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Block SOR methods for fuzzy linear systems

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Abstract In this paper, the block SOR iterative methods are studied for $n \times n$ fuzzy linear systems and the corresponding convergence theorems are also given out. We know that the coefficient matrix S of the augmented system $SX = Y$ is consistently ordered when S_1 is nonsingular, and in this case the optimal parameter ω of the block SOR method is obtained. Numerical examples are presented to illustrate the theory.

Keywords Block SOR method · Fuzzy linear systems · Optimal parameter

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1 Introduction

Systems of simultaneous linear equations play a major role in various areas such as mathematics, physics, statistics, engineering and social sciences. Since many problems are too complex to be defined in precise terms, i.e. some of the system's parameters and measurements are represented by fuzzy rather than crisp numbers, it is

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immensely important to development mathematical model and numerical procedure that would appropriately treat general fuzzy linear systems and solve them.

The fuzzy data, which can formulate uncertainty in actual environment, and operations-research algorithms have been designed for them since the concept of *fuzzy number* and arithmetic operations with these numbers are investigated by Zadeh [8–10].

A general model for solving an $n \times n$ fuzzy linear system which coefficient matrix is crisp and the right hand side is arbitrary fuzzy vector were first proposed by Friedman et al. [3]. They used the embedding method [6] and replaced the original $n \times n$ fuzzy linear system by a $2n \times 2n$ crisp linear system, i.e. solving the $n \times n$ fuzzy linear system is equal to solving a $2n \times 2n$ crisp linear system. The $2n \times 2n$ crisp linear system is large and sparse [3]. Direct methods, such as Gauss elimination etc. can solve these problems. However, iterative methods are effective and then become more attractive than direct methods because of storage requirements and preservation of sparsity. So we can use iterative methods solving such $2n \times 2n$ crisp linear systems efficiently. The iterative procedure have been investigated in many papers [2, 4, 5, 11] for solving $n \times n$ fuzzy linear system.

In this paper, we consider the block SOR iterative procedure for solving $n \times n$ fuzzy linear system, which are efficient and practical because the procedure only require the nonsingularity of the coefficient matrix of a fuzzy linear system while the point iterative methods require the diagonal entries of the coefficient matrix are nonzero. The structure of this paper is organized as follows:

In Sect. 2 we recall preliminaries for an $n \times n$ fuzzy linear system and a general model for solving the system. The block SOR iterative methods are discussed in Sect. 3 and Sect. 4. In Sect. 5 the optimal parameter in a special case are obtained. Numerical examples are given to illustrate our theory in Sect. 6 and conclusion in Sect. 7.

2 Preliminaries

An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements (see [6]):

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over [0, 1],
2. $\overline{u}(r)$ is a bounded left continuous nonincreasing function over [0, 1],
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

A crisp number α can be simply expressed as $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$.

The addition and scalar multiplication of fuzzy numbers previously defined can be described as follows, for arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and real number λ,

(a) $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$;

(b) $\lambda u = \begin{cases} (\lambda \underline{u}(r), \lambda \overline{u}(r)), & \lambda \geq 0, \\ (\lambda \overline{u}(r), \lambda \underline{u}(r)), & \lambda < 0. \end{cases}$

Definition 2.1 ([3]) The $n \times n$ linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\ \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n, \end{cases} \tag{1}$$

where the coefficient matrix $A = (a_{ij})$, $1 \le i, j \le n$ is a crisp matrix and y_i, $1 \le i \le n$ are fuzzy numbers, is called a *fuzzy linear system* (FLS).

Definition 2.2 ([3]) A fuzzy number vector $X = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$ given by

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)), \quad 1 \le i \le n,\ 0 \le r \le 1,$$

is called a *solution* of the fuzzy linear system (1) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_i. \end{cases} \tag{2}$$

Following Friedman et al. [3], the system (1) can be extended to a $2n \times 2n$ crisp linear system

$$SX = Y, \tag{3}$$

where $S = (s_{kl})$, s_{kl} are determined as follows

$$\begin{aligned} a_{ij} \ge 0 \quad &\Rightarrow \quad s_{ij} = a_{ij}, \qquad s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 \quad &\Rightarrow \quad s_{i,j+n} = -a_{ij}, \quad s_{i+n,j} = -a_{ij}, \end{aligned} \qquad 1 \le i, j \le n,$$

and any s_{kl} which is not determined by the above items is zero, $1 \le k, l \le 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \qquad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}.$$

In terms of [3], we know that S has the following structure

$$S = \begin{matrix} & n & n \\ n & \multicolumn{2}{c}{\multirow{2}{*}{}} \end{matrix} \begin{array}{c} \\ n \\ n \end{array} \begin{array}{c} \begin{matrix} \;\; n & \quad n \end{matrix} \\ \begin{bmatrix} S_1 \ge 0 & S_2 \ge 0 \\ S_2 \ge 0 & S_1 \ge 0 \end{bmatrix} \end{array},$$

where $S_1, S_2 \geq 0$, $A = S_1 - S_2$, and (3) can be rewritten as follows

$$\begin{cases} S_1\underline{X} - S_2\overline{X} = \underline{Y}, \\ S_2\underline{X} - S_1\overline{X} = -\overline{Y}, \end{cases} \tag{4}$$

where

$$\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix}, \quad \overline{X} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix}, \quad \underline{Y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_n \end{bmatrix}, \quad \overline{Y} = \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \vdots \\ \overline{y}_n \end{bmatrix}.$$

The following theorem implies that when FLS (1) has a unique solution.

Theorem 2.3 ([3]) *The matrix S is nonsingular if and only if the matrices $A = S_1 - S_2$ and $S_1 + S_2$ are both nonsingular.*

Under the conditions of Theorem 2.3, the solution vector of (3)

$$X = S^{-1}Y \tag{5}$$

is thus unique but may still not be an *appropriate fuzzy vector*. By Theorem 2 of [4], we know that S^{-1} has the same structure as S, i.e.

$$S^{-1} = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}.$$

The following result provides a sufficient condition for the unique solution to be a fuzzy vector.

Theorem 2.4 ([1]) *The unique solution X of* (5) *is a fuzzy vector for arbitrary fuzzy vector Y, if S^{-1} is nonnegative.*

Restricting the discussion to triangular fuzzy numbers, i.e. $\underline{y}_i(r), \overline{y}_i(r)$ and consequently $\underline{x}_i(r), \overline{x}_i(r)$ are all linear functions of r, and having calculated X which solves (3), we can define the fuzzy solution to the original system given by (1) as follows.

Definition 2.5 Let $X = \{(\underline{x}_i(r), -\overline{x}_i(r)), 1 \leq i \leq n\}$ denote the unique solution of (3). The fuzzy number vector $U = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \leq i \leq n\}$ defined by

$$\underline{u}_i(r) = \min\left\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\right\},$$

$$\overline{u}_i(r) = \max\left\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\right\}$$

is called the *fuzzy solution* of $SX = Y$. If $(\underline{x}_i(r), \overline{x}_i(r))$, $1 \leq i \leq n$ are all fuzzy numbers then $\underline{u}_i(r) = \underline{x}_i(r)$, $\overline{u}_i(r) = \overline{x}_i(r)$, $1 \leq i \leq n$ and U is called a *strong fuzzy solution*, otherwise, U is called a *weak fuzzy solution*.

3 Block SOR methods for FLS

For nonsingular system (3), where S is nonsingular, that is $A = S_1 - S_2$ and $S_1 + S_2$ are nonsingular, we can use the following splitting

$$S = D - L - U,$$

where

$$D = \begin{bmatrix} S_1 - S_2 & 0 \\ 0 & S_1 - S_2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ -S_2 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -S_2 & -S_2 \\ 0 & -S_2 \end{bmatrix}, \tag{6}$$

or

$$D = \begin{bmatrix} S_1 - S_2 & 0 \\ 0 & S_1 - S_2 \end{bmatrix}, \quad L = \begin{bmatrix} -S_2 & 0 \\ -S_2 & -S_2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -S_2 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

or

$$D = \begin{bmatrix} S_1 + S_2 & 0 \\ 0 & S_1 + S_2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ -S_2 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} S_2 & -S_2 \\ 0 & S_2 \end{bmatrix}, \tag{8}$$

or

$$D = \begin{bmatrix} S_1 + S_2 & 0 \\ 0 & S_1 + S_2 \end{bmatrix}, \quad L = \begin{bmatrix} S_2 & 0 \\ -S_2 & S_2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -S_2 \\ 0 & 0 \end{bmatrix}, \tag{9}$$

or

$$D = \begin{bmatrix} S_1 - S_2 & 0 \\ 0 & S_1 + S_2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ -S_2 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -S_2 & -S_2 \\ 0 & S_2 \end{bmatrix}, \tag{10}$$

or

$$D = \begin{bmatrix} S_1 - S_2 & 0 \\ 0 & S_1 + S_2 \end{bmatrix}, \quad L = \begin{bmatrix} -S_2 & 0 \\ -S_2 & S_2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -S_2 \\ 0 & 0 \end{bmatrix}, \tag{11}$$

or

$$D = \begin{bmatrix} S_1 + S_2 & 0 \\ 0 & S_1 - S_2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ -S_2 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} S_2 & -S_2 \\ 0 & -S_2 \end{bmatrix}, \tag{12}$$

or

$$D = \begin{bmatrix} S_1 + S_2 & 0 \\ 0 & S_1 - S_2 \end{bmatrix}, \quad L = \begin{bmatrix} S_2 & 0 \\ -S_2 & -S_2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -S_2 \\ 0 & 0 \end{bmatrix}. \tag{13}$$

If S_1 is nonsingular, we may use splitting:

$$\begin{aligned} S &= \begin{bmatrix} S_1 & 0 \\ 0 & S_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -S_2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -S_2 \\ 0 & 0 \end{bmatrix} \\ &\equiv D - L - U. \end{aligned} \tag{14}$$

For the augmented linear system $SX = Y$, the SOR method is defined as:

$$X^{(k+1)} = (D - \omega L)^{-1}[(1-\omega)D + \omega U]X^{(k)} + \omega Y, \quad k = 0, 1, \ldots$$

based on the splitting $S = D - L - U$, where ω is the nonzero real relaxation parameter.

For splittings (6)–(14), we have the following different block SOR iterative schemes.

(1) For splitting (6), the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D - \omega L)^{-1}Y, \quad k = 0, 1, \ldots, \tag{15}$$

where $X_k = \left[\begin{smallmatrix} X_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$\begin{aligned} H_\omega &= (D - \omega L)^{-1}((1-\omega)D + \omega U) \\ &= \begin{bmatrix} S_1 - S_2 & 0 \\ \omega S_2 & S_1 - S_2 \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} (1-\omega)(S_1 - S_2) - \omega S_2 & -\omega S_2 \\ 0 & (1-\omega)(S_1 - S_2) - \omega S_2 \end{bmatrix}. \end{aligned}$$

(2) For splitting (7), the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D - \omega L)^{-1}Y, \quad k = 0, 1, \ldots, \tag{16}$$

where $X_k = \left[\begin{smallmatrix} X_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$\begin{aligned} H_\omega &= (D - \omega L)^{-1}((1-\omega)D + \omega U) \\ &= \begin{bmatrix} S_1 - S_2 + \omega S_2 & 0 \\ \omega S_2 & S_1 - S_2 + \omega S_2 \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} (1-\omega)(S_1 - S_2) & -\omega S_2 \\ 0 & (1-\omega)(S_1 - S_2) \end{bmatrix}. \end{aligned}$$

(3) For splitting (8), the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D - \omega L)^{-1}Y, \quad k = 0, 1, \ldots, \tag{17}$$

where $X_k = \left[\begin{smallmatrix} X_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$\begin{aligned} H_\omega &= (D - \omega L)^{-1}((1-\omega)D + \omega U) \\ &= \begin{bmatrix} S_1 + S_2 & 0 \\ \omega S_2 & S_1 + S_2 \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} (1-\omega)(S_1 + S_2) + \omega S_2 & -\omega S_2 \\ 0 & (1-\omega)(S_1 + S_2) + \omega S_2 \end{bmatrix}. \end{aligned}$$

(4) For splitting (9), the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D-\omega L)^{-1}Y, \quad k = 0, 1, \ldots, \tag{18}$$

where $X_k = \left[\begin{smallmatrix} \underline{X}_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$\begin{aligned} H_\omega &= (D-\omega L)^{-1}((1-\omega)D+\omega U) \\ &= \begin{bmatrix} S_1+S_2-\omega S_2 & 0 \\ \omega S_2 & S_1+S_2-\omega S_2 \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} (1-\omega)(S_1+S_2) & -\omega S_2 \\ 0 & (1-\omega)(S_1+S_2) \end{bmatrix}. \end{aligned}$$

(5) For splitting (10), the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D-\omega L)^{-1}Y, \quad k = 0, 1, \ldots, \tag{19}$$

where $X_k = \left[\begin{smallmatrix} \underline{X}_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$\begin{aligned} H_\omega &= (D-\omega L)^{-1}((1-\omega)D+\omega U) \\ &= \begin{bmatrix} S_1-S_2 & 0 \\ \omega S_2 & S_1+S_2 \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} (1-\omega)(S_1-S_2)-\omega S_2 & -\omega S_2 \\ 0 & (1-\omega)(S_1+S_2)+\omega S_2 \end{bmatrix}. \end{aligned}$$

(6) For splitting (11), the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D-\omega L)^{-1}Y, \quad k = 0, 1, \ldots, \tag{20}$$

where $X_k = \left[\begin{smallmatrix} \underline{X}_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$\begin{aligned} H_\omega &= (D-\omega L)^{-1}((1-\omega)D+\omega U) \\ &= \begin{bmatrix} S_1-S_2+\omega S_2 & 0 \\ \omega S_2 & S_1+S_2-\omega S_2 \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} (1-\omega)(S_1-S_2) & -\omega S_2 \\ 0 & (1-\omega)(S_1+S_2) \end{bmatrix}. \end{aligned}$$

(7) For splitting (12),the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D-\omega L)^{-1}Y, \quad k = 0, 1, \ldots, \tag{21}$$

where $X_k = \left[\begin{smallmatrix} \underline{X}_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$H_\omega = (D-\omega L)^{-1}((1-\omega)D+\omega U)$$

$$= \begin{bmatrix} S_1 + S_2 & 0 \\ \omega S_2 & S_1 - S_2 \end{bmatrix}^{-1} \times \begin{bmatrix} (1-\omega)(S_1+S_2) + \omega S_2 & -\omega S_2 \\ 0 & (1-\omega)(S_1 - S_2) - \omega S_2 \end{bmatrix}.$$

(8) For splitting (13), the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D - \omega L)^{-1} Y, \quad k = 0, 1, \ldots, \tag{22}$$

where $X_k = \left[\begin{smallmatrix} X_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$H_\omega = (D - \omega L)^{-1}((1-\omega)D + \omega U) = \begin{bmatrix} S_1 + (1-\omega) S_2 & 0 \\ \omega S_2 & S_1 - (1+\omega) S_2 \end{bmatrix}^{-1} \times \begin{bmatrix} (1-\omega)(S_1+S_2) & -\omega S_2 \\ 0 & (1-\omega)(S_1 - S_2) \end{bmatrix}.$$

(9) For splitting (14), the scheme is:

$$X_{k+1} = H_\omega X_k + \omega(D - \omega L)^{-1} Y, \quad k = 0, 1, \ldots, \tag{23}$$

where $X_k = \left[\begin{smallmatrix} X_k \\ -\overline{X}_k \end{smallmatrix}\right]$, and

$$H_\omega = (D - \omega L)^{-1}((1-\omega)D + \omega U) = \begin{bmatrix} S_1 & 0 \\ \omega S_2 & S_1 \end{bmatrix}^{-1} \begin{bmatrix} (1-\omega) S_1 & -\omega S_2 \\ 0 & (1-\omega) S_1 \end{bmatrix}.$$

4 Convergence analysis of block SOR for FLS

In this section, we mainly discuss the convergence of all the above mentioned SOR methods. As we know, the SOR iterative scheme

$$X_{k+1} = H_\omega X_k + \omega(D - \omega L)^{-1} Y$$

converges if and only if the spectral radius $\rho(H_\omega)$ of the iterative matrix H_ω is less than 1. Thus we will investigate the intervals for parameter ω of all these methods such that $\rho(H_\omega) < 1$. First we give the functional relations between the eigenvalues $\lambda(H_\omega)$ of the iterative matrix H_ω for all the iterative schemes (15)–(23) and the parameter ω.

4.1 The functional relations between ω and $\lambda(H_\omega)$

Theorem 4.1.1 *For SOR scheme* (15), *the eigenvalues* λ (*except for* $\lambda = 1 - \omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $(S_1 - S_2)^{-1} S_2$ *are of the functional* *relationship*

$$(1-\omega-\lambda)^2 = 2\omega(1-\omega-\lambda)\mu + \omega^2(\lambda-1)\mu^2. \tag{24}$$

Proof Let λ be a nonzero eigenvalue of the iterative matrix H_ω with its eigenvector $(x^T, y^T)^T$. Then from

$$H_\omega \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

we have

$$\begin{pmatrix} (1-\omega)(S_1-S_2)-\omega S_2 & -\omega S_2 \\ 0 & (1-\omega)(S_1-S_2)-\omega S_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} S_1-S_2 & 0 \\ \omega S_2 & S_1-S_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

That is to say, it holds that

$$\begin{aligned} (1-\omega-\lambda)(S_1-S_2)x &= \omega S_2 x + \omega S_2 y, \\ (1-\omega-\lambda)(S_1-S_2)y &= \lambda\omega S_2 x + \omega S_2 y. \end{aligned} \tag{25}$$

Since $S_1 - S_2$ is nonsingular and when $\lambda \neq 1-\omega$, it then follows from (25) that

$$(1-\omega-\lambda)^2 x = 2\omega(1-\omega-\lambda)(S_1-S_2)^{-1}S_2 x + \omega^2(\lambda-1)((S_1-S_2)^{-1}S_2)^2 x.$$

Assume that μ is an eigenvalue of $(S_1-S_2)^{-1}S_2$, then we get

$$(1-\omega-\lambda)^2 = 2\omega(1-\omega-\lambda)\mu + \omega^2(\lambda-1)\mu^2.$$

From the functional relationship (24), we know that if μ is an eigenvalue of the matrix $(S_1-S_2)^{-1}S_2$, then the λ determined by (24) is an eigenvalue of H_ω. Conversely, if λ is an eigenvalue of H_ω such that $\lambda \neq 1-\omega$, and μ satisfies (24), then μ is an eigenvalue of $(S_1-S_2)^{-1}S_2$. □

For the SOR iterative schemes (16)–(23), similar functional relations between the parameter ω and the eigenvalues $\lambda(H_\omega)$ can be obtained in the same manner. Hence we will state them below without proofs.

Theorem 4.1.2 *For SOR scheme* (16), *the eigenvalues* λ (*except for* $\lambda = 1-\omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $(S_1-S_2)^{-1}S_2$ *are of the functional relationship*

$$(1-\omega-\lambda)^2 = 2\lambda\omega(1-\omega-\lambda)\mu + \lambda\omega^2(\lambda-1)\mu^2. \tag{26}$$

Theorem 4.1.3 *For SOR scheme* (17), *the eigenvalues* λ (*except for* $\lambda = 1-\omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $(S_1+S_2)^{-1}S_2$ *are of the functional relationship*

$$(1-\omega-\lambda)^2 = -2\omega(1-\omega-\lambda)\mu + \omega^2(\lambda-1)\mu^2. \tag{27}$$

Theorem 4.1.4 *For SOR scheme* (18), *the eigenvalues* λ (*except for* $\lambda = 1-\omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $(S_1+S_2)^{-1}S_2$ *are of the functional relationship*

$$(1-\omega-\lambda)^2 = -2\lambda\omega(1-\omega-\lambda)\mu - \lambda\omega^2(\lambda-1)\mu^2. \tag{28}$$

Theorem 4.1.5 *For SOR scheme* (19), *the eigenvalues* λ (*except for* $\lambda = 1-\omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $(S_1-S_2)^{-1}S_2(S_1+S_2)^{-1}S_2$ *are of the functional relationship*

$$(1-\omega-\lambda)^2 = 2\omega(1-\omega-\lambda)\mu + \omega^2(\lambda+1)\mu. \tag{29}$$

Theorem 4.1.6 *For SOR scheme* (20), *the eigenvalues* λ (*except for* $\lambda = 1-\omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $(S_1+S_2)^{-1}S_2(S_1-S_2)^{-1}S_2$ *are of the functional relationship*

$$(1-\omega-\lambda)^2 = 2\lambda\omega(1-\omega-\lambda)\mu + \lambda\omega^2(\lambda+1)\mu. \tag{30}$$

Theorem 4.1.7 *For SOR scheme* (21), *the eigenvalues* λ (*except for* $\lambda = 1-\omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $(S_1+S_2)^{-1}S_2(S_1-S_2)^{-1}S_2$ *are of the functional relationship*

$$(1-\omega-\lambda)^2 = 2\omega(1-\omega-\lambda)\mu + \omega^2(\lambda+1)\mu. \tag{31}$$

Theorem 4.1.8 *For SOR scheme* (22), *the eigenvalues* λ (*except for* $\lambda = 1-\omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $(S_1-S_2)^{-1}S_2(S_1+S_2)^{-1}S_2$ *are of the functional relationship*

$$(1-\omega-\lambda)^2 = 2\lambda\omega(1-\omega-\lambda)\mu + \lambda\omega^2(\lambda+1)\mu. \tag{32}$$

Theorem 4.1.9 *For SOR scheme* (23), *the eigenvalues* λ (*except for* $\lambda = 1-\omega$) *of the iterative matrix* H_ω *and the eigenvalues* μ *of* $S_1^{-1}S_2$ *are of the functional relationship*

$$(1-\omega-\lambda)^2 = \lambda\omega^2\mu^2. \tag{33}$$

4.2 The convergence of the iteration

Now we give the convergence theorems of the iterative schemes (15)–(23), first we quote the following useful result [7, pp. 171–172]:

Lemma 4.2.1 *Both roots of real quadratic equation* $x^2 - bx + c = 0$ *are less than one in modulus if and only if* $|c| < 1$ *and* $|b| < 1 + c$.

In the following we denote $\mu_* = \min\{\mu\}$ and $\mu^* = \max\{\mu\}$ without other statements.

Theorem 4.2.2 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $(S_1 - S_2)^{-1}S_2$ *are real and* $\mu > -\frac{1}{2}$, *then the block SOR method* (15) *is convergent if and only if one of the following cases holds*:

Case I: $-\frac{1}{2} < \mu < 0$.

$$0 < \omega < 2;$$

Case II: $\mu \geq 0$.

$$0 < \omega < \frac{2}{1+\mu^*}.$$

Proof The relation (24) is the following real quadratic equation:

$$\lambda^2 - [2(1-\omega) - 2\omega\mu + \omega^2\mu^2]\lambda + [(1-\omega) - \omega\mu]^2 = 0. \tag{34}$$

For (34), from Lemma 4.2.1, we have

$$\begin{cases} [(1-\omega) - \omega\mu]^2 < 1, \\ \left|2(1-\omega) - 2\omega\mu + \omega^2\mu^2\right| < 1 + [(1-\omega) - \omega\mu]^2. \end{cases} \tag{35}$$

Solving (35), we obtain:

$$0 < \omega(1+\mu) < 2, \tag{36}$$

and

$$\begin{cases} \omega^2(2\mu+1) > 0, \\ \omega^2(2\mu^2 + 2\mu + 1) - 4\omega(1+\mu) + 4 > 0. \end{cases} \tag{37}$$

The (37) is obviously satisfied when $\mu > -\frac{1}{2}$, it follows from (36) that $0 < \omega < \frac{2}{1+\mu^*}$ when $\mu \geq 0$, case II holds.

Meanwhile we note that $\lambda = 1 - \omega$ is also an eigenvalue of H_ω. In fact, from (25) it follows:

$$\begin{cases} \omega S_2 x + \omega S_2 y = 0, \\ (1-\omega)\omega S_2 x + \omega S_2 y = 0 \end{cases}$$

so we have $S_2 x = 0$ and $S_2 y = 0$, then $x \in N(S_2)$ and $y \in N(S_2)$, where $N(S_2)$ is the null space of the matrix S_2. Hence, $\lambda = 1 - \omega$ is an eigenvalue of H_ω with the corresponding eigenvector $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$ and at least one of x, y are nonzero vectors, where $x, y \in N(S_2)$.

Hence $|1 - \omega| < 1$ i.e. $0 < \omega < 2$, compiling the above results, then case I holds.

For ω not satisfy Case I or Case II, we can easily prove that the spectrum radius of H_ω is more than 1, i.e. the iterative scheme is not convergent. □

Similarly, we have the following theorems without proofs.

Theorem 4.2.3 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $(S_1 - S_2)^{-1}S_2$ *are real, when* $\mu > -\frac{1}{2}$

and $\mu \neq 1$, *then the block SOR method* (16) *is convergent if and only if one of the following cases holds*:

Case I: $-\frac{1}{2} < \mu \leq 0$.

$$0 < \omega < \frac{2}{1-\mu_*};$$

Case II: $\mu > 0$, $\mu \neq 1$.

$$0 < \omega < 2.$$

Theorem 4.2.4 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $(S_1 + S_2)^{-1} S_2$ *are real and* $\mu < \frac{1}{2}$, *the block SOR method* (17) *is convergent if and only if*

Case I: $0 < \mu < \frac{1}{2}$.

$$0 < \omega < 2;$$

Case II: $\mu \leq 0$.

$$0 < \omega < \frac{2}{1-\mu_*}.$$

Theorem 4.2.5 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $(S_1 + S_2)^{-1} S_2$ *are real, when* $\mu < \frac{1}{2}$ *and* $\mu \neq -1$, *then the block SOR method* (18) *is convergent if and only if one of the following cases holds*:

Case I: $0 < \mu < \frac{1}{2}$.

$$0 < \omega < \frac{2}{1+\mu^*};$$

Case II: $\mu \leq 0$, $\mu \neq -1$.

$$0 < \omega < 2.$$

Theorem 4.2.6 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $(S_1 - S_2)^{-1} S_2 (S_1 + S_2)^{-1} S_2$ *are real and* $\mu > -1$, *then the block SOR method* (19) *is convergent if and only if one of the following cases holds*:

Case I: $-1 < \mu \leq 0$.

$$0 < \omega < 2;$$

Case II: $0 < \mu < 1$.

$$0 < \omega < \frac{2}{1+2\mu^*};$$

Case III: $\mu \geq 1$.

$$0 < \omega < 1 - \frac{\sqrt{{\mu_*}^2 - 1}}{1+\mu^*}.$$

Theorem 4.2.7 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $(S_1+S_2)^{-1}S_2(S_1-S_2)^{-1}S_2$ *are real and,* $\mu > -1$, *then the block SOR method* (20) *is convergent if and only if one of the following cases holds*:

Case I: $-1 < \mu < 0$.

$$0 < \omega < \frac{2}{1-2\mu_*};$$

Case II: $0 \le \mu \le 1$.

$$0 < \omega < 2;$$

Case III: $\mu > 1$.

$$0 < \omega < 1 + \frac{\sqrt{\mu^{*2}-1}}{1-\mu_*}.$$

Theorem 4.2.8 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $(S_1+S_2)^{-1}S_2(S_1-S_2)^{-1}S_2$ *are real,* $\mu > -1$, *then the block SOR method* (21) *vis convergent if and only if one of the following conditions holds*:

Case I: $-1 < \mu \le 0$.

$$0 < \omega < 2;$$

Case II: $0 < \mu < 1$.

$$0 < \omega < \frac{2}{1+2\mu^*};$$

Case III: $\mu \ge 1$.

$$0 < \omega < 1 - \frac{\sqrt{\mu_*{}^2-1}}{1+\mu^*}.$$

Theorem 4.2.9 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $(S_1-S_2)^{-1}S_2(S_1+S_2)^{-1}S_2$ *are real,* $\mu > -1$, *then the block SOR method* (22) *is convergent if and only if one of the following conditions holds*:

Case I: $-1 < \mu < 0$.

$$0 < \omega < \frac{2}{1-2\mu_*};$$

Case II: $0 \le \mu \le 1$.

$$0 < \omega < 2;$$

Case III: $\mu > 1$.

$$0 < \omega < 1 + \frac{\sqrt{\mu^{*2}-1}}{1-\mu_*}.$$

Theorem 4.2.10 *Suppose that the system* (3) *is nonsingular and* S_2 *in augmented system* (3) *is singular, all the eigenvalues* μ *of* $S_1^{-1}S_2$ *are real, when* $-1 < \mu < 1$, *then the block SOR methods* (23) *is convergent if and only if*

$$0 < \omega < 2.$$

5 The optimal parameter for the iterative scheme (23)

For the iterative schemes (15)–(22), it is very difficult to obtain the optimal parameter, but for scheme (23), we can easily give the optimal parameter ω.

Theorem 5.1 *For scheme* (23), *and under the assumption of Theorem* 4.2.10, *then the optimum parameter*

$$\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\mu^{*2}}}$$

and the optimum spectral radius is

$$\rho(H_{\omega_{\text{opt}}}) = \min_{\omega}\{\rho(H_\omega)\} = \omega_{\text{opt}} - 1.$$

Proof Since the coefficient matrix S of the augmented system $SX = Y$ is consistently ordered when S_1 is nonsingular, it follows from a simple application of Young's results [7] that the value of ω that minimizes $\rho(H_\omega)$ is

$$\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\mu^{*2}}}$$

yielding $\rho(H_{\omega_{\text{opt}}}) = \min_\omega\{\rho(H_\omega)\} = \omega_{\text{opt}} - 1$. □

In the next section we will give some numerical examples for illustrating the methods in this paper. For this purpose, we present a stopping criterion with tolerance $\varepsilon > 0$ as follows

$$\|X_{k+1} - X_k\| \le \varepsilon.$$

Since the fuzzy number we will use is trapezoidal fuzzy number, the norm of

$$X = \begin{bmatrix} \underline{X} \\ -\overline{X} \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix} = \begin{bmatrix} x_{1a} + x_{1b}r \\ x_{2a} + x_{2b}r \\ \vdots \\ x_{2na} + x_{2nb}r \end{bmatrix},$$

where x_{ia} and x_{ib} are crisp numbers, $i = 1, \dots, 2n$, $0 \le r \le 1$, can be defined as

$$\|X\| = \max_i \{|x_{ia}|, |x_{ib}|\}. \qquad (*)$$

6 Numerical example

We use the following examples to illustrate our theory, computed by Maple.

Example 6.1 Consider the 2×2 fuzzy linear system

$$\begin{cases} x_1 - x_2 = (r, 2 - r), \\ x_1 + 3x_2 = (4 + r, 7 - 2r). \end{cases}$$

The extended 4×4 matrix is

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right], \quad \text{which is nonsingular,}$$

where $S_1 = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 3 \end{smallmatrix}\right]$ is nonsingular as well, and the exact solution is

$$\begin{cases} x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (1.375 + 0.625r, 2.875 - 0.875r), \\ x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (0.875 + 0.125r, 1.375 - 0.375r), \end{cases}$$

which is a strong fuzzy solution.

Applying the iterative schemes provided in Sect. 3 on the systems of Example 6.1 with $X_0 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, we have the iteration results in Table 1 (using the vector norm as $(*)$, IT denotes the iterations).

Example 6.2 Consider the 25×25 fuzzy linear system

$$\begin{cases} x_1 - x_{13} = (r, 2 - r), \\ \vdots \\ x_{12} - x_{13} = (r, 2 - r), \\ x_{13} = (r, 2 - r), \\ x_{14} - x_{13} = (r, 2 - r), \\ \vdots \\ x_{25} - x_{13} = (r, 2 - r). \end{cases}$$

Table 1

Scheme	Convergence interval	ω	IT	Approximate solution ($\varepsilon = 10^{-5}$)
(15)	$0 < \omega < 2$	$1/2$	30	$x_1 = (1.3749 + 0.6251r, 2.8751 - 0.8751r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$
(16)	$0 < \omega < \frac{8}{5}$	$1/4$	55	$x_1 = (1.3747 + 0.6252r, 2.8753 - 0.8752r)$ $x_2 = (0.8751 + 0.1249r, 1.3749 - 0.3749r)$
(17)	$0 < \omega < \frac{4}{3}$	$1/3$	25	$x_1 = (1.3748 + 0.6250r, 2.8747 - 0.8749r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$
(18)	$0 < \omega < 2$	$1/3$	25	$x_1 = (1.3747 + 0.6250r, 2.8748 - 0.8750r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$
(19)	$0 < \omega < \frac{8}{5}$	$6/5$	25	$x_1 = (1.3750 + 0.6250r, 2.8750 - 0.8750r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$
(20)	$0 < \omega < 2$	$9/5$	50	$x_1 = (1.3749 + 0.6250r, 2.8749 - 0.8750r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$
(21)	$0 < \omega < \frac{8}{5}$	$3/5$	21	$x_1 = (1.3749 + 0.6250r, 2.8751 - 0.8750r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$
(22)	$0 < \omega < 2$	$2/3$	12	$x_1 = (1.3749 + 0.6250r, 2.8751 - 0.8750r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$
(23)	$0 < \omega < 2$	$1/2$	25	$x_1 = (1.3750 + 0.6250r, 2.8750 - 0.8750r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$
		$1(\omega_{\text{opt}})$	6	$x_1 = (1.3750 + 0.6250r, 2.8750 - 0.8750r)$ $x_2 = (0.8750 + 0.1250r, 1.3750 - 0.3750r)$

The extended 50×50 matrix is $S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}$ which is nonsingular, where

$$S_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

is also nonsingular, and the exact solution is

$$\begin{aligned} x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25} \\ = (2, 2), \\ x_{13} = (r, 2 - r), \end{aligned}$$

which is a strong fuzzy solution.

Applying the iterative schemes provided in Sect. 3 on the systems of Example 6.2 with $X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we have the iteration results in Table 2 (using the vector norm as ($*$), IT denotes the iterations).

Table 2

Scheme	Convergence interval	ω	IT	Approximate solution ($\varepsilon = 10^{-5}$)
(15)	$0 < \omega < 2$	1/2	23	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (2.0000 + 0.0000r, 2.0000 - 0.0000r)$ $x_{13} = (1.0000r, 2.0000 - 1.0000r)$
(16)	$0 < \omega < 2$	1/4	49	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (2.0000 + 0.0000r, 2.0000 - 0.0000r)$ $x_{13} = (1.0000r, 2.0000 - 1.0000r)$
(17)	$0 < \omega < 2$	1/3	36	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (2.0000 - 0.0000r, 2.0000 + 0.0000r)$ $x_{13} = (1.0000r, 2.0000 - 1.0000r)$
(18)	$0 < \omega < 2$	1/3	36	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (2.0000 + 0.0000r, 2.0000 + 0.0000r)$ $x_{13} = (1.0000r, 2.0000 - 1.0000r)$
(19)	$0 < \omega < 2$	6/5	10	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (2.0000 - 0.0000r, 2.0000 - 0.0000r)$ $x_{13} = (1.0000r, 2.0000 - 1.0000r)$
(20)	$0 < \omega < 2$	9/5	71	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (2.0000 + 0.0000r, 2.0000)$ $x_{13} = (1.0000r, 2.0000 - 1.0000r)$
(21)	$0 < \omega < 2$	3/5	12	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (1.9994, 2.0006 - 0.0004r)$ $x_{13} = (1.0000r, 2.0000 - 1.0000r)$
(22)	$0 < \omega < 2$	2/3	11	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (1.9997 + 0.0001r, 2.0001 - 0.0001r)$ $x_{13} = (1.0000r, 2.0001 - 1.0000r)$
(23)	$0 < \omega < 2$	1/2	19	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (1.9999 + 0.0000r, 2.0000 - 0.0000r)$ $x_{13} = (1.0000r, 2.0000 - 1.0000r)$
		$1(\omega_{\text{opt}})$	3	$x_1 = \cdots = x_{12} = x_{14} = \cdots = x_{25}$ $= (2, 2)$ $x_{13} = (r, 2 - r)$

7 Conclusion

We present the block SOR iterative methods for the $n \times n$ fuzzy linear system and obtain the necessary and sufficient convergence conditions of the iterative schemes. If the extended matrix S by Friedman et al. [4] is nonsingular, then for any initial vector X_0, the SOR iterations will converge to $X = \left[\begin{smallmatrix} \underline{X} \\ -\overline{X} \end{smallmatrix}\right]$, the unique solution of $SX = Y$. The numerical examples show that the methods are effective and applicable for solving the fuzzy linear system.

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