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Strong convergence theorems for generalized asymptotically quasi-nonexpansive mappings

Jing Zhao · Songnian He · Guangxuan Liu

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Abstract The purpose of this paper is to prove strong convergences of a modified implicit iteration process to a common fixed point for a finite family of generalized asymptotically quasi-nonexpansive mappings. The results presented in this paper improve and extend Shahzad and Zegeye's corresponding results (Shahzad and Zegeye in Appl. Math. Comput. 189:1058–1065, 2007)

Keywords Implicit iterative · Generalized asymptotically quasi-nonexpansive · Strong convergence · Common fixed point

Mathematics Subject Classification (2000) 47H09 · 47H10 · 47J05 · 54H25

1 Introduction and preliminaries

Let E be a real Banach space, K be a nonempty closed convex subset of E. Let T be a self-mapping of K. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T)=\{x\in K : Tx=x\}$. T is said to be nonexpansive if $\|Tx-Ty\|\le\|x-y\|$ for all $x,y\in K$. T is said to be asymptotically nonexpansive if there exists a sequence $\{u_n\}\subset[0,\infty)$ with $\lim_{n\to\infty}u_n=0$ such that

$$\|T^n x-T^n y\|\le(1+u_n)\|x-y\| \quad \forall n\ge 1,$$

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J. Zhao (✉) · S. He · G. Liu
College of Science, Civil Aviation University of China, Tianjin 300300, People's Republic of China
e-mail: zhaojing200103@163.com

S. He
e-mail: hesongnian2003@yahoo.com.cn

G. Liu
e-mail: guangxuanliu@163.com

for all $x, y \in K$. T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\forall x \in K$, the following inequality holds:

$$\|T^n x - x^*\| \leq (1 + u_n)\|x - x^*\| \quad \forall x^* \in F(T), \quad \forall n \geq 1.$$

It is clear from this definition that every asymptotically nonexpansive mapping with a fixed point is asymptotically quasi-nonexpansive. T is said to be generalized asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exist sequences of real numbers $\{u_n\}, \{c_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = \lim_{n\to\infty} c_n = 0$ such that $\forall x \in K$, the following inequality holds:

$$\|T^n x - x^*\| \leq \|x - x^*\| + u_n\|x - x^*\| + c_n \quad \forall x^* \in F(T), \quad \forall n \geq 1. \tag{1.1}$$

If in definition (1.1), $c_n = 0$ for all $n \geq 1$ then T becomes asymptotically quasi-nonexpansive mapping and hence the class of generalized asymptotically quasi-nonexpansive mappings includes the class of asymptotically quasi-nonexpansive mappings. The following example shows that the inclusion is proper. Let $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and define (see [5]) $Tx = \frac{x}{2}\sin(\frac{1}{x})$ if $x \neq 0$ and $Tx = 0$ if $x = 0$. Then $T^n x \to 0$ uniformly but T is not Lipschitzian. Notice that $F(T) = \{0\}$. For each fixed n, define $f_n(x) = \|T^n x\| - \|x\|$ for $x \in K$. Set $c_n = \sup_{x\in K}\{f_n(x), 0\}$. Then $\lim_{n\to\infty} c_n = 0$ and

$$\|T^n x\| \leq \|x\| + c_n.$$

This show that T is a generalized asymptotically quasi-nonexpansive but it is not asymptotically quasi-nonexpansive and asymptotically nonexpansive because it is not Lipschitzian. T is said to be uniformly *L-Lipschitzian* if there exists a constant L such that $\forall x, y \in K$, the following inequality holds:

$$\|T^n x - T^n y\| \leq L\|x - y\| \quad \forall n \geq 1.$$

In 2001, Xu and Ori [8] introduced implicit iteration scheme and proved weak convergence theorem for approximating common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in Hilbert spaces, which is defined as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_{n \bmod N} x_n, \quad n \geq 1.$$

In 2002, Zhou and Chang [9] introduced implicit iteration scheme and proved weak and strong convergence theorems for approximating common fixed points of a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$ in Banach spaces, which is generated as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T^n_{n \bmod N} x_n, \quad n \geq 1.$$

In [6], Sun defined an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \ldots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in K$, as follows:

$$
\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1^2 x_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N}) T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1}) T_1^3 x_{2N+1}, \\
&\vdots
\end{aligned}
$$

which can be written in the following compact form:

$$
x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_i^k x_n, \quad \forall n \geq 1, \tag{1.2}
$$

where $n = (k-1)N + i$, $T_n = T_{n(\bmod N)} = T_i$, $i \in I$. Sun proved the strong convergence of this process to a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings in Banach space.

More recently, Shahzad and Zegeye [5] proved the strong convergence of the implicit iteration process (1.2) to a common fixed point for the finite family of generalized asymptotically quasi-nonexpansive mappings in Banach space.

In this paper, we introduce a new implicit iteration process

$$
\begin{cases} x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + \gamma_n T_i^k x_n, \quad \forall n \geq 1, \\ \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1], \end{cases} \tag{1.3}
$$

where $n = (k-1)N + i$, $T_n = T_{n(\bmod N)} = T_i$ and $i \in I$, for common fixed points of a finite family of generalized asymptotically quasi-nonexpansive mapping $\{T_i : i \in I\}$ and prove strong convergence theorems. Our results improve and extend the corresponding ones of Shahzad and Zegeye [5].

Throughout the paper, we shall assume that $(I - tT_i^p)^{-1}$ exists for all $t \in (0, 1)$, $i = 1, 2, \dots, N$ and all $p \geq 1$. Then for an initial point $x_0 \in K$, the implicit iteration scheme (1.3) can be employed for the approximation of common fixed points of a finite family of generalized asymptotically quasi-nonexpansive mappings.

Now, we give some definitions and lemmas for our main results.

Let $d(x, F)$ denote the distance of x to a set $F \subset K$, i.e. $d(x, F) = \inf\{d(x, y) : y \in F\}$.

A mapping $T : K \to K$ is said to be *semi-compact* if, for any bounded sequence $\{x_n\} \subset K$ with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $x^* \in K$.

A family $\{T_i : i \in I\}$ of N self-mappings of K with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy

(1) condition (B) on K [2] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ and all $x \in K$ such that

$$\max_{1 \le l \le N} \{\|x - T_l x\|\} \ge f(d(x, F));$$

(2) condition (C) on K [2] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ and all $x \in K$ such that

$$\frac{1}{N} \sum_{l=1}^{N} \|x - T_l x\| \ge f(d(x, F));$$

(3) condition $(\overline{C})$ on K [1] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ and all $x \in K$ such that

$$\|x - T_l x\| \ge f(d(x, F))$$

for at least one $T_l,\ l = 1, \dots, N$. Note that conditions (B) and $(\overline{C})$ are equivalent (see [1]).

Lemma 1.1 [3] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \le (1 + \delta_n)a_n + b_n, n \ge 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists. Moreover, if there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} \to 0$ as $j \to \infty$, then $a_n \to 0$ as $n \to \infty$.*

Lemma 1.2 [4] *Let E be a uniformly convex Banach space and a, b be two constants with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a, b]$ is a real sequence and $\{x_n\}$, $\{y_n\}$ are two sequences in E. Then the conditions*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = d, \qquad \limsup_{n\to\infty} \|x_n\| \le d, \qquad \limsup_{n\to\infty} \|y_n\| \le d,$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \ge 0$ is a constant.

Lemma 1.3 [7] *Let $p > 1$ and $R > 1$ be two fixed numbers and E be a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| \le R\}$ and $\lambda \in [0, 1]$, where $W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.*

2 Main results

Theorem 2.1 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N generalized asymptotically quasi-nonexpansive self-mappings of K with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$ for all $i \in I$. Suppose $F = \bigcap_{i=1}^{N} F(T_i) \ne \emptyset$ is closed. Let $\{\alpha_n + \beta_n\}_{n\ge1} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \ge 1$. Starting from arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (1.3). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof We only prove the sufficiency. Let $x^* \in F$. Then from (1.3) we obtain that

$$\begin{aligned}\|x_n - x^*\| &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_{n-1} + \gamma_n T_i^k x_n - x^*\| \\ &\le \alpha_n \|x_{n-1} - x^*\| + \beta_n \|T_i^k x_{n-1} - x^*\| + \gamma_n \|T_i^k x_n - x^*\| \\ &\le \alpha_n \|x_{n-1} - x^*\| + \beta_n [(1 + u_{ik})\|x_{n-1} - x^*\| + c_{ik}] \\ &\quad + \gamma_n [(1 + u_{ik})\|x_n - x^*\| + c_{ik}] \\ &\le \alpha_n \|x_{n-1} - x^*\| + (\beta_n + u_{ik})\|x_{n-1} - x^*\| + \beta_n c_{ik} \\ &\quad + (\gamma_n + u_{ik})\|x_n - x^*\| + \gamma_n c_{ik} \\ &= (\alpha_n + \beta_n + u_{ik})\|x_{n-1} - x^*\| + \gamma_n \|x_n - x^*\| \\ &\quad + u_{ik}\|x_n - x^*\| + \beta_n c_{ik} + \gamma_n c_{ik}.\end{aligned}$$

From the above inequality we get

$$(1 - \gamma_n)\|x_n - x^*\| \le (\alpha_n + \beta_n + u_{ik})\|x_{n-1} - x^*\| + u_{ik}\|x_n - x^*\| + (\beta_n + \gamma_n)c_{ik}.$$

It follows from the condition $\delta \le \alpha_n + \beta_n \le 1 - \delta$ that $\delta \le 1 - \gamma_n \le 1 - \delta$. So we have

$$\begin{aligned}\|x_n - x^*\| &\le \left(1 + \frac{u_{ik}}{1 - \gamma_n}\right)\|x_{n-1} - x^*\| + \frac{u_{ik}}{1 - \gamma_n}\|x_n - x^*\| + \frac{1 - \alpha_n}{1 - \gamma_n} c_{ik} \\ &\le \left(1 + \frac{u_{ik}}{\delta}\right)\|x_{n-1} - x^*\| + \frac{u_{ik}}{\delta}\|x_n - x^*\| + \frac{1}{\delta} c_{ik}.\end{aligned}$$

Thus,

$$\left(1 - \frac{u_{ik}}{\delta}\right)\|x_n - x^*\| \le \left(1 + \frac{u_{ik}}{\delta}\right)\|x_{n-1} - x^*\| + \frac{1}{\delta} c_{ik}. \tag{2.1}$$

Since $\sum_{k=1}^{\infty} u_{ik} < \infty$ for all $i \in I$, we have that $\lim_{k\to\infty} u_{ik} = 0$ and hence there exists a natural number n_1 such that $u_{ik} < \frac{\delta}{2}$ for $k \ge \frac{n_1}{N} + 1$ or $n > n_1$. Thus, from (2.1), for $n > n_1$ we obtain that

$$\begin{aligned}\|x_n - x^*\| &\le \frac{\delta + u_{ik}}{\delta - u_{ik}}\|x_{n-1} - x^*\| + \frac{1}{\delta - u_{ik}} c_{ik} \\ &= \left(1 + \frac{2u_{ik}}{\delta - u_{ik}}\right)\|x_{n-1} - x^*\| + \frac{c_{ik}}{\delta - u_{ik}}.\end{aligned} \tag{2.2}$$

Let $v_{ik} = \frac{2u_{ik}}{\delta - u_{ik}}$ and $w_{ik} = \frac{c_{ik}}{\delta - u_{ik}}$. Then for $n > n_1$, we see that $v_{ik} < \frac{4}{\delta} u_{ik}$ and $w_{ik} < \frac{2}{\delta} c_{ik}$ from $u_{ik} < \frac{\delta}{2}$. Now from (2.2), for $n > n_1$, we get that

$$\|x_n - x^*\| \le (1 + v_{ik})\|x_{n-1} - x^*\| + w_{ik}. \tag{2.3}$$

Since $\sum_{k=1}^{\infty} u_{ik} < \infty$ and $\sum_{k=1}^{\infty} c_{ik} < \infty$, it follows that $\sum_{k=1}^{\infty} v_{ik} < \infty$ and $\sum_{k=1}^{\infty} w_{ik} < \infty$ for all $i \in I$. Thus, by Lemma 1.1 we have that $\lim_{n\to\infty} \|x_n - x^*\|$

exists and hence $\{x_n\}$ is a bounded sequence. Since, for $n > n_1$,

$$d(x_n, F) \leq (1 + v_{ik})d(x_{n-1}, F) + w_{ik}$$

and by assumption $\liminf_{n\to\infty} d(x_n, F) = 0$ we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Notice that from (2.3) for any $x^* \in F$ we have

$$\begin{aligned} \|x_{n+m} - x^*\| &\leq \exp\left\{\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{ik}\right\}\|x_n - x^*\| + \exp\left\{\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{ik}\right\}\sum_{i=1}^{N}\sum_{k=n}^{\infty} w_{ik} \\ &< M\left(\|x_n - x^*\| + \sum_{i=1}^{N}\sum_{k=n}^{\infty} w_{ik}\right) \end{aligned} \tag{2.4}$$

for all natural numbers m, n, where $M = \exp\{\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{ik}\} + 1 < \infty$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{k=1}^{\infty} w_{ik} < \infty$ for all $i \in I$, given any $\epsilon > 0$, there exists a natural number $n_0 > n_1$ such that $d(x_n, F) < \frac{\epsilon}{4M}$ and $\sum_{i=1}^{N}\sum_{k=n}^{\infty} w_{ik} < \frac{\epsilon}{4M}$ for all $n \geq n_0$. So we can find $y^* \in F$ such that $\|x_{n_0} - y^*\| \leq \frac{\epsilon}{4M}$. Hence, for all $n \geq n_0$ and $m \geq 1$, we have that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - y^*\| + \|x_n - y^*\| \\ &< M\left(\|x_{n_0} - y^*\| + \sum_{i=1}^{N}\sum_{k=n_0}^{\infty} w_{ik}\right) + M\left(\|x_{n_0} - y^*\| + \sum_{i=1}^{N}\sum_{k=n_0}^{\infty} w_{ik}\right) \\ &< M\left(\frac{\epsilon}{4M} + \frac{\epsilon}{4M} + \frac{\epsilon}{4M} + \frac{\epsilon}{4M}\right) = \epsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} x_n = z^*$. Since K is closed we have $z^* \in K$. It remains to show that $z^* \in F$. Notice that

$$\|z^* - x^*\| \leq \|z^* - x_n\| + \|x_n - x^*\|$$

for all $x^* \in F, n \geq 1$. So we obtain that

$$0 \leq d(z^*, F) \leq \|z^* - x_n\| + d(x_n, F)$$

for all $n \geq 1$. Since $\lim_{n\to\infty} x_n = z^*$ and $\lim_{n\to\infty} d(x_n, F) = 0$, we have that $d(z^*, F) = 0$. From assumption that F is closed we have that $z^* \in F$. This completes the proof. □

The following corollary follows from Theorem 2.1

Corollary 2.2 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N generalized asymptotically quasi-nonexpansive self-mappings of K with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$ for all $i \in I$. Suppose $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is closed. Let $\{\alpha_n + \beta_n\}_{n\geq 1} \subset [\delta, 1-\delta]$*

for some $\delta \in (0,1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$. *Starting from arbitrary* $x_0 \in K$, *define the sequence* $\{x_n\}$ *by* (1.3). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in I\}$ *if and only if there exists some subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$ *which converges to* $x^* \in F$.

Corollary 2.3 *Let* E *be a real Banach space and* K *be a nonempty closed convex subset of* E. *Let* $\{T_i : i \in I\}$ *be* N *asymptotically quasi-nonexpansive self-mappings of* K *with* $\{u_{in}\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *for all* $i \in I$. *Suppose* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n + \beta_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$. *Starting from arbitrary* $x_0 \in K$, *define the sequence* $\{x_n\}$ *by* (1.3). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in I\}$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof The corollary follows from Theorem 2.1 with $c_{in} = 0$ for all $n \geq 1$. □

Corollary 2.4 *Let* E *be a real Banach space and* K *be a nonempty closed convex subset of* E. *Let* $\{T_i : i \in I\}$ *be* N *asymptotically nonexpansive self-mappings of* K *with* $\{u_{in}\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *for all* $i \in I$. *Suppose* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n + \beta_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$. *Starting from arbitrary* $x_0 \in K$, *define the sequence* $\{x_n\}$ *by* (1.3). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mapping* $\{T_i : i \in I\}$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof The corollary follows from Theorem 2.1 with $c_{in} = 0$ for all $n \geq 1$. □

Corollary 2.5 (Shahzad and Zegeye [5, Theorem 3.3]) *Let* E *be a real Banach space and* K *be a nonempty closed convex subset of* E. *Let* $\{T_i : i \in I\}$ *be* N *generalized asymptotically quasi-nonexpansive self-mappings of* K *with* $\{u_{in}\}, \{c_{in}\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *and* $\sum_{n=1}^{\infty} c_{in} < \infty$ *for all* $i \in I$. *Suppose* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *is closed. Let* $\{\alpha_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$. *Starting from arbitrary* $x_0 \in K$, *define the sequence* $\{x_n\}$ *by* (1.2). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mapping* $\{T_i : i \in I\}$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof Put $\beta_n = 0$ for $n \geq 1$ in (1.3). Then by Theorem 2.1 the corollary follows. □

Lemma 2.6 *Let* E *be a real uniformly convex Banach space and* K *be a nonempty closed convex subset of* E. *Let* $\{T_i : i \in I\}$ *be* N *uniformly* L-*Lipschitzian, generalized asymptotically quasi-nonexpansive self-mappings of* K *with* $\{u_{in}\}, \{c_{in}\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *and* $\sum_{n=1}^{\infty} c_{in} < \infty$ *for all* $i \in I$. *Suppose* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}_{n\geq 1} \subset [\delta_1, 1-\delta_1]$ *for some* $\delta_1 \in (0,1)$, $\{\alpha_n + \beta_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$. *Starting from arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by* (1.3). *Then* $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ *for all* $l \in I$.

Proof Notice that $\{x_n\}$ is bounded (see proof of Theorem 2.1). So, there exist $R > 0$ such that $x_n \in B_R(0)$ for all $n \geq 1$. For given $x^* \in F$, we have $\lim_{n\to\infty} \|x_n - x^*\| = d$

(see proof of Theorem 2.1). Then we obtain that

$$\begin{aligned}
&\lim_{n\to\infty}\|x_n-x^*\|\\
&\quad=\lim_{n\to\infty}\left\|(1-\gamma_n)\left[\frac{\alpha_n}{1-\gamma_n}x_{n-1}+\frac{\beta_n}{1-\gamma_n}T_i^kx_{n-1}-x^*\right]+\gamma_n(T_i^kx_n-x^*)\right\|=d.
\end{aligned}$$

Again,

$$\begin{aligned}
&\limsup_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}x_{n-1}+\frac{\beta_n}{1-\gamma_n}T_i^kx_{n-1}-x^*\right\|\\
&\quad\le\limsup_{n\to\infty}\left(\frac{\alpha_n}{1-\gamma_n}\|x_{n-1}-x^*\|+\frac{\beta_n}{1-\gamma_n}\|T_i^kx_{n-1}-x^*\|\right)\\
&\quad\le\limsup_{n\to\infty}\left\{\frac{\alpha_n}{1-\gamma_n}\|x_{n-1}-x^*\|+\frac{\beta_n}{1-\gamma_n}[(1+u_{ik})\|x_{n-1}-x^*\|+c_{ik}]\right\}\\
&\quad\le\limsup_{n\to\infty}\left\{\frac{\alpha_n}{1-\gamma_n}\|x_{n-1}-x^*\|+\frac{\beta_n}{1-\gamma_n}(1+u_{ik})\|x_{n-1}-x^*\|+c_{ik}\right\}\le d
\end{aligned}$$

and

$$\limsup_{n\to\infty}\|T_i^kx_n-x^*\|\le\limsup_{n\to\infty}[(1+u_{ik})\|x_n-x^*\|+c_{ik}]\le d.$$

By Lemma 1.2 and the condition $0<\delta<1-\gamma_n<1-\delta<1$, we get that

$$\lim_{n\to\infty}\left\|\left(\frac{\alpha_n}{1-\gamma_n}x_{n-1}+\frac{\beta_n}{1-\gamma_n}T_i^kx_{n-1}-x^*\right)-(T_i^kx_n-x^*)\right\|=0.$$

So $\lim_{n\to\infty}\|\alpha_nx_{n-1}+\beta_nT_i^kx_{n-1}-(1-\gamma_n)T_i^kx_n\|=0$. Because

$$\begin{aligned}
&\|\alpha_nx_{n-1}+\beta_nT_i^kx_{n-1}-(1-\gamma_n)T_i^kx_n\|\\
&\quad=\|x_n-\gamma_nT_i^kx_n-(1-\gamma_n)T_i^kx_n\|=\|x_n-T_i^kx_n\|,\quad\forall n\ge1,
\end{aligned}$$

we know that

$$\lim_{n\to\infty}\|x_n-T_i^kx_n\|=0. \tag{2.5}$$

From (1.3) and Lemma 1.3 we have that

$$\begin{aligned}
&\|x_n-x^*\|^p=\|\alpha_nx_{n-1}+\beta_nT_i^kx_{n-1}+\gamma_nT_i^kx_n-x^*\|^p\\
&\quad=\left\|\alpha_n(x_{n-1}-x^*)+(1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n}T_i^kx_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^kx_n-x^*\right]\right\|^p\\
&\quad\le\alpha_n\|x_{n-1}-x^*\|^p+(1-\alpha_n)\left\|\frac{\beta_n}{1-\alpha_n}T_i^kx_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^kx_n-x^*\right\|^p\\
&\qquad-W_p(\alpha_n)g\left(\left\|\frac{\beta_n}{1-\alpha_n}T_i^kx_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^kx_n-x^*-x_{n-1}+x^*\right\|\right).
\end{aligned}$$

So that using (2.3), we obtain

$$
\begin{aligned}
&2\delta_1^{p+1} g\left(\left\|\frac{\beta_n}{1-\alpha_n}T_i^k x_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^k x_n - x_{n-1}\right\|\right)\\
&\le \alpha_n\|x_{n-1}-x^*\|^p+(1-\alpha_n)\left\|\frac{\beta_n}{1-\alpha_n}(T_i^k x_{n-1}-x^*)+\frac{\gamma_n}{1-\alpha_n}(T_i^k x_n - x^*)\right\|^p\\
&\quad -\|x_n-x^*\|^p\\
&\le \alpha_n\|x_{n-1}-x^*\|^p+(1-\alpha_n)\frac{\beta_n}{1-\alpha_n}\|T_i^k x_{n-1}-x^*\|^p\\
&\quad +(1-\alpha_n)\frac{\gamma_n}{1-\alpha_n}\|T_i^k x_n - x^*\|^p-\|x_n-x^*\|^p\\
&\le \alpha_n\|x_{n-1}-x^*\|^p+\beta_n[(1+u_{ik})\|x_{n-1}-x^*\|+c_{ik}]^p\\
&\quad +\gamma_n[(1+u_{ik})\|x_n-x^*\|+c_{ik}]^p-\|x_n-x^*\|^p\\
&\le \alpha_n\|x_{n-1}-x^*\|^p+\beta_n[\|x_{n-1}-x^*\|+u_{ik}\|x_{n-1}-x^*\|+c_{ik}]^p\\
&\quad +\gamma_n[(1+u_{ik})(1+v_{ik})\|x_{n-1}-x^*\|+(1+u_{ik})w_{ik}+c_{ik}]^p-\|x_n-x^*\|^p
\end{aligned}
$$

which implies (for $p=2$) that

$$
\begin{aligned}
&2\delta_1^3 g\left(\left\|\frac{\beta_n}{1-\alpha_n}T_i^k x_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^k x_n - x_{n-1}\right\|\right)\\
&\le \alpha_n\|x_{n-1}-x^*\|^2+\beta_n\|x_{n-1}-x^*\|^2+\gamma_n\|x_{n-1}-x^*\|^2\\
&\quad +(u_{ik}+c_{ik}+v_{ik}+w_{ik})V-\|x_n-x^*\|^2
\end{aligned}
$$

for some constant $V>0$. This gives that

$$
\begin{aligned}
&2\delta_1^3 g\left(\left\|\frac{\beta_n}{1-\alpha_n}T_i^k x_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^k x_n - x_{n-1}\right\|\right)\\
&\le \|x_{n-1}-x^*\|^2-\|x_n-x^*\|^2+(u_{ik}+c_{ik}+v_{ik}+w_{ik})V.
\end{aligned}
$$

Since $\sum_{k=1}^{\infty}(u_{ik}+c_{ik}+v_{ik}+w_{ik})<\infty$ for all $i\in I$ and $\lim_{n\to\infty}\|x_n-x^*\|$ exists (see proof of Theorem 2.1), we obtain that

$$
\sum_{n=1}^{\infty} g\left(\left\|\frac{\beta_n}{1-\alpha_n}T_i^k x_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^k x_n - x_{n-1}\right\|\right)<\infty.
$$

Hence

$$
\lim_{n\to\infty} g\left(\left\|\frac{\beta_n}{1-\alpha_n}T_i^k x_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^k x_n - x_{n-1}\right\|\right)=0.
$$

But g is strictly increasing, continuous and $g(0)=0$. Therefore,

$$\lim_{n\to\infty}\left\|\frac{\beta_n}{1-\alpha_n}T_i^k x_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^k x_n - x_{n-1}\right\| = 0.$$

So

$$\begin{aligned}\lim_{n\to\infty}\|x_n - x_{n-1}\| &= \lim_{n\to\infty}\|\alpha_n x_{n-1}+\beta_n T_i^k x_{n-1}+\gamma_n T_i^k x_n - x_{n-1}\|\\ &= \lim_{n\to\infty}(1-\alpha_n)\left\|\frac{\beta_n}{1-\alpha_n}T_i^k x_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_i^k x_n - x_{n-1}\right\| = 0.\end{aligned}$$

Also

$$\lim_{n\to\infty}\|x_n - x_{n+l}\| = 0 \tag{2.6}$$

for all $l\in I$. Clearly, $i = n \bmod N = (n-N)\bmod N$. So $T_n = T_{n-N}$ and we have that

$$\begin{aligned}&\|T_i^k x_n - T_n x_n\| = \|T_n^k x_n - T_n x_n\|\\ &\quad\le \|T_n^k x_n - T_n^k x_{n-N}\| + \|T_n^k x_{n-N} - T_n x_{n-N}\| + \|T_n x_{n-N} - T_n x_n\|\\ &\quad= \|T_n^k x_n - T_n^k x_{n-N}\| + \|T_{n-N}^k x_{n-N} - T_{n-N} x_{n-N}\| + \|T_n x_{n-N} - T_n x_n\|.\end{aligned}$$

Hence, for $n > N$,

$$\begin{aligned}&\|x_n - T_n x_n\|\\ &\quad\le \|x_n - T_i^k x_n\| + \|T_i^k x_n - T_n x_n\|\\ &\quad\le \|x_n - T_i^k x_n\| + \|T_n^k x_n - T_n^k x_{n-N}\| + \|T_{n-N}^k x_{n-N} - T_{n-N} x_{n-N}\|\\ &\quad\quad + \|T_n x_{n-N} - T_n x_n\|.\end{aligned}$$

Since each T_i is uniformly L-*Lipschitzian*, from (2.5) and (2.6), we conclude that

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{2.7}$$

Also for all $l\in I$

$$\|x_n - T_{n+l}x_n\| \le \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\|.$$

Since each T_i is uniformly L-*Lipschitzian*, from (2.6) and (2.7), we obtain that

$$\lim_{n\to\infty}\|x_n - T_{n+l}x_n\| = 0$$

for all $l\in I$. Consequently, we have $\lim_{n\to\infty}\|x_n - T_l x_n\| = 0$ for all $l\in I$. This completes the proof. □

Theorem 2.7 *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i\in I\}$ be N uniformly L-Lipschitzian, generalized asymptotically quasi-nonexpansive self-mappings of K with $\{u_{in}\}, \{c_{in}\} \subset$*

$[0, \infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *and* $\sum_{n=1}^{\infty} c_{in} < \infty$ *for all* $i \in I$. *Suppose* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and there exists one member* T *in* $\{T_i : i \in I\}$ *which either is semi-compact or satisfies condition* $(\overline{C})$. *Let* $\{\alpha_n\}_{n\geq 1} \subset [\delta_1, 1-\delta_1]$ *for some* $\delta_1 \in (0, 1)$, $\{\alpha_n + \beta_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0, 1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$. *Starting from arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by* (1.3). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in I\}$.

Proof Without loss of generality, we may assume that T_1 either is semi-compact or satisfies condition $(\overline{C})$. By Lemma 2.6, we have $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for all $l \in I$. If T_1 is semi-compact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in K$ as $j \to \infty$. Now Lemma 2.6 guarantees that $\lim_{j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0$ for all $l \in I$ and so $\|x^* - T_l x^*\| = 0$ for all $l \in I$. This implies that $x^* \in F$. So $\liminf_{n\to\infty} d(x_n, F) = 0$. From Theorem 2.1, we know that $\{x_n\}$ converges strongly to some common fixed point in F. If T_1 satisfies condition $(\overline{C})$, then we have $\liminf_{n\to\infty} d(x_n, F) = 0$. Now apply Theorem 2.1. This completes the proof. □

Corollary 2.8 *Let* E *be a real uniformly convex Banach space and* K *be a nonempty closed convex subset of* E. *Let* $\{T_i : i \in I\}$ *be* N *uniformly* L*-Lipschitzian, asymptotically quasi-nonexpansive self-mappings of* K *with* $\{u_{in}\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *for all* $i \in I$. *Suppose* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and there exists one member* T *in* $\{T_i : i \in I\}$ *which either is semi-compact or satisfies condition* $(\overline{C})$. *Let* $\{\alpha_n\}_{n\geq 1} \subset [\delta_1, 1-\delta_1]$ *for some* $\delta_1 \in (0, 1)$, $\{\alpha_n + \beta_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0, 1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$. *Starting from arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by* (1.3). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in I\}$.

Corollary 2.9 *Let* E *be a real uniformly convex Banach space and* K *be a nonempty closed convex subset of* E. *Let* $\{T_i : i \in I\}$ *be* N *asymptotically nonexpansive self-mappings of* K *with* $\{u_{in}\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *for all* $i \in I$. *Suppose* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and there exists one member* T *in* $\{T_i : i \in I\}$ *which either is semi-compact or satisfies condition* $(\overline{C})$. *Let* $\{\alpha_n\}_{n\geq 1} \subset [\delta_1, 1-\delta_1]$ *for some* $\delta_1 \in (0, 1)$, $\{\alpha_n + \beta_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0, 1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$. *Starting from arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by* (1.3). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in I\}$.

Corollary 2.10 (Shahzad and Zegeye [5, Theorem 3.8]) *Let* E *be a real uniformly convex Banach space and* K *be a nonempty closed convex subset of* E. *Let* $\{T_i : i \in I\}$ *be* N *uniformly* L*-Lipschitzian, generalized asymptotically quasi-nonexpansive self-mappings of* K *with* $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *and* $\sum_{n=1}^{\infty} c_{in} < \infty$ *for all* $i \in I$. *Suppose* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and there exists one member* T *in* $\{T_i : i \in I\}$ *which either is semi-compact or satisfies condition* $(\overline{C})$. *Let* $\{\alpha_n\}_{n\geq 1} \subset [\delta_1, 1-\delta_1]$ *for some* $\delta_1 \in (0, 1)$. *Starting from arbitrary* $x_1 \in K$, *define the sequence* $\{x_n\}$ *by* (1.2). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in I\}$.

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Jing Zhao received her BS and master's degree from Harbin Normal University. She is working in Civil Aviation University of China now. Her research interests focus on functional analysis, fixed point theory, geometry of Banach space, etc.

Songnian He received his Ph.D. at Shanghai University of Science and Technology. He once researched at Nanjing University as a post Ph.D. He is now working at Civil Aviation University of China as a professor. His research areas include numerical methods of partial differential equations, approximation theory of real variable functions functional analysis, etc.

Guangxuan Liu received her master's degree from Nankai University. She is working in Civil Aviation University of China now. Her research interests focus on fixed point theory, probability theory and mathematical statistics, etc.