J Appl Math Comput (2008) 28: 123–131
DOI 10.1007/s12190-008-0081-0

JAMC

Solution of twelfth-order boundary value problems by variational iteration technique

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Received: 3 February 2008 / Revised: 15 April 2008 / Published online: 9 May 2008

Abstract In this paper, we implement a relatively new analytical technique which is called the variational iteration method for solving the twelfth-order boundary value problems. The analytical results of the problems have been obtained in terms of convergent series with easily computable components. Comparisons are made to verify the reliability and accuracy of the proposed algorithm. Several examples are given to check the efficiency of the suggested technique. The fact that variational iteration method solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Keywords Variational iteration method · Nonlinear problems · Higher order boundary value problems · Error estimates

Mathematics Subject Classification (2000) 65N10

1 Introduction

In this paper, we consider the general twelfth-order boundary value problem of the type:

$$y^{(xii)}(x) = f(x, y), \quad a < x < b, \tag{1}$$

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with boundary conditions

$$\begin{aligned} &y(a)=A_0, \quad y^{(2)}(a)=A_1, \quad y^{(4)}(a)=A_2,\\ &y^{(6)}(a)=A_3, \quad y^{(8)}(a)=A_4, \quad y^{(10)}(a)=A_5,\\ &y(b)=B_0, \quad y^{(2)}(b)=B_1, \quad y^{(4)}(b)=B_2,\\ &y^{(6)}(b)=B_3, \quad y^{(8)}(b)=B_4, \quad y^{(10)}(b)=B_5, \end{aligned}$$

where $f=f(x,y)$ is assumed real and as many times differentiable as required for $x\in[0,b]$ and A_i, $i=0,1,2,3,4,5$ and B_i, $i=0,1,2,3,4,5$ are real finite constants. A class of characteristic-value problems of higher order (as higher as twenty four) is known to arise in hydrodynamic and hydro magnetic stability. In addition, it is well known that when a uniform magnetic field is applied across the fluid in the same direction as that of gravity than instability may sets in as over stability which may be modeled by twelfth-order boundary value problems, see [1–5, 14, 17, 18, 23–25] and the references therein. The literature of numerical analysis contains little on the solution of the twelfth-order boundary value problems. Research in this direction may be considered in its early stages [1–5]. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are contained in a comprehensive survey by Agarwal [1]. The boundary value problems of higher order have been investigated due to their mathematical importance and the potential for applications in hydrodynamic and hydro magnetic stability; see [1–5, 24, 25]. Several techniques including finite-difference, polynomial and non polynomial spline and decomposition have been employed for solving such problems, see [2, 5, 23–25] and the references therein. All these techniques have their inbuilt deficiencies, like divergence of the results at the points adjacent to the boundary and calculation of the so-called Adomian's polynomials. Moreover, the performance of most of the methods used so far is well known that they provide the solution at grid points only. Recently, Noor and Mohyud-Din employed homotopy perturbation method, variational iteration method and a new technique variational iteration decomposition method (which is an elegant combination of variational iteration method and the decomposition method) for solving higher-order boundary value problems, see [14, 17–19, 21]. Inspired and motivated by the ongoing research in this area, we apply the variational iteration method (VIM) for solving the twelfth-order boundary value problems. It is worth mentioning that our proposed technique can handle any boundary value problem with a set of boundary conditions defined at any order derivatives.

He [6–10] developed the variational iteration method (VIM) for solving linear, nonlinear, initial and boundary value problems. It is worth mentioning that the origin of variational iteration method is traced back to Inokuti, Sekine and Mura [11], but the real potential of the VIM was explored by He. The basic motivation of this paper is to apply the variational iteration method (VIM) for solving the twelfth-order boundary value problems. It is shown that the variational iteration method provides the solution in a rapid convergent series with easily computable components. We write the correct functional for the twelfth-order boundary value problems and calculate the Lagrange multiplier optimally via variational theory. The use of Lagrange multiplier reduces the successive application of the integral operator and minimizes the computational work. Moreover, the selection of the initial value is done by introducing an essential modification which increases the efficiency of the proposed algorithm. The VIM solves effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. Several examples are given to illustrate the reliability and performance of the proposed method.

2 Variational iteration method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{2}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method [6–13, 15–18, 20, 22, 26], we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds, \tag{3}$$

where λ is a Lagrange multiplier [6–11], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0$; (3) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [6–11]. In this method, it is required first to determine the Lagrange multiplier λ optimally via variational theory. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n.$$

For the convergence criteria and error estimates of variational iteration method, see Ramos [22].

3 Numerical applications

In this section, we apply the variational iteration method developed by He for solving the twelfth-order boundary value problems. We introduce a slight modification in the selection of the initial value which increases the efficiency of the proposed iterative scheme. For the sake of comparison, we take the same examples as in [24, 25].

Example 3.1 Consider the following nonlinear boundary value problem of twelfth order,

$$y^{(xii)}(x) = 2e^x y^2(x) + y'''(x), \quad 0 < x < 1$$

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = y^{(x)}(0) = 1,$$
$$y(1) = y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = y^{(x)}(1) = e^{-1}.$$

The exact solution of the problem is

$$y(x) = e^{-x}.$$

The correct functional is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(\frac{d^{12}y_n}{dx^{12}} - 2e^x \tilde{y}_n^2(x) - \tilde{y}_n'''(x)\right)ds.$$

Making the correct functional stationary, the Lagrange multiplier is identified as $\lambda = \frac{1}{11!}(s-x)^{11}$, [26], we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{11!}(s-x)^{11}\left(\frac{d^{12}y_n}{dx^{12}} - 2e^x \tilde{y}_n^2(x) - \tilde{y}_n'''(x)\right)ds.$$

Using the initial conditions, we obtain

$$\begin{aligned} y_{n+1}(x) &= 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 \\ &\quad + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} \\ &\quad + \int_0^x \frac{1}{11!}(s-x)^{11}\left(\frac{d^{12}y_n}{dx^{12}} - 2e^x \tilde{y}_n^2(x) - \tilde{y}_n'''(x)\right)ds, \end{aligned}$$

where

$$A = y'(0), \qquad B = y^{(3)}(0), \qquad C = y^{(5)}(0),$$
$$D = y^{(7)}(0), \qquad E = y^{(9)}(0), \qquad F = y^{(11)}(0).$$

Consequently, we obtain the following approximants

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 \\ &\quad + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} + \frac{1}{239500800}x^{12} + \cdots, \\ y_2(x) &= 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 \\ &\quad + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} + \frac{1}{239500800}x^{12} \\ &\quad + \frac{1}{12!}Bx^{12} + \frac{1}{13!}x^{13} + \cdots, \\ &\ \vdots \end{aligned}$$

Table 1 (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	0.904837418	0.904837579	−1.61E-7
0.2	0.818730753	0.818731060	−3.07E-7
0.3	0.740818221	0.740818643	−4.22E-7
0.4	0.670320046	0.670320543	−4.97E-7
0.5	0.606530659	0.606531182	−5.22E-7
0.6	0.548811636	0.548812133	−4.97E-7
0.7	0.496585304	0.496585726	−4.22E-7
0.8	0.449328964	0.449329271	−3.07E-7
0.9	0.406569659	0.406569821	−1.61E-7
1.0	0.367879441	0.367879441	2.00E-10

*Error = Exact solution − Series solution

The series solution is given as

$$y(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} + \frac{1}{12!}(2+B)x^{12} + \cdots.$$

Imposing the boundary conditions at $x = 1$ will yield

$$A = -0.9999983604, \quad B = -1.600016174, \quad C = -0.9998407313,$$
$$D = -1.001558298, \quad E = -0.9851011393, \quad F = -1.132112472.$$

Consequently, the series solution is given by

$$\begin{aligned} y(x) = {} & 1 - 0.9999983604x + \frac{1}{2!}x^2 - 0.1666693624x^3 + \frac{1}{4!}x^4 \\ & - 0.008332006094x^5 + \frac{1}{6!}x^6 - 0.0001987218845x^7 + \frac{1}{8!}x^8 \\ & - 2.715 \times 10^{-9}x^9 + \frac{1}{10!}x^{10} \\ & - 2.836 \times 10^{-8}x^{11} + 2.087 \times 10^{-9}x^{11} + 2.087 \times 10^{-9}x^{12} + \cdots, \end{aligned}$$

which is in full agreement with [24].

Table 1 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

Example 3.2 Consider the following nonlinear boundary value problem of twelfth order,

$$y^{(xii)}(x) = \frac{1}{2}e^{-x}y^2(x), \quad 0 < x < 1,$$

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = y^{(x)}(0) = 2,$$
$$y(1) = y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = y^{(x)}(1) = 2e.$$

The exact solution of the problem is

$$y(x) = 2e^x.$$

The correct functional is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(\frac{d^{12}y_n}{dx^{12}} - \frac{1}{2}e^{-x}\tilde{y}_n^2(x)\right)ds.$$

Making the correct functional stationary, the Lagrange multiplier is identified as $\lambda = \frac{1}{11!}(s-x)^{11}$, [24], we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{11!}(s-x)^{11}\left(\frac{d^{12}y_n}{dx^{12}} - \frac{1}{2}e^{-x}\tilde{y}_n^2(x)\right)ds,$$

$$\begin{aligned} y_{n+1}(x) &= 2 + Ax + \frac{2}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{2}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{2}{6!}x^6 + \frac{1}{7!}Dx^7 \\ &\quad + \frac{2}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{2}{10!}x^{10} + \frac{1}{11!}Fx^{11} \\ &\quad + \int_0^x \frac{1}{11!}(s-x)^{11}\left(\frac{d^{12}y_n}{dx^{12}} - \frac{1}{2}e^{-x}\tilde{y}_n^2(x)\right)ds, \end{aligned}$$

where

$$A = y'(0), \qquad B = y^{(3)}(0), \qquad C = y^{(5)}(0),$$
$$D = y^{(7)}(0), \qquad E = y^{(9)}(0), \qquad F = y^{(11)}(0).$$

Consequently, we obtain the following approximants:

$$y_0(x) = 2,$$

$$\begin{aligned} y_1(x) &= 2 + Ax + x^2 + \frac{1}{3!}Bx^3 + \frac{2}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{2}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{2}{8!}x^8 \\ &\quad + \frac{1}{9!}Ex^9 + \frac{2}{10!}x^{10} + \frac{1}{11!}Fx^{11} + \frac{2}{12!}x^{12} - \frac{2}{13!}x^{13} + O(x^{14}), \end{aligned}$$

$$\begin{aligned} y_2(x) &= 2 + Ax + x^2 + \frac{1}{3!}Bx^3 + \frac{2}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{2}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{2}{8!}x^8 \\ &\quad + \frac{1}{9!}Ex^9 + \frac{2}{10!}x^{10} + \frac{1}{11!}Fx^{11} + \frac{2}{12!}x^{12} - \frac{2}{13!}x^{13} + \frac{2}{13!}Ax^{13} + \cdots, \end{aligned}$$

$$\vdots$$

Table 2 (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	2.000000000	2.000000000	0.00000000000
0.1	2.2103418362	2.2105496398	0.0002078037
0.2	2.4428055163	2.4431996667	0.0003941504
0.3	2.6997176152	2.7002578996	0.0005402845
0.4	2.9836493953	2.9842815093	0.0006321141
0.5	3.2974425414	3.2981039180	0.0006613766
0.6	3.6442376008	3.6448637056	0.0006261048
0.7	4.0275054149	4.0280359083	0.0005304933
0.8	4.4510818570	4.4514661221	0.0003842651
0.9	4.9192062223	4.9194078617	0.0002016394
1.0	5.4365636569	5.4365636559	0.0002016394

*Error = Exact solution − Series solution

The series solution is given as

$$y(x) = 2 + Ax + \frac{2}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{2}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{2}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{2}{10!}x^{10} + \frac{1}{11!}Fx^{11} + \frac{2}{12!}x^{12} + \frac{2}{13!}(A-1)x^{13} + \cdots,$$

which is in full agreement with [25].

Imposing the boundary conditions at $x = 1$ will yield

$$A = 2.002114383, \quad B = 1.986107510, \quad C = 2.333176702,$$
$$D = 3.001515917, \quad E = 1.986107510, \quad F = 2.102525395.$$

The series solution is given as

$$y(x) = 2 + 2.002114783x + x^2 + 0.329322880x^3 + \frac{1}{12}x^4 + 0.01944313918x^5 + \frac{1}{720}x^6 + 0.0005955388724x^7 + \frac{1}{20160}x^8 + 0.5473179867 \times 10^{-5}x^9 + \frac{1}{1814400}x^{10} + 0.5267269408 \times 10^{-7}x^{11} + \frac{1}{239500800}x^{12} + 0.3218601046 \times 10^{-9}x^{13} + \cdots.$$

Table 2 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

4 Conclusions

In this paper, we have used the variational iteration method (VIM) for solving boundary value problems for twelfth-order. The method is applied in a direct way without using linearization, perturbation or restrictive assumptions. It may be concluded that VIM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result, the size reduction amounts to the improvement of performance of approach. The fact that the VIM solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Acknowledgements The authors are highly grateful to the Referee for his/her constructive comments. We would like to thank Dr. S.M. Junaid Zaidi, Rector CIIT for providing excellent research environment and facilities.

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