution to the test problem is not available, so for all integers $N, 2N \in R_N = [32, 64, 128, 256, 512]$ and for a finite set of values $\varepsilon \in R_\varepsilon = [2^{-25}, 2^{-2}]$, we compute the maximum point-wise two-mesh difference for $j = 1, 2$,

$$D_{\varepsilon,j}^N = \begin{cases} \max |(D^- U_j^N - \widetilde{D}^- U_j^{2N})(x_i)|, & \text{for } 1 \leq i \leq N/2, \\ \max |\varepsilon(D^- U_j^N - \widetilde{D}^- U_j^{2N})(x_i)|, & \text{for } N/2+1 \leq i \leq N. \end{cases}$$

From these values the ε-uniform maximum pointwise two-mesh difference $D_j^N = \max_{\varepsilon\in R_\varepsilon} D_{\varepsilon,j}^N$, for $j = 1, 2$ are formed for each available value of N satisfying N,

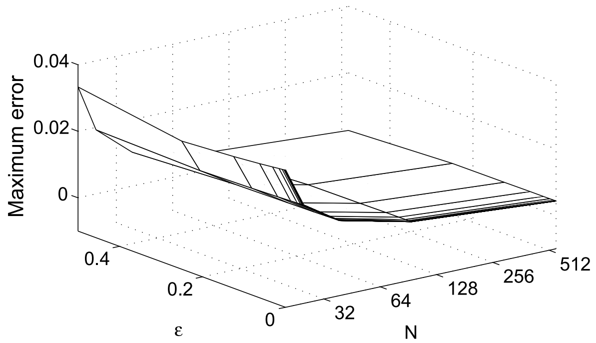

Fig. 1 Surface plot of the maximum pointwise errors as a function of N and ε for the solution U_1

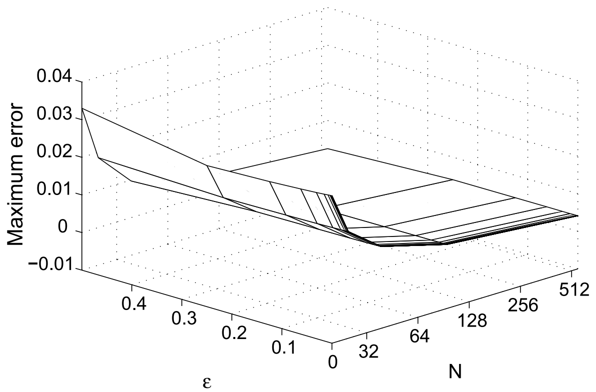

Fig. 2 Surface plot of the maximum pointwise errors as a function of N and ε for the solution U_2

Table 1 Values of D_1^N, p_1^N and D_2^N, p_2^N for the scaled derivative components $\varepsilon u_1'$ and $\varepsilon u_2'$ respectively in the fine mesh region

ε	Number of mesh points N				
	32	64	128	256	512
D_1^N	4.5385e−1	4.1819e−1	3.2969e−1	2.2917e−1	1.4563e−1
p_1^N	**1.1806e−1**	**3.4305e−1**	**5.2469e−1**	**6.5411e−1**	–
D_2^N	4.2683e−1	3.9350e−1	3.1035e−1	2.1579e−1	1.3716e−1
p_2^N	**1.1730e−1**	**3.4247e−1**	**5.2427e−1**	**6.5377e−1**	–

$2N \in R_N$. Approximations to the ε-uniform order of local convergence are defined, for all $N, 4N \in R_N$, by

$$p_j^N = \log_2\left(\frac{D_j^N}{D_j^{2N}}\right), \quad j = 1, 2.$$

Surface plots of the maximum error for the solution of the above test problem are presented. In Figs. 1 and 2, respectively we observe that as $\varepsilon \to 0$, the maximum error for the numerical approximation U_1 and U_2 to the exact solution u_1 and u_2 respectively decreases and stabilize at a constant value. Tables 1 and 2 present ε-uniform maximum pointwise two-mesh difference and ε-uniform order of local convergence

Table 2 Values of D_1^N, p_1^N and D_2^N, p_2^N for the derivative components u_1' and u_2' respectively in the coarse mesh region

ε	Number of mesh points N				
	32	64	128	256	512
D_1^N	1.7708e−3	9.0431e−4	4.5664e−4	2.2941e−4	1.1497e−4
p_1^N	**9.6951e−1**	**9.8576e−1**	**9.9313e−1**	**9.9667e−1**	–
D_2^N	4.5907e−4	2.5116e−4	1.3119e−4	6.7024e−5	3.3871e−5
p_2^N	**8.7011e−1**	**9.3695e−1**	**9.6891e−1**	**9.8463e−1**	–

to the scaled derivatives in the fine mesh region and the non scaled derivative in the coarse mesh region.

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