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The $(\frac{G'}{G})$-expansion method and its applications to some nonlinear evolution equations in the mathematical physics

E.M.E. Zayed

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Abstract In the present paper, we construct the traveling wave solutions involving parameters for some nonlinear evolution equations in the mathematical physics via the (2 + 1)-dimensional Painlevé integrable Burgers equations, the (2 + 1)-dimensional Nizhnik-Novikov-Vesselov equations, the (2 + 1)-dimensional Boiti-Leon-Pempinelli equations and the (2 + 1)-dimensional dispersive long wave equations by using a new approach, namely the $(\frac{G'}{G})$-expansion method, where $G = G(\xi)$ satisfies a second order linear ordinary differential equation. When the parameters are taken special values, the solitary waves are derived from the traveling waves. The traveling wave solutions are expressed by hyperbolic, trigonometric and rational functions.

Keywords The $(\frac{G'}{G})$-expansion method · Traveling wave solutions · The Painlevé integrable Burgers equations · The Nizhnik-Novikov-Vesselov equations · The Boiti-Leon-Pempinelli equations · The dispersive long wave equations

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1 Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors (see for example [1–41]) who are interested in nonlinear physical phenom-

E.M.E. Zayed (✉)
Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt
e-mail: emezayed@hotmail.com

Present address:
E.M.E. Zayed
Mathematics Department, Faculty of Science, Taif University, P.O. Box 888, El-Taif, El-Hawiyah, Kingdom of Saudi Arabia

ena. Many powerful methods have been presented such as the inverse scattering transform [3], the Backlund transform [15, 17], the generalized Riccati equation [18, 25], the Jacobi elliptic function expansion [7, 12, 26, 30, 33], the extended tanh-function method [1, 8, 31, 32, 39], the F-expansion method [2, 19–21, 36], the exp-function expansion method [5, 9, 28, 37, 38], the sub-ODE method [13, 22], the extended sinh-cosh and sine-cosine methods [23], the complex hyperbolic function method [34], the truncated Painlevé expansion [35] and others.

In the present paper, we shall use a new method which is called the $(\frac{G'}{G})$-expansion method [4, 24, 40, 41]. This method is firstly proposed by which the traveling wave solutions of the nonlinear evolution equations are obtained. The main idea of this method is that the traveling wave solutions of the nonlinear evolution equations can be expressed by polynomials in $(\frac{G'}{G})$ where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$ while λ, μ, V are constants and $' = \frac{d}{d\xi}$. The degree of these polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. The coefficients of these polynomials can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method. This new method will play an important role in expressing the traveling wave solutions in terms of hyperbolic, trigonometric and rational functions for the nonlinear evolution equations in the mathematical physics via the $(2+1)$-dimensional Painlevé integrable Burgers equations, the $(2+1)$-dimensional Nizhnik-Novikov-Vesselov equations, the $(2+1)$-dimensional Boiti-Leon-Pempinelli equations and the $(2+1)$-dimensional dispersive long wave equations.

2 Description of the $(\frac{G'}{G})$-expansion method

Suppose we have the following nonlinear partial differential equation:

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{xy}, u_{yy}, u_{yt} \dots) = 0, \tag{2.1}$$

where $u = u(x, y, t)$ is an unknown function, P is a polynomial in $u(x, y, t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. In the following we give the main steps of the $(\frac{G'}{G})$-expansion method:

Step 1. The traveling wave variable

$$u(x, y, t) = u(\xi), \qquad \xi = x + y - Vt, \tag{2.2}$$

permits us reducing (2.1) to an ODE for $u = u(\xi)$ in the form

$$P(u, -Vu', u', V^2u'', -Vu'', u'', \dots) = 0, \tag{2.3}$$

where V is a constant.

Step 2. Suppose the solution of (2.3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi)=\sum_{i=0}^{n}\alpha_i\left(\frac{G'}{G}\right)^i, \tag{2.4}$$

where $G=G(\xi)$ satisfies the following second order linear ordinary differential equation:

$$G''+\lambda G'+\mu G=0, \tag{2.5}$$

while α_i, λ and μ are constants to be determined provided $\alpha_n \neq 0$. The positive integer "n" can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in (2.3).

Step 3. Substituting (2.4) into (2.3) and using (2.5), collecting all terms with the same power of $(\frac{G'}{G})$ together and then equating each coefficient of the resulted polynomial to zero, yield a set of algebraic equations for α_i, V, λ and μ.

Step 4. Since the general solution of (2.5) has been well known for us, then substituting α_i, V and the general solution of (2.5) into (2.4) we have more traveling wave solutions of the nonlinear partial differential equation (2.1).

3 Some applications

In this section, we apply the $(\frac{G'}{G})$-expansion method to construct the traveling wave solutions for the Painlevé integrable Burgers equations, the Nizhnik-Novikov-Vesselov equations, the Boiti-Leon-Pempinelli equations and the dispersive long wave equations which are very important nonlinear evolution equations in the mathematical physics and have been paid attention by many researchers.

3.1 Example 1. The $(2+1)$-dimensional Painlevé integrable Burgers equations

We start with the $(2+1)$-dimensional Painlevé integrable Burgers equations [18] in the forms

$$-u_t+uu_y+\alpha vu_x+\beta u_{yy}+\alpha\beta u_{xx}=0, \tag{3.1}$$

$$u_x-v_y=0, \tag{3.2}$$

where α and β are non-zero constants. This system of equations was derived from the generalized Painlevé integrability classification. Some explicitly exact solutions of this system have been obtained via variable separation approach [10, 18].

Let us now solve the system (3.1) and (3.2) by the $(\frac{G'}{G})$ expansion method. To this end, we see that the following traveling wave variables:

$$u(x,y,t)=u(\xi), \qquad v(x,y,t)=v(\xi), \qquad \xi=x+y-Vt, \tag{3.3}$$

permit us converting (3.1) and (3.2) into

$$-Vu'+uu'+\alpha vu'+\beta u''+\alpha\beta u''=0, \tag{3.4}$$

and

$$C_1 + u - v = 0, \tag{3.5}$$

where $C_1$is an integration constant. Suppose the solutions of the ODE (3.4) can be expressed by polynomials in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{3.6}$$

and

$$v(\xi) = \sum_{i=0}^{m} \beta_i \left(\frac{G'}{G}\right)^i, \tag{3.7}$$

where α_i and $\beta_i\,(i = 0, 1, 2, \ldots, n)$ are arbitrary constants provided $\alpha_n, \beta_n \neq 0$, while $G(\xi)$ satisfies the second order linear ODE (2.5).

Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in (3.4), we get

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \tag{3.8}$$

and

$$v(\xi) = \beta_1 \left(\frac{G'}{G}\right) + \beta_0, \tag{3.9}$$

where $\alpha_1, \beta_1 \neq 0$. Substituting (3.8) and (3.9) into (3.4) and (3.5), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting them to zero. Consequently, we have a system of algebraic equations which can be solved. After some reduction, we get

$$\begin{aligned} \alpha_1 &= \beta_1 = 2\beta, \\ V &= \alpha_0 + \alpha\beta_0 - \beta\lambda(1+\alpha), \\ C_1 &= \beta_0 - \alpha_0. \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.8) and (3.9) yields

$$u(\xi) = 2\beta \left(\frac{G'}{G}\right) + \alpha_0, \tag{3.11}$$

and

$$v(\xi) = 2\beta \left(\frac{G'}{G}\right) + \beta_0, \tag{3.12}$$

where

$$\xi = x + y - t[\alpha_0 + \alpha\beta_0 - \beta\lambda(1+\alpha)]. \tag{3.13}$$

Solving (2.5), we deduce after some reduction that

$$\frac{G'}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}\right) - \frac{\lambda}{2}, \quad (3.14)$$

where A and B are arbitrary constants.

Substituting (3.14) into (3.11) and (3.12), we deduce the following traveling wave solutions:

Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \beta\sqrt{\lambda^2 - 4\mu}\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}\right) + \alpha_0 - \beta\lambda, \quad (3.15)$$

and

$$v(\xi) = \beta\sqrt{\lambda^2 - 4\mu}\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}\right) + \beta_0 - \beta\lambda. \quad (3.16)$$

Case 2. If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = \beta\sqrt{4\mu - \lambda^2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right) + \alpha_0 - \beta\lambda, \quad (3.17)$$

and

$$v(\xi) = \beta\sqrt{4\mu - \lambda^2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right) + \beta_0 - \beta\lambda. \quad (3.18)$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = 2\beta\left(\frac{B}{A + B\xi}\right) + \alpha_0 - \beta\lambda, \quad (3.19)$$

and

$$v(\xi) = 2\beta\left(\frac{B}{A + B\xi}\right) + \beta_0 - \beta\lambda. \quad (3.20)$$

In particular, if $A = 0$, $B \neq 0$, $\lambda > 0$, $\mu = 0$, then we deduce from (3.15) and (3.16) that

$$u(\xi) = \beta\lambda\tanh\left(\frac{\lambda}{2}\xi\right) + \alpha_0 - \beta\lambda, \quad (3.21)$$

and

$$v(\xi)=\beta\lambda\tanh\left(\frac{\lambda}{2}\xi\right)+\beta_0-\beta\lambda, \tag{3.22}$$

which represent the solitary wave solutions of the $(2+1)$-dimensional Painlevé integrable Burgers equations (3.1) and (3.2).

3.2 Example 2. The $(2+1)$-dimensional Nizhnik-Novikov-Vesselov equations

In this subsection, we study the following $(2+1)$-dimensional Nizhnik-Novikov-Vesselov equations [6, 11, 14, 27, 29]:

$$\begin{aligned} u_t+ku_{xxx}+ru_{yyy}+su_x+qu_y-3k(uv)_x-3r(uw)_y&=0,\\ u_x-v_y&=0,\\ u_y-w_x&=0, \end{aligned} \tag{3.23}$$

where k, r, s and q are constants. Boiti et al. [6] solved this system of equations via the inverse scattering transformation. It is well known [16] that the system (3.23) is an isotropic Lax integrable extension of the well known $(1+1)$-dimensional KdV equations and has physical significance.

Let us now solve the system (3.23) by the proposed new method. To this end, we see that the traveling wave variables (3.3) permit us converting (3.23) into

$$\begin{aligned} C_1+(s+q-V)u+(k+r)u''-3kuv-3ruw&=0,\\ C_2+u-v&=0,\\ C_3+u-w&=0, \end{aligned} \tag{3.24}$$

where C_i $(i=1,2,3)$ are constants of integration. Considering the homogeneous balance between highest order derivatives and nonlinear terms in (3.24), we get

$$u(\xi)=\alpha_2\left(\frac{G'}{G}\right)^2+\alpha_1\left(\frac{G'}{G}\right)+\alpha_0, \tag{3.25}$$

$$v(\xi)=\beta_2\left(\frac{G'}{G}\right)^2+\beta_1\left(\frac{G'}{G}\right)+\beta_0, \tag{3.26}$$

and

$$w(\xi)=\gamma_2\left(\frac{G'}{G}\right)^2+\gamma_1\left(\frac{G'}{G}\right)+\gamma_0, \tag{3.27}$$

where $\alpha_2, \beta_2, \gamma_2\neq 0$. Substituting (3.25)–(3.27) into (3.24), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting them to zero. Consequently, we have a system of algebraic equations which can be solved. After some reduction, we get

$$\alpha_2=\beta_2=\gamma_2=2,$$

$$
\begin{aligned}
&\alpha_1 = \beta_1 = \gamma_1 = 2\lambda,\\
&\beta_0 = 0,\\
&V = q + s - 3r\gamma_0 + (k + r)(8\mu - \lambda^2 - 3\alpha_0),\\
&C_1 = (k + r)(4\mu^2 + 2\mu\lambda^2 - 8\mu\alpha_0 + \alpha_0\lambda^2 + 3\alpha_0^2),\\
&C_2 = -\alpha_0,\\
&C_3 = \gamma_0 - \alpha_0.
\end{aligned}
\tag{3.28}
$$

Substituting (3.28) into (3.25)–(3.27) yields

$$u(\xi) = 2\left(\frac{G'}{G}\right)^2 + 2\lambda\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.29}$$

$$v(\xi) = 2\left(\frac{G'}{G}\right)^2 + 2\lambda\left(\frac{G'}{G}\right), \tag{3.30}$$

and

$$w(\xi) = 2\left(\frac{G'}{G}\right)^2 + 2\lambda\left(\frac{G'}{G}\right) + \gamma_0, \tag{3.31}$$

where

$$\xi = x + y - [q + s - 3r\gamma_0 + (k + r)(8\mu - \lambda^2 - 3\alpha_0)]t.$$

On substituting (3.14) into (3.29)–(3.31), we deduce the following traveling wave solutions:

Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$
\begin{aligned}
u(\xi) = {}& \frac{1}{2}(\lambda^2 - 4\mu)\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}\right)^2\\
&+ \alpha_0 - \frac{\lambda^2}{2},
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
v(\xi) = {}& \frac{1}{2}(\lambda^2 - 4\mu)\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}\right)^2\\
&- \frac{\lambda^2}{2},
\end{aligned}
\tag{3.33}
$$

and

$$
\begin{aligned}
w(\xi) = {}& \frac{1}{2}(\lambda^2 - 4\mu)\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}\right)^2\\
&+ \gamma_0 - \frac{\lambda^2}{2}.
\end{aligned}
\tag{3.34}
$$

Case 2 . If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = \frac{1}{2}(4\mu - \lambda^2)\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)^2 + \alpha_0 - \frac{\lambda^2}{2}, \tag{3.35}$$

$$v(\xi) = \frac{1}{2}(4\mu - \lambda^2)\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)^2 - \frac{\lambda^2}{2}, \tag{3.36}$$

and

$$w(\xi) = \frac{1}{2}(4\mu - \lambda^2)\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)^2 + \gamma_0 - \frac{\lambda^2}{2}. \tag{3.37}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = 2\left(\frac{B}{A + B\xi}\right)^2 + \alpha_0 - \frac{\lambda^2}{2}, \tag{3.38}$$

$$v(\xi) = 2\left(\frac{B}{A + B\xi}\right)^2 - \frac{\lambda^2}{2}, \tag{3.39}$$

and

$$w(\xi) = 2\left(\frac{B}{A + B\xi}\right)^2 + \gamma_0 - \frac{\lambda^2}{2}. \tag{3.40}$$

In particular, if $A = 0$, $B \neq 0$, $\lambda > 0$, $\mu = 0$, then we deduce from (3.32)–(3.34) that

$$u(\xi) = -\frac{1}{2}\lambda^2 \operatorname{sech}^2\left(\frac{\lambda}{2}\xi\right) + \alpha_0, \tag{3.41}$$

$$v(\xi) = -\frac{1}{2}\lambda^2 \operatorname{sech}^2\left(\frac{\lambda}{2}\xi\right), \tag{3.42}$$

and

$$w(\xi) = -\frac{1}{2}\lambda^2 \operatorname{sech}^2\left(\frac{\lambda}{2}\xi\right) + \gamma_0, \tag{3.43}$$

which represent the solitary wave solutions of the (2 + 1)-dimensional Nizhnik-Novikov-Vesselov equations (3.23).

3.3 Example 3. The $(2+1)$-dimensional Boiti-Leon-Pempinelli equations

In this subsection, we consider the $(2+1)$-dimensional Boiti-Leon-Pempinelli equations [6, 11, 14, 27] in the form:

$$u_{ty} - (u^2 - u_x)_{xy} - 2v_{xxx} = 0, \tag{3.44}$$

and

$$v_t - v_{xx} - 2uv_x = 0. \tag{3.45}$$

The integrability of the system (3.44) and (3.45) was established by Hong et al. [10]. Boiti et al. [6] presented the Backlund transformation of this system to find its solutions.

Let us now solve this system by the proposed method. To this end, we see that the traveling wave variables (3.3) permit us converting (3.44) and (3.45) into

$$-Vu'' - 2(uu')' + u''' - 2v''' = 0, \tag{3.46}$$

and

$$Vv' + v'' + 2uv' = 0. \tag{3.47}$$

Considering the homogeneous balance between highest order derivatives and nonlinear terms in (3.46) and (3.47), we deduce that the solutions $u(\xi)$ and $v(\xi)$ respectively have the same forms (3.8) and (3.9) of example 1. Substituting (3.8) and (3.9) into (3.46) and (3.47), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting them to zero. Consequently, we have a system of algebraic equations which can be solved, to get

$$\alpha_1 = \beta_1 = 1, \qquad V = \lambda - 2\alpha_0.$$

Now, we have

$$u(\xi) = \left(\frac{G'}{G}\right) + \alpha_0, \tag{3.48}$$

and

$$v(\xi) = \left(\frac{G'}{G}\right) + \beta_0, \tag{3.49}$$

where

$$\xi = x + y - (\lambda - 2\alpha_0)t.$$

On substituting (3.14) into (3.48) and (3.49), we deduce the following traveling wave solutions:

Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}\right) + \alpha_0 - \frac{\lambda}{2}, \tag{3.50}$$

and

$$v(\xi)=\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi+B\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi+B\cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}\right)$$
$$+\beta_0-\frac{\lambda}{2}. \tag{3.51}$$

Case 2. If $\lambda^2-4\mu<0$, then we have

$$u(\xi)=\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)$$
$$+\alpha_0-\frac{\lambda}{2}, \tag{3.52}$$

and

$$v(\xi)=\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)$$
$$+\beta_0-\frac{\lambda}{2}. \tag{3.53}$$

Case 3. If $\lambda^2-4\mu=0$, then we have

$$u(\xi)=\left(\frac{B}{A+B\xi}\right)+\alpha_0-\frac{\lambda}{2}, \tag{3.54}$$

and

$$v(\xi)=\left(\frac{B}{A+B\xi}\right)+\beta_0-\frac{\lambda}{2}. \tag{3.55}$$

In particular, if $A=0$, $B\neq 0$, $\lambda>0$, $\mu=0$, then we deduce from (3.50) and (3.51) that:

$$u(\xi)=\frac{1}{2}\lambda\tanh\left(\frac{\lambda}{2}\xi\right)+\alpha_0-\frac{\lambda}{2}, \tag{3.56}$$

and

$$v(\xi)=\frac{1}{2}\lambda\tanh\left(\frac{\lambda}{2}\xi\right)+\beta_0-\frac{\lambda}{2}, \tag{3.57}$$

which represent the solitary wave solutions of the $(2+1)$-dimensional Boiti-Leon-Pempinelli equations (3.44) and (3.45).

3.4 Example 4. The $(2+1)$-dimensional dispersive long wave equations

In this subsection, we consider the $(2+1)$-dimensional dispersive long wave equations [23, 27, 39] in the form

$$u_{ty} + v_{xx} + \frac{1}{2}(u^2)_{xy} = 0, \tag{3.58}$$

and

$$v_t + (uv + u + u_{xy})_x = 0. \tag{3.59}$$

This system of equations was first obtained by Boiti et al. [6] as compatibility condition for a weak Lax pair. The solutions of the system (3.58) and (3.59) including Jacobi elliptic function solutions, soliton-like solutions, periodic formal solutions and rational function solutions are found in [23, 39].

In order to solve this system by the proposed method, we see that the traveling wave variables (3.3) permit us converting (3.58) and (3.59) into

$$-Vu'' + v'' + uu'' + (u')^2 = 0, \tag{3.60}$$

and

$$C_1 - Vv + (uv) + u + u'' = 0, \tag{3.61}$$

where C_1 is an integration constant. The homogeneous balance of the highest derivatives and nonlinear terms in (3.58) and (3.59) yields the following solutions:

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.62}$$

and

$$v(\xi) = \beta_2\left(\frac{G'}{G}\right)^2 + \beta_1\left(\frac{G'}{G}\right) + \beta_0, \tag{3.63}$$

where α_1 and β_2 are nonzero constants. Substituting (3.62) and (3.63) into (3.60) and (3.61), collecting all terms with the same power of $(\frac{G'}{G})$ together and setting them to zero. Consequently, we have a system of algebraic equations which can be solved. After some reduction, we get

$$\begin{aligned} &\alpha_1 = \pm 2, \qquad \beta_2 = -2, \\ &\beta_1 = -2\lambda, \qquad \beta_0 = -(2\mu + 1), \\ &C_1 = \pm(\lambda \mp \alpha_0), \qquad V = \alpha_0 \mp \lambda. \end{aligned} \tag{3.64}$$

From (3.62)–(3.64), we have

$$u(\xi) = \pm 2\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.65}$$

and

$$v(\xi) = -2\left(\frac{G'}{G}\right)^2 - 2\lambda\left(\frac{G'}{G}\right) - (2\mu + 1), \tag{3.66}$$

where

$$\xi = x + y - (\alpha_0 \mp \lambda)t.$$

Substituting (3.14) into (3.65) and (3.66), we deduce the following traveling wave solutions:

Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \pm\sqrt{(\lambda^2 - 4\mu)}\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}\right) + \alpha_0 \mp \lambda,$$

and

$$v(\xi) = -\frac{1}{2}(\lambda^2 - 4\mu)\left(\frac{A\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}{A\sinh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi + B\cosh\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\xi}\right)^2 + \frac{\lambda^2}{2} - (2\mu + 1). \tag{3.67}$$

Case 2. If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = \pm\sqrt{(4\mu - \lambda^2)}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi)}\right) + \alpha_0 \mp \lambda, \tag{3.68}$$

and

$$v(\xi) = -\frac{1}{2}(4\mu - \lambda^2)\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi)}\right)^2 + \frac{\lambda^2}{2} - (2\mu + 1). \tag{3.69}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = \pm\left(\frac{2B}{A + B\xi}\right) + \alpha_0 \mp \lambda, \tag{3.70}$$

and

$$v(\xi) = -2\left(\frac{B}{A + B\xi}\right)^2 + \frac{\lambda^2}{2} - (2\mu + 1). \tag{3.71}$$

In particular, if $A = 0$, $B \neq 0$, $\lambda > 0$, $\mu = 0$, then we deduce from (3.67) and (3.67) that

$$u(\xi) = \pm\lambda\tanh\left(\frac{\lambda}{2}\xi\right) + \alpha_0 \mp \lambda, \tag{3.72}$$

and

$$v(\xi) = \frac{1}{2}\lambda^2 \operatorname{sech}^2\left(\frac{\lambda}{2}\xi\right) - 1, \tag{3.73}$$

which represent the solitary wave solutions of the $(2+1)$-dimensional dispersive long wave equations (3.58) and (3.59).

4 Conclusions

In this paper, we have seen that three types of traveling wave solutions in terms of hyperbolic, trigonometric and rational functions for Painlevé integrable Burgers equations, Nizhnik-Novikov-Vesselov equations, Boiti-Leon-Pempinelli equations and dispersive long wave equations are successfully found out by using the $(\frac{G'}{G})$-expansion method. These equations are very difficult to be solved by traditional methods. On comparing between these methods, we conclude that the $(\frac{G'}{G})$-expansion method is more powerful, effective and convenient. The performance of this method is reliable, simple and gives many new solutions. The $(\frac{G'}{G})$-expansion method has more advantages: It is direct and concise. It is also a standard and computerizable method which allows us to solve complicated nonlinear evolution equations in the mathematical physics.

Note that the nonlinear evolution equations proposed in the present paper are difficult and more general than the nonlinear evolution equations discussed in [4, 24, 40, 41]. Therefore, the solutions of the proposed nonlinear evolution equations in this paper have many potential applications in physics.

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