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Oscillation theorems for fourth order functional differential equations

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Abstract The authors investigate the oscillatory behavior of all solutions of the fourth order functional differential equations $\frac{d^3}{dt^3}(a(t)(\frac{dx(t)}{dt})^\alpha) + q(t)f(x[g(t)]) = 0$ and $\frac{d^3}{dt^3}(a(t)(\frac{dx(t)}{dt})^\alpha) = q(t)f(x[g(t)]) + p(t)h(x[\sigma(t)])$ in the case where $\int^\infty a^{-1/\alpha}(s)ds < \infty$. The results are illustrated with examples.

Keywords Functional differential equation · Oscillation · Nonoscillation · Comparison

Mathematics Subject Classification (2000) 34C10

1 Introduction

In this paper, we are concerned with some criteria for the oscillation of all solutions of the fourth order functional differential equations

$$\frac{d^3}{dt^3}\left(a(t)\left(\frac{dx(t)}{dt}\right)^\alpha\right) + q(t)f(x[g(t)]) = 0 \tag{1.1}$$

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and

$$\frac{d^3}{dt^3}\left(a(t)\left(\frac{dx(t)}{dt}\right)^{\alpha}\right) = q(t)f(x[g(t)]) + p(t)h(x[\sigma(t)]), \tag{1.2}$$

where the following conditions are always assumed to hold:

(i) α is the ratio of two positive odd integers;
(ii) $a(t)$, $p(t)$, and $q(t) \in C([t_0, \infty), (0, \infty))$ and

$$\int^{\infty} a^{-1/\alpha}(s)ds < \infty; \tag{1.3}$$

(iii) $g(t), \sigma(t) \in C^1([t_0, \infty), \mathbb{R}), \mathbb{R} = (-\infty, \infty),\ g(t) < t, \sigma(t) > t, g'(t) \geq 0$, and $\sigma'(t) \geq 0$ for $t \geq t_0$, and $\lim_{t\to\infty} g(t) = \infty$;
(iv) $f, h \in C(\mathbb{R}, \mathbb{R}), xf(x) > 0, xh(x) > 0, f'(x) \geq 0$, and $h'(x) \geq 0$ for $x \neq 0$, and f and h satisfy

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for } xy > 0 \tag{1.4}$$

and

$$-h(-xy) \geq h(xy) \geq h(x)h(y) \quad \text{for } xy > 0. \tag{1.5}$$

By a *solution* of (1.1) (respectively, (1.2)) we mean a function $x : [T_x, \infty) \to \mathbb{R}$, $T_x \geq t_0$ such that $x(t), a(t)(x'(t))^{\alpha}, \big(a(t)(x'(t))^{\alpha}\big)'$ and $\big(a(t)(x'(t))^{\alpha}\big)''$ are continuously differentiable and satisfy (1.1) (respectively, (1.2)) on $[T_x, \infty)$. Our attention will be restricted to these solutions $x(t)$ of (1.1) and (1.2) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. Such a solution is said to be *oscillatory* if it has a sequence of zeros tending to infinity, and *nonoscillatory* otherwise.

The problem of determining the oscillation and nonoscillation of solutions of functional differential equations has been a very active area in the last three decades, and many references and summaries of known results can be found in the monographs by Agarwal et al. [1–3] and Györi and Ladas [4]. Much of the literature on the subject has been concerned with equations of the form (1.1) and (1.2) with $\alpha = 1$ and $a(t) = 1$ and/or higher order functional differential equations. For typical results, we refer to reader to [1–10] and the references cited therein. There is however much current interest (see [5–8]) in the study of the oscillation properties of solutions of (1.1) and (1.2) when $\alpha \neq 1$ and

$$\int^{\infty} a^{-1/\alpha}(s)ds = \infty. \tag{1.6}$$

In the present paper, we proceed further in this direction to establish some new criteria for the oscillation of (1.1) and (1.2) when the function $a(t)$ satisfies condition (1.3). We note that some of the results of this paper do hold for (1.1) and (1.2) even if condition (1.6) holds.

2 Oscillation of equation (1.1)

In this section, we shall establish sufficient conditions for the oscillation of (1.1). For $t \geq t_0 \geq 0$, we let

$$A[t, t_0] = \int_{t_0}^{t} \left(\frac{s^2}{a(s)}\right)^{1/\alpha} ds \quad \text{and} \quad m(t) = \int_{t}^{\infty} a^{-1/\alpha}(s)ds.$$

Our first result is the following.

Theorem 2.1 *Let conditions* (i)–(iv), (1.3), *and* (1.4) *hold and assume that there exists a nondecreasing function* $\xi(t) \in C([t_0, \infty), \mathbb{R})$ *such that* $g(t) \leq \xi(t) < t$ *for* $t \geq t_0$. *Consider the following three first order delay differential equations*

$$y'(t) + cq(t)f(A[g(t), t_1])f\left(y^{1/\alpha}[g(t)]\right) = 0 \tag{2.1}$$

for any constant $0 < c < 1$;

$$z'(t) + bq(t)f\left(a^{-1/\alpha}[g(t)]g(t)\right) f\left(\left(\frac{[\xi(t) - g(t)]^2}{2!}\right)^{1/\alpha}\right) f\left(z^{1/\alpha}[\xi(t)]\right) = 0 \tag{2.2}$$

for any constant $0 < b < 1$; *and*

$$w'(t) + dq(t)f(m[g(t)])f\left(g^{1/\alpha}(t)\right) f\left([\xi(t) - g(t)]^{1/\alpha}\right) f\left(w^{1/\alpha}[\xi(t)]\right) = 0 \tag{2.3}$$

for any constant $0 < d < 1$ *and with*

$$\int^{\infty} \left(\frac{1}{a(s)} \int_{t_0}^{s} \int_{t_0}^{u} \int_{t_0}^{v} q(\tau)f(m[g(\tau)])d\tau dv du\right)^{1/\alpha} ds = \infty. \tag{2.4}$$

If (2.1)–(2.3) *are oscillatory, then* (1.1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1), say, $x(t) > 0$ and $x[g(t)] > 0$ for $t \geq t_0 \geq 0$. Then there exists $t_1 \geq t_0$ such that one of the following four possible cases holds:

(I) $\left(a(t)(x'(t))^{\alpha}\right)'' > 0$, $\left(a(t)(x'(t))^{\alpha}\right)' > 0$ and $x'(t) > 0$ for $t \geq t_1$;
(II) $\left(a(t)(x'(t))^{\alpha}\right)'' > 0$, $\left(a(t)(x'(t))^{\alpha}\right)' < 0$ and $x'(t) > 0$ for $t \geq t_1$;
(III) $\left(a(t)(x'(t))^{\alpha}\right)'' > 0$, $\left(a(t)(x'(t))^{\alpha}\right)' < 0$ and $x'(t) < 0$ for $t \geq t_1$;
(IV) $\left(a(t)(x'(t))^{\alpha}\right)'' < 0$, $\left(a(t)(x'(t))^{\alpha}\right)' < 0$ and $x'(t) < 0$ for $t \geq t_1$.

We also note that the following four cases:

$\left(a(t)(x'(t))^{\alpha}\right)'' > 0$, $\left(a(t)(x'(t))^{\alpha}\right)' > 0$ and $x'(t) < 0$ for $t \geq t_1$;
$\left(a(t)(x'(t))^{\alpha}\right)'' < 0$, $\left(a(t)(x'(t))^{\alpha}\right)' > 0$ and $x'(t) > 0$ for $t \geq t_1$;
$\left(a(t)(x'(t))^{\alpha}\right)'' < 0$, $\left(a(t)(x'(t))^{\alpha}\right)' > 0$ and $x'(t) < 0$ for $t \geq t_1$; and
$\left(a(t)(x'(t))^{\alpha}\right)'' < 0$, $\left(a(t)(x'(t))^{\alpha}\right)' < 0$ and $x'(t) > 0$ for $t \geq t_1$

can be disregarded. We will now consider the four cases in order.

Case (I). There exist $t_2 \geq t_1$ and a constant $k, 0 < k < 1$ such that

$$y(t) \geq kty'(t) \quad \text{for } t \geq t_2,$$

where $y(t) = (a(t)(x'(t))^\alpha)'$ for $t \geq t_2$. Integrating both sides of the above inequality from t_2 to t, we obtain

$$x'(t) \geq \overline{c}\left(\frac{t^2}{a(t)}\right)^{1/\alpha} (y'(t))^{1/\alpha} \quad \text{for } t \geq t_3 \geq t_2, \tag{2.5}$$

where $\overline{c}$ is a constant with $0 < \overline{c} < 1$. Integrating (2.5) from t_3 to t, we have

$$x(t) \geq c\left(\int_{t_3}^{t}\left(\frac{s^2}{a(s)}\right)^{1/\alpha} ds\right)(y'(t))^{1/\alpha} \quad \text{for } t \geq t_4 \geq t_3,$$

where c is a constant, $0 < c < 1$. Now, there exists $t_5 \geq t_4$ such that

$$x[g(t)] \geq cA[g(t), t_3]z^{1/\alpha}[g(t)] \quad \text{for } t \geq t_5, \tag{2.6}$$

where $z(t) = y'(t)$ for $t \geq t_5$. Using (2.6) and (1.4) in (1.1), we have

$$z'(t) + f(c)q(t)f(A[g(t), t_3])f\big(z^{1/\alpha}[g(t)]\big) \leq 0 \quad \text{for } t \geq t_5. \tag{2.7}$$

Integrating (2.7) from t to $u \geq t \geq t_5$ and letting $u \to \infty$, we obtain

$$z(t) \geq \int_t^\infty f(c)q(s)f(A[g(s), t_3])f\big(z^{1/\alpha}[g(s)]\big)ds.$$

The function $z(t)$ is obviously strictly decreasing on $[t_5, \infty)$. Hence, by Theorem 1 in [10], we conclude that there exists a positive solution $y(t)$ of (2.1) with $\lim_{t\to\infty} y(t) = 0$, which is a contradiction.

Case (II). There exist $t_2 \geq t_1$ and a constant $k, 0 < k < 1$ such that

$$x(t) \geq ktx'(t) = k\big(ta^{-1/\alpha}(t)\big)y^{1/\alpha}(t) \quad \text{for } t \geq t_2,$$

where $y(t) = a(t)(x'(t))^\alpha$ for $t \geq t_2$. Also, there exists $t_3 \geq t_2$ such that

$$x[g(t)] \geq k\big(g(t)a^{-1/\alpha}[g(t)]\big)y^{1/\alpha}[g(t)] \quad \text{for } t \geq t_3. \tag{2.8}$$

Substituting (2.8) into (1.1), we have

$$y'''(t) + f(k)q(t)f\big(g(t)a^{-1/\alpha}[g(t)]\big)f\big(y^{1/\alpha}[g(t)]\big) \leq 0 \quad \text{for } t \geq t_3. \tag{2.9}$$

Clearly, $y(t) > 0$, $y'(t) < 0$, and $y''(t) > 0$ for $t \geq t_3$. By Taylor's theorem, there exists $t_4 \geq t_3$ such that

$$y[g(t)] \geq \left(\frac{[\xi(t) - g(t)]^2}{2!}\right)z[\xi(t)] \quad \text{for } t \geq t_4, \tag{2.10}$$

where $z(t)=y''(t)$ for $t\geq t_4$. Now using (2.10) and (1.4) in (2.9), we obtain

$$z'(t)+f(k)q(t)f\left(g(t)a^{-1/\alpha}[g(t)]\right)f\left(\left(\frac{[\xi(t)-g(t)]^2}{2!}\right)^{1/\alpha}\right)f\left(z^{1/\alpha}[\xi(t)]\right)\leq 0$$

for $t\geq t_4$. The remainder of the proof in this case is similar to that of Case (I) above.

Case (III). For $s\geq t\geq t_1$, we can easily see that

$$-a(s)(x'(s))^{\alpha}\geq -a(t)(x'(t))^{\alpha},$$

and so

$$-x'(s)\geq a^{-1/\alpha}(s)\left(y^{1/\alpha}(t)\right),$$

where $y(t)=-a(t)(x'(t))^{\alpha}$ for $t\geq t_1$. Integrating the above inequality from t to $u\geq t\geq t_1$ and letting $u\to\infty$, we obtain

$$x(t)\geq m(t)y^{1/\alpha}(t)\quad \text{for } t\geq t_1. \tag{2.11}$$

Now there exists $t_2\geq t_1$ such that

$$x[g(t)]\geq m[g(t)]y^{1/\alpha}[g(t)]\quad \text{for } t\geq t_2, \tag{2.12}$$

and substituting (2.12) into (1.1) yields

$$y'''(t)\geq q(t)f(m[g(t)])f\left(y^{1/\alpha}[g(t)]\right)\quad \text{for } t\geq t_2. \tag{2.13}$$

Clearly, $y(t)>0, y'(t)>0$ and $y''(t)<0$ for $t\geq t_2$. Thus, there exist a constant $k, 0<k<1$, and $t_3\geq t_2$ such that

$$y[g(t)]\geq kg(t)y'[g(t)]\quad \text{for } t\geq t_3. \tag{2.14}$$

Using (2.14) and (1.4) in (2.13), we have

$$z''(t)\geq f\left(k^{1/\alpha}\right)q(t)f(m[g(t)])f\left(g^{1/\alpha}(t)\right)f\left(z^{1/\alpha}[g(t)]\right)\quad \text{for } t\geq t_3, \tag{2.15}$$

where $z(t)=y'(t)$ for $t\geq t_3$. It is easy to see that $z(t)>0$ and $z'(t)<0$ for $t\geq t_3$. For $s\geq t\geq t_3$, we see that

$$z(s)\geq (t-s)(-z'(t)).$$

Replacing s and t by $g(t)$ and $\xi(t)$ respectively, we have

$$z[g(t)]\geq (\xi(t)-g(t))(-z'[\xi(t)])\quad \text{for } t\geq t_4\geq t_3, \tag{2.16}$$

and so (2.15) yields

$$w'(t)+f\left(k^{1/\alpha}\right)q(t)f(m[g(t)])f\left(g^{1/\alpha}(t)\right)f\left([\xi(t)-g(t)]^{1/\alpha}\right)f\left(w^{1/\alpha}[\xi(t)]\right)\leq 0$$

for $t\geq t_4$, where $w(t)=-z'(t)$. The rest of the proof is similar to that of Case (I) above.

Case (IV). As in the proof of Case (III), we obtain (2.13) for $t \geq t_2$. Now, there exist constants $b > 0$ and $t_3 \geq t_2$ such that

$$x[g(t)] \geq bm[g(t)] \quad \text{for } t \geq t_3. \tag{2.17}$$

Using (2.17) and (1.4) in (1.1), we have

$$-\left(a(t)(x'(t))^{\alpha}\right)''' = q(t)f(x[g(t)]) \geq f(b)q(t)f(m[g(t)]) \quad \text{for } t \geq t_3. \tag{2.18}$$

Integrating (2.18) three times from t_3 to t, we obtain

$$-a(t)(x'(t))^{\alpha} \geq f(b)\int_{t_3}^{t}\int_{t_3}^{u}\int_{t_3}^{v} q(s)f(m[g(s)])ds\,dv\,du,$$

or

$$-x'(t) \geq \overline{c}\left(\frac{1}{a(t)}\int_{t_3}^{t}\int_{t_3}^{u}\int_{t_3}^{v} q(s)f(m[g(s)])ds\,dv\,du\right)^{1/\alpha}, \tag{2.19}$$

where $\overline{c} = f^{1/\alpha}(b)$. Integrating (2.19) from t_3 to t, yields

$$\begin{aligned}\infty &> x(t_3) \geq x(t_3) - x(t)\\ &\geq \overline{c}\int_{t_3}^{t}\left(\frac{1}{a(\tau)}\int_{t_3}^{\tau}\int_{t_3}^{u}\int_{t_3}^{v} q(s)f(m[g(s)])ds\,dv\,du\right)^{1/\alpha} d\tau \to \infty \quad \text{as } t \to \infty,\end{aligned}$$

which is a contradiction and completes the proof of the theorem. □

Notice that in the proof of Theorem 2.1, condition (1.3) is only needed in order for $m(t)$ to be well defined. This is the case in parts (III) and (IV). Thus, we have the following result.

Theorem 2.2 *Let conditions* (i)–(iv), (1.4), *and* (1.6) *hold and assume that there exists a nondecreasing function* $\xi(t) \in C([t_0,\infty),\mathbb{R})$ *such that* $g(t) \leq \xi(t) < t$ *for* $t \geq t_0$. *If both* (2.1) *and* (2.2) *are oscillatory, then* (1.1) *is oscillatory.*

We can combine (2.1) and (2.2) into the single equation

$$y'(t) + \overline{Q}(t)f\left(y^{1/\alpha}[\xi(t)]\right) = 0, \tag{2.20}$$

where

$$\begin{aligned}\overline{Q}(t) = \min\Bigg\{ & cq(t)f(A[g(t),t_1]),\, bq(t)f\left(g(t)a^{-1/\alpha}[g(t)]\right)\\ & \times f\left(\left(\frac{[\xi(t)-g(t)]^2}{2!}\right)^{1/\alpha}\right)\Bigg\},\end{aligned}$$

$0 < b$, $c < 1$ are arbitrary constants, and $t \geq t_1 \geq t_0$. Thus, Theorem 2.2 takes the following form.

Theorem 2.3 *Let the hypotheses of Theorem 2.2 hold and let (2.1) and (2.2) be replaced by (2.20). Then the conclusion of Theorem 2.2 holds.*

We may proceed further and replace the (2.1)–(2.3) by the equation

$$y'(t) + Q(t) f\left(y^{1/\alpha}[\xi(t)]\right) = 0, \tag{2.21}$$

where

$$Q(t) = \min\left\{\overline{Q}(t), dq(t) f(m[g(t)]) f\left(g^{1/\alpha}(t)\right) f\left((\xi(t) - g(t))^{1/\alpha}\right)\right\},$$

$0 < d < 1$ is an arbitrary constant, and $t \geq t_1 \geq t_0$. In this case, we have the following result.

Theorem 2.4 *Let the hypotheses of Theorem 2.1 hold and let (2.1)–(2.3) be replaced by (2.21). Then the conclusion of Theorem 2.1 holds.*

Next, we shall state some sufficient conditions for the oscillation of (2.21) (see, for example, [3, 4]).

(2.21) is oscillatory if one of the following conditions holds:

(I_1) $\frac{f(u^{1/\alpha})}{u} \geq 1$ for $u \neq 0$, and

$$\limsup_{t\to\infty} \int_{\xi(t)}^{t} Q(s)ds > 1, \tag{2.22}$$

or

$$\liminf_{t\to\infty} \int_{\xi(t)}^{t} Q(s)ds > \frac{1}{e}. \tag{2.23}$$

(I_2) $\frac{u}{f(u^{1/\alpha})} \to 0$ as $u \to 0$, and

$$\limsup_{t\to\infty} \int_{\xi(t)}^{t} Q(s)ds > 0. \tag{2.24}$$

(I_3) $\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty$, and

$$\int^{\infty} Q(s) = \infty. \tag{2.25}$$

Remarks

1. We may replace (2.20) of Theorem 2.3 or (2.21) of Theorem 2.4 by any of the above conditions and obtain new criteria for the oscillation of (1.1). The details are left to the reader.
2. In (1.1), if we let $f(x) = x^{\beta}$, where β is the ratio of two positive odd integers, then condition (1.4) is automatically satisfied. Also, we see that the results of this section remain valid if the argument $g(t)$ is of mixed type. The details are left to the reader.

3. Clearly the condition "there exists a nondecreasing function $\xi(t) \in C([t_0, \infty), \mathbb{R})$ such that $g(t) \leq \xi(t) < t$ for $t \geq t_0$" is not restrictive; it is easily satisfied by taking $\xi(t) = \max_{t_0 \leq s \leq t} g(s)$.

The following examples illustrate the above results.

Example 2.1 Consider the fourth order functional differential equation

$$\frac{d^3}{dt^3}\left(t^6\left(\frac{dx(t)}{dt}\right)^3\right) + \frac{1}{\sqrt{t}}x^3[\sqrt{t}] = 0, \quad t \geq 5. \tag{2.26}$$

Here $\alpha = 3$, $f(x) = x^3$, $a(t) = t^6$, $q(t) = 1/\sqrt{t}$ and $g(t) = \sqrt{t}$. We let $\xi(t) = 2\sqrt{t}$ for $t \geq 5$. Clearly,

$$\int_5^\infty a^{-1/\alpha}(s)ds = \int_5^\infty s^{-2}ds = \frac{1}{5} < \infty$$

and

$$m(t) = \int_t^\infty a^{-1/\alpha}(s)ds = \frac{1}{t} \quad \text{for } t \geq 5.$$

Now it is easy to calculate the function $Q(t)$ in (2.21) and find

$$Q(t) = \frac{c}{t} \quad \text{for any constant } c > 0 \text{ and } t \geq 5.$$

By applying Theorem 2.4 together with condition (I_1) above, we conclude that (2.26) is oscillatory.

Example 2.2 Consider the fourth order functional differential equation

$$\left(t^{10}(x'(t))^5\right)''' + \frac{1}{\sqrt{t}}x^3\left[\sqrt{t}\right] = 0, \quad t \geq 5. \tag{2.27}$$

Here, $\alpha = 5$, $f(x) = x^3$, $a(t) = t^{10}$, $q(t) = 1/\sqrt{t}$ and $g(t) = \sqrt{t}$. As in Example 2.1, we let $\xi(t) = 2\sqrt{t}$ for $t \geq 5$. It is easy to check that all conditions of Theorem 2.4 together with condition (I_2) or (I_3) are satisfied and conclude that (2.27) is oscillatory.

From the above discussion, we see that the special case of (1.1), namely,

$$\left(a(t)(x'(t))^\alpha\right)''' + q(t)x^\beta\left[\sqrt{t}\right] = 0, \quad t \geq 5, \tag{2.28}$$

where $a(t)$ satisfies condition (1.3) and β is the ratio of positive odd integers, is oscillatory if condition (2.4) holds and for every constant $0 < c < 1$ and $t_1 \geq 5$, the equation

$$y'(t) + cq(t)Q^*(t)y^{\beta/\alpha}\left[2\sqrt{t}\right] = 0, \quad t \geq 5, \beta \leq \alpha,$$

is oscillatory, where $\xi(t) = 2\sqrt{t}$ and for $t \geq 5$,

$$Q^*(t) = \min\left\{\left(\int_{t_1}^{\sqrt{t}} \left(\frac{s^2}{a(s)}\right)^{1/\alpha} ds\right)^{\beta}, \left(\frac{\sqrt{t}}{a(\sqrt{t})}\right)^{\beta} t^{\frac{\beta}{\alpha}}, \left(\int_{\sqrt{t}}^{\infty} a^{-1/\alpha}(s)ds\right)^{\beta} t^{\frac{\beta}{\alpha}}\right\}.$$

3 Oscillation of equation (1.2)

In this section, we study the oscillatory character of (1.2) and present the following result.

Theorem 3.1 *Let conditions* (i)–(iv) *and* (1.3)–(1.5) *hold and assume that there exist nondecreasing functions* $\xi(t)$, $\eta(t)$ *and* $\rho(t) \in C([t_0, \infty), \mathbb{R})$ *such that* $g(t) < \xi(t) < \eta(t) < t$ *and* $\sigma(t) > \rho(t) > t$ *for* $t \geq t_0$. *Consider the first order advanced differential equation*

$$y'(t) - p(t)h\left(\int_{\rho(t)}^{\sigma(t)} \left(\frac{[u-\rho(t)]^2}{2!a(u)}\right)^{1/\alpha} du\right) h\left(y^{1/\alpha}[\rho(t)]\right) = 0, \tag{3.1}$$

and the first order delay equations

$$z'(t) + q(t) f\left(\frac{\xi(t) - g(t)}{a^{1/\alpha}[\xi(t)]}\right) f\left(\left(\frac{[\eta(t) - \xi(t)]^2}{2!}\right)^{1/\alpha}\right) f\left(z^{1/\alpha}[\eta(t)]\right) = 0, \tag{3.2}$$

$$w'(t) + cq(t) f\left(\int_{t_1}^{g(t)} \left(\frac{s}{a(s)}\right)^{1/\alpha} ds\right) f\left((\xi(t) - g(t))^{1/\alpha}\right) f\left(w^{1/\alpha}[\xi(t)]\right) = 0, \tag{3.3}$$

and

$$v'(t) + \overline{c}q(t) f(m[g(t)]) f\left(g^{2/\alpha}(t)\right) f\left(v^{1/\alpha}[g(t)]\right) = 0, \tag{3.4}$$

where $0 < c, \overline{c} < 1$ *are constants. If* (3.1)–(3.4) *are oscillatory, then* (1.2) *is oscillatory.*

Proof As in Theorem 2.1, let $x(t)$ be a nonoscillatory solution of (1.2), say, $x(t) > 0$, $x[g(t)] > 0$, and $x[\sigma(t)] > 0$ for $t \geq t_0 \geq 0$. Then there exists $t_1 \geq t_0$ such that one of the following four possibilities holds:

(I) $\left(a(t)(x'(t))^{\alpha}\right)'' > 0$, $\quad \left(a(t)(x'(t))^{\alpha}\right)' > 0$ and $x'(t) > 0$ for $t \geq t_1$;
(II) $\left(a(t)(x'(t))^{\alpha}\right)'' < 0$, $\quad \left(a(t)(x'(t))^{\alpha}\right)' > 0$ and $x'(t) < 0$ for $t \geq t_1$;
(III) $\left(a(t)(x'(t))^{\alpha}\right)'' < 0$, $\quad \left(a(t)(x'(t))^{\alpha}\right)' > 0$ and $x'(t) > 0$ for $t \geq t_1$;
(IV) $\left(a(t)(x'(t))^{\alpha}\right)'' < 0$, $\quad \left(a(t)(x'(t))^{\alpha}\right)' < 0$ and $x'(t) < 0$ for $t \geq t_1$.

We also note that the following four cases

$$\begin{array}{llll}
\left(a(t)(x'(t))^{\alpha}\right)'' > 0, & \left(a(t)(x'(t))^{\alpha}\right)' > 0 \text{ and } x'(t) < 0 & \text{for } t \ge t_1; & \\
\left(a(t)(x'(t))^{\alpha}\right)'' > 0, & \left(a(t)(x'(t))^{\alpha}\right)' < 0 \text{ and } x'(t) > 0 & \text{for } t \ge t_1; & \\
\left(a(t)(x'(t))^{\alpha}\right)'' > 0, & \left(a(t)(x'(t))^{\alpha}\right)' < 0 \text{ and } x'(t) < 0 & \text{for } t \ge t_1; & \text{and} \\
\left(a(t)(x'(t))^{\alpha}\right)'' < 0, & \left(a(t)(x'(t))^{\alpha}\right)' < 0 \text{ and } x'(t) > 0 & \text{for } t \ge t_1 &
\end{array}$$

can be disregarded. Next, we consider each of the four cases.

Case (I). Set $y(t) = a(t)(x'(t))^{\alpha}$ for $t \ge t_1$. Then, $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$, and $y'''(t) \ge 0$ for $t \ge t_1$. For $t \ge s \ge t_1$, we have

$$y(t) \ge \frac{(t-s)^2}{2!} y''(s), \quad \text{or}$$

$$x'(t) \ge \left(\frac{(t-s)^2}{2!a(t)}\right)^{1/\alpha} (y''(s))^{1/\alpha}.$$

It is easy to see that there exists $t_2 \ge t_1$ such that

$$x[\sigma(t)] \ge \left(\int_{\rho(t)}^{\sigma(t)} \left(\frac{[u-\rho(t)]^2}{2!a(u)}\right)^{1/\alpha} du\right) (y''[\rho(t)])^{1/\alpha} \quad \text{for } t \ge t_2. \tag{3.5}$$

Using (3.5) and (1.5) in (1.2), we obtain

$$z'(t) \ge p(t) h\left(\int_{\rho(t)}^{\sigma(t)} \left(\frac{[u-\rho(t)]^2}{2!a(u)}\right)^{1/\alpha} du\right) h\left(z^{1/\alpha}[\rho(t)]\right) \quad \text{for } t \ge t_2,$$

where $z(t) = y''(t)$. By results in [3] or [9], we arrive at the desired contradiction.

Case (II). For $t \ge s \ge t_1$, we have

$$x(s) \ge (t-s)(-x'(t)).$$

Replacing s and t by $g(t)$ and $\xi(t)$ respectively, we obtain

$$x[g(t)] \ge \frac{(\xi(t) - g(t))}{a^{1/\alpha}[\xi(t)]} y^{1/\alpha}[\xi(t)] \quad \text{for } t \ge t_2 \ge t_1, \tag{3.6}$$

where $y(t) = -a(t)(x'(t))^{\alpha}$. Using (3.6) and (1.4) in (1.2), we obtain

$$y'''(t) + f\left(\frac{\xi(t) - g(t)}{a^{1/\alpha}[\xi(t)]}\right) q(t) f\left(y^{1/\alpha}[\xi(t)]\right) \le 0 \quad \text{for } t \ge t_2. \tag{3.7}$$

Clearly, we have $y(t) > 0$, $y'(t) < 0$, and $y''(t) > 0$ for $t \ge t_2$. As above, we see that there exists a $t_3 \ge t_2$ such that

$$y[\xi(t)] \ge \left(\frac{[\eta(t) - \xi(t)]^2}{2!}\right) y''[\eta(t)] \quad \text{for } t \ge t_3. \tag{3.8}$$

Using (3.8) and (1.4) in (3.7), we then have

$$z'(t)+q(t)f\left(\frac{\xi(t)-g(t)}{a^{1/\alpha}[\xi(t)]}\right)f\left(\left(\frac{[\eta(t)-\xi(t)]^2}{2!}\right)^{1/\alpha}\right)f\left(z^{1/\alpha}[\eta(t)]\right)\le 0,$$

where $z(t)=y''(t)$ for $t\ge t_3$. The rest of the proof is similar to that of Case (I) in Theorem 2.1.

Case (III). There exist $t_2\ge t_1$ and a constant $k, 0<k<1$, such that $y(t)\ge kty'(t)$ for $t\ge t_2$, where $y(t)=a(t)(x'(t))^{\alpha}$ for $t\ge t_2$. Thus,

$$x'(t)\ge\left(k\frac{t}{a(t)}\right)^{1/\alpha}(y'(t))^{1/\alpha}\quad\text{for } t\ge t_2.$$

Integrating from t_2 to t, we have

$$x(t)\ge\overline{c}\left(\int_{t_2}^{t}\left(\frac{s}{a(s)}\right)^{1/\alpha}ds\right)(y'(t))^{1/\alpha}\quad\text{for } t\ge t_2,$$

where $\overline{c}=k^{1/\alpha}$. Now there exists $t_3\ge t_2$, such that

$$x[g(t)]\ge\overline{c}\left(\int_{t_2}^{g(t)}\left(\frac{s}{a(s)}\right)^{1/\alpha}ds\right)(y'[g(t)])^{1/\alpha}\quad\text{for } t\ge t_3, \tag{3.9}$$

and using (3.9) and (1.4) in (1.2), we have

$$z''(t)\ge\tilde{c}q(t)f\left(\int_{t_2}^{g(t)}\left(\frac{s}{a(s)}\right)^{1/\alpha}ds\right)f\left(z^{1/\alpha}[g(t)]\right)\quad\text{for } t\ge t_3, \tag{3.10}$$

where $z(t)=y'(t)$ and $\tilde{c}=f(\overline{c})$. Clearly, we see that $z(t)>0$ and $z'(t)<0$ for $t\ge t_3$. Thus, for $t\ge s\ge t_3$, we have

$$z(s)\ge(t-s)(-z'(t)).$$

Replacing s and t by $g(t)$ and $\xi(t)$ respectively, we have

$$z[g(t)]\ge(\xi(t)-g(t))(-z'[\xi(t)])\quad\text{for } t\ge t_4\ge t_3. \tag{3.11}$$

By (3.11) and (1.4), (3.10) yields

$$w'(t)+\tilde{c}q(t)f\left(\int_{t_2}^{t}\left(\frac{s}{a(s)}\right)^{1/\alpha}ds\right)f\left([\xi(t)-g(t)]^{1/\alpha}\right)f\left(w^{1/\alpha}[\xi(t)]\right)\le 0$$

for $t\ge t_4$, where $w(t)=-z'(t)$. The remainder of the proof is similar to that of Case (I) in Theorem 2.1.

Case (IV). For $s \geq t \geq t_1$, we have

$$-a(s)(x'(s))^{\alpha} \geq -a(t)(x'(t))^{\alpha}, \quad \text{or}$$
$$-x'(s) \geq \left(a^{-1/\alpha}(s)\right)\left(-a(t)(x'(t))^{\alpha}\right)^{1/\alpha}.$$

Integrating the above from t to $u \geq t \geq t_1$ and letting $u \to \infty$, we have

$$x(t) \geq m(t) y^{1/\alpha}(t) \quad \text{for } t \geq t_1,$$

where $y(t) = -a(t)(x'(t))^{\alpha}$ for $t \geq t_1$. There exists a $t_2 \geq t_1$ such that

$$x[g(t)] \geq m[g(t)] y^{1/\alpha}[g(t)] \quad \text{for } t \geq t_2, \tag{3.12}$$

so from (3.12) and (1.4), (1.2) becomes

$$y'''(t) + q(t) f(m[g(t)]) f\left(y^{1/\alpha}[g(t)]\right) \leq 0 \quad \text{for } t \geq t_2. \tag{3.13}$$

Clearly, $y(t) > 0$, $y'(t) > 0$ and $y''(t) > 0$ for $t \geq t_2$. so there exist $t_3 \geq t_2$ and a constant k, $0 < k < 1$ such that

$$y'(t) \geq k t y''(t) \quad \text{for } t \geq t_3.$$

Integrating the above inequality from t_3 to t, we have

$$y(t) \geq \frac{k}{2} t^2 y''(t) \quad \text{for } t \geq t_3,$$

and so there exists $t_4 \geq t_3$ such that

$$y[g(t)] \geq \frac{k}{2} g^2(t) z[g(t)] \quad \text{for } t \geq t_4, \tag{3.14}$$

where $z(t) = y''(t)$ for $t \geq t_4$. Substituting (3.14) into (3.13), we obtain

$$z'(t) + c q(t) f(m[g(t)]) f\left(g^{2/\alpha}(t)\right) f\left(z^{1/\alpha}[g(t)]\right) \leq 0 \quad \text{for } t \geq t_4,$$

where $c = f((k/2)^{\alpha})$. The rest of the proof is similar to that of Case (I) in Theorem 2.1 This completes the proof of the theorem. □

We note that if condition (1.3) is replaced by condition (1.6), then from the proof of Theorem 3.1, we see that the Cases (I)–(III) hold, while Case (IV) does not. In this situation, we obtain the following result.

Theorem 3.2 *Let conditions* (i)–(iv) *and* (1.4)–(1.6) *hold and suppose that there exist nondecreasing functions* $\xi(t)$, $\eta(t)$ *and* $\rho(t) \in C([t_0, \infty), \mathbb{R})$ *such that* $g(t) < \xi(t) < \eta(t) < t$ *and* $\sigma(t) > \rho(t) > t$ *for* $t \geq t_0$. *If the* (3.1)–(3.3) *are oscillatory, then* (1.2) *is oscillatory.*

Now for $t \geq t_1 \geq t_0$, we let

$$P(t) = p(t)\left(\int_{\rho(t)}^{\sigma(t)} \left(\frac{[u-\rho(t)]^2}{2!a(u)}\right)^{1/\alpha} du\right),$$

and

$$\tilde{Q}(t) = \min\left\{q(t) f\left(\frac{\xi(t)-g(t)}{a^{1/\alpha}[\xi(t)]}\right) f\left(\left(\frac{[\eta(t)-\xi(t)]^2}{2!}\right)^{1/\alpha}\right),\right.$$
$$cq(t) f\left(\int_{t_1}^{g(t)} \left(\frac{s}{a(s)}\right)^{1/\alpha} ds\right) f\left((\xi(t)-g(t))^{1/\alpha}\right),$$
$$\left.\overline{c}q(t) f(m[g(t)]) f\left(g^{2/\alpha}(g(t))\right)\right\},$$

for any constants $0 < c, \overline{c} < 1$. In Theorem 3.1, (3.1) takes the form

$$y'(t) - P(t)h\left(y^{1/\alpha}[\rho(t)]\right) = 0 \tag{3.15}$$

and combining (3.2)–(3.4) into one equation yields

$$z'(t) + \tilde{Q}(t) f\left(z^{1/\alpha}[\eta(t)]\right) = 0. \tag{3.16}$$

In Sect. 2, we stated some known sufficient conditions for the oscillation of (3.16). Now, we do the same for (3.15).

(3.15) is oscillatory if one of the following conditions holds:

(II$_1$) $\frac{f(u^{1/\alpha})}{u} \geq 1$ for $u \neq 0$, and

$$\liminf_{t\to\infty} \int_t^{\rho(t)} P(s)ds > \frac{1}{e}. \tag{3.17}$$

(II$_2$) $\int^{\pm\infty} \frac{du}{h(u^{1/\alpha})} < \infty$, and

$$\int^{\infty} \tilde{Q}(s)ds = \infty. \tag{3.18}$$

We can apply these sufficient conditions for the oscillation of (3.15) and (3.16) to obtain new criteria for the oscillation of (1.2). The details are left to the reader.

Remarks

1. The results of this section can be obtained for equations of the form

$$\left(a(t)(x'(t))^{\alpha}\right)''' = q(t) f(x[g(t)]),$$

where $\alpha, a(t), q(t)$ and $f(x)$ are as in (i)–(iv) and $g(t) \in C([t_0, \infty), \mathbb{R})$, $g'(t) \geq 0$, $\lim_{t\to\infty} g(t) = \infty$, and $g(t)$ is of mixed type, say, $g(t) = t - \sin t$.

2. Conditions (1.4) and (1.5) are automatic if we let $f(x)=x^{\beta}$ and $h(x)=x^{\gamma}$, where β and γ are the ratios of positive odd integers. As an illustrative example, we consider the equation

$$\left(t^{6}(x'(t))^{3}\right)'''=t^{\beta/6}x^{\beta}\left[\sqrt{t}\right]+t^{\gamma/3}x^{\gamma}[2t]\quad \text{for } t\geq 5. \tag{3.19}$$

Here, $a(t)=t^{6}$, $p(t)=t^{\gamma/3}, q(t)=t^{\beta/6}, g(t)=\sqrt{t}, \sigma(t)=2t$ and $\alpha=3, \beta$ and γ are the ratios of positive odd integers and $\beta\leq\alpha\leq\gamma$. For $t\geq 5$, we let $\xi(t)=3\sqrt{t}/2, \eta(t)=2\sqrt{t}$ and $\rho(t)=3t/2$. It is easy to check that all conditions of Theorem 3.1 together with the conditions (I_1)–(I_3) and (II_1)–(II_2) are satisfied and hence we conclude that (3.19) is oscillatory.

3. The results of this paper can be extended to neutral equations of the form

$$\frac{d^{3}}{dt^{3}}\left(a(t)\left(\frac{d}{dt}[x(t)+c(t)x[\tau(t)]]\right)^{\alpha}\right)+q(t)f(x[g(t)])=0$$

and

$$\frac{d^{3}}{dt^{3}}\left(a(t)\left(\frac{d}{dt}[x(t)+c(t)x[\tau(t)]]\right)^{\alpha}\right)=q(t)f(x[g(t)])+p(t)h(x[\sigma(t)]),$$

where the functions $a(t), p(t), q(t), g(t), \sigma(t), f(x)$ and $h(x)$ and α are as in (1.1) and (1.2), $c(t)$ and $\tau(t)\in C([t_0,\infty),\mathbb{R})$ and $\lim_{t\to\infty}\tau(t)=\infty$. We may assume that $0<c(t)<1$, $c(t)>1$, or $c(t)<-1$ for $t\geq t_0$. The details are left to the reader.

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