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A class of multi-period semi-variance portfolio selection with a four-factor futures price model

Wei Yan · Shurong Li

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Abstract Considering the stochastic exchange rate, a four-factor futures model with the underling asset, convenience yield, instantaneous risk free interest rate and exchange rate, is established. These processes follow jump-diffusion processes (Wiener process and Poisson process). The corresponding partial differential equation (PDE) of the futures price is derived. The general solution with parameters of the PDE is drawn. The weight least squares approach is applied to obtain the parameters of above PDE. Variance is substituted by semi-variance in Markovitz's portfolio selection model. Therefore, a class of multi-period semi-variance model is formulated originally. A hybrid genetic algorithm (GA) with particle swarm optimizer (PSO) is proposed to solve the multi-period semi-variance model. Finally, an example, which are fuel futures in Shanghai exchange market, is selected to demonstrate the effectiveness of above models and methods.

Keywords Four-factor model · Multi-period semi-variance portfolio · Exchange rate · Futures · Hybrid GA with PSO

Mathematics Subject Classification (2000) 90C90 · 60G35

1 Introduction

Futures contracts become popular more recently in the financial market. Hence, the pricing theory of futures is an important research field. The trend of prices are influenced easily by many market and non-market factors, such as relationship of supply

W. Yan (✉) · S. Li
College of Information and Control Engineering, China University of Petroleum, Dongying, Shandong 257061, China
e-mail: yanwei_123456@yahoo.com.cn

and demand, political situation, military conflict, the raid of terrorism and speculation and so on. Therefore, the research of futures price is very important topic, although it is an extremely difficult field. The models of futures prices have been studied for the past few years. Ross [1] presented a one-factor mathematical model, and then Schwartz [2] developed his work. By considering the stochastic convenience yield, Gibson and Schwartz [3] put forward a two-factor mathematical model. Consequently, Schwartz and Smith [4], Bjerksund [5] also investigated the two-factor model. And then, Cortazar and Schwartz [6], Schwartz [2] builded a three-factor mathematical model by considering the stochastic interest rate.

For those models above, they are assumed that all stochastic processes follow continuous-time processes and they only contain Brownian motions (Wiener processes). However, as pointed by Merton [7], Jarrow and Rudd [8], the actual spot price is not continuous because there is some external information affecting it. In fact, in addition to the continuous process based on Brownian motion, the analysis of price evolution does reveal sudden and rare breaks logically accounted for by exogenous events on information. Such behaviour from probabilistic point of view is naturally modelled by a point process. This process governed by Brownian motion and point process is called jump-diffusion process which is the discontinuous price process. Some concrete results, such as Cox and Ross [9] proved that there were some jump processes in reality.

This paper is concerned with the futures prices process which contains not only Brownian motions but also jump processes according to the actual prices of stocks and the normality and stability of the financial market. The jump process is described by a Poisson process. However, another factor which is exchange rate, plays the vital role in the futures price. There has been few researches considering the futures price with this factor. Considering the stochastic exchange rate, a four-factor futures model with the underling asset, convenience yield, instantaneous risk free interest rate and exchange rate, is formulated originally. The corresponding partial differential equation (PDE) of the futures price is derived. The general solution with parameters of the PDE is presented. The weight least squares approach which is different from [10], is applied to obtain the parameters of above PDE.

One of the frequent questions in finance is how to allocate a certain amount of money in different assets. The earliest approach to consider the optimal portfolio problem is so called mean-variance. It was pioneered by Markowitz (1952) [12] and it has been playing a critical role in the theory of portfolio selection. Gradually, Markowitz measured the risk with variance. It has also gained widespread acceptance as a practical tool for portfolio optimization. However, it is a single period model which makes an one-off decision at the beginning of the period and holds on until the end of the period. While it was natural to extend Markowitz's work to multi-period portfolio selections (see [13–17]).

In above literatures, they often used variance as a measure of risk. Then, the traditional analysis of valuation uncertainty, with its emphasis on variance of property values were extended to a more relevant risk measure from the point of view of most lenders: semi-variance of property values, or more specifically, "downside risk". Markowitz [18] suggested the semi-variance as a measure of risk. Empirical studies by Lanzilotti [19], Swalm [20], Mao [21] and Petty et al. [22] all found that business executives generally consider risk to be the probability of not meeting a target rate or return. The concept of downside risk apparently has considerable impact on the practitioner's view of risk. Therefore, the semi-variance is more consistent than the variance with business executives' concept of risk. Following Mao's [23] work, Hogan and Warren [24] developed a mean semi-variance model.

However, there has been few literatures on multi-period semi-variance portfolio selection. In this paper, we formulate a class of multi-period semi-variance model originally based on the above researches. In the model, the objective function is non-smooth in some points. So many methods of optimization, which depend on gradient information, can not solve the problem.

GA is a kind of evolutionary algorithm. It has been successfully applied to many research fields. However, there still exists some limitations in general GA. Therefore, some other algorithms, such as the simulated annealing (SA) and ant colony optimization (ACO) [25, 26], are applied to improve GA's performance. This paper presents a hybrid GA which makes use of the position displacement strategy of the particle swarm optimization (PSO) as a mutation operator. The new mutation operator shares the history information in the population. It improves the convergence speed and the accuracy of the general GA.

The outline of this paper is as follows. In Sect. 2, we describe a four-factor futures price model with exchange rate in financial market. In Sect. 3, the analytical result with constructive method according to the features of this PDE, is obtained. Section 4 is devoted to estimating the optimal parameters of the four-factor model. In Sect. 5, we formulate a class of multi-period Semi-variance portfolio model. In Sect. 6, the hybrid GA with PSO, which makes use the position displacement strategy of the PSO as a mutation operation, is presented. In Sect. 7, the above methods are applied to the fuel futures in Shanghai exchange market. Section 8 concludes the paper.

2 A four-factor futures price model with stochastic exchange rate

In this section, we consider a general financial market firstly. Then we formulate a mathematical model. The three-factor model is given by Schwartz [2] and Cortazar et al. [10]. The four factors include the underling asset price S, the instantaneous convenience yield δ, instantaneous risk interest rate r and exchange rate η. These factors are assumed to follow the joint stochastic processes,

$$\begin{cases} \mathrm{d}S = (\mu + r - \delta + f\eta)S\mathrm{d}t + \sigma_1 S\mathrm{d}W + \varphi_1 S\mathrm{d}N = (\mu + r - \delta + f\eta + v\varphi_1)S\mathrm{d}t \\ \qquad + \sigma_1 S\mathrm{d}W + \varphi_1 S\mathrm{d}M \\ \mathrm{d}\delta = \kappa(\alpha - \delta)\mathrm{d}t + \sigma_2\mathrm{d}W + \varphi_2\mathrm{d}N = [\kappa(\alpha - \delta) + v\varphi_2]\mathrm{d}t + \sigma_2\mathrm{d}W + \varphi_2\mathrm{d}M \\ \mathrm{d}r = a(m - r)\mathrm{d}t + \sigma_3\mathrm{d}W + \varphi_3\mathrm{d}N = [a(m - r) + v\varphi_3]\mathrm{d}t + \sigma_3\mathrm{d}W + \varphi_3\mathrm{d}M \\ \mathrm{d}\eta = b(n - e)\mathrm{d}t + \sigma_4\mathrm{d}W + \varphi_4\mathrm{d}N = [b(n - \eta) + v\varphi_4]\mathrm{d}t + \sigma_4\mathrm{d}W + \varphi_4\mathrm{d}M \end{cases} \quad (1)$$

where four main impact factors including the instantaneous convenience yield δ, instantaneous risk interest rate r, exchange rate η and other factor μ are considered in the underling asset price S; f is a parameter which adjusts the exchange rate; α, m, n are the appreciation rates; κ, a, b are adjusting parameters;

$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ are the volatility coefficients. Let us consider a financial market which is subject to uncertainty that enters through the components of $\mathcal{R}$-value Brownian motion $W := W(t)$ on its canonical space $(\Omega^W, \mathcal{L}^W, \mathrm{P}^W)$ and the component of a one-dimensional Poisson process $N := N(t)$ on its canonical space $(\Omega^N, \mathcal{L}^N, \mathrm{P}^N)$ with a fixed constant intensity equal to $v(v \geq 0)$. By Gong [11], the compensated Poisson process defined by $M := M(t) = N(t) - vt$ is a martingale. We denote the P^W-augmentation of $\sigma(W(s) : 0 \leq s \leq t)$ by$\{\mathcal{L}_t^W\}$ and the P^N-augmentation of $\sigma(N(s) : 0 \leq s \leq t)$ by $\{\mathcal{L}_t^N\}$. Let $(\Omega, \mathcal{F}, \mathrm{P}) := (\Omega^W \times \Omega^N, \mathcal{F}^W \otimes \mathcal{F}^N, \mathrm{P}^W \otimes \mathrm{P}^N)$ denote the product space. On this space, $W(t)$ and $N(t)$ are independent.

Applying Ito formula [11] to the futures price $F(S, \delta, r, \eta, T)$, it is presented as follows,

$$
\begin{aligned}
\mathrm{d}F = \Big[& -F_T + F_S(\mu + r - \delta + f\eta)S + F_\delta \kappa(\alpha - \delta) + F_r a(m - r) + F_\eta b(n - \eta) \\
& + \frac{1}{2}F_{SS}\sigma_1^2 S^2 + \frac{1}{2}F_{\delta\delta}\sigma_2^2 + \frac{1}{2}F_{rr}\sigma_3^2 + \frac{1}{2}F_{\eta\eta}\sigma_4^2 + F_{S\delta}\sigma_1\sigma_2 S + F_{Sr}\sigma_1\sigma_3 S \\
& + F_{S\eta}\sigma_1\sigma_4 S + F_{\delta r}\sigma_2\sigma_3 + F_{\delta\eta}\sigma_2\sigma_4 + F_{r\eta}\sigma_3\sigma_4 \Big] \mathrm{d}t \\
& + (\sigma_1 S F_S + \sigma_2 F_\delta + \sigma_3 F_r + \sigma_4 F_\eta)\mathrm{d}W \\
& + \big[F(S + S\varphi_1, \delta + \varphi_2, r + \varphi_3, \eta + \varphi_4) - F\big]\mathrm{d}N \\
= \Big\{ & -F_T + F_S(\mu + r - \delta + f\eta)S + F_\delta \kappa(\alpha - \delta) + F_r a(m - r) + F_\eta b(n - \eta) \\
& + \frac{1}{2}F_{SS}\sigma_1^2 S^2 + \frac{1}{2}F_{\delta\delta}\sigma_2^2 + \frac{1}{2}F_{rr}\sigma_3^2 + \frac{1}{2}F_{\eta\eta}\sigma_4^2 + F_{S\delta}\sigma_1\sigma_2 S + F_{Sr}\sigma_1\sigma_3 S \\
& + F_{S\eta}\sigma_1\sigma_4 S + F_{\delta r}\sigma_2\sigma_3 + F_{\delta\eta}\sigma_2\sigma_4 + F_{r\eta}\sigma_3\sigma_4 \\
& + v[F(S + S\varphi_1, \delta + \varphi_2, r + \varphi_3, \eta + \varphi_4) - F] \Big\} \mathrm{d}t \\
& + (\sigma_1 S F_S + \sigma_2 F_\delta + \sigma_3 F_r + \sigma_4 F_\eta)\mathrm{d}W \\
& + \big[F(S + S\varphi_1, \delta + \varphi_2, r + \varphi_3, \eta + \varphi_4) - F\big]\mathrm{d}M
\end{aligned} \tag{2}
$$

where $T = \tau - t$ is the time to maturity and τ is maturity time.

By introducing the market price of risk (see [3]), we can get the following equation:

$$
\mu + r - \delta + f\eta + v\varphi_1 + \delta - r = \overline{\lambda}\sigma_1 + \lambda\varphi_1 \tag{3}
$$

where $\overline{\lambda}$ denotes the market prices per unit of W, and λ denotes the market prices per unit of M.

The no-arbitrage condition leads to the following relationship between the total (expected) return of futures price μ_F and its risk exposure (see [3]),

$$\mu_F = \overline{\lambda}\frac{\sigma_1 SF_S + \sigma_2 F_\delta + \sigma_3 F_r + \sigma_4 F_\eta}{F} + \lambda\frac{F(S+S\varphi_1, \delta+\varphi_2, r+\varphi_3, \eta+\varphi_4) - F}{F} \tag{4}$$

where

$$\begin{aligned}\mu_F = \Big\{&-F_T + F_S(\mu + r - \delta + fe)S + F_\delta \kappa(\alpha-\delta) + F_r a(m-r) + F_\eta b(n-\eta)\\ &+ \frac{1}{2}F_{SS}\sigma_1^2 S^2 + \frac{1}{2}F_{\delta\delta}\sigma_2^2 + \frac{1}{2}F_{rr}\sigma_3^2 + \frac{1}{2}F_{\eta\eta}\sigma_4^2 + F_{S\delta}\sigma_1\sigma_2 S + F_{Sr}\sigma_1\sigma_3 S\\ &+ F_{S\eta}\sigma_1\sigma_4 S + F_{\delta r}\sigma_2\sigma_3 + F_{\delta\eta}\sigma_2\sigma_4 + F_{r\eta}\sigma_3\sigma_4\\ &+ v\big[F(S+S\varphi_1, \delta+\varphi_2, r+\varphi_3, \eta+\varphi_4) - F\big]\Big\}\Big/F\end{aligned} \tag{5}$$

It is obtained the following equation from (3)–(5).

$$\begin{aligned}&\frac{1}{2}F_{SS}\sigma_1^2 S^2 + \frac{1}{2}F_{\delta\delta}\sigma_2^2 + \frac{1}{2}F_{rr}\sigma_3^2 + \frac{1}{2}F_{\eta\eta}\sigma_4^2 + F_{S\delta}\sigma_1\sigma_2 S + F_{Sr}\sigma_1\sigma_3 S + F_{S\eta}\sigma_1\sigma_4 S\\ &+ F_{\delta r}\sigma_2\sigma_3 + F_{\delta\eta}\sigma_2\sigma_4 + F_{r\eta}\sigma_3\sigma_4 + F_S(r - \delta - v\varphi_1 + \lambda\varphi_1)S\\ &+ F_\delta\left[\kappa(\alpha-\delta) - \frac{\sigma_2}{\sigma_1}(v\varphi_1 - \lambda\varphi_1 + \mu + fe)\right]\\ &+ F_r\left[a(m-r) - \frac{\sigma_3}{\sigma_1}(v\varphi_1 - \lambda\varphi_1 + \mu + f\eta)\right]\\ &+ F_\eta\left[b(n-\eta) - \frac{\sigma_4}{\sigma_1}(v\varphi_1 - \lambda\varphi_1 + \mu + f\eta)\right]\\ &+ (v-\lambda)\big[F(S+S\varphi_1, \delta+\varphi_2, r+\varphi_3, \eta+\varphi_4) - F\big] - F_T = 0\end{aligned} \tag{6}$$

Moreover, $F(S, \delta, r, \eta, 0) = S$

3 Analytical solution to the model

Firstly, set

$$g_0 = \lambda\varphi_1 - v\varphi_1, \qquad \hat{\alpha} = \alpha - \frac{\sigma_2}{\kappa\sigma_1}(v\varphi_1 - \lambda\varphi_1 + \mu),$$

$$m^* = m - \frac{\sigma_3}{a\sigma_1}(v\varphi_1 - \lambda\varphi_1 + \mu), \qquad n^* = n - \frac{\sigma_4}{b\sigma_1}(v\varphi_1 - \lambda\varphi_1 + \mu),$$

$$C_2 = \frac{\sigma_2 f}{\kappa\sigma_1}, \qquad C_3 = \frac{\sigma_3 f}{a\sigma_1}, \qquad C_4 = \frac{\sigma_4 f}{b\sigma_1}.$$

And then, (6) is equivalent to

$$\frac{1}{2}F_{SS}\sigma_1^2S^2+\frac{1}{2}F_{\delta\delta}\sigma_2^2+\frac{1}{2}F_{rr}\sigma_3^2+\frac{1}{2}F_{\eta\eta}\sigma_4^2+F_{S\delta}\sigma_1\sigma_2S+F_{Sr}\sigma_1\sigma_3S+F_{S\eta}\sigma_1\sigma_4S$$
$$+F_{\delta r}\sigma_2\sigma_3+F_{\delta\eta}\sigma_2\sigma_4+F_{r\eta}\sigma_3\sigma_4+F_S(r-\delta+g_0)S+F_\delta\kappa(\hat{\alpha}-\delta-C_2\eta)$$
$$+F_ra(m^*-r-C_3\eta)+F_\eta b(n^*-\eta-C_4\eta)$$
$$+(v-\lambda)\big[F(S+S\varphi_1,\delta+\varphi_2,r+\varphi_3,\eta+\varphi_4)-F\big]-F_T=0 \tag{7}$$

Similar to a result in Schwartz's research (see [2]), it is assumed that the analytical solution of the above PDE has the following form.

$$\begin{aligned}F(S,\delta,r,\eta,T)=S\exp\bigg\{&-\delta\frac{1-e^{-\kappa T}}{\kappa}+r\frac{1-e^{-aT}}{a}+A(T)+B(T)\eta\\&+\frac{(\kappa\hat{\alpha}+\sigma_1\sigma_2)[(1-e^{-\kappa T})-\kappa T]}{\kappa^2}\\&-\frac{\sigma_2^2[4(1-e^{-\kappa T})-(1-e^{-2\kappa T})-2\kappa T]}{4\kappa^3}\\&-\frac{(am^*+\sigma_1\sigma_3)[(1-e^{-aT})-aT]}{a^2}\\&-\frac{\sigma_3^2[4(1-e^{-aT})-(1-e^{-2aT})-2aT]}{4a^3}\\&+\bigg[\frac{(1-e^{-\kappa T})+(1-e^{-aT})-(1-e^{-(\kappa+a)T})}{\kappa a(\kappa+a)}\\&+\frac{\kappa^2(1-e^{-aT})+a^2(1-e^{-\kappa T})-\kappa a^2T-a\kappa^2T}{\kappa^2a^2(\kappa+a)}\bigg]\cdot\sigma_2\sigma_3\bigg\}\end{aligned} \tag{8}$$

Substituting the above equation into (3.2), we can get $A(T)$ and $B(T)$ as follows:

$$\begin{aligned}A(T)=&\frac{1-e^{-2\kappa T}}{2\kappa}H_1+\frac{1-e^{-\kappa T}}{\kappa}H_2+\frac{1-e^{-2aT}}{2a}H_3+\frac{1-e^{-aT}}{a}H_4\\&+\frac{1-e^{-(\kappa+a)T}}{\kappa+a}H_5+\frac{1-e^{-2b(1+C_4)T}}{2b(1+C_4)}H_6+\frac{1-e^{-b(1+C_4)T}}{b(1+C_4)}H_7\\&+\frac{1-e^{-[b(1+C_4)\kappa]T}}{b(1+C_4)+\kappa}H_8+\frac{1-e^{-[b(1+C_4)+a]T}}{b(1+C_4)+a}H_9+H_{10}\cdot T\\&+(v-\lambda)\bigg\{\int_0^T(1+\varphi_1)\exp\bigg[-\varphi_2\frac{1-e^{-\kappa s}}{\kappa}+\varphi_3\frac{1-e^{-as}}{a}+\varphi_4B(s)\bigg]\mathrm{d}s-T\bigg\}\end{aligned}$$

$$B(T)=g_1e^{-\kappa T}+g_2e^{-aT}+g_3e^{-b(1+C_4)T}+g_4$$

where $g_1=-\frac{C_2}{b(1+C_4)-\kappa}$, $g_2=\frac{C_3}{b(1+C_4)-a}$, $g_3=\frac{C_2}{b(1+C_4)-\kappa}-\frac{C_3}{b(1+C_4)-a}-\frac{C_2-C_3}{b(1+C_4)}$, $g_4=\frac{C_2-C_3}{b(1+C_4)}$, $H_1=\frac{g_1\sigma_2\sigma_4}{\kappa}+\frac{g_1^2\sigma_4^2}{2}$, $H_2=\frac{g_4\sigma_2\sigma_4}{\kappa}+g_1g_4\sigma_4^2+g_1\sigma_1\sigma_4-\frac{g_1\sigma_2\sigma_4}{\kappa}+$

$\frac{g_1\sigma_3\sigma_4}{a} + g_1 bn^*$, $H_3 = -\frac{g_2\sigma_3\sigma_4}{\kappa} + \frac{g_2^2\sigma_4^2}{2}$, $H_4 = -\frac{g_4\sigma_3\sigma_4}{a} + g_2 g_4\sigma_4^2 + g_2\sigma_1\sigma_4 - \frac{g_2\sigma_2\sigma_4}{\kappa} + \frac{g_2\sigma_3\sigma_4}{a} + g_2 bn^*$, $H_5 = \frac{g_2\sigma_2\sigma_4}{\kappa} - \frac{g_1\sigma_3\sigma_4}{a} + g_1 g_2\sigma_4^2$, $H_6 = \frac{g_3^2\sigma_4^2}{2}$, $H_7 = g_3 g_4\sigma_4^2 + g_3\sigma_1\sigma_4 - \frac{g_3\sigma_2\sigma_4}{\kappa} + \frac{g_3\sigma_3\sigma_4}{a} + g_2 bn^*$, $H_8 = \frac{g_3\sigma_2\sigma_4}{\kappa} + g_1 g_3\sigma_4^2$, $H_9 = -\frac{g_3\sigma_3\sigma_4}{a} + g_2 g_3\sigma_4^2$, $H_{10} = \frac{g_4^2\sigma_4^2}{2} + g_4\sigma_1\sigma_4 - \frac{g_4\sigma_2\sigma_4}{\kappa} + \frac{g_4\sigma_3\sigma_4}{a} + g_0 + g_4 bn^*$.

4 Estimating the parameters of model

The weight least squares approach is applied to estimating the parameters of model in this section. From Ito formula, we can get the following equation.

$$\begin{aligned} \mathrm{d}\ln S &= \left(\mu + r - \delta + f\eta - \frac{1}{2}\sigma_1^2\right)\mathrm{d}t + \sigma_1 \mathrm{d}W + \ln(1+\varphi_1)\mathrm{d}N \\ &= \left[\mu + r - \delta + f\eta - \frac{1}{2}\sigma_1^2 + v\ln(1+\varphi_1)\right]\mathrm{d}t + \sigma_1 \mathrm{d}W + \ln(1+\varphi_1)\mathrm{d}M \end{aligned}$$

Moreover, we discretize $\ln S$, δ, r and η as follows.

$$\begin{cases} \ln S_t = \ln S_{t-1} + [\mu + r_{t-1} - \delta_{t-1} + f\eta_{t-1} - \frac{1}{2}\sigma_1^2 + v\ln(1+\varphi_1)]\Delta t + w_1(t) \\ \qquad + m_1(t) \\ \delta_t = (1-\kappa\Delta t)\delta_{t-1} + (\kappa\alpha + v\varphi_2)\Delta t + w_2(t) + m_2(t) \\ r_t = (1-a\Delta t)r_{t-1} + (am + v\varphi_3)\Delta t + w_3(t) + m_3(t) \\ \eta_t = (1-b\Delta t)\eta_{t-1} + (bn + v\varphi_4)\Delta t + w_4(t) + m_4(t) \end{cases} \tag{9}$$

where $w_1(t)$, $w_2(t)$, $w_3(t)$ and $w_4(t)$ are Gauss White Noises which means are zero and variances are $\sigma_1^2\Delta t$, $\sigma_2^2\Delta t$, $\sigma_3^2\Delta t$ and $\sigma_4^2\Delta t$ respectively; $m_1(t)$, $m_2(t)$, $m_3(t)$ and $m_4(t)$ are Poisson White Noises which means are zero and variances are $[\ln(1+\varphi_1)]^2 v\Delta t$, $\varphi_2^2 v\Delta t$, $\varphi 3^2 v\Delta t$ and $\varphi_4^2 v\Delta t$ respectively.

To find the optimal set of parameter values, the following algorithm would be applied to the process of identification. (It is assumed that $v = 1$ for simplifying the estimation):

Step 1. Initialize $\Omega = \{\kappa, \alpha, a, m, b, n, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \lambda, \mu, f\}$.

Step 2. Obtain spot price S, convenience yield δ and instantaneous risk interest rate r in N dates respectively by solving the following optimization problem:

$$\min_{S,\delta,r} \sum_{j=1}^{Q_i} \left[\ln \hat{F}_{ij}(S,\delta,r,\eta,\tau_i - t_j) - \ln F_{ij}\right]^2, \quad i = 1,\ldots,N$$

where M_i is the number of contracts with different maturities, $\hat{F}_{ij}$ is the model price, F_{ij} is the observed market price at date i for maturity j.

Step 3. Calculate the following values ($b_i, i = 1, \dots, 11$) according to *Step* 2 and (8).

$$\begin{cases} \mu - \frac{1}{2}\sigma_1^2 + v\ln(1+\varphi_1) = b_1, \\ \sigma_1^2 + [\ln(1+\varphi_1)]^2 = b_2, \\ 1 - \kappa\Delta t = b_3, \\ \kappa\alpha + v\varphi_2 = b_4, \\ \sigma_2^2 + \varphi_2^2 = b_5, \\ 1 - a\Delta t = b_6, \\ am + v\varphi_3 = b_7, \\ \sigma_3^2 + \varphi_3^2 = b_8, \\ 1 - b\Delta t = b_9, \\ bn + v\varphi_4 = b_{10}, \\ \sigma_4^2 + \varphi_4^2 = b_{11} \end{cases} \tag{10}$$

Step 4. Set ω_i and solve the following minimization problem.

$$\min_{\Omega} \quad \sum_{i=1}^{N}\sum_{j=1}^{Q_i} \omega_i \cdot \left[\ln \hat{F}_{ij}(S, \delta, r, \eta, \tau_i - t_j) - \ln F_{ij}\right]^2$$

s.t. (10) holds

where ω_i denotes the weight of ith observed data.

Step 5. If Ω is convergent, quit the iterative operation; if not, return to *Step* 2.

Remark 1 The weight least squares approach which is different form the method in [10], is applied to solve the Ω. In [10], all data plays the same role in the optimization process. However, the more the data is closed to now, the more important it plays. The above method avoids this problem in this paper.

5 Multi-period semi-variance portfolio model

In this section, a class of multi-period semi-variance portfolio model will be formulated. We consider a financial market in which N risky assets are traded. An investor will allocate his/her wealth in every beginning of T periods respectively.

Suppose that x_t^i denotes the weight of investment in assets i ($i = 1, 2, \dots, N$) at the beginning of period t ($t = 1, 2, \dots, T$). Then the wealth is increased by $\sum_{i=1}^{n} R_t^i x_t^i$ from the beginning to the end of ith period, where R_t^i is the return of the ith asset at period t.

Let v_t be the growth rate from the beginning of period 1 to the beginning of period t. Therefore, we can get the following equations:

$$v_1 = 1, \tag{11}$$

$$v_{t+1} = v_t\left(\sum_{i=1}^{N} R_t^i x_t^i\right), \quad t = 1, 2, \ldots, T \tag{12}$$

Let $R_t = [R_t^1, R_t^2, \ldots, R_t^N]'$ and $x_t = [x_t^1, x_t^2, \ldots, x_T^N]'$ (M' denotes the transpose of any matrix or vector M). We can rewrite (12) as follows:

$$v_{t+1} = v_t R_t' x_t, \quad t = 1, 2, \ldots, T. \tag{13}$$

According to the theory of dynamic mean semi-variance(M-SV) criterion, the investor seeks an optimal investment strategy $\pi = \{x_1, x_2, \ldots, x_T\}'$ to minimize the semi-variance $\mathrm{SVar}(R_{T+1}, x_{T+1}, d)$ when the growth rate of the wealth v_{T+1} is given. It is shown as follows.

$$\mathrm{SVar}(R_{T+1}, x_{T+1}, d) = \sum_{i=1}^{N}\left[(R_{T+1}^i x_{T+1}^i - d)^-\right]^2 \tag{14}$$

where

$$v_{T+1} = d \quad \text{and} \quad (R_{T+1}^i x_{T+1}^i - d)^- = \begin{cases} 0, & \text{if } R_{T+1}^i x_{T+1}^i > d, \\ -(R_{T+1}^i x_{T+1}^i - d), & \text{if } R_{T+1}^i x_{T+1}^i \le d \end{cases}$$

The relationship between growth rate and risk is efficient frontier in M-SV criterion.

As above discussion mentioned, we can get the following optimization model.

$$\begin{cases} \min\ \mathrm{SVar}(R_{T+1}, x_{T+1}, d) = \min\ \sum_{i=1}^{N}\left[(R_{T+1}^i x_{T+1}^i - d)^-\right]^2, \\ \text{s.t.} \quad v_{t+1} = v_t\left(\sum_{i=1}^{N} R_t^i x_t^i\right), \\ \qquad v_{T+1} = d, \\ \qquad \sum_{i=1}^{N} x_t^i = 1, \quad -1 \le x_t^i \le 1,\ t = 1, \ldots, T, \\ \qquad v_1 = 1 \end{cases} \tag{15}$$

From (15), we can see that the objective function is nonsmooth in some points. Therefore, a hybrid GA with PSO is proposed to solve the problem in the next section.

6 The hybrid GA with PSO

The particle swarm optimizer (PSO) is a population based algorithm. It was presented by Kennedy and Eberhart [27]. The swarm is initialized firstly in a set of randomly generated potential solutions, and then performs the search for the optimal solution iteratively. Let $x_i = [x_i^1, x_i^2, \ldots, x_i^m]$ be the ith particle in a D-dimensional space. In the PSO, the particles are manipulated according to the following equation:

$$\begin{cases} v_i = w \cdot v_i + c_1 \cdot rand \cdot (pbest_i - x_i) + c_2 \cdot rand \cdot (gbest_i - x_i), \\ x_i = x_1 + v_i, \end{cases} \tag{16}$$

where c_1 and c_2 are two positive constants called acceleration factors, w is the inertia weight, $rand$ is a uniformly distributed number in [0, 1], v_i is the velocity vector, $pbest_i$ ($pbest_i = [pbest_i^1, pbest_i^2, \ldots, pbest_i^m]$) is the best position of each individual and $gbest_i(gbest_i = [gbest_i^1, gbest_i^2, \ldots, gbest_i^m])$ is the best solution of the swarm. Krohling [28] proposed a new position displacement principle. He suggested that the updating of the velocity vector is independent of the history velocity vector. According to his work, the particles are manipulated as follows:

$$\begin{cases} v_1 = |randn| \cdot (pbest_i - x_i) + c_2 \cdot |Randn| \cdot (gbest_i - x_i), \\ x_i = x_1 + v_i, \end{cases} \tag{17}$$

where $|randn|$ and $|Randn|$ are positive random numbers generated according to the absolute value of the Gaussian probability distribution $N(0, 1)$.

Angeline [29] pointed out that the position displacement of particles was a kind of mutation operation. We make use of the position displacement as a mutation operator to modify the GA. During the mutation operation, the position of those individuals selected for mutation is modified via a vector that depends on the personal best position and the current global best position. If GA adopts the new mutation operator, it can use the history information of the chromosomes and share information among the population. The mutation will no longer be stochastic and the local search will be more efficient. At the same time, it keeps the GA's good property in global search.So the hybrid GA can provide more efficient exploration and exploitation ability.

But the velocity vector has to be treated specially during the mutation operation. After selection and crossover, the chromosomes are different from their parents. The history velocity vectors present the parents' information and will not make sense any more. So during the mutation, (16) is not suitable. But (17) is available for the operation. The hybrid GA with PSO is described as follows:

Step 1. Initialize the population in the search space.
Step 2. Record every individual's best solution $pbest_i$ and the best solution among the population $gbest_i$.
Step 3. Perform selection operation.
Step 4. According to the crossover probability pc, perform crossover operation.
Step 5. Select individuals for mutation and employ the results coming from *Step* 2 to perform mutation operation according (17).
Step 6. If the optimum is found, quit the iterative operation; if not, return to *Step* 2.

7 A numerical example

In this section, the above methods will be applied to an experiment. All the computations are performed in MATLAB on a Pentium PC under the Windows XP environment.

7.1 Forecasting futures price

The data used to test the models consist of daily observations. There are 240 daily prices corresponding to all futures contracts traded at the Shanghai Futures Exchange from 12th February 2007 to 30th March 2007. Five different maturities which are May, June, July, August and September, are used in the estimation. We used exchange rate of Dollar against the yuan in the estimation procedure.

The optimal parameter values are displayed in Table 1. We can obtain the fitting figures in Figs. 1–5. The FpM is model price and Fp is observed market price. The

Table 1 Optimal parameter values of fuel futures model

Parameter	κ	α	a	m	b	n
Value	180.7915	−0.0454	289.6735	0.0068	112.9608	0.1292
Parameter	σ_1	σ_2	σ_3	σ_4	φ_1	φ_2
Value	0.2503	0.2514	0.6091	0.0001	0.252	0.2492
Parameter	φ_3	φ_4	λ	μ	f	
Value	0.6138	0.0014	0.2512	0.2469	0.2399	

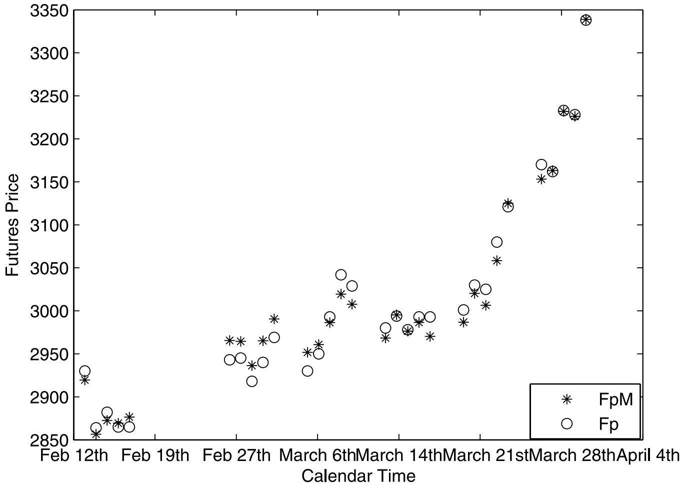

Fig. 1 Fitting graph with maturity of May

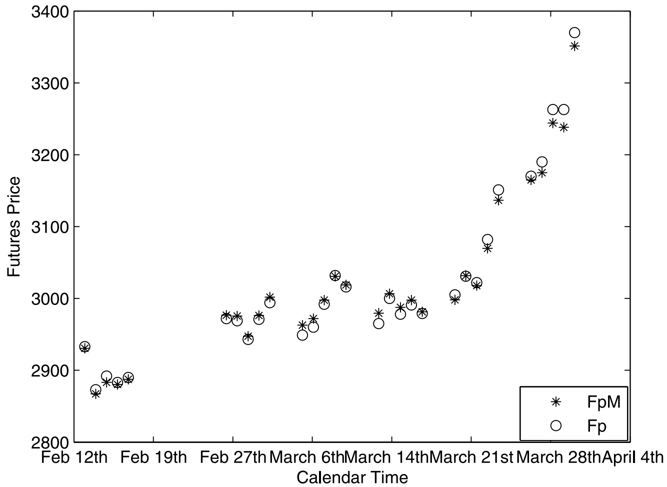

Fig. 2 Fitting graph with maturity of June

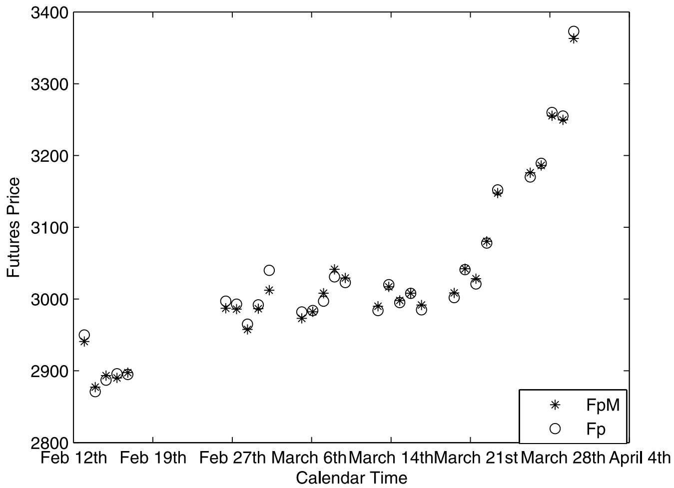

Fig. 3 Fitting graph with maturity of July

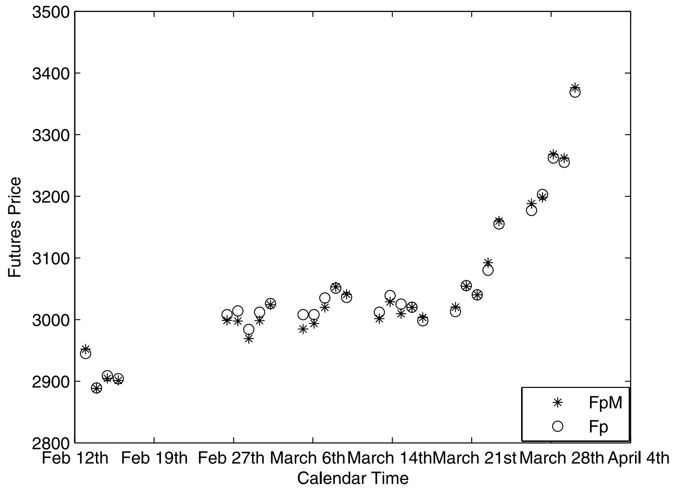

Fig. 4 Fitting graph with maturity of August

forecasting futures prices and relative errors are shown in Tables 2 and 3 respectively.

From the results, we can see that the four-factor futures model are effectiveness. The forecasting results correspond to reality. The four-factor futures model can be applied to the other futures markets.

7.2 Optimal portfolio with multi-period semi-variance model

We use the multi-period semi-variance model (15) to obtain the optimal portfolio with the hybrid GA with PSO. This is a 7-period case in the above example. The

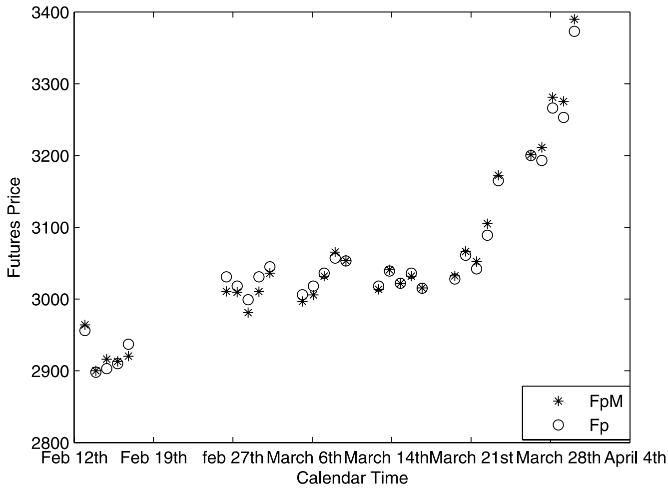

Fig. 5 Fitting graph with maturity of September

Table 2 Forecasting futures prices of different maturities

	April 2nd	April 3rd	April 4th	April 5th	April 6th	April 9th	April 10th
May	3336.837	3343.291	3348.708	3353.606	3358.246	3361.804	3366.252
June	3351.544	3358.027	3363.467	3368.387	3373.047	3376.621	3381.089
July	3365.838	3372.348	3377.813	3382.753	3387.433	3391.022	3395.509
August	3380.673	3387.212	3392.7	3397.663	3402.363	3405.968	3410.475
September	3395.573	3402.141	3407.653	3412.638	3417.359	3420.979	3425.506

Table 3 Relative errors of forecasting futures prices (%)

	April 2nd	April 3rd	April 4th	April 5th	April 6th	April 9th	April 10th
May	0.0251	0.3991	0.6827	−0.8982	−0.7904	−2.5564	−0.5538
June	0.3456	0.7509	1.1569	−0.7546	−0.0875	−1.4988	0.7776
July	0.7434	1.2717	1.9563	−0.2432	0.3981	−1.0498	1.4493
August	1.3088	1.6875	2.313	0.0784	0.9903	−0.3229	2.11
September	1.5762	1.9215	2.795	0.5195	1.556	0.1751	2.5601

data in (15) can be obtained from Table 2. In the example, we use 26 different value about d which is from 1.01 to 1.026. After calculation, we obtain the efficient frontier under multi-period semi-variance criterion as shown in Fig. 6. Moreover, Tables 4–6 display the optimal investment strategies while $d = 1.013$, $d = 1.018$ and $d = 1.023$ respectively.

From the results, the approaches can be implemented easily in the real market. The investors can work out the strategies according to the above optimal portfolio.

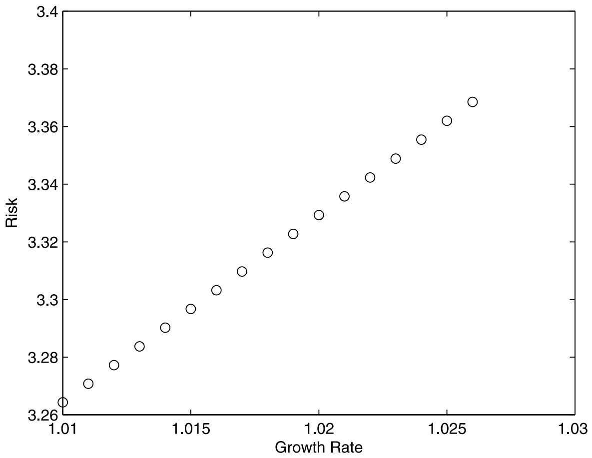

Fig. 6 Efficient frontier

Table 4 Optimal investment strategies of $d = 1.013$

	April 2nd	April 3rd	April 4th	April 5th	April 6th	April 9th	April 10th
May	0.2854	−0.9979	0.9979	−0.9979	0.9979	0.0035	0.2
June	0.1549	0.9979	−0.9979	1	0.9595	0.9979	0.2001
July	−0.2709	−0.9979	0.9979	0.9979	0.9979	−0.0014	0.2
August	0.6297	1	1	0.9979	−0.9823	0.9979	0.2003
September	0.201	0.9979	−0.9979	0.9979	−0.973	−0.9979	0.1996

Table 5 Optimal investment strategies of $d = 1.018$

	April 2nd	April 3rd	April 4th	April 5th	April 6th	April 9th	April 10th
May	0.9623	−0.9977	0.96	−0.7035	0.9244	−0.0128	0.2
June	−0.4306	0.9977	−0.9791	0.5654	0.9824	0.9289	0.2
July	−0.5452	−0.9977	0.9977	−0.7203	0.982	0.1073	0.2
August	0.5739	1	1	0.9299	−0.9486	0.9289	0.2
September	0.4396	0.9977	−0.9786	0.9297	−0.9402	−0.9524	0.2

Table 6 Optimal investment strategies of $d = 1.023$

	April 2nd	April 3rd	April 4th	April 5th	April 6th	April 9th	April 10th
May	0.9138	0.9413	0.9808	−0.762	0.6912	−0.7302	0.2
June	−0.7201	0.7112	−0.7649	0.8482	0.8984	0.7184	0.2
July	−0.7488	−0.7118	0.7112	−0.7715	0.9635	1	0.2
August	0.7915	−0.6512	0.8293	0.9812	−0.779	0.7182	0.2
September	0.7636	0.7103	−0.7564	0.7041	−0.7742	−0.7065	0.2

8 Conclusions

This paper is concerned with the futures prices process so that a four-factor futures price model with the underling asset, convenience yield, instantaneous risk interest rate and exchange rate, is formulated originally. These processes follow jump-diffusion processes. The corresponding partial differential equation (PDE) of the futures price is derived. The general solution with parameters of the PDE is presented. The weight least squares approach is applied to obtain the parameters of above PDE. Moreover, a class of multi-period semi-variance model is formulated originally. A numerical method of hybrid GA with PSO is proposed to solve the model. Finally, the numerical results show that the models and the methods are feasible and effective.

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Wei Yan is actually preparing his Ph.D. under the direction of Shurong Li at College of Information and Control Engineering, China University of Petroleum. His research interests are stochastic optimal control theory and financial engineering.

Shurong Li earned his Ph.D. at the Institute of Systems Science, Chinese Academy of Sciences. Since 1998 he has been at the China University of Petroleum (Professor). His research interests are the optimal control, numerical computing, nonlinear control and system engineering.