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On the non-vanishing of theta lifting of Bianchi modular forms to Siegel modular forms

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Abstract
In this paper we study the theta lifting of a weight 2 Bianchi modular form $\mathcal{F}$ of level $\Gamma_0(\mathfrak{n})$ with $\mathfrak{n}$ square-free to a weight 2 holomorphic Siegel modular form. Motivated by Prasanna's work for the Shintani lifting, we define the local Schwartz function at finite places using a quadratic Hecke character χ of square-free conductor $\mathfrak{f}$ coprime to level $\mathfrak{n}$. Then, at certain 2 by 2 g matrices β related to $\mathfrak{f}$, we can express the Fourier coefficient of this theta lifting as a multiple of $L(\mathcal{F}, \chi, 1)$ by a non-zero constant. If the twisted L-value is known to be non-vanishing, we can deduce the non-vanishing of our theta lifting.

Keywords Theta lifting · Bianchi modular form · Siegel modular form · L-function

Mathematics Subject Classification 11F30 · 11F27 · 11F41 · 11F67

1 Introduction

Shimura initiated the systematic study of holomorphic modular forms of half-integral weight and provided a correspondence between certain modular forms of even weight and modular forms of half-integral weight. Later, in the other direction, Shintani [21] described a method in terms of weighted periods of holomorphic cusp forms to construct modular forms of half-integral weight. Waldspurger showed in [23] a proportional relation between special values of L-functions attached to an eigenform of even weight and the square of the square-free Fourier coefficients of the Shintani lifting. For the special case of modular forms on the full modular group, Kohnen-Zagier [14] proved a simple version of Waldspurger's theorem with the constant of proportionality given explicitly. Inspired by their work we will analyse the theta lifting of Bianchi modular forms to Siegel modular forms and investigate the relationship between Fourier coefficients of this lifting and special L-values attached to the Bianchi modular forms. This can be used to describe the non-vanishing of the theta lifting, which is an open problem in general.

To construct the theta lifting of a weight 2 Bianchi modular form $\mathcal{F}$ for level $\Gamma_0(\mathfrak{n})$ with $\mathfrak{n}$ a square-free ideal for an imaginary quadratic field F of class number one, following [3] and

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[13] we consider the 4-dimensional rational quadratic space V given by Hermitian matrices with entries in F. Its associated symmetric space D is isomorphic to the upper half space $\mathbb{H}_3$. In our theta integral we use the differential form $\eta_{\mathcal{F}}$ attached to $\mathcal{F}$ defined on the arithmetic quotient $\Gamma\backslash D$. We choose the Schwartz form $\varphi = \varphi_\infty^{\mathrm{KM}}\varphi_f$ defined on a pair of vectors in V so that the theta kernel is given by

$$\theta(g,h,z) := \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in V^2} \omega(g,h)\varphi(\mathbf{x}_1,\mathbf{x}_2;z) \quad \text{for } g \in \mathrm{Sp}_4 \subset \mathrm{GL}_4, h \in \mathrm{SO}(3,1).$$

Then the theta lifting is constructed as

$$\Theta_\varphi(\eta_{\mathcal{F}})(g) = \int_{\Gamma\backslash D} \eta_{\mathcal{F}}(z) \wedge \theta(g,h,z)$$

which by results of Kudla and Millson turns out to be a weight 2 holomorphic Siegel modular form. Its Fourier coefficient at a 2×2 symmetric matrix $\beta > 0$ is given here by

$$\sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma\backslash\Omega_\beta} \varphi_f(\mathbf{x}_1,\mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_{\mathcal{F}}$$

where $\Omega_\beta := \{(\mathbf{x}_1,\mathbf{x}_2) \in V^2 : ((\mathbf{x}_i,\mathbf{x}_j)) = \beta\}$ and $U(\mathbf{x}_1,\mathbf{x}_2) := \mathrm{Span}\{\mathbf{x}_1,\mathbf{x}_2\} \subset V$. For an auxiliary quadratic Hecke character χ with its conductor coprime to $\mathfrak{n}$ we define the Schwartz form as $\varphi = \varphi_\infty^{\mathrm{KM}}\varphi_f^\chi$. The choice of the Schwartz function φ_f^χ in Sect. 4 is crucial for us to get the period integral related to some twisted L-values. With this choice we take certain $\beta > 0$ (again depending on the conductor of χ) at which the coefficient of $\Theta_\varphi(\eta_{\mathcal{F}})$ is expressed as the above weighted sum of period integrals over infinite geodesics joining two cusps. By [24, Theorem 1.8], the period integral over infinite geodesics ending in ∞ can be related to $L(\mathcal{F},\chi,1)$. We apply Atkin-Lehner operators to transform other infinite geodesics and reduce to this case. In Sect. 5 I compute the coefficient at such a β as a multiple of $L(\mathcal{F},\chi,1)$ by a non-vanishing number. By Friedberg-Hoffstein's theorem [8, Theorem B], we can deduce that there always exists a character χ such that the twisted L-value is non-vanishing which implies the non-vanishing of the corresponding theta lifting.

Theorem 1.1 *(Theorem 5.17) Let $F = \mathbb{Q}(\sqrt{d})$ (square-free $d < 0$) be an imaginary quadratic field of class number one and denote by $\mathcal{O}$ its ring of integers. Consider a weight 2 Bianchi cusp form $\mathcal{F}$ of level $\Gamma_0(\mathfrak{n})$ with $\mathfrak{n}$ a square-free ideal away from ramified primes in $F/\mathbb{Q}$. Given a square-free product m of split or inert primes in $F/\mathbb{Q}$ such that $(m,\mathfrak{n}) = 1$, for $\mathfrak{m} = m\mathcal{O}$ we choose a quadratic Hecke character $\chi_{\mathfrak{m}}$ of conductor $\mathfrak{m}\sqrt{d}$. Then, at certain $\beta > 0$ related to m, the Fourier coefficient of the theta lifting $\Theta_\varphi(\eta_{\mathcal{F}})$ can be computed as*

$$(*)\cdot L(\mathcal{F},\chi_{\mathfrak{m}},1)$$

where the non-zero constant $(*)$ is given explicitly in (38).

Constructions of congruences between automorphic forms are one of important tools, particularly those between cusp forms and theta lifts on $\mathrm{GSp}_4(\mathbb{A})$. [17] gave an explicit construction of the theta lifts of automorphic forms for imaginary quadratic fields to further study the congruence on $\mathrm{GSp}_4(\mathbb{A})$. Different to our approach to construct the Schwartz form (defining the finite part), its choice is to define a distinguished infinite part while at each finite place the local Schwartz function is the characteristic function in Sect. 4.3.

It is known that Böcherer formulates an equality between sums of Fourier coefficients of Siegel modular forms and certain L-values. [5] proved a precise formula relating the Bessel period of lifted automorphic forms on $\mathrm{GSp}_4(\mathbb{A})$ to central L-values, where the Bessel period is the Fourier coefficient considered by Böcherer. We have not calculated the Bessel period of our theta lifts but in the future we expect to relate some combinations of the Fourier coefficients of our lifts to special L-values. This would lead us to investigate the connections between our calculations and those on the Bessel period, and how our further result would be related to Böcherer's conjecture.

2 Binary Hermitian form

In this section we recall some basics from linear algebra about Hermitian matrices and Hermitian binary forms from [7, Chapter 9].

For an complex matrix A, the matrix $\bar{A}$ is obtained from A by applying complex conjugation to all entries and the matrix A^t is the transpose of A. An $n \times n$ matrix A with complex entries is called *Hermitian* if $\bar{A}^t = A$. By the definition we see that an Hermitian matrix is unchanged by taking its conjugate transpose. Note that any Hermitian matrix must have real diagonal entries.

Let R be a subring of $\mathbb{C}$ with $R = \bar{R}$. We write $\mathcal{H}(R)$ for the set of Hermitian 2×2 matrices with entries in R, i.e.

$$\mathcal{H}(R) = \{A \in M_2(R) \mid \bar{A}^t = A\}.$$

Every $f \in \mathcal{H}(R)$ defines a binary Hermitian form with coefficients in R. If $f = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$ then the associated binary Hermitian form is the semi quadratic map $f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined by

$$f(u, v) = (u, v) \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} (\bar{u}, \bar{v})^t = au\bar{u} + bu\bar{v} + \bar{b}\bar{u}v + dv\bar{v}.$$

We shall often call an element $f \in \mathcal{H}(R)$ a binary hermitian form with coefficients in R. The discriminant $\Delta(f)$ of $f \in \mathcal{H}(R)$ is defined as $\Delta(f) = \det(f)$. Set $|a| = (a\bar{a})^{1/2}$ for $a \in \mathbb{C}$ where $^-$ denotes the complex conjugation. We define the $\mathrm{GL}_2(R)$-action on $\mathcal{H}(R)$ given by the formula

$$\sigma \cdot f = (|\det(\sigma)|^{-1/2}\sigma) f (|\det(\bar{\sigma}^t)|^{-1/2}\bar{\sigma}^t) = |\det(\sigma)|^{-1}\sigma f \bar{\sigma}^t \tag{1}$$

for $\sigma \in \mathrm{GL}_2(R)$ and $f \in \mathcal{H}(R)$. If $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(R)$ we have

$$\sigma \cdot f = |\det(\sigma)|^{-1} \begin{pmatrix} (\alpha, \beta) f (\bar{\alpha}, \bar{\beta})^t & (\alpha, \beta) f (\gamma, \delta)^t \\ (\gamma, \delta) f (\bar{\alpha}, \bar{\beta})^t & (\gamma, \delta) f (\bar{\gamma}, \bar{\delta})^t \end{pmatrix}.$$

Note that $\Delta(\sigma \cdot f) = \Delta(f)$ for every $\sigma \in \mathrm{GL}_2(R)$ and $f \in \mathcal{H}(R)$. Two elements $f, g \in \mathcal{H}(R)$ are called $\mathrm{GL}_2(R)$-equivalent if $g = \sigma \cdot f$ for some $\sigma \in \mathrm{GL}_2(R)$; $\mathrm{SL}_2(R)$-equivalence is defined analogously.

A binary Hermitian form $f \in \mathcal{H}(R)$ is positive definite if $f(u, v) > 0$ for all $(u, v) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$. If $-f$ is positive definite f is called negative definite. If $\Delta(f) < 0$ then f is called indefinite.

We define

$$\mathcal{H}^+(R) = \{f \in \mathcal{H}(R) \mid f \text{ is positive definite}\}$$
$$\mathcal{H}^-(R) = \{f \in \mathcal{H}(R) \mid f \text{ is indefinite}\}.$$

Clearly the group $\mathrm{GL}_2(R)$ leaves the $\mathcal{H}^{\pm}$ invariant. It is easy to see that $f \in \mathcal{H}^{+}(R)$ if and only if $a > 0$ and $\Delta(f) > 0$. The group $\mathbb{R}_{>0}$ acts on $\mathcal{H}^{+}(\mathbb{C})$ by scalar multiplication. Similarly $\mathbb{R}^{\times}$ acts on $\mathcal{H}^{-}(\mathbb{C})$. We define

$$\tilde{\mathcal{H}}^{+}(\mathbb{C}) := \mathcal{H}^{+}(\mathbb{C})/\mathbb{R}_{>0}, \quad \tilde{\mathcal{H}}^{-}(\mathbb{C}) := \mathcal{H}^{-}(\mathbb{C})/\mathbb{R}^{\times}.$$

For $f \in \mathcal{H}^{\pm}(\mathbb{C})$, $[f]$ stands for the class of f in $\tilde{\mathcal{H}}^{\pm}(\mathbb{C})$. The action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathcal{H}^{\pm}(\mathbb{C})$ clearly induces an action of $\mathrm{GL}_2(\mathbb{C})$ on $\tilde{\mathcal{H}}^{\pm}(\mathbb{C})$. The centre of $\mathrm{SL}_2(\mathbb{C})$ acts trivially on $\mathcal{H}(\mathbb{C})$, so we get an induced action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{H}(\mathbb{C})$ and $\tilde{\mathcal{H}}^{\pm}(\mathbb{C})$.

Recall the upper half space $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$, elements of which can be written as (z, r) with $z = x + iy$ for $x, y \in \mathbb{R}$, $r \in \mathbb{R}_{>0}$.

Definition 2.1 The map $\Phi : \mathcal{H}^{+}(\mathbb{C}) \to \mathbb{H}_3$ is defined as

$$\phi : f = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \to \frac{b}{d} + \frac{\sqrt{\Delta(f)}}{d} \cdot j$$

In fact ϕ induces a map $\phi : \tilde{\mathcal{H}}^{+}(\mathbb{C}) \to \mathbb{H}_3$.

This map is a bijection since for a point $(z, r) \in \mathbb{H}_3$ there exists $f = \begin{pmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{pmatrix}$ such that $\phi(f) = z + rj \in \mathbb{H}_3$. Therefore, this map gives a one to one correspondence between equivalence classes of positive definite Hermitian forms and points in the upper half space. Note that Φ is the analogue of identification of the set of equivalence classes of binary positive definite quadratic forms with points of the upper half plane.

Proposition 2.2 *The map* $\phi : \tilde{\mathcal{H}}^{+}(\mathbb{C}) \to \mathbb{H}_3$ *is a* $\mathrm{PSL}_2(\mathbb{C})$*-equivariant bijection; that is* $\phi(\sigma \cdot f) = \sigma \cdot \phi(f)$ *for every* $\sigma \in \mathrm{PSL}_2(\mathbb{C})$ *and* $f \in \tilde{\mathcal{H}}^{+}$.

Proof See [7, Proposition 9.1.2, Chapter 9]. □

Definition 2.3 For a binary Hermitian form $f = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \in \mathcal{H}^{-}(\mathbb{C})$ we define

$$\psi(f) = \{z + rj \in \mathbb{H}_3 \mid a - \bar{b}z - b\bar{z} + dz\bar{z} + r^2 d = 0\}$$

and $\mathbf{G} = \{\psi(f) \mid f \in \mathcal{H}^{-}(\mathbb{C})\}$ which is a set of geodesic planes in $\mathbb{H}_3$.

Remark 2.4 This map ψ is slightly different to the map in [7, Definition 1.3, Chapter 9] which is given by

$$f \mapsto \{z + rj \in \mathbb{H}_3 : a + \bar{b}z + b\bar{z} + dz\bar{z} + r^2 d = 0\}.$$

The above map ψ is chosen for us to prove Proposition 3.4. In addition we will consider the cycle D_U for positive definite U generated by f with $f \in \mathcal{H}^{-}(\mathbb{C})$.

If $d \neq 0$ then $\psi(f)$ is the following geodesic hemisphere

$$\psi(f) = \{z + rj \in \mathbb{H}_3 \mid |dz - b|^2 + |d|^2 r^2 = -\Delta(f)\}.$$

If $d = 0$ then $\psi(f)$ is a vertical plane. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathbf{G}$ by its induced action on subsets of $\mathbb{H}_3$. Clearly ψ induces a map $\psi : \tilde{\mathcal{H}}^{-}(\mathbb{C}) \to \mathbf{G}$.

Proposition 2.5 *The map* $\psi : \tilde{\mathcal{H}}^{-}(\mathbb{C}) \to \mathbf{G}$ *is a* $\mathrm{PSL}_2(\mathbb{C})$*-equivariant bijection; that is* $\psi(\sigma \cdot f)) = \sigma \cdot \psi(f)$ *for every* $\sigma \in \mathrm{PSL}_2(\mathbb{C})$ *and* $f \in \tilde{\mathcal{H}}^{-}(\mathbb{C})$.

Proof We will prove the equivariance property only for the generators of $\mathrm{PSL}_2(\mathbb{C})$.

Let $\sigma = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ where $\beta \in \mathbb{C}$. Then

$$\sigma \cdot f = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{\beta} & 1 \end{pmatrix} = \begin{pmatrix} a + \beta\bar{b} + \bar{\beta}b + \beta\bar{\beta}d & b + \beta d \\ \bar{b} + \bar{\beta}d & d \end{pmatrix}.$$

It follows that

$$\psi(\sigma \cdot f) = \{z + rj \in \mathbb{H}_3 \mid a + \beta\bar{b} + \bar{\beta}b + \beta\bar{\beta}d - (\bar{b} + \bar{\beta}d)z - (b + \beta d)\bar{z} + dz\bar{z} + r^2 d = 0\}.$$

On the other hand, for $z + rj \in \psi(f)$, we have $\sigma \cdot (z + rj) = (z + \beta) + rj \in \mathbb{H}_3$. Setting $z^{'} = z + \beta$ and $r^{'} = r$, we observe that

$$a - \bar{b}(z^{'} - \beta) - b(\bar{z}^{'} - \bar{\beta}) + d(z^{'} - \beta)(\bar{z}' - \bar{\beta}) + r'^2 d = 0.$$

Then it is not hard to see that $\psi(\sigma \cdot f)) = \sigma \cdot \psi(f)$ for $\sigma = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$.

In the same way we prove this property for $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We have

$$\sigma \cdot f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & -\bar{b} \\ -b & a \end{pmatrix}.$$

It follows that

$$\psi(\sigma \cdot f) = \{z + rj \in \mathbb{H}_3 \mid d + bz + \bar{b}\bar{z} + az\bar{z} + r^2 a = 0\}.$$

For $z + rj \in \psi(f)$, we have $z^{'} + r^{'}j = \sigma \cdot (z + rj) = -\frac{\bar{z}}{|z|^2 + r^2} + \frac{r}{|z|^2 + r^2} j$. Then $|z'|^2 + r'^2 = \frac{1}{|z|^2 + r^2}$. It follows that $z = -\frac{\bar{z}'}{|z'|^2 + r'^2}$ and $r = \frac{r'}{|z'|^2 + r'^2}$. Hence the following identity holds

$$a + \bar{b}\frac{\bar{z}'}{|z'|^2 + r'^2} + b\frac{z}{|z'|^2 + r'^2} + d\frac{z'\bar{z}'}{(|z'|^2 + r'^2)^2} + d\frac{r'^2}{(|z'|^2 + r'^2)^2} = 0.$$

Then it is not hard to see that $\psi(\sigma \cdot f)) = \sigma \cdot \psi(f)$ for $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. □

3 Orthogonal group of sign (3, 1) and cycles

In this section we recall some basic aspects on orthogonal groups of signature (3,1) and cycles in this case from [3, Sect. 4].

Let $F = \mathbb{Q}(\sqrt{d})$ $(d < 0)$ be an imaginary quadratic field of class number 1. Denote by $\mathcal{O}$ by its ring of integers. For an ideal $\mathfrak{n} \subset \mathcal{O}$ put

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) : c \in \mathfrak{n} \right\}.$$

Assume that the four-dimensional space V over $\mathbb{Q}$ is given by the hermitian matrices

$$V = \{\mathbf{x} \in M_2(F) : \mathbf{x}^t = \bar{\mathbf{x}}\},$$

with quadratic form

$$\mathbf{x} \mapsto -\det(\mathbf{x})$$

and corresponding bilinear form

$$(\mathbf{x}, \mathbf{y}) \mapsto -\frac{1}{2}\,\mathrm{tr}(\mathbf{x}\mathbf{y}^*),$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{2}$$

Note that this bilinear form is preserved under the action of $\mathrm{GL}_2(\mathbb{C})$ where its action is given in (1); that is, for $g \in \mathrm{GL}_2(\mathbb{C})$,

$$\begin{aligned}(g \cdot \mathbf{x}, g \cdot \mathbf{y}) &= (|\det(g)|^{-1} g\mathbf{x}\bar{g}^t, |\det(g)|^{-1} g\mathbf{y}\bar{g}^t) \\ &= -\frac{1}{2}\operatorname{tr}(|\det(g)|^{-1} g\mathbf{x}\bar{g}^t |\det(g)|(\bar{g}^t)^{-1}\mathbf{y}^{-1}g^{-1}\det(|\det(g)|^{-1}g\mathbf{y}\bar{g}^t)) \\ &= -\frac{1}{2}\operatorname{tr}(g\mathbf{x}\mathbf{y}^{-1}\det(\mathbf{y})g^{-1}) = (\mathbf{x}, \mathbf{y}). \end{aligned} \tag{3}$$

We fix an orthogonal basis of $V(\mathbb{Q})$ given by $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{smallmatrix}\right)$ and $e_4 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = Z_0$ such that the discriminant of V is d. The basis of $Z_0^{\perp}$ can be identified with $\{e_1, e_2, e_3\}$.

The symmetric space in this case can be realized as

$$D = \{Z \in V(\mathbb{R}) : (Z, Z_0) < 0\},$$

which is isomorphic to hyperbolic 3-space $\mathbb{H}_3$. The isomorphism can be given by

$$\mu : z + rj \in \mathbb{H}_3 \longmapsto \frac{1}{r}\begin{pmatrix} |z|^2 + r^2 & z \\ \bar{z} & 1 \end{pmatrix}. \tag{4}$$

The GL_2-action on the Hermitian form defined as in (1) induces that on $\mathbb{H}_3$ in the following. For $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ we have $g \cdot \frac{1}{r}\left(\begin{smallmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{smallmatrix}\right) = \frac{1}{r'}\left(\begin{smallmatrix} |z'|^2+r'^2 & z' \\ \bar{z}' & 1 \end{smallmatrix}\right)$; expand the LHS,

$$\begin{aligned} g \cdot \frac{1}{r}\left(\begin{smallmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{smallmatrix}\right) &= r^{-1}|\det(g)|^{-1}\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\left(\begin{smallmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{smallmatrix}\right) \\ &= r^{-1}|\det(g)|^{-1}\left(\begin{smallmatrix} \alpha|z|^2+\alpha r^2+\beta\bar{z} & \alpha z+\beta \\ \gamma|z|^2+\gamma r^2+\delta\bar{z} & \gamma z+\delta \end{smallmatrix}\right)\left(\begin{smallmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{smallmatrix}\right) \\ &= r^{-1}|\det(g)|^{-1}\left(\begin{smallmatrix} \alpha\bar{\alpha}|z|^2+\alpha\bar{\alpha}r^2+\bar{\alpha}\beta\bar{z}+\alpha\bar{\beta}z+\beta\bar{\beta} & \alpha\bar{\gamma}|z|^2+\alpha\bar{\gamma}r^2+\beta\bar{\gamma}\bar{z}+\alpha\bar{\delta}z+\beta\bar{\delta} \\ \bar{\alpha}\gamma|z|^2+\bar{\alpha}\gamma r^2+\bar{\alpha}\delta\bar{z}+\bar{\beta}\gamma z+\bar{\beta}\delta & \gamma\bar{\gamma}|z|^2+\gamma\bar{\gamma}r^2+\bar{\gamma}\delta\bar{z}+\gamma\bar{\delta}z+\delta\bar{\delta} \end{smallmatrix}\right), \end{aligned}$$

and then

$$z' = \frac{(\alpha z + \beta)(\bar{\gamma}\bar{z} + \bar{\delta}) + \alpha\bar{\gamma}r^2}{|\gamma z + \delta|^2 + |\gamma|^2 r^2}, r' = \frac{|\alpha\delta - \beta\gamma| r}{|\gamma z + \delta|^2 + |\gamma|^2 r^2}.$$

By (4), we can define the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{H}_3$ to be as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot (z, r) = \left(\frac{(\alpha z + \beta)(\bar{\gamma}\bar{z} + \bar{\delta}) + \alpha\bar{\gamma}r^2}{|\gamma z + \delta|^2 + |\gamma|^2 r^2}, \quad \frac{|\alpha\delta - \beta\gamma| r}{|\gamma z + \delta|^2 + |\gamma|^2 r^2}\right). \tag{5}$$

Proposition 3.1 *The above map μ as in (4) intertwines the $\mathrm{GL}_2(\mathbb{C})$-action on $V(\mathbb{R})$ and $\mathbb{H}_3$; that is $\mu(g \cdot (z, r)) = g \cdot \mu(z, r)$ for $g \in \mathrm{GL}_2(\mathbb{C})$.*

Proof See [26, Proposition 4.2.1]. □

The set $\mathrm{Iso}(V)$ of all isotropic lines (1-dimensional $\mathbf{x} \in V$ such that $q(\mathbf{x}) = 0$) in $V(\mathbb{Q})$ can be identified with $\mathbb{P}^1(F) = F \cup \infty$ ($\infty = [1 : 0]$). Assume that the cusp ∞ corresponds to the isotropic line spanned by $u_\infty = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$. Given an element $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(F)$ transforming the cusp ∞ to another cusp $\kappa = [\alpha : \gamma]$, we can see that

$$g \cdot u_\infty = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar\alpha & \bar\gamma\\ \bar\beta & \bar\delta\end{pmatrix} = \begin{pmatrix}\alpha\bar\alpha & \alpha\bar\gamma\\ \bar\alpha\gamma & \gamma\bar\gamma\end{pmatrix}.$$

Hence we can identify the cusp with the isotropic line by means of the map

$$\nu : [a : b] \longmapsto \operatorname{span}\begin{pmatrix}a\bar a & a\bar b\\ \bar a b & b\bar b\end{pmatrix} \in \mathrm{Iso}(V). \tag{6}$$

Proposition 3.2 *The above map ν satisfies*

$$\nu(g \cdot [a : b]) = g \cdot \nu([a : b])$$

for $g \in \mathrm{GL}_2(F)$ and $[a : b] \in \mathbb{P}^1(F)$.

Proof See [26, Proposition 4.2.2]. □

Let $U \subset V$ be a $\mathbb{Q}$-subspace with $\dim_{\mathbb{Q}} U = 2$ such that $(\ )|_U$ is positive definite; say $U = \langle \mathbf{x}_1, \mathbf{x}_2\rangle$. Define the special cycle as

$$D_U = \{Z \in D : Z \perp U\}.$$

Let Γ be a torsion-free congruence subgroup of $\mathrm{GL}(V)$ preserving the quadratic form on V and let Γ_U be the stabilizer of U in $\Gamma \cap \mathrm{SO}_0(3,1)(\mathbb{R})$, where SO_0 is the identity component of SO. We denote the image of the quotient $\Gamma_U\backslash D_U$ in $\Gamma\backslash D$ by C_U. The stabilizer Γ_U is either trivial (if the orthogonal complement $U^\perp \subset V$ is split over $\mathbb{Q}$) or infinite cyclic (if $U^\perp$ is non-split over $\mathbb{Q}$) (see [11, Lemma 4.2]). If Γ_U is infinite, then C_U is a closed geodesic in $\Gamma\backslash D$, while C_U is infinite if Γ_U is trivial (see [3, Sect. 4.3]).

Lemma 3.3 *For $U = \langle \mathbf{x}_1, \mathbf{x}_2\rangle \subset V$ as above. Then the following two statements are equivalent:*

(1) $U^\perp$ is split over $\mathbb{Q}$,
(2) $\mathrm{disc}(U) \in -d(\mathbb{Q}^\times)^2$.

Proof For an arbitrary subspace U of a non-degenerate quadratic space V we have $\dim(V) = \dim(U) + \dim(U^\perp)$. Thus $U^\perp$ is also 2-dimensional. By assumption $U^\perp$ is a hyperbolic plane. By Witt's Theorem (a 2-dimensional quadratic space over a field F is a hyperbolic plane if and only if its discriminant lies in $-(F^\times)^2$) we have $\mathrm{disc}(U^\perp) \in -(\mathbb{Q}^\times)^2$. Thus $\mathrm{disc}(U) \in -d(\mathbb{Q}^\times)^2$ as $\mathrm{disc}(V) = \mathrm{disc}(U)\mathrm{disc}(U^\perp)$.

Conversely suppose $\mathrm{disc}(U) \in -d(\mathbb{Q}^\times)^2$. Again by $\mathrm{disc}(V) = \mathrm{disc}(U)\mathrm{disc}(U^\perp)$, we have $\mathrm{disc}(U^\perp) \in -(\mathbb{Q}^\times)^2$ implying that $U^\perp$ is split over $\mathbb{Q}$. □

We orient D_U by requiring that a tangent vector $v \in T_Z(D_U) \simeq Z^\perp \cap U^\perp$ followed by $Z^\perp \cap U$ gives a properly oriented basis of $T_Z(D) \simeq Z^\perp$. Then $\langle Z^\perp \cap U^\perp, Z^\perp \cap U, Z\rangle$ has the same orientation as $\langle e_1, e_2, e_3, e_4\rangle$, i.e. the determinant of the base change is positive.

For $\beta = \beta^t \in M_2(\mathbb{Q})$ a positive definite symmetric matrix, let

$$\Omega_\beta = \left\{(\mathbf{x}_1, \mathbf{x}_2) \in V^2(\mathbb{Q}) : ((\mathbf{x}_i, \mathbf{x}_i)) = \beta\right\}.$$

Consider the subspace $U(\mathbf{x}_1, \mathbf{x}_2) := \langle \mathbf{x}_1, \mathbf{x}_2\rangle \subset V$. For a fixed cusp κ_i corresponding to the isotropic line l_{κ_i}, we write

$$\Omega_{\beta,\kappa_i} = \{(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_\beta : U(\mathbf{x}_1, \mathbf{x}_2) \perp l_{\kappa_i}\}.$$

From now on, fix a β such that $\mathrm{disc}(U) \in -d(\mathbb{Q}^\times)^2$, i.e. $\det\beta \in -d(\mathbb{Q}^\times)^2$. Let $(\mathbf{x}_1,\mathbf{x}_2) \in \Omega_\beta$ and $U = U(\mathbf{x}_1,\mathbf{x}_2) = \langle \mathbf{x}_1,\mathbf{x}_2\rangle$. Given a vector $\mathbf{x} \in U$, by Lemma 3.3, it is orthogonal to two isotropic lines l_{κ_1} and l_{κ_2} generated by u_{κ_1} and u_{κ_2} respectively associated to two cusps κ_1 and κ_2. Again, if these two cusps are not equivalent with respect to Γ, we can give a positive orientation to U to distinguish the cusps in the sense that the new base $\langle u_{\kappa_1},\mathbf{x}_1,\mathbf{x}_2,u_{\kappa_2}\rangle$ preserves the orientation of $\langle e_1,e_2,e_3,e_4\rangle$. For a fixed cusp κ_i corresponding to the isotropic line l_{κ_i}, we write

$$\Omega_{\beta,\kappa_i,+} = \{(\mathbf{x}_1,\mathbf{x}_2)\in\Omega_{\beta,\kappa_i} : \langle u_{\kappa_i},u_{\kappa_j}\rangle \perp U(\mathbf{x}_1,\mathbf{x}_2), \langle u_{\kappa_i},\mathbf{x}_1,\mathbf{x}_2,u_{\kappa_j}\rangle \text{has a positive orientation}\}.$$

It should be mentioned here that $\langle u_{\kappa_i},\mathbf{x}_1,\mathbf{x}_2,u_{\kappa_j}\rangle$ and $\langle u_{\kappa_i},-\mathbf{x}_1,-\mathbf{x}_2,u_{\kappa_j}\rangle$ have the same orientation which means that we need to count $(\mathbf{x}_1,\mathbf{x}_2)$ and $(-\mathbf{x}_1,-\mathbf{x}_2)$ simultaneously in $\Omega_{\beta,\kappa_i,+}$. Alternatively, the following Lemma 3.7 describes the orientations associated to two pairs $(\mathbf{x}_1,\mathbf{x}_2)$ and $(-\mathbf{x}_1,-\mathbf{x}_2)$ in $\Omega_{\beta,\infty,+}$ in a different way. Note that the stabilizer $\Gamma_{\kappa_i} \subset \Gamma_0(\mathfrak{n})$ of the cusp κ_i acts on $\Omega_{\beta,\kappa_i,+}$ as $\mathrm{GL}_2(\mathbb{C})$ preserves bilinear forms and the orientation.

Proposition 3.4 *For* $\det\beta \in -d(\mathbb{Q}^\times)^2$*, we have*

$$\Gamma\backslash\Omega_\beta = \sum_{\kappa_i\in\Gamma\backslash\mathbb{P}^1(F)} \Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}.$$

Proof Given a representative $[(\mathbf{x}_1,\mathbf{x}_2)]$ in $\Gamma\backslash\Omega_\beta$ such that $U(\mathbf{x}_1,\mathbf{x}_2)\perp u_{\kappa_i}$, we consider its Γ-orbit $\Gamma\cdot(\mathbf{x}_1,\mathbf{x}_2)$. The corresponding D_U for $U=\langle\Gamma\cdot(\mathbf{x}_1,\mathbf{x}_2)\rangle$ has the image $C_{(\mathbf{x}_1,\mathbf{x}_2)}$ in $\Gamma\backslash\mathbb{H}_3$ under the natural projection $\mathbb{H}_3\to\Gamma\backslash\mathbb{H}_3$. For a $\gamma\in\Gamma$ we have $U_{\Gamma\cdot(\mathbf{x}_1,\mathbf{x}_2)}\perp\gamma\cdot u_{\kappa_i}$. By Proposition 3.1, we know that $\gamma\cdot u_{\kappa_i}=u_{\gamma\cdot\kappa_i}$. It follows that $\gamma\cdot(\mathbf{x}_1,\mathbf{x}_2)$ lies in $\Omega_{\beta,\gamma\cdot\kappa_i,+}$. Thus, modulo the Γ-action, we have a well-defined map:

$$\iota:\Gamma\backslash\Omega_\beta \longrightarrow \coprod_{\kappa_i\in\Gamma\backslash\mathbb{P}^1(F)} \Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}.$$

If two pairs $(\mathbf{x}_1,\mathbf{x}_2)$ and $(\mathbf{y}_1,\mathbf{y}_2)$ are not Γ-equivalent then they are not Γ_{κ_i}-equivalent since $\Gamma_{\kappa_i}\subset\Gamma$. Hence this map is injective.

We will show that the inverse map ι^{-1} is injective in the following. For $\mathbf{x}=\begin{pmatrix} a & b\\ \bar b & d\end{pmatrix}$, we calculate its orthogonal complement in $\mathbb{H}_3$, due to the isomorphism (4),

$$\mathbf{x}^\perp\cap\mathbb{H}_3 = \{z+rj\in\mathbb{H}_3 : d(|z|^2+r^2)-b\bar z-\bar b z+a=0\}=\psi(\mathbf{x})$$

where ψ is defined as in Definition 2.3. Observe that $\mathbf{x}_1^\perp\cap\mathbf{x}_2^\perp\cap\mathbb{H}_3=\psi(\mathbf{x}_1)\cap\psi(\mathbf{x}_2)$ of which one boundary point on the complex plane is κ_i. Suppose that two pairs $(\mathbf{x}_1,\mathbf{x}_2)$ and $(\mathbf{y}_1,\mathbf{y}_2)$ are not Γ_{κ_i}-equivalent in $\Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}$. Note that $\psi(\mathbf{x}_1)\cap\psi(\mathbf{x}_2)$ and $\psi(\mathbf{y}_1)\cap\psi(\mathbf{y}_2)$ have a boundary point in common, the cusp κ_i. Assume that there exists an element $\gamma\in\Gamma$ such that $\gamma\cdot(\mathbf{x}_1,\mathbf{x}_2)=(\mathbf{y}_1,\mathbf{y}_2)$. Then, by Proposition 2.5, we have $\gamma\cdot\psi(\mathbf{x}_1)=\psi(\mathbf{y}_1)$ and $\gamma\cdot\psi(\mathbf{x}_2)=\psi(\mathbf{y}_2)$. It is easy to observe that

$$\gamma\cdot(\psi(\mathbf{x}_1)\cap\psi(\mathbf{x}_2))=\gamma\cdot\psi(\mathbf{x}_1)\cap\gamma\cdot\psi(\mathbf{x}_2)=\psi(\mathbf{y}_1)\cap\psi(\mathbf{y}_2).$$

It follows that γ must be in Γ_{κ_i}, which is a contradiction to that $(\mathbf{x}_1,\mathbf{x}_2)$ and $(\mathbf{y}_1,\mathbf{y}_2)$ are not Γ_{κ_i}-equivalent. So such a γ does not exist. We have proven the injectivity of ι^{-1}. □

Set $\det(\beta)\in -d(\mathbb{Q}^\times)^2$. It is easy to observe that, for the cusp ∞,

$$\Omega_{\beta,\infty}=\left\{\left(\begin{pmatrix} a_1 & b_1\\ \bar b_1 & 0\end{pmatrix},\begin{pmatrix} a_2 & b_2\\ \bar b_2 & 0\end{pmatrix}\right)\in\Omega_\beta : \det(\beta)\in -d(\mathbb{Q}^\times)^2, a_1,a_2\in\mathbb{Q}, b_1,b_2\in F\right\}.$$

Setting $(\mathbf{x}_1,\mathbf{x}_2)\in\Omega_{\beta,\infty}=\left(\begin{pmatrix}a_1 & b_1\\ \bar b_1 & 0\end{pmatrix},\begin{pmatrix}a_2 & b_2\\ \bar b_2 & 0\end{pmatrix}\right)$, the associated Gram matrix is of form

$$\beta=\begin{pmatrix}(\mathbf{x}_1,\mathbf{x}_1) & (\mathbf{x}_1,\mathbf{x}_2)\\ (\mathbf{x}_1,\mathbf{x}_2) & (\mathbf{x}_2,\mathbf{x}_2)\end{pmatrix}=\begin{pmatrix}b_1\bar b_1 & \frac{1}{2}(b_1\bar b_2+\bar b_1 b_2)\\ \frac{1}{2}(b_1\bar b_2+\bar b_1 b_2) & b_2\bar b_2\end{pmatrix}.$$

of which the determinant is

$$\det(\beta)=\mathrm{disc}(U(\mathbf{x}_1,\mathbf{x}_2))=-\frac{1}{4}(b_1\bar b_2-\bar b_1 b_2)^2.$$

We are not interested in the case when $b_1\bar b_2\in\mathbb{Q}$ since then $\det(\beta)=0$.

Let $U=U(\mathbf{x}_1,\mathbf{x}_2)$ for $(\mathbf{x}_1,\mathbf{x}_2)\in\Omega_{\beta,\infty}$. We will calculate its corresponding special cycle D_U in the following. Given a point $z+rj\in\mathbb{H}_3$ identified with $\frac{1}{r}\begin{pmatrix}|z|^2+r^2 & z\\ \bar z & 1\end{pmatrix}$, we compute

$$\frac{1}{r}\begin{pmatrix}|z|^2+r^2 & z\\ \bar z & 1\end{pmatrix}\begin{pmatrix}a_1 & b_1\\ \bar b_1 & 0\end{pmatrix}^*=\frac{1}{r}\begin{pmatrix}|z|^2+r^2 & z\\ \bar z & 1\end{pmatrix}\begin{pmatrix}0 & -b_1\\ -\bar b_1 & a_1\end{pmatrix}=\frac{1}{r}\begin{pmatrix}-\bar b_1 z & *\\ * & -b_1\bar z+a_1\end{pmatrix}.$$

Thus we have

$$\mathbf{x}_1^{\perp}=\{z+rj\in\mathbb{H}_3: a_1-b_1\bar z-\bar b_1 z=0\},$$

and similarly,

$$\mathbf{x}_2^{\perp}=\{z+rj\in\mathbb{H}_3: a_2-b_2\bar z-\bar b_2 z=0\}.$$

Then, solving above equations, we can deduce that the special cycle D_U consists of the infinite geodesic line joining two cusps ∞ and

$$z_U=\frac{a_2b_1-a_1b_2}{b_1\bar b_2-\bar b_1 b_2}. \tag{7}$$

Lemma 3.5 *Suppose that $\mathcal{O}=\mathbb{Z}[\omega]$ with ω is either $\sqrt{d}$ or $\frac{1+\sqrt{d}}{2}$ and denote the stabilizer of the cusp ∞ by $\Gamma_\infty=\left\{\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}:\alpha\in\mathcal{O}\right\}$. Denote*

$$L_{\infty,\dagger}=\left\{\left(\begin{pmatrix}a_1 & b_1\\ \bar b_1 & 0\end{pmatrix},\begin{pmatrix}a_2 & b_2\\ \bar b_2 & 0\end{pmatrix}\right): a_1,a_2\in\mathbb{Z}, b_1,b_2\in\mathcal{O},\ \text{the condition } \dagger \text{ holds}\right\}$$

where the condition † is given by

$$\begin{pmatrix}b_1\\ b_2\end{pmatrix}=m\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1\\ \omega\end{pmatrix} \tag{8}$$

with $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$, $\alpha\delta-\beta\gamma=\pm1$ and $m\in F$. Then the cusp z_U associated to $\Gamma_\infty\backslash L_{\infty,\dagger}$ runs through all the representatives in $(\overline{m}\sqrt{d_F}|)^{-1}\mathcal{O}/\mathcal{O}$.

Proof Write $U=U(\mathbf{x}_1,\mathbf{x}_2)$. The Γ_∞-action on $(\mathbf{x}_1,\mathbf{x}_2)$ is given explicitly by

$$\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\cdot(\mathbf{x}_1,\mathbf{x}_2)=\left(\begin{pmatrix}a_1+\alpha\bar b_1+\bar\alpha b_1 & b_1\\ \bar b_1 & 0\end{pmatrix},\begin{pmatrix}a_2+\alpha\bar b_2+\bar\alpha b_2 & b_2\\ \bar b_2 & 0\end{pmatrix}\right).$$

Under the Γ_∞-action, the cusp z_U becomes $z_U^{'}$; that is

$$z_U^{'}=\frac{(a_2+\alpha\bar b_2+\bar\alpha b_2)b_1-(a_1+\alpha\bar b_1+\bar\alpha b_1)b_2}{b_1\bar b_2-\bar b_1 b_2}$$

$$=\frac{a_2b_1+\alpha\bar{b}_2b_1-a_1b_2-\alpha\bar{b}_1b_2}{b_1\bar{b}_2-\bar{b}_1b_2}=z_U+\alpha.$$

By our assumption, the cusp z_U can be rewritten as

$$\begin{aligned}z_U&=\frac{ma_2(\alpha+\beta\omega)-ma_1(\gamma+\delta\omega)}{-m\overline{m}(\alpha+\beta\omega)(\gamma+\delta\bar{\omega})+m\overline{m}(\alpha+\beta\bar{\omega})(\gamma+\delta\omega)}\\&=\frac{a_2(\alpha+\beta\omega)-a_1(\gamma+\delta\omega)}{-\overline{m}(\alpha\delta-\beta\gamma)(\bar{\omega}-\omega)}=\frac{a_2(\alpha+\beta\omega)-a_1(\gamma+\delta\omega)}{\overline{m}\sqrt{d_F}}\end{aligned}$$

of which the numerator ranges over the whole $\mathcal{O}$.

Thus, modulo the Γ_∞-action, the corresponding cusp z_U runs through all the representatives in $(\overline{m}\sqrt{d_F})^{-1}\mathcal{O}/\mathcal{O}$. □

Remark 3.6 Let m be a square-free product of split or inert primes.

(1) Let $d\equiv 1 \bmod 4$ and then $d_F=d$. The above z_U ranges over $(m\sqrt{d})^{-1}\mathcal{O}$. Writing $\mathfrak{f}=(m\sqrt{d})\mathcal{O}$, we have

$$z_U\mathfrak{f}=\left(\frac{a_2b_1-a_1b_2}{b_1\bar{b}_2-\bar{b}_1b_2}\right)\mathfrak{f}=\left(\frac{a_2b_1-a_1b_2}{m}\right)\mathcal{O}.$$

(2) Let $d\equiv 2,3 \bmod 4$ and then $d_F=4d$. Note that in this case prime 2 is ramified in $F=\mathbb{Q}(\sqrt{d})$. Rewrite (8) above as

$$\begin{pmatrix}b_1\\b_2\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\begin{pmatrix}1\\\omega\end{pmatrix}$$

with $\alpha,\beta,\gamma,\delta\in\mathbb{Z},\alpha\delta-\beta\gamma=\pm1$. Then the above z_U ranges over $(m\sqrt{d})^{-1}\mathcal{O}$. Writing $\mathfrak{f}=(m\sqrt{d})\mathcal{O}$, we have

$$z_U\mathfrak{f}=\left(\frac{a_2b_1-a_1b_2}{b_1\bar{b}_2-\bar{b}_1b_2}\right)\mathfrak{f}=\left(\frac{2(a_2b_1-a_1b_2)}{m}\right)\mathcal{O}.$$

In Sect. 4 we will define the Schwartz function evaluated at $\frac{a_2b_1-a_1b_2}{m}$ or $\frac{2(a_2b_1-a_1b_2)}{m}$ depending on d.

Let $U=\langle\mathbf{x}_1,\mathbf{x}_2\rangle=\left\langle\begin{pmatrix}a_1&b_1\\\bar{b}_1&0\end{pmatrix},\begin{pmatrix}a_2&b_2\\\bar{b}_2&0\end{pmatrix}\right\rangle$ where $a_1,a_2\in\mathbb{Q}$ and $b_1,b_2\in F^\times$. We have seen that D_U consists of the infinite geodesic line joining the cusps ∞ and z_U as in (7). Choose a point $Z=z_U+rj$ on D_U and then the orientation of $T_Z(D_U)$ depends on the sign of $\mathrm{Im}(b_1\bar{b}_2)$ (assuming $\mathrm{Im}(b_1\bar{b}_2)\neq0$) by the following lemma.

Lemma 3.7 *Let U, D_U, Z be as above. Then the sign of $\mathrm{Im}(b_1\bar{b}_2)$ (assuming $\mathrm{Im}(b_1\bar{b}_2)\neq0$) determines the orientation of $T_Z(D_U)$.*

Proof Here we sketch the proof and for more details see [26, Lemma 4.2.7].

Let $Z=z_U+rj$ be a point on D_U which can be identified with $\frac{1}{r}\begin{pmatrix}z_U\bar{z}_U+r^2&z_U\\\bar{z}_U&1\end{pmatrix}$.

Suppose that $\begin{pmatrix}\alpha&\beta\\\bar{\beta}&\delta\end{pmatrix}\in Z^\perp$ and we compute, recalling $*$ action in (2)

$$\begin{pmatrix}\alpha&\beta\\\bar{\beta}&\delta\end{pmatrix}\begin{pmatrix}z_U\bar{z}_U+r^2&z_U\\\bar{z}_U&1\end{pmatrix}^*=\begin{pmatrix}\alpha-\beta\bar{z}_U&*\\ *&-\bar{\beta}z_U+\delta(z_U\bar{z}_U+r^2)\end{pmatrix}.$$

It follows that

$$Z^{\perp}=\left\{\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \delta\end{pmatrix}:\alpha-\beta\bar{z}_U-\bar{\beta}z_U+\delta(z_U\bar{z}_U+r^2)=0\right\}.$$

We describe the subspace $Z^{\perp}\cap U$ as

$$\begin{aligned}Z^{\perp}\cap U&=\left\{\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & 0\end{pmatrix}:\alpha-\beta\bar{z}_U-\bar{\beta}z_U=0,\alpha\in\mathbb{R},\beta\in\mathbb{C}\right\}\\&=\left\{\begin{pmatrix}\beta\bar{z}_U+\bar{\beta}z_U & \beta\\ \bar{\beta} & 0\end{pmatrix}\right\}\end{aligned}$$

where $\beta\in\operatorname{Span}\{b_1,b_2\}$. We see that

$$\begin{aligned}Z^{\perp}\cap U=\Big\langle&\frac{1}{2}a_1(e_1+e_4)+\frac{1}{2}(b_1+\bar{b}_1)e_2-\frac{1}{2}i(b_1-\bar{b}_1)e_3,\\&\frac{1}{2}a_2(e_1+e_4)+\frac{1}{2}(b_2+\bar{b}_2)e_2-\frac{1}{2}i(b_2-\bar{b}_2)e_3\Big\rangle\end{aligned}$$

is identical to $U=\left\langle\begin{pmatrix}a_1 & b_1\\ \bar{b}_1 & 0\end{pmatrix},\begin{pmatrix}a_2 & b_2\\ \bar{b}_2 & 0\end{pmatrix}\right\rangle$.

Similarly, we have that

$$\begin{aligned}&Z^{\perp}\cap U^{\perp}\\&=\left\langle\frac{\varepsilon}{2}(z_U\bar{z}_U-r^2-1)e_1+\frac{1}{2}(z_U+\bar{z}_U)e_2-\frac{1}{2}i(z_U-\bar{z}_U)+e_3\frac{\varepsilon}{2}(z_U\bar{z}_U-r^2+1)e_4\right\rangle\end{aligned}$$

where $\varepsilon=\pm1$ describes the orientation of $T_Z(D_U)\simeq Z^{\perp}\cap U^{\perp}$. If the cycle D_U is directed from z_U to ∞ then we take $\varepsilon=-1$. For a different direction of D_U we take $\varepsilon=1$.

Then we consider the base change, which describes the orientation related to introducing $\Omega_{\beta,\kappa_i,+}$, given by

$$\begin{pmatrix}Z^{\perp}\cap U^{\perp}\\ Z^{\perp}\cap U\\ Z\end{pmatrix}=M\begin{pmatrix}e_1\\ e_2\\ e_3\\ e_4\end{pmatrix}$$

where the determinnant of M can ba calculated as

$$\det M=-\varepsilon r\frac{1}{2}i(b_1\bar{b}_2-\bar{b}_1b_2)=\frac{1}{2}\varepsilon r\operatorname{Im}(b_1\bar{b}_2)>0.$$

It is obvious that the sign of $\operatorname{Im}(b_1\bar{b}_2)$ determines the orientation ε of $T_Z(D_U)$. □

4 Schwartz function

In this paper we need to consider the pair $\mathrm{Sp}_4\times\mathrm{SO}(3,1)$ to construct the theta liftings of weight 2 Bianchi modular forms. In the following Subsects. 4.1, 4.2 and 4.3, we define local Schwartz functions at split primes dividing m, inert primes dividing m and ramified primes away from 2 respectively. In Sect. 5 we will construct the theta lifting of a weight 2 Bianchi modular form of $\Gamma_0(\mathfrak{n})$ with square-free $\mathfrak{n}$ coprime to $(m|d_F|)$. To avoid the vanishing of our theta lifting, in Subsect. 4.4 we define the local Schwartz function at each place dividing $N(\mathfrak{n})$

(norm of $\mathfrak{n}$) and ramified prime 2, to be different to the characteristic function of integral lattice. In Subsect. 4.5 we consider all other finite places.

Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field and denote by $\mathcal{O}$ its ring of integers. Choose $m \in \mathbb{Z}$ as a square-free product of inert or split primes and put $\mathfrak{m} = m\mathcal{O}$. Let $\chi_\mathfrak{m}$ be a finite order Hecke character of conductor $\mathfrak{m}\sqrt{d}$. Denote by $\widetilde{\chi}_\mathfrak{m}$ the induced idelic one and by $\widetilde{\chi}_{\mathfrak{m},v}$ its local component. In this section we will define a Schwartz function $\varphi^{\chi_\mathfrak{m}}$ related to this character $\chi_\mathfrak{m}$.

We first describe how to localise the quadratic space in the following proposition. In Sect. 3 we have chosen the rational quadratic space of dimension 4 such that $(V(\mathbb{Q}), q) \simeq (\mathcal{H}(F), -\det)$. Thus, to extend it to the 4-dimensional quadratic space over p-adic numbers $\mathbb{Q}_p$, we can consider $\mathcal{H}(F) \otimes \mathbb{Q}_p$. Following [20, p.273], there are two four dimensional quadratic spaces over $\mathbb{Q}_p$ with discriminant $\mathfrak{d} \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ up to isometry. If $\mathfrak{d} = 1$, it is isometric to $M_{2\times 2}(\mathbb{Q}_p)$ equipped with the determinant; if $\mathfrak{d} \neq 1$, it is isometric to

$$V_1(\mathbb{Q}_p) = \left\{ \begin{pmatrix} e & f\sqrt{\mathfrak{d}} \\ g\sqrt{\mathfrak{d}} & \bar{e} \end{pmatrix} : f, g \in \mathbb{Q}_p, e \in \mathbb{Q}_p(\sqrt{\mathfrak{d}}) \right\} \subset M_2\left(\mathbb{Q}_p(\sqrt{\mathfrak{d}})\right)$$

equipped with the determinant.

Proposition 4.1 *For a prime p, the four dimensional quadratic space over $\mathbb{Q}_p$ is isometric to either $(V_1(\mathbb{Q}_p), \det)$ when p is inert or ramified in $F/\mathbb{Q}$, or $(M_2(\mathbb{Q}_p), \det)$ when p splits in $F/\mathbb{Q}$.*

Proof Given a diagonal quadratic form $Q = \sum_{i=1}^n a_i x_i^2$ with $a_i \in \mathbb{Q}_p^\times$, we define the Hasse invariant as $c_p(Q) = c(Q) = \prod_{i<j}(a_i, a_j)_p = \pm 1$ where $(\ )$ denotes the Hilbert symbol. The non-degenerate quadratic spaces over $\mathbb{Q}_p$ $(p < \infty)$ are in 1-1 correspondence with the triples $(n, \mathfrak{d}, c)$, where n is the dimension, $\mathfrak{d}$ is the discriminant, and c is the Hasse invariant [4, Theorem 1.1, Chapter 4].

Let p be inert in $F/\mathbb{Q}$ which implies that $\sqrt{d} \notin \mathbb{Q}_p$ and that $F \otimes \mathbb{Q}_p = \mathbb{Q}_p(\sqrt{d})$. Then we have

$$\mathcal{H}(F) \otimes \mathbb{Q}_p = \left\{ \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} : a, d \in \mathbb{Q}_p, b \in \mathbb{Q}_p(\sqrt{d}) \right\} = \mathcal{H}(F \otimes \mathbb{Q}_p),$$

where $^-$ denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_p(\sqrt{d})/\mathbb{Q}_p)$. Equipping $\mathcal{H}(F) \otimes \mathbb{Q}_p$ with the quadratic form being $-\det$ and choosing an orthogonal basis $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $e_4 = \begin{pmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{pmatrix}$, we have an associated diagonal form $Q = -x_1^2 + x_2^2 + x_3^2 - dx_4^2$. It follows that $\mathfrak{d} = d$ and $c = (-1, -d)_p = 1$ since $p \nmid d$. Similarly, for $V_1(\mathbb{Q}_p)$ with the discriminant $\mathfrak{d} = d$, choosing an orthogonal basis $e_1' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_2' = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}$, $e_3' = \begin{pmatrix} 0 & \sqrt{d} \\ \sqrt{d} & 0 \end{pmatrix}$ and $e_4' = \begin{pmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{pmatrix}$ in $V_1(\mathbb{Q}_p)$ above, we have a diagonal form $Q' = x_1^2 - dx_2^2 - dx_3^2 + dx_4^2$. Then $\mathfrak{d}' = d^3$ and $c' = (1, -d)_p^2 (1, d)_p (-d, -d)_p (-d, d)_p^2 = 1$ since $p \nmid d$. Thus, we can deduce that $(\mathcal{H}(F) \otimes \mathbb{Q}_p, -\det) \simeq (V_1(\mathbb{Q}_p), \det)$ if p is inert in $F/\mathbb{Q}$.

Let p split in $F/\mathbb{Q}$ such that $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Then d has a square root α in the ring $\mathbb{Z}_p$ of p-adic integers by Hensel's lemma. It is known that $F \otimes \mathbb{Q}_p = F_\mathfrak{p} \times F_{\bar{\mathfrak{p}}}$ where $F_\mathfrak{p}, F_{\bar{\mathfrak{p}}}$ are both isomorphic to $\mathbb{Q}_p$. Consider the map $\mathcal{H}(F) \otimes \mathbb{Q}_p \to M_2(\mathbb{Q}_p)$ via $\left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \otimes x \mapsto \left(\begin{smallmatrix} a_p x & b_\mathfrak{p} x \\ b_{\bar{\mathfrak{p}}} x & d_p x \end{smallmatrix}\right)$ where the subscripts $p, \mathfrak{p}, \bar{\mathfrak{p}}$ denote images under $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, $F \hookrightarrow F_\mathfrak{p}$ and $F \hookrightarrow F_{\bar{\mathfrak{p}}}$ respectively. Note that $\bar{b}_\mathfrak{p}, b_{\bar{\mathfrak{p}}}$ have the same image in $\mathbb{Q}_p$. It is not hard to see the map $\mathcal{H}(F) \otimes \mathbb{Q}_p \to M_2(\mathbb{Q}_p)$ is surjective: for any element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot \lambda$ with $\lambda \in \mathbb{Q}_p$ we can find its preimage $\begin{pmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{pmatrix} \otimes \alpha^{-1}\lambda$ in $\mathcal{H}(F) \otimes \mathbb{Q}_p$; for $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \lambda$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot \lambda$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot \lambda$, we can find their preimages $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \lambda$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes \lambda$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes \lambda$ respectively in $\mathcal{H}(F) \otimes \mathbb{Q}_p$. Then $\mathcal{H}(F) \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$ as they are both 4-dimensional over $\mathbb{Q}_p$. In fact $M_2(\mathbb{Q}_p)$ equipped with the determinant is one isometric class of four dimensional quadratic spaces of discriminant 1 [20, p.273]. Again, equipping $\mathcal{H}(F) \otimes \mathbb{Q}_p$ with minus determinant and choosing an orthogonal basis $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $e_4 = \begin{pmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{pmatrix}$, we have an associated diagonal form $Q = -x_1^2 + x_2^2 + x_3^2 - dx_4^2$. It follows that $\mathfrak{d} = d$ (square in $\mathbb{Z}_p$) and $c = (-1, -d)_p = 1$. Choosing an orthogonal basis $e_1^{'} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_2^{'} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e_3^{'} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $e_4^{'} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in $M_2(\mathbb{Q}_p)$ we have $\mathfrak{d}^{'} = 1$ and $c^{'} = 1$. Thus we can deduce that $(\mathcal{H}(F) \otimes \mathbb{Q}_p, -\det) \simeq (M_2(\mathbb{Q}_p), \det)$ if p splits in $F/\mathbb{Q}$.

Suppose that p is ramified in $F/\mathbb{Q}_p$ and then we have $F \otimes \mathbb{Q}_p = \mathbb{Q}_p(\sqrt{d})$. As in the inert case, $\mathcal{H}(F) \otimes \mathbb{Q}_p = \mathcal{H}(F \otimes \mathbb{Q}_p)$. Corresponding to $(\mathcal{H}(F) \otimes \mathbb{Q}_p, -\det)$ the Hasse invariant $c = (-1, -d)_p = (-1)^{\frac{p-1}{2}}$. For $(V_1(\mathbb{Q}_p), \det)$ we calculate

$$c^{'} = (1, d)_p(-d, -d)_p = (1, -d)_p(1, -1)_p(1, -d)_p(-d, -d)_p = (-d, -d)_p = (-1)^{\frac{p-1}{2}}$$

where the last equality holds as d is square-free and divisible by q. Thus, if p is ramified we have $(\mathcal{H}(F \otimes \mathbb{Q}_p), -\det) \simeq (V_1(\mathbb{Q}_p), \det)$. □

Fix the additive character $\psi = \prod_q \psi_q$ of $\mathbb{A}_{\mathbf{Q}}$ given by $\psi_\infty(x) = \exp(2\pi i x)$ and, for every rational prime q, $\psi_q(x) = \exp(2\pi i \mathrm{Fr}_q(q^{-1}x))$ for $x \in \mathbb{Q}_q$, where $\mathrm{Fr}_q(x)$ denotes the fractional part of x. In this case we have $\ker(\psi_q) = q\mathbb{Z}_q$. Note that computing the congruence subgroup of Sp_4 under which φ_q is invariant is related to $\ker(\psi_q)$.

Let $\chi_V(x) = (x, \det V) : \mathbb{Q}^\times \backslash \mathbb{A}^\times \mapsto \{\pm 1\}$ be the quadratic Hecke character associated to V. For $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) \in V(\mathbb{Q})^2$, represent the associated Gram matrix by

$$(\mathbf{X}, \mathbf{X}) = ((\mathbf{x}_i, \mathbf{x}_j)) \in \mathrm{Sym}_2(\mathbb{Q}),$$

where ( ) is the symmetric bilinear form on $V(\mathbb{Q})^2$. Let $\omega = \omega_{\psi_q}$ denote the Weil representation on the Schwartz space $S(V^2)$ characterized (for every q) by the following actions of $\mathrm{Sp}_4 \times \mathrm{SO}(3, 1)$ locally on $\varphi_q \in S((V \otimes \mathbb{Q}_q)^2)$:

$$\omega(1, h)\varphi_q(\mathbf{X}) = \varphi_q(h^{-1}\mathbf{X}), \tag{9}$$

$$\omega\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1\right)\varphi_q(\mathbf{X}) = \psi_q\left(\frac{1}{2}\mathrm{tr}(b(\mathbf{X}, \mathbf{X}))\right)\varphi_q(\mathbf{X}), \tag{10}$$

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, 1\right)\varphi_q(\mathbf{X}) = \chi_{V,q}(\det(a))|\det(a)|_q^2\varphi_q(\mathbf{X}a), \tag{11}$$

$$\omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\varphi_q(\mathbf{X}) = \gamma\hat{\varphi}_q(\mathbf{X}). \tag{12}$$

Here the Fourier transform is defined by

$$\hat{\varphi}_q(\mathbf{X}) = \int_{(V \otimes \mathbb{Q}_q)^2} \varphi_q(\mathbf{Y})\psi_q(\mathrm{tr}(\mathbf{X}, \mathbf{Y}))d\mathbf{Y}$$

and γ is a certain complex number of absolute value 1.

4.1 At split prime dividing *m*

Let $q|m$ be a split prime such that $(q) = \mathfrak{q}\bar{\mathfrak{q}}$. According to Proposition 4.1, there is an isomorphism $(\mathcal{H}(F) \otimes \mathbb{Q}_q, -\det) \simeq (V_1(\mathbb{Q}_q), \det)$ for q split in $F/\mathbb{Q}$ given by

$$\begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where c is the image of $\bar{b} \in F$ under $F \hookrightarrow F_{\mathfrak{q}} \simeq \mathbb{Q}_q$.

Definition 4.2 (1) Suppose that $d \equiv 1 \bmod 4$. The local Schwartz function $\varphi_q^{\chi_{\mathfrak{m}}}$ at q is vanishing unless

$$a_i \in \mathbb{Z}_q, b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, c_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, d_i \in \mathbb{Z}_q, a_2d_1 - a_1d_2 \in q\mathbb{Z}_q,$$

in which case

$$\varphi_q^{\chi_{\mathfrak{m}}}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = \begin{cases} (\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})\left(\frac{a_2b_1-a_1b_2}{m} + \frac{c_2d_1-c_1d_2}{m}\right), & \text{if } \frac{a_2b_1-a_1b_2}{m} + \frac{c_2d_1-c_1d_2}{m} \in \mathcal{O}_{\mathfrak{q}}^\times \times \mathcal{O}_{\bar{\mathfrak{q}}}^\times, \\ 0, & \text{if } \frac{a_2b_1-a_1b_2}{m} + \frac{c_2d_1-c_1d_2}{m} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}} \text{ or } \bar{\mathfrak{q}}\mathcal{O}_{\bar{\mathfrak{q}}}. \end{cases}$$

where $(\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})(-) = \widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}(-)\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}}(-)$. Note that $b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, c_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ is equivalent to $b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}} \times \bar{\mathfrak{q}}\mathcal{O}_{\bar{\mathfrak{q}}}$.

(2) Suppose that $d \equiv 2, 3 \bmod 4$. Replace above $\frac{a_2b_1-a_1b_2}{m}$ by $\frac{2(a_2b_1-a_1b_2)}{m}$ and $\frac{c_2d_1-c_1d_2}{m}$ by $\frac{2(c_2d_1-c_1d_2)}{m}$ as discussed in Remark 3.6

In the following we will check the invariance properties of this local Schwartz function under some congruence subgroups of Sp_4 and $\mathrm{SO}(3, 1)$ in details in case of $d \equiv 1 \bmod 4$ and the other case can be treated similarly. We need to calculate the transformation properties (9), (10), (11) and (12). For simplicity we write $\varphi_q = \varphi_q^{\chi_{\mathfrak{m}}}$ and $\chi = \chi_{\mathfrak{m}}$ in the following computation.

Set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right)$. For $a_i \in \mathbb{Z}_q, b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, c_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, d_i \in \mathbb{Z}_q$, it is not difficult to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\varphi_q(\mathbf{X}) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for } u \in M_2(q\mathbb{Z}_q) \tag{13}$$

as $\psi_q\left(\frac{1}{2}\mathrm{tr}(u(\mathbf{X}, \mathbf{X}))\right)$ is trivial for such $(\mathbf{x}_1, \mathbf{x}_2)$ and u.

Set $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2) = \left(\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}\right)$. Inspired by Prasanna's computations in the proof of [19, Proposition 3.4], we will calculate the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y}) = \int_{\substack{a_i \in \mathbb{Z}_q, b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}} \\ c_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, d_i \in \mathbb{Z}_q}} \psi_q(\mathrm{tr}(\mathbf{X}, \mathbf{Y}))\varphi_q(\mathbf{X})d\mathbf{X} \quad \text{for } i = 1, 2,$$

where

$$\mathrm{tr}(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2}(a_1\delta_1 - b_1\gamma_1 - c_1\beta_1 + d_1\alpha_1 + a_2\delta_2 - b_2\gamma_2 - c_2\beta_2 + d_2\alpha_2).$$

Denote by $\lambda_{\sqrt{d}}$ the image of $\sqrt{d}$ in $\mathcal{O}_{\mathfrak{q}}$. By the above definition, φ_q is invariant under the transformations $a_i \mapsto a_i + q$, $b_i \mapsto b_i + q^2$, $b_i \mapsto b_i + q^2\lambda_{\sqrt{d}}$ (or $b_i \mapsto b_i + q^2\frac{1+\lambda_{\sqrt{d}}}{2}$)

and $d_i \mapsto d_i + q$. Sending $a_i \mapsto a_i + q$, we will have $\psi_q(-q\delta_i)$, factored out of the above integral, which has to be trivial for the non-vanishing of $\hat{\varphi}_q$. So for $\hat{\varphi}_q(\mathbf{Y})$ non-vanishing we need $\delta_i \in \mathbb{Z}_q$. Sending $b_1 \mapsto b_1 + q^2$, we get $\psi_q(q^2(\gamma_1 + \beta_1))$. For $b_1 \mapsto b_1 + q^2\lambda_{\sqrt{d}}$ and $b_1 \mapsto b_1 + q^2\frac{1+\lambda_{\sqrt{d}}}{2}$, we get $\psi_q(q^2(\gamma_1 - \beta_1)\lambda_{\sqrt{d}})$ and $\psi_q(\frac{1}{2}q^2(\gamma_1+\beta_1)+\frac{1}{2}q^2(\gamma_1-\beta_1)\lambda_{\sqrt{d}})$ respectively. For $\hat{\varphi}_q(\mathbf{Y})$ non-vanishing we need $\gamma_1 + \beta_1, \gamma_1 - \beta_1 \in q^{-1}\mathbb{Z}_q$ which implies $\beta_1, \gamma_1 \in q^{-1}\mathbb{Z}_q$. Repeating the same argument we can deduce that for the non-vanishing of $\hat{\varphi}_q$ the following conditions must be satisfied, for $i = 1, 2$,

$$\alpha_i \in \mathbb{Z}_q, \beta_i \in q^{-1}\mathbb{Z}_q, \gamma_i \in q^{-1}\mathbb{Z}_q, \delta_i \in \mathbb{Z}_q.$$

It follows that $\omega\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\varphi_q(\mathbf{Y})$ is vanishing unless $\alpha_i \in \mathbb{Z}_q$, $\beta_i \in q^{-1}\mathbb{Z}_q$, $\gamma_i \in q^{-1}\mathbb{Z}_q$ and $\delta_i \in \mathbb{Z}_q$.

Recall from (10) that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\hat{\varphi}_q(\mathbf{Y}) = \psi_q\left(\frac{1}{2}\,\mathrm{tr}(u(\mathbf{Y},\mathbf{Y}))\right)\hat{\varphi}_q(\mathbf{Y}).$$

Set $\alpha_i \in \mathbb{Z}_q, \beta_i \in q^{-1}\mathbb{Z}_q, \gamma_i \in q^{-1}\mathbb{Z}_q, \delta_i \in \mathbb{Z}_q$. Then when $u \in M_2(q^3\mathbb{Z}_q)$ we can see that $\psi_q\left(\frac{1}{2}\,\mathrm{tr}(u(\mathbf{Y},\mathbf{Y}))\right)$ is trivial. Thus we can deduce that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{Y}) = \omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{Y}) \quad \text{for } u \in M_2(q^3\mathbb{Z}_q),\ u = u^t$$

which implies

$$\omega\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right)\varphi_q(\mathbf{Y}) = \varphi_q(\mathbf{Y}) \quad \text{for } u \in M_2(q^3\mathbb{Z}_q),\ u = u^t. \tag{14}$$

For $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_q)$, we compute

$$\begin{aligned}\mathbf{X}a &= (\mathbf{x}_1, \mathbf{x}_2)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \left(\begin{pmatrix} \alpha a_1 + \gamma a_2 & \alpha b_1 + \gamma b_2 \\ \alpha c_1 + \gamma c_2 & \alpha d_1 + \gamma d_2 \end{pmatrix}, \begin{pmatrix} \beta a_1 + \delta a_2 & \beta b_1 + \delta b_2 \\ \beta c_1 + \delta c_2 & \beta d_1 + \delta d_2 \end{pmatrix}\right) =: \left(\begin{pmatrix} a_1' & b_1' \\ c_1' & d_1' \end{pmatrix}, \begin{pmatrix} a_1' & b_1' \\ c_1' & d_1' \end{pmatrix}\right)\end{aligned}$$

due to which we obtain that

$$\begin{aligned}\frac{a_2'b_1' - a_1'b_2'}{m} &= \frac{(\beta a_1 + \delta a_2)(\alpha b_1 + \gamma b_2) - (\alpha a_1 + \gamma a_2)(\beta b_1 + \delta b_2)}{m} \\ &= \frac{(\alpha\delta - \beta\gamma)a_2b_1 - (\alpha\delta - \beta\gamma)a_1b_2}{m} = \det(a)\cdot\frac{a_2b_1 - a_1b_2}{m}\end{aligned}$$

and similarly that $\frac{c_2'd_1' - c_1'd_2'}{m} = \det(a)\cdot\frac{c_2d_1 - c_1d_2}{m}$. Then from (11) we see that if $\det(a) \in \mathbb{Z}_q^\times$ then

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{pmatrix}\right)\varphi_q(\mathbf{X}) = \chi_{V,q}(\det(a))|\det(a)|_q^2(\widetilde{\chi}_{\mathfrak{q}}\widetilde{\chi}_{\bar{\mathfrak{q}}})(\det(a))\varphi_q(\mathbf{X}). \tag{15}$$

Combining (13), (14) and (15), we have proved the following lemma:

Lemma 4.3 *We have*

$$\omega(k_1)\varphi_q = \chi_{V,q}(\det(A))|\det(A)|_q^2(\widetilde{\chi}_{\mathfrak{q}}\widetilde{\chi}_{\bar{\mathfrak{q}}})(\det(A))\varphi_q$$

for

$$k_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : B \in M_2(q\mathbb{Z}_q), C \in M_2(q^3\mathbb{Z}_q) \right\}.$$

Proof The assertion follows from the Iwahori decomposition of Sp_4. □

We next discuss the action of $\mathrm{SO}(3,1)(\mathbb{Q}_q)$ on φ_q characterised by

$$\omega(1,h)\varphi_q(\mathbf{x}_1,\mathbf{x}_2) = \varphi_q(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2) \quad \text{for } h \in \mathrm{SO}(3,1)(\mathbb{Q}_q).$$

Recall from [26, Sect. 1.2], for odd split prime q we have the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{Q}_q \times \mathbb{Q}_q) \xrightarrow{\Lambda} \mathrm{SO}(3,1)(\mathbb{Q}_q) \to \mathbb{Q}_q^\times/\mathbb{Q}_q^{\times 2} \to 1$$

from which we can deduce the isomorphism $\mathrm{PSL}_2(\mathbb{Q}_q \times \mathbb{Q}_q) \simeq \mathrm{SO}^+(3,1)(\mathbb{Q}_q)$ where $\mathrm{SO}^+ := \mathrm{Im}(\Lambda)$. In this case of split q, we want to check congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z}_q \times \mathbb{Z}_q)$ under which the Schwartz function φ_q is invariant. Recall from [20, Sect. 2] that $h^{-1}\mathbf{x}_i := h_1^{-1}\mathbf{x}_i{}^t(h_2^{-1})^*$ for $h = (h_1,h_2) \in \mathrm{PSL}_2(\mathbb{Z}_q) \times \mathrm{PSL}_2(\mathbb{Z}_q) = \mathrm{PSL}_2(\mathbb{Z}_q \times \mathbb{Z}_q)$.

Lemma 4.4 *For $h = (h_1,h_2) \in \mathrm{PSL}_2(\mathbb{Z}_q) \times \mathrm{PSL}_2(\mathbb{Z}_q)$ satisfying*

$$h_i \in \overline{\Gamma}(q) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}_q) : \pm \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod q \right\},$$

we have that

$$\omega(1,h)\varphi_q(\mathbf{x}_1,\mathbf{x}_2) = \varphi_q(\mathbf{x}_1,\mathbf{x}_2). \tag{16}$$

Proof Set

$$h_j^{-1} = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} \text{ and } \mathbf{x}_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad \text{for } i,j \in \{1,2\}$$

with $\alpha_j, \delta_j \equiv 1 \bmod q$ and $\beta_j, \gamma_j \equiv 0 \bmod q$.

First we assume that $a_i, d_i \in \mathbb{Z}_q$ and $b_i, c_i \in q\mathbb{Z}_q$ so that φ_q is non-vanishing on $(\mathbf{x}_1,\mathbf{x}_2)$. We compute

$$\begin{pmatrix} a_i' & b_i' \\ c_i' & d_i' \end{pmatrix} := h_1^{-1}\mathbf{x}_i{}^t(h_2^{-1})^*$$

$$= \begin{pmatrix} \delta_2(\alpha_1 a_i + \beta_1 c_i) - \beta_2(\alpha_1 b_i + \beta_1 d_i) & -\gamma_2(\alpha_1 a_i + \beta_1 c_i) + \alpha_2(\alpha_1 b_i + \beta_1 d_i) \\ \delta_2(\gamma_1 a_i + \delta_1 c_i) - \beta_2(\gamma_1 b_i + \delta_1 d_i) & -\gamma_2(\gamma_1 a_i + \delta_1 c_i) + \alpha_2(\gamma_1 b_i + \delta_1 d_i) \end{pmatrix}.$$

It is not hard to observe that$b_i', c_i' \in q\mathbb{Z}_q$ as $b_i, c_i, \beta_j, \gamma_j \in q\mathbb{Z}_q$, and

$$a_2' d_1' - a_1' d_2' \equiv \alpha_1\alpha_2\delta_1\delta_2(a_2 d_1 - a_1 d_2) \equiv a_2 d_1 - a_1 d_2 \mod q.$$

Modulo q^2, we have

$$\begin{aligned} a_2' b_1' - a_1' b_2' &\\ &\equiv \delta_2\alpha_1 a_2(-\gamma_2\alpha_1 a_1 + \alpha_2\alpha_1 b_1 + \alpha_2\beta_1 d_1) - \delta_2\alpha_1 a_1(-\gamma_2\alpha_1 a_2 + \alpha_2\alpha_1 b_2 + \alpha_2\beta_1 d_2) \\ &\equiv \alpha_1^2\alpha_2\delta_2(a_2 b_1 - a_1 b_2) + \alpha_1\alpha_2\beta_1\delta_2(a_2 d_1 - a_1 d_2). \end{aligned}$$

It follows that

$$\frac{a_2'b_1' - a_1'b_2'}{m} \equiv \frac{a_2b_1 - a_1b_2}{m} \mod q.$$

Similarly, we obtain that modulo q^2

$$\begin{aligned} c_2'd_1' - c_1'd_2' \equiv& \alpha_2\delta_1 d_1(\delta_2\gamma_1 a_2 + \delta_2\delta_1 c_2 - \beta_2\delta_1 d_2) - \alpha_2\delta_1 d_2(\delta_2\gamma_1 a_1 + \delta_2\delta_1 c_1 - \beta_2\delta_1 d_1) \\ \equiv& \alpha_2\delta_1^2\delta_2(c_2d_1 - c_1d_2) + \alpha_2\gamma_1\delta_1\delta_2(a_2d_1 - a_1d_2). \end{aligned}$$

and

$$\frac{c_2'd_1' - c_1'd_2'}{m} \equiv \frac{c_2d_1 - c_1d_2}{m} \mod q.$$

Therefore, when φ_q is non-vanishing, we can deduce that

$$\omega(1,h)\varphi_q((\mathbf{x}_1,\mathbf{x}_2)) = \varphi_q((\mathbf{x}_1,\mathbf{x}_2)) \quad \text{for } h = (h_1,h_2) \in \overline{\Gamma}(q) \times \overline{\Gamma}(q).$$

When φ_q is vanishing, we consider that $b_1 \in \mathbb{Z}_q^\times$ and other cases that b_2, d_1 or d_2 in $\mathbb{Z}_q^\times$ can be treated similarly. For $h^{-1} = \left(\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}\right) \in \overline{\Gamma}(q)$, it is observed that

$$b_1' = -\gamma_2(\alpha_1 a_1 + \beta_1 c_1) + \alpha_2(\alpha_1 b_1 + \beta_1 d_1) \in \mathbb{Z}_q^\times$$

which makes φ_q vanish on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$. Now we have proven this lemma. □

4.2 At inert prime dividing *m*

Let q be an inert prime dividing $\mathfrak{m}$ such that $(q) = \mathfrak{q}$. According to Proposition 4.1, there is an isomorphism $(\mathcal{H}(F) \otimes \mathbb{Q}_q, -\det) \simeq (V_1(\mathbb{Q}_q\sqrt{d}), \det)$ for q inert in $F/\mathbb{Q}$ given by

$$\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \longmapsto \begin{pmatrix} b & a\sqrt{d} \\ c\sqrt{d} & \bar{b} \end{pmatrix} \quad \text{for } a, c \in \mathbb{Q}_q, b \in \mathbb{Q}_q(\sqrt{d})$$

where $^-$ on the right hand side denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_q(\sqrt{d})/\mathbb{Q}_q)$.

Definition 4.5 (1) Suppose that $d \equiv 1 \bmod 4$. The local Schwartz function $\varphi_q^{\chi_\mathfrak{m}}$ at q is vanishing unless, for $i = 1, 2$,

$$a_i \in \mathbb{Z}_q, b_i \in \mathfrak{q}\mathcal{O}_\mathfrak{q}, c_i \in \mathbb{Z}_q \text{ and } a_2c_1 - a_1c_2 \in q\mathbb{Z}_q,$$

in which case

$$\begin{aligned} &\varphi_q^{\chi_\mathfrak{m}}\left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix}\right) \\ &= \begin{cases} \widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\left(\frac{a_2b_1 - a_1b_2}{m} + \frac{\bar{b}_2c_1 - \bar{b}_1c_2}{m}\right), & \text{if } \frac{a_2b_1 - a_1b_2}{m} + \frac{\bar{b}_2c_1 - \bar{b}_1c_2}{m} \in \mathcal{O}_\mathfrak{q}^\times, \\ 0, & \text{if } \frac{a_2b_1 - a_1b_2}{m} + \frac{\bar{b}_2c_1 - \bar{b}_1c_2}{m} \in \mathfrak{q}\mathcal{O}_\mathfrak{q}. \end{cases} \end{aligned}$$

(2) Suppose that $d \equiv 2, 3 \bmod 4$. We replace above $\frac{a_2b_1 - a_1b_2}{m}$ by $\frac{2(a_2b_1 - a_1b_2)}{m}$ and $\frac{\bar{b}_2c_1 - \bar{b}_1c_2}{m}$ by $\frac{2(\bar{b}_2c_1 - \bar{b}_1c_2)}{m}$ as discussed in Remark 3.6.

In the following we will check the invariance properties of this local Schwartz function with respect to $\mathrm{Sp}_4 \times \mathrm{SO}(3,1)$ in detail in case of $d \equiv 1 \bmod 4$ and the other case can be treated similarly. For simplicity we write $\varphi_q = \varphi_q^{\chi_\mathfrak{m}}$ and $\chi = \chi_\mathfrak{m}$.

Set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix} \right)$. For $a_i \in \mathbb{Z}_q$, $b_i \in \mathfrak{q}\mathcal{O}_\mathfrak{q}$ and $c_i \in \mathbb{Z}_q$, it is easy to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for } u \in M_2(q\mathbb{Z}_q). \tag{17}$$

Set $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2) = \left(\begin{pmatrix} \beta_1 & \alpha_1\sqrt{d} \\ \gamma_1\sqrt{d} & \bar{\beta}_1 \end{pmatrix}, \begin{pmatrix} \beta_2 & \alpha_2\sqrt{d} \\ \gamma_2\sqrt{d} & \bar{\beta}_2 \end{pmatrix} \right)$. Consider the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y}) = \int \psi_q(\mathrm{tr}(\mathbf{X}, \mathbf{Y}))\varphi_q(\mathbf{X})d\mathbf{X}$$

where

$$\mathrm{tr}(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2}(b_1\bar{\beta}_1 + \bar{b}_1\beta_1 - a_1\gamma_1 d - \alpha_1 c_1 d + b_2\bar{\beta}_2 + \bar{b}_2\beta_2 - a_2\gamma_2 d - \alpha_2 c_2 d).$$

By the definition, φ_q is invariant under the transformations $a_i \mapsto a_i + q$, $b_i \mapsto b_i + q^2$, $b_i \mapsto b_i + q^2\sqrt{d}$ (or $b_i \mapsto b_i + q^2 \cdot \frac{1+\sqrt{d}}{2}$) and $c_i \mapsto c_i + q$. Repeating arguments in the previous subsection, we can observe that the Fourier transform $\hat{\varphi}(\mathbf{y}_1, \mathbf{y}_2)$ is vanishing unless, for $i = 1, 2$,

$$\alpha_i \in \mathbb{Z}_q;\ \beta, \bar{\beta} \in \mathfrak{q}^{-1}\mathcal{O}_\mathfrak{q}(\text{as } \beta_i + \bar{\beta}_i \in q^{-1}\mathbb{Z}_q, \beta_i - \bar{\beta}_i \in q^{-1}\sqrt{d}\mathbb{Z}_q);\ \gamma_i \in \mathbb{Z}_q.$$

It follows that, for $u \in M_2(q^3\mathbb{Z}_q)$ such that $u = u^t$,

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2) = \omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2)$$

which implies

$$\omega\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2) = \varphi_q(\mathbf{y}_1, \mathbf{y}_2) \quad \text{for } u \in M_2(q^3\mathbb{Z}_q),\ u = u^t. \tag{18}$$

For $a \in \mathrm{GL}_2(\mathbb{Z}_q)$, write $\mathbf{X}a := \left(\begin{pmatrix} b_1' & a_1'\sqrt{d} \\ c_1'\sqrt{d} & \bar{b}_1' \end{pmatrix}, \begin{pmatrix} b_2' & a_2'\sqrt{d} \\ c_2'\sqrt{d} & \bar{b}_2' \end{pmatrix} \right)$. We have $a_2'b_1' - a_1'b_2' = \det(a)(a_2b_1 - a_1b_2)$, $\bar{b}_2'c_1' - \bar{b}_1'c_2' = \det(a)(\bar{b}_2c_1 - \bar{b}_1c_2)$ and $a_2'c_1' - a_1'c_2' = \det(a)(a_2c_1 - a_1c_2)$. So for $\det(a) \in \mathbb{Z}_q^\times$, we obtain

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, 1\right)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \chi_{V,q}(\det(a))|\det(a)|_q^2\widetilde{\chi}_\mathfrak{q}(\det(a))\varphi_q(\mathbf{x}_1, \mathbf{x}_2). \tag{19}$$

Combining (17),(18) and (19), we can deduce the following lemma:

Lemma 4.6 *We have*

$$\omega(k_2)\varphi_q = \chi_{V,q}(\det(A))|\det(A)|_q^2\widetilde{\chi}_\mathfrak{q}(\det(A)\varphi_q$$

for

$$k_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : B \in M_2(q\mathbb{Z}_q), C \in M_2(q^3\mathbb{Z}_q) \right\}.$$

We next discuss the action of $\mathrm{SO}(3,1)(\mathbb{Q}_q)$ on φ_q characterised by

$$\omega(1,h)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(h^{-1}\mathbf{x}_1,h^{-1}\mathbf{x}_2)\quad \text{for } h\in \mathrm{SO}(3,1)(\mathbb{Q}_q).$$

Due to the exceptional isomorphism $\mathrm{PSL}_2(\mathbb{Q}_q(\sqrt{d}))\simeq \mathrm{SO}^+(3,1)(\mathbb{Q}_q)$ as in [26, Sect. 1.2], in this case we check the invariance property under some congruence subgroups of $\mathrm{PSL}_2(\mathcal{O}_\mathfrak{q})$. Here we have that $h^{-1}\mathbf{x}_i:=h^{-1}\mathbf{x}_i(\bar{h}^{-1})^*$ for $i=1,2$ where $^-$ denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_q(\sqrt{d})/\mathbb{Q}_q)$ (see [20, Sect. 2]).

Lemma 4.7 *For*

$$h\in\overline{\Gamma}(q)=\left\{\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in \mathrm{PSL}_2(\mathcal{O}_\mathfrak{q}):\pm\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\equiv\pm\begin{pmatrix}1&0\\ 0&1\end{pmatrix}\bmod q\right\},$$

we have that

$$\omega(1,h)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2). \tag{20}$$

Proof Set

$$h^{-1}=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\ \text{and}\ \mathbf{x}_i=\begin{pmatrix}b_i&a_i\sqrt{d}\\ c_i\sqrt{d}&\bar{b}_i\end{pmatrix}.$$

with $\alpha,\delta\equiv 1 \bmod q$ and $\beta,\gamma\equiv 0\bmod q$.

First we assume $a_i\in\mathbb{Z}_q$, $\bar{b}_i\in\mathfrak{q}\mathcal{O}_\mathfrak{q}$,$c_i\in\mathbb{Z}_q$ so that φ_q is non-vanishing on $(\mathbf{x}_1,\mathbf{x}_2)$. Writing $h^{-1}\mathbf{x}_i=\begin{pmatrix}b_i'&a_i'\sqrt{d}\\ c_i'\sqrt{d}&\bar{b}_i'\end{pmatrix}$, we compute

$$\begin{aligned}\begin{pmatrix}b_i'&a_i'\sqrt{d}\\ c_i'\sqrt{d}&\bar{b}_i'\end{pmatrix}&=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\begin{pmatrix}b_i&a_i\sqrt{d}\\ c_i\sqrt{d}&\bar{b}_i\end{pmatrix}\begin{pmatrix}\bar{\delta}&-\bar{\beta}\\ -\bar{\gamma}&\bar{\alpha}\end{pmatrix}\\
&=\begin{pmatrix}\alpha b_i+\beta c_i\sqrt{d}&\alpha a_i\sqrt{d}+\beta\bar{b}_i\\ \gamma b_i+\delta c_i\sqrt{d}&\gamma a_i\sqrt{d}+\delta\bar{b}_i\end{pmatrix}\begin{pmatrix}\bar{\delta}&-\bar{\beta}\\ -\bar{\gamma}&\bar{\alpha}\end{pmatrix}\\
&=\begin{pmatrix}\bar{\delta}(\alpha b_i+\beta c_i\sqrt{d})-\bar{\gamma}(\alpha a_i\sqrt{d}+\beta\bar{b}_i)&-\bar{\beta}(\alpha b_i+\beta c_i\sqrt{d})+\bar{\alpha}(\alpha a_i\sqrt{d}+\beta\bar{b}_i)\\ \bar{\delta}(\gamma b_i+\delta c_i\sqrt{d})-\bar{\gamma}(\gamma a_i\sqrt{d}+\delta\bar{b}_i)&-\bar{\beta}(\gamma b_i+\delta c_i\sqrt{d})+\bar{\alpha}(\gamma a_i\sqrt{d}+\delta\bar{b}_i)\end{pmatrix}.\end{aligned}$$

It is not hard to observe that $a_i'\in\mathbb{Z}_q$,$b_i'\in\mathfrak{q}\mathcal{O}_\mathfrak{q}$, $c_i'\in\mathbb{Z}_q$ and that $a_2'c_1'-a_1'c_2'\equiv a_2c_1-a_1c_2$ mod q. Then we expand

$$\begin{aligned}&a_2'b_1'-a_1'b_2'\\
&=\left(\frac{\bar{\alpha}\beta\bar{b}_2-\bar{\beta}\alpha b_2}{\sqrt{d}}-\bar{\beta}\beta c_2+\bar{\alpha}\alpha a_2\right)(\bar{\delta}(\alpha b_1+\beta c_1\sqrt{d})-\bar{\gamma}(\alpha a_1\sqrt{d}+\beta\bar{b}_1))\\
&\quad-\left(\frac{\bar{\alpha}\beta\bar{b}_1-\bar{\beta}\alpha b_1}{\sqrt{d}}-\bar{\beta}\beta c_1+\bar{\alpha}\alpha a_1\right)(\bar{\delta}(\alpha b_2+\beta c_2\sqrt{d})-\bar{\gamma}(\alpha a_2\sqrt{d}+\beta\bar{b}_2)),\end{aligned}$$

and, modulo q^2, we get

$$a_2'b_1'-a_1'b_2'\equiv\alpha^2\bar{\alpha}\bar{\delta}(a_2b_1-a_1b_2)+\alpha\bar{\alpha}\beta\bar{\delta}\sqrt{d}(a_2c_1-a_1c_2).$$

Similarly, we have, modulo q^2,

$$\begin{aligned}&\bar{b}_2'c_1'-\bar{b}_1'c_2'\\
&\equiv\delta\bar{\delta}c_1(-\bar{\beta}\delta c_2\sqrt{d}+\bar{\alpha}(\gamma a_2\sqrt{d}+\delta\bar{b}_2))-\delta\bar{\delta}c_2(-\bar{\beta}\delta c_1\sqrt{d}+\bar{\alpha}(\gamma a_1\sqrt{d}+\delta\bar{b}_1))\end{aligned}$$

$$\equiv \delta^2 \bar{\alpha}\bar{\delta}(\bar{b}_2 c_1 - \bar{b}_1 c_2) + \bar{\alpha}\gamma\delta\bar{\delta}\sqrt{d}(a_2 c_1 - a_1 c_2).$$

Then we can deduce that

$$\frac{a_2^{'} b_1^{'} - a_1^{'} b_2^{'}}{m} + \frac{\bar{b}_2^{'} c_1^{'} - \bar{b}_1^{'} c_2^{'}}{m} \equiv \frac{a_2 b_1 - a_1 b_2}{m} + \frac{\bar{b}_2 c_1 - \bar{b}_1 c_2}{m} \mod q$$

which implies that

$$\varphi_q \left(\frac{a_2^{'} b_1^{'} - a_1^{'} b_2^{'}}{m} + \frac{\bar{b}_2^{'} c_1^{'} - \bar{b}_1^{'} c_2^{'}}{m} \right) = \varphi_q \left(\frac{a_2 b_1 - a_1 b_2}{m} + \frac{\bar{b}_2 c_1 - \bar{b}_1 c_2}{m} \right).$$

Next we assume $b_1 \in \mathcal{O}_{\mathfrak{q}}^{\times}$ so that φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$. It follows that $\bar{\delta}\alpha b_1 \in \mathcal{O}_{\mathfrak{q}}^{\times}$ and then $b_1^{'} \in \mathcal{O}_{\mathfrak{q}}^{\times}$ which makes φ_q is vanishing on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$. Other cases that $\bar{b}_1^{'}, b_2^{'}, \bar{b}_2^{'} \in \mathcal{O}_{\mathfrak{q}}^{\times}$ can be treated in the same way and recall that $a_2^{'} c_1^{'} - a_1^{'} c_2^{'} \equiv a_2 c_1 - a_1 c_2 \mod q$. Hence, if φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$, so is that on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$.

Now we have proven this lemma. □

4.3 At ramified prime away from 2

Let q be a ramified prime away from 2 such that $(q) = \mathfrak{q}^2$. According to Proposition 4.1, there is an isomorphism $(\mathcal{H}(F) \otimes \mathbb{Q}_q, -\det) \simeq (V_1(\mathbb{Q}_q), \det)$ for q ramified in $F/\mathbb{Q}$ given by

$$\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \longmapsto \begin{pmatrix} b & a\sqrt{d} \\ c\sqrt{d} & \bar{b} \end{pmatrix} \quad \text{for } a, c \in \mathbb{Q}_q, b \in \mathbb{Q}_q(\sqrt{d})$$

where $^-$ on the right hand side denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_q(\sqrt{d})/\mathbb{Q}_q)$. Note that when $d \equiv 2, 3 \pmod 4$ the prime 2 is ramified and at the ramified 2 the local Schwartz function is defined in the next subsection.

Definition 4.8 (1) Suppose that $d \equiv 1 \bmod 4$. The local Schwartz function $\varphi_q^{\chi_{\mathfrak{m}}}$ at q is vanishing unless, for $i = 1, 2$,

$$a_i \in \mathbb{Z}_q, c_i \in \mathbb{Z}_q, b_i \in \mathcal{O}_{\mathfrak{q}}, b_1\bar{b}_2 - \bar{b}_1 b_2 \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}},$$

in which case

$$\varphi_q^{\chi_{\mathfrak{m}}} \left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix} \right)$$
$$\begin{cases} \widetilde{\chi}_{\mathfrak{m},\mathfrak{q}} \left(\frac{a_2 b_1 - a_1 b_2}{m} + \frac{\bar{b}_2 c_1 - \bar{b}_1 c_2}{m} \right), & \text{if } \frac{a_2 b_1 - a_1 b_2}{m} + \frac{\bar{b}_2 c_1 - \bar{b}_1 c_2}{m} \in \mathcal{O}_{\mathfrak{q}}^{\times}, \\ 0, & \text{if } \frac{a_2 b_1 - a_1 b_2}{m} + \frac{\bar{b}_2 c_1 - \bar{b}_1 c_2}{m} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}. \end{cases}$$

(2) Suppose that $d \equiv 2, 3 \bmod 4$. We replace above $\frac{a_2 b_1 - a_1 b_2}{m}$ by $\frac{2(a_2 b_1 - a_1 b_2)}{m}$ and $\frac{\bar{b}_2 c_1 - \bar{b}_1 c_2}{m}$ by $\frac{2(\bar{b}_2 c_1 - \bar{b}_1 c_2)}{m}$ as discussed in Remark 3.6.

In the following we will check the invariance properties of this local Schwartz function with respect to $\mathrm{Sp}_4 \times \mathrm{SO}(3, 1)$ in case of $d \equiv 1 \bmod 4$ and the other case can be treated similarly. For simplicity we write $\varphi_q = \varphi_q^{\chi_{\mathfrak{m}}}$ and $\chi = \chi_{\mathfrak{m}}$.

Set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix} \right)$. For $a_i, c_i \in \mathbb{Z}_q$ and $b_i, \bar{b}_i \in \mathcal{O}_{\mathfrak{q}}$, it is easy to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for } u \in M_2(q\mathbb{Z}_q). \tag{21}$$

Set $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2) = \left(\begin{pmatrix} \beta_1 & \alpha_1\sqrt{d} \\ \gamma_1\sqrt{d} & \bar{\beta}_1 \end{pmatrix}, \begin{pmatrix} \beta_2 & \alpha_2\sqrt{d} \\ \gamma_2\sqrt{d} & \bar{\beta}_2 \end{pmatrix} \right)$. Consider the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y}) = \int \psi_q(\mathrm{tr}(\mathbf{X}, \mathbf{Y}))\varphi_q(\mathbf{X})d\mathbf{X}$$

where

$$\mathrm{tr}(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2}(b_1\bar{\beta}_1 + \bar{b}_1\beta_1 - a_1\gamma_1 d - \alpha_1 c_1 d + b_2\bar{\beta}_2 + \bar{b}_2\beta_2 - a_2\gamma_2 d - \alpha_2 c_2 d).$$

By the definition, φ_q is invariant under the transformations $a_i \mapsto a_i + q$, $b_i \mapsto b_i + q$, $b_i \mapsto b_i + \sqrt{d}$ and $c_i \mapsto c_i + q$. Repeating arguments in the previous subsection, we can observe that the Fourier transform $\hat{\varphi}(\mathbf{y}_1, \mathbf{y}_2)$ is vanishing unless, for $i = 1, 2$,

$$\alpha_i, \gamma_i \in q^{-1}\mathbb{Z}_q \text{ and } \beta_i, \bar{\beta}_i \in \mathcal{O}_{\mathfrak{q}} (\text{as } \beta_i + \bar{\beta}_i \in \mathbb{Z}_q, \beta_i - \bar{\beta}_i \in \sqrt{d}\mathbb{Z}_q).$$

It follows that, for $u \in M_2(q^2\mathbb{Z}_q)$ such that $u = u^t$,

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2) = \omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2)$$

which implies

$$\omega\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2) = \varphi_q(\mathbf{y}_1, \mathbf{y}_2) \quad \text{for } u \in M_2(q^2\mathbb{Z}_q),\ u = u^t. \tag{22}$$

For $a \in \mathrm{GL}_2(\mathbb{Z}_q)$ with $\det(a) \in \mathbb{Z}_q^\times$, we also have

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, 1\right)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \chi_{V,q}(\det(a))|\det(a)|_q^2\widetilde{\chi}_{\mathfrak{q}}(\det(a))\varphi_q(\mathbf{x}_1, \mathbf{x}_2). \tag{23}$$

Again, combining (21),(22) and (23), we can deduce the following lemma:

Lemma 4.9 *We have*

$$\omega(k_3)\varphi_q = \chi_{V,q}(\det(A))|\det(A)|_q^2\widetilde{\chi}_{\mathfrak{q}}(\det(A))\varphi_q$$

for

$$k_3 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : B \in M_2(q\mathbb{Z}_q), C \in M_2(q^2\mathbb{Z}_q) \right\}.$$

We next discuss the action of $\mathrm{SO}(3, 1)(\mathbb{Q}_q)$ on φ_q characterised by

$$\omega(1, h)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \varphi_q(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2) \quad \text{for } h \in \mathrm{SO}(3, 1)(\mathbb{Q}_q).$$

In this case, we check the invariance property under congruence subgroups of $\mathrm{PSL}_2(\mathcal{O}_{\mathfrak{q}})$ and have that $h^{-1}\mathbf{x}_i := h^{-1}\mathbf{x}_i(\bar{h}^{-1})^*$ for $i = 1, 2$ where $^-$ denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_q(\sqrt{d})/\mathbb{Q}_q)$ (see [20, Sect. 2]).

Lemma 4.10 *For*

$$h \in \overline{\Gamma}(\mathfrak{q}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_\mathfrak{q}) : \pm \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{q} \right\},$$

we have that

$$\omega(1, h)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2). \tag{24}$$

Proof Set

$$h^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } \mathbf{x}_i = \begin{pmatrix} b_i & a_i\sqrt{d} \\ c_i\sqrt{d} & \bar{b}_i \end{pmatrix}.$$

with $\alpha, \delta \equiv 1 \bmod \mathfrak{q}$ and $\beta, \gamma \equiv 0 \bmod \mathfrak{q}$.

First we assume $a_i, c_i \in \mathbb{Z}_q$ and $b_i, \bar{b}_i \in \mathcal{O}_\mathfrak{q}$ so that φ_q is non-vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$. Writing $h^{-1}\mathbf{x}_i = \begin{pmatrix} b_i' & a_i'\sqrt{d} \\ c_i'\sqrt{d} & \bar{b}_i' \end{pmatrix}$, we compute

$$\begin{aligned}
\begin{pmatrix} b_i' & a_i'\sqrt{d} \\ c_i'\sqrt{d} & \bar{b}_i' \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} b_i & a_i\sqrt{d} \\ c_i\sqrt{d} & \bar{b}_i \end{pmatrix} \begin{pmatrix} \bar{\delta} & -\bar{\beta} \\ -\bar{\gamma} & \bar{\alpha} \end{pmatrix} \\
&= \begin{pmatrix} \alpha b_i + \beta c_i\sqrt{d} & \alpha a_i\sqrt{d} + \beta\bar{b}_i \\ \gamma b_i + \delta c_i\sqrt{d} & \gamma a_i\sqrt{d} + \delta\bar{b}_i \end{pmatrix} \begin{pmatrix} \bar{\delta} & -\bar{\beta} \\ -\bar{\gamma} & \bar{\alpha} \end{pmatrix} \\
&= \begin{pmatrix} \bar{\delta}(\alpha b_i + \beta c_i\sqrt{d}) - \bar{\gamma}(\alpha a_i\sqrt{d} + \beta\bar{b}_i) & -\bar{\beta}(\alpha b_i + \beta c_i\sqrt{d}) + \bar{\alpha}(\alpha a_i\sqrt{d} + \beta\bar{b}_i) \\ \bar{\delta}(\gamma b_i + \delta c_i\sqrt{d}) - \bar{\gamma}(\gamma a_i\sqrt{d} + \delta\bar{b}_i) & -\bar{\beta}(\gamma b_i + \delta c_i\sqrt{d}) + \bar{\alpha}(\gamma a_i\sqrt{d} + \delta\bar{b}_i) \end{pmatrix}.
\end{aligned}$$

It is not hard to observe that $a_i', c_i' \in \mathbb{Z}_q, b_i', \bar{b}_i' \in \mathcal{O}_\mathfrak{q}$ and $b_1'\bar{b}_2' - \bar{b}_1'b_2' \equiv b_1\bar{b}_2 - \bar{b}_1 b_2 \bmod \mathfrak{q}$. Modulo $\mathfrak{q}$, we have

$$\begin{aligned}
&a_2'b_1' - a_1'b_2' \\
\equiv&\bar{\delta}\alpha b_1 \left(\frac{\bar{\alpha}\beta\bar{b}_2 - \bar{\beta}\alpha b_2}{\sqrt{d}} + \bar{\alpha}\alpha a_2 \right) - \bar{\delta}\alpha b_2 \left(\frac{\bar{\alpha}\beta\bar{b}_1 - \bar{\beta}\alpha b_1}{\sqrt{d}} + \bar{\alpha}\alpha a_1 \right) \\
\equiv&\alpha^2\bar{\alpha}\bar{\delta}(a_2 b_1 - a_1 b_2) - \alpha\bar{\alpha}\beta\bar{\delta}/\sqrt{d}(b_1\bar{b}_2 - \bar{b}_1 b_2),
\end{aligned}$$

and

$$\begin{aligned}
&c_2'\bar{b}_1' - c_1'\bar{b}_2' \\
\equiv&\bar{\alpha}\delta\bar{b}_1 \left(\frac{\bar{\delta}\gamma b_2 - \delta\bar{\gamma}\bar{b}_2}{\sqrt{d}} + \bar{\delta}\delta c_2 \right) - \bar{\alpha}\delta\bar{b}_2 \left(\frac{\bar{\delta}\gamma b_1 - \delta\bar{\gamma}\bar{b}_1}{\sqrt{d}} + \bar{\delta}\delta c_1 \right) \\
\equiv&\delta^2\bar{\alpha}\bar{\delta}(c_2\bar{b}_1 - c_1\bar{b}_2) + \bar{\alpha}\gamma\delta\bar{\delta}/\sqrt{d}(b_1\bar{b}_2 - \bar{b}_1 b_2).
\end{aligned}$$

So, modulo $\mathfrak{q}$, we get

$$\frac{a_2'b_1' - a_1'b_2'}{m} \equiv \alpha^2\bar{\alpha}\bar{\delta}\frac{a_2 b_1 - a_1 b_2}{m} \quad \text{and} \quad \frac{c_2'\bar{b}_1' - c_1'\bar{b}_2'}{m} \equiv \delta^2\bar{\alpha}\bar{\delta}\frac{c_2\bar{b}_1 - c_1\bar{b}_2}{m}.$$

It follows that

$$\varphi_q\left(\frac{a_2'b_1' - a_1'b_2'}{m} + \frac{c_2'\bar{b}_1' - c_1'\bar{b}_2'}{m} \right) = \varphi_q\left(\frac{a_2 b_1 - a_1 b_2}{m} + \frac{c_2\bar{b}_1 - c_1\bar{b}_2}{m} \right).$$

Next we assume $b_1 \in \mathfrak{q}^{-1}\mathcal{O}_\mathfrak{q}^\times$ so that φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$. It follows that $\bar{\delta}\alpha b_1 \in \mathfrak{q}^{-1}\mathcal{O}_\mathfrak{q}^\times$ and then $b_1' \in \mathfrak{q}^{-1}\mathcal{O}_\mathfrak{q}^\times$ which makes φ_q be vanishing on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$. Other cases for $a_i, c_i, \bar{b}_1, b_2, \bar{b}_2$ can be treated in the same way. Hence, if φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$, so is that on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$.

Now we have proven this lemma. □

4.4 At places dividing $N(\mathfrak{n})$ and ramified 2

In this subsection we consider the local Schwartz function at finite places dividing $N(\mathfrak{n})$ and at ramified prime 2 (when $d \equiv 2, 3 \bmod 4$). For a place q and an integral lattice X on V, we put $X_q = X \otimes_{\mathbb{Z}} \mathbb{Z}_q$.

Definition 4.11 (1) Let $q|N(\mathfrak{n})$ be split with $(q) = \mathfrak{q}\bar{\mathfrak{q}}$.

- Suppose that $(\mathfrak{n}, (q)) = \mathfrak{q}$. Define the local Schwartz function φ_q to be the characteristic function of

$$\left\{ \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in X_q^2 : b_1c_2 + c_1b_2 \in \mathcal{O}_{\mathfrak{q}}^{\times}, d_i \in q\mathbb{Z}_q \right\}.$$

- Suppose that $(\mathfrak{n}, (q)) = \bar{\mathfrak{q}}$. Define φ_q to be the characteristic function of

$$\left\{ \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in X_q^2 : b_1c_2 + c_1b_2 \in \mathcal{O}_{\bar{\mathfrak{q}}}^{\times}, d_i \in q\mathbb{Z}_q \right\}.$$

- Suppose that $(\mathfrak{n}, (q)) = (q)$. Define φ_q to be the characteristic function of

$$\left\{ \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in X_q^2 : b_1c_2 + c_1b_2 \in \mathcal{O}_{\mathfrak{q}}^{\times} \times \mathcal{O}_{\bar{\mathfrak{q}}}^{\times}, d_i \in q\mathbb{Z}_q \right\}.$$

(2) At inert place $q|\mathfrak{n}$ with $(q) = \mathfrak{q}$, we define φ_q to be the characteristic function of

$$\left\{ \left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix} \right) \in X_q^2 : b_1\bar{b}_2 + \bar{b}_1b_2 \in \mathcal{O}_{\mathfrak{q}}^{\times}, c_i \in q\mathbb{Z}_q \right\}$$

(3) If 2 is ramified with $(2) = \mathfrak{q}_2^2$, we define φ_2 to be the characteristic function of

$$\left\{ \left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix} \right) : a_i, c_i \in \mathbb{Z}_2, b_i \in \frac{1}{2}\mathcal{O}_{\mathfrak{q}_2}, b_1\bar{b}_2 + \bar{b}_1b_2 \in \frac{1}{2}\mathcal{O}_{\mathfrak{q}_2}^{\times} \right\}.$$

Note that if we take the local Schwartz function at finite places dividing $N(\mathfrak{n})$ and ramified 2 as the characteristic function of integral lattice, the resulting theta lifting would be vanishing.

In the following we will check the invariance properties of this local Schwartz function with respect to $\mathrm{Sp}_4 \times \mathrm{SO}(3, 1)$.

Lemma 4.12 *(1) For φ_q as in above Definition 4.11 (1) and (2), We have*

$$\omega(k_4)\varphi_q = \varphi_q$$

for

$$k_4 \in \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : A \in \begin{pmatrix} \mathbb{Z}_q & q\mathbb{Z}_q \\ q\mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix}, B \in M_2(q\mathbb{Z}_q), C \in M_2(q\mathbb{Z}_q) \right\}.$$

(2) At ramified 2, we have

$$\omega(k_5)\varphi_2 = \varphi_2$$

for

$$k_5 \in \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : A \in \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ 2\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}, B \in M_2(2^4\mathbb{Z}_2) \right\}.$$

Proof (1) We prove this lemma in details only for split q with $(\mathfrak{n}, (q)) = \mathfrak{q}$ and other cases can be treated similarly. Set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \right) \in X_q^2$. It is not difficult to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \varphi_q(\mathbf{X}) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for } u \in M_2(q\mathbb{Z}_q).$$

Set $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2) = \left(\left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{smallmatrix}\right) \right)$. Consider the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y}) = \int_{X_{q,v}^2} \psi_q(\mathrm{tr}(\mathbf{X}, \mathbf{Y})) \varphi_q(\mathbf{X}) d\mathbf{X} \quad \text{for } i = 1, 2,$$

where

$$\mathrm{tr}(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2}(a_1\delta_1 - b_1\gamma_1 - c_1\beta_1 + d_1\alpha_1 + a_2\delta_2 - b_2\gamma_2 - c_2\beta_2 + d_2\alpha_2).$$

By the above definition, φ_q is invariant under the transformations $a_i \mapsto a_i + \mathbb{Z}_q$, $b_i \mapsto b_i + q$, $b_i \mapsto b_i + q\lambda_{\sqrt{d}}$ (or $b_i \mapsto b_i + q\frac{1+\lambda\sqrt{d}}{2}$) and $d_i \mapsto d_i + q$. Note that $b_1c_2 + c_1b_2 \in \mathcal{O}_{\mathfrak{q}}^\times$ is not preserved under $b_i \mapsto b_i + \mathfrak{q}$ or $b_i \mapsto b_i + \bar{\mathfrak{q}}$. Then we can deduce that $\omega\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) \varphi_q(\mathbf{Y})$ is vanishing unless $\alpha_i \in \mathbb{Z}_q$, $\beta_i \in \mathbb{Z}_q$, $\gamma_i \in \mathbb{Z}_q$ and $\delta_i \in q\mathbb{Z}_q$. It follows that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \varphi_q(\mathbf{Y}) = \omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \varphi_q(\mathbf{Y}) \quad \text{for } u \in M_2(q\mathbb{Z}_q),\ u = u^t$$

which implies

$$\omega\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right) \varphi_q(\mathbf{Y}) = \varphi_q(\mathbf{Y}) \quad \text{for } u \in M_2(q\mathbb{Z}_q),\ u = u^t.$$

For $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_q)$, set $\left(\begin{pmatrix} a_1' & b_1' \\ c_1' & d_1' \end{pmatrix}, \begin{pmatrix} a_2' & b_2' \\ c_2' & d_2' \end{pmatrix} \right) := (\mathbf{x}_1, \mathbf{x}_2)a$. It is clear that

$$d_1' = \alpha d_1 + \beta d_2 \text{ and } d_2' = \gamma d_1 + \delta d_2$$

lie in $q\mathbb{Z}_q$. Also we have

$$b_1'c_2' + c_1'b_2' = (\alpha b_1 + \gamma b_2)(\beta c_1 + \delta c_2) + (\beta b_1 + \delta b_2)(\alpha c_1 + \gamma c_2).$$

If $\beta, \gamma \equiv 0 \bmod q$ and $\det(a) \in \mathbb{Z}_q^\times$, then $b_1'c_2' + c_1'b_2' \in \mathcal{O}_{\mathfrak{q}}^\times$. We can deduce that

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}\right) \varphi_q(\mathbf{X}) = \varphi_q(\mathbf{X}) \quad \text{for } a \in \begin{pmatrix} \mathbb{Z}_q & q\mathbb{Z}_q \\ q\mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix} \text{ and } \det(a) \in \mathbb{Z}_q^\times.$$

(2) Set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\left(\begin{smallmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{smallmatrix}\right) \right)$. For $a_i, c_i \in \mathbb{Z}_2$ and $b_i \in \frac{1}{2}\mathcal{O}_{\mathfrak{q}_2}$, it is easy to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for } u \in M_2(2^4\mathbb{Z}_2).$$

Set $\mathbf{Y}=(\mathbf{y}_1,\mathbf{y}_2)=\left(\begin{pmatrix}\beta_1 & \alpha_1\sqrt{d}\\ \gamma_1\sqrt{d} & \bar{\beta}_1\end{pmatrix},\begin{pmatrix}\beta_2 & \alpha_2\sqrt{d}\\ \gamma_2\sqrt{d} & \bar{\beta}_2\end{pmatrix}\right)$. Consider the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y})=\int\psi_q(\mathrm{tr}(\mathbf{X},\mathbf{Y}))\varphi_q(\mathbf{X})d\mathbf{X}$$

where

$$\mathrm{tr}(\mathbf{X},\mathbf{Y})=-\frac{1}{2}(b_1\bar{\beta}_1+\bar{b}_1\beta_1-a_1\gamma_1 d-\alpha_1 c_1 d+b_2\bar{\beta}_2+\bar{b}_2\beta_2-a_2\gamma_2 d-\alpha_2 c_2 d).$$

By the definition, φ_q is invariant under the transformations $a_i\mapsto a_i+\mathbb{Z}_2, b_i\mapsto b_i+\mathbb{Z}_2$, $b_i\mapsto b_i+\mathbb{Z}_2\sqrt{d}$ and $c_i\mapsto c_i+\mathbb{Z}_2$. Repeating arguments in the previous subsection, we can observe that the Fourier transform $\hat{\varphi}(\mathbf{y}_1,\mathbf{y}_2)$ is vanishing unless, for $i=1,2$,

$$\alpha_i,\gamma_i\in 4\mathbb{Z}_2 \text{ and } \beta_i\in 4\mathcal{O}_{\mathfrak{q}_2}(\text{as } \beta_i+\bar{\beta}_i\in 4\mathbb{Z}_q,\ \beta_i-\bar{\beta}_i\in 4\sqrt{d}\mathbb{Z}_q).$$

It follows that, for $u\in M_2(\mathbb{Z}_2)$ such that $u=u^t$,

$$\omega\left(\begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2)=\omega\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2)$$

which implies

$$\omega\left(\begin{pmatrix}1&0\\u&1\end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2)=\varphi_q(\mathbf{y}_1,\mathbf{y}_2)\quad\text{for } u\in M_2(\mathbb{Z}_2),\ u=u^t.$$

For $a=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in \mathrm{GL}_2(\mathbb{Z}_q)$, set $\left(\begin{pmatrix}b_1' & a_1'\sqrt{d}\\ c_1'\sqrt{d} & \bar{b}_1'\end{pmatrix},\begin{pmatrix}b_2' & a_2'\sqrt{d}\\ c_2'\sqrt{d} & \bar{b}_2'\end{pmatrix}\right):=(\mathbf{x}_1,\mathbf{x}_2)a$. We have

$$b_1'=\alpha b_1+\gamma b_2 \text{ and } b_2'=\beta b_1+\delta b_2$$

and then

$$b_1'\bar{b}_2'+\bar{b}_1'b_2'=(\alpha b_1+\gamma b_2)(\beta\bar{b}_1+\delta\bar{b}_2)+(\alpha\bar{b}_1+\gamma\bar{b}_2)(\beta b_1+\delta b_2).$$

If $\beta,\gamma\in 2\mathbb{Z}_2$ and $\alpha\gamma\in\mathbb{Z}_2^{\times}$, we have

$$\omega\left(\begin{pmatrix}a&0\\0&{}^t a^{-1}\end{pmatrix},1\right)\varphi_2(\mathbf{x}_1,\mathbf{x}_2)=\varphi_2(\mathbf{x}_1,\mathbf{x}_2).$$

□

Lemma 4.13 *(1) Let $q|N(\mathfrak{n})$ split with $(q)=\mathfrak{q}\bar{\mathfrak{q}}$.*

- *Suppose that $(\mathfrak{n},(q))=\mathfrak{q}$. We have*

$$\omega(1,h_1)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)$$

for $h_1=h_{1,1}\times h_{1,2}$ with

$$h_{1,i}\in\overline{\Gamma}(\mathfrak{q})=\left\{\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in\mathrm{PSL}_2(\mathcal{O}_{\mathfrak{q}}):\pm\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\equiv\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\bmod \mathfrak{q}\right\}.$$

- *If $(\mathfrak{n},(q))=\bar{\mathfrak{q}}$, we have*

$$\omega(1,h_2)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)\quad\textit{for } h_2\in\overline{\Gamma}(\bar{\mathfrak{q}}).$$

- *If $(\mathfrak{n},(q))=(q)$, we have*

$$\omega(1,h_3)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)\quad\textit{for } h_3\in\overline{\Gamma}(q).$$

(2) For inert $q|N(\mathfrak{n})$ *with* $(q)=\mathfrak{q}$, *we have*

$$\omega(1,h_4)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)\quad \text{for } h_4\in\overline{\Gamma}(\mathfrak{q}).$$

(3) For 2 ramified with $(2)=\mathfrak{q}_2^2$, *we have*

$$\omega(1,h_5)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)\quad \text{for } h_5\in\overline{\Gamma}(\mathfrak{q}_2).$$

Proof In part (1), we prove the first statement and other cases can be treated similarly. Let $(q)=\mathfrak{q}\bar{\mathfrak{q}}$. Set $h_j^{-1}=\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{pmatrix}$ and $\mathbf{x}_i=\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}$ with $\alpha_j,\delta_j\equiv 1 \bmod \mathfrak{q}$ and $\beta_j,\gamma_j\equiv 0 \bmod \mathfrak{q}$. We compute

$$\begin{aligned}\begin{pmatrix}a_i' & b_i'\\ c_i' & d_i'\end{pmatrix} &:= h_1^{-1}\mathbf{x}_i{}^t(h_2^{-1})^*\\ &=\begin{pmatrix}\delta_2(\alpha_1a_i+\beta_1c_i)-\beta_2(\alpha_1b_i+\beta_1d_i) & -\gamma_2(\alpha_1a_i+\beta_1c_i)+\alpha_2(\alpha_1b_i+\beta_1d_i)\\ \delta_2(\gamma_1a_i+\delta_1c_i)-\beta_2(\gamma_1b_i+\delta_1d_i) & -\gamma_2(\gamma_1a_i+\delta_1c_i)+\alpha_2(\gamma_1b_i+\delta_1d_i)\end{pmatrix}\\ &\equiv\begin{pmatrix}* & b_i\\ c_i & d_i\end{pmatrix}\mod \mathfrak{q}.\end{aligned}$$

So the conditions on $b_1c_2+b_2c_1$ and d_i' for φ_q non-vanishing are preserved.

Let q be an inert prime. Set $h^{-1}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\mathbf{x}_i=\begin{pmatrix}b_i & a_i\sqrt{d}\\ c_i\sqrt{d} & \bar{b}_i\end{pmatrix}$ with $\alpha,\delta\equiv 1 \bmod q$ and $\beta,\gamma\equiv 0 \bmod q$. It suffices to show that

$$\begin{aligned}\begin{pmatrix}b_i' & a_i'\sqrt{d}\\ c_i'\sqrt{d} & \bar{b}_i'\end{pmatrix}&=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}b_i & a_i\sqrt{d}\\ c_i\sqrt{d} & \bar{b}_i\end{pmatrix}\begin{pmatrix}\bar{\delta} & -\bar{\beta}\\ -\bar{\gamma} & \bar{\alpha}\end{pmatrix}\\ &=\begin{pmatrix}\bar{\delta}(\alpha b_i+\beta c_i\sqrt{d})-\bar{\gamma}(\alpha a_i\sqrt{d}+\beta\bar{b}_i) & *\\ \bar{\delta}(\gamma b_i+\delta c_i\sqrt{d})-\bar{\gamma}(\gamma a_i\sqrt{d}+\delta\bar{b}_i) & -\bar{\beta}(\gamma b_i+\delta c_i\sqrt{d})+\bar{\alpha}(\gamma a_i\sqrt{d}+\delta\bar{b}_i)\end{pmatrix}\\ &\equiv\begin{pmatrix}b_i & *\\ c_i\sqrt{d} & \bar{b}_i\end{pmatrix}\mod q.\end{aligned}$$

It is clear that if φ_q is vanishing on $(\mathbf{x}_1,\mathbf{x}_2)$, then so is $\omega(1,h_j)\varphi_q$ on $(\mathbf{x}_1,\mathbf{x}_2)$ in the same way as dicussed in previous subsections. Similarly at ramified 2 we obtain the same result. □

4.5 At other finite places

We consider non-archimedean places away from $m|d_F|N(\mathfrak{n})$. For such a place q and an integral lattice X on V, we put $X_q=X\otimes_{\mathbb{Z}}\mathbb{Z}_q$. Define its dual lattice

$$X_q^\sharp=\{\mathbf{x}\in V\otimes\mathbb{Q}_q:(\mathbf{x},\mathbf{y})\in\mathbb{Z}_q\ \forall\mathbf{y}\in X_q\}$$

and let (q^{-l_q}) be the $\mathbb{Z}_q$-module generated by $\{(\mathbf{x},\mathbf{x}):\mathbf{x}\in X_q^\sharp\}$. In [3, Lemma 27], it is shown that $l_q=0$ at these places. At each place q, we define the local Schwartz function φ_q to be the characteristic function of X_q^2. Note that φ_q is invariant under $\mathrm{PSL}_2(\mathbb{Z}_q)\times\mathrm{PSL}_2(\mathbb{Z}_q)$ for split q and $\mathrm{PSL}_2(\mathcal{O}_\mathfrak{q})$ for inert or ramified q due to $l_q=0$ (see [3, Sect. 5.2]).

Lemma 4.14 *([25, Lemma 2.1]) At non-archimedean* $q\nmid m|d_F|N(\mathfrak{n})$ *we have*

$$\omega(\sigma)\varphi_q=\chi_{V,q}(\det A)\varphi_q\quad \text{for } \sigma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}_4(\mathbb{Z}_q).$$

5 Theta lift and Fourier coefficient

Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field of class number 1. Denote its ring of integers by $\mathcal{O} = \mathbb{Z}[\omega]$ and the discriminant by d_F. Let m be a product of distinct inert or split primes as introduced in Sect. 4 and choose a quadratic Hecke character $\chi_{\mathfrak{m}}$ ($\mathfrak{m} = m\mathcal{O}$) of conductor $\mathfrak{f} = \sqrt{d}\mathfrak{m}$. Let $\mathfrak{n}$ be square-free and coprime to $(m|d_F|)$. Suppose that $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2) : \mathbb{H}_3 \to \mathbb{C}^3$ is a weight 2 cusp form for $\Gamma_0(\mathfrak{n})$ with the corresponding $\Gamma_0(\mathfrak{n})$-invariant differential $\eta_{\mathcal{F}}$ of the form $-\mathcal{F}_0\frac{dz}{r} + \mathcal{F}_1\frac{dr}{r} + \mathcal{F}_2\frac{d\bar{z}}{r}$ on $\mathbb{H}_3$. More detailed discussion on Bianchi modular form can be found in [26, Chapter 1].

Remark 5.1 (1) Suppose that $d \equiv 1 \bmod 4$ with $d_F = d$. We choose the local Schwartz function $\varphi_q^{\chi_{\mathfrak{m}}}$ at each place q dividing $m|d_F|$ as defined in Definition 4.2, 4.5 and 4.8. At each place q dividing $N(\mathfrak{n})$, the local Schwartz function $\varphi_q^{\mathfrak{n}}$ is chosen to be as in Definition 4.11. For all other finite places we take the local Schwartz function as in Sect. 4.5.

(2) Suppose that $d \equiv 2, 3 \bmod 4$ with $d_F = 4d$. We choose the local Schwartz function $\varphi_q^{\chi_{\mathfrak{m}}}$ at each place q dividing m as defined in Definition 4.2 and 4.5, and that at ramified place away from 2 as in Definition 4.8. At each place q dividing $2N(\mathfrak{n})$, the local Schwartz functions $\varphi_q^{\mathfrak{n}}$ and φ_2 are chosen to be as in Definition 4.11. For all other places we take the local Schwartz function as in Sect. 4.5.

It has been shown in Lemma 4.4, 4.7, 4.10 and 4.13 that the local Schwartz function φ_v at each place v dividing $m|d_F|N(\mathfrak{n})$ is invariant under the action of the principal congruence subgroup $\overline{\Gamma}(\mathfrak{q}_v) \subset \mathrm{SO}^+(3,1)(V(\mathbb{Q}_v))$. We now consider a $\overline{\Gamma}_0(\mathfrak{q}_v)$-invariant local Schwartz function φ_v^{new} at these places defined by

$$\varphi_v^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) = \sum_{[\gamma] \in \overline{\Gamma}_0(\mathfrak{q}_v)/\overline{\Gamma}(\mathfrak{q}_v)} \omega(1, \gamma)\varphi_v(\mathbf{x}_1, \mathbf{x}_2)$$

where the sum is taken over all the representatives of $\overline{\Gamma}_0(\mathfrak{q}_v)/\overline{\Gamma}(\mathfrak{q}_v)$. With this new local Schwartz function we know that φ_f^{new} is invariant under $\overline{\Gamma}_0(\mathfrak{f}\mathfrak{n})$ when $d \equiv 1 \bmod 4$ or $\overline{\Gamma}_0(\mathfrak{f}\mathfrak{n}\mathfrak{q}_2)$ when $d \equiv 2, 3 \bmod 4$.

Give the Schwartz form $\varphi_2 \in S(V(\mathbb{R})^2) \otimes \Omega^2(D)$ as constructed in [13, Sect. 5], [9, Sect. 4] and [10, Sect. 5], and the above finite Schwartz function φ_f^{new} on $V(\mathbb{A}_f)^2$, we now consider a Schwartz form

$$\varphi(\mathbf{X}, z) := \varphi_2 \otimes \varphi_f^{\mathrm{new}} \in S(V(\mathbb{A})^2) \otimes \Omega^2(D) \quad \text{for } \mathbf{X} \in V(\mathbb{A})^2, z \in D.$$

Then we consider the theta series in this case is given by

$$\theta(g', \varphi_f^{\mathrm{new}}, z) := \sum_{\mathbf{X} \in V(\mathbb{Q})^2} \omega(g')\varphi(\mathbf{X}, z) \quad \text{for } g' \in \mathrm{Sp}_4(\mathbb{A})$$

which defines a closed differential 2-form on $\overline{\Gamma}_0(2mN)\backslash D$.

Following [13, Theorem 1], the theta lifting of $\mathcal{F}$, which is a holomophic Siegel modular form of weight 2, is given by

$$\Theta_\varphi(\eta_{\mathcal{F}})(g') := \int_{\Gamma\backslash D} \eta_{\mathcal{F}}(z) \wedge \theta(g', \varphi_f^{\mathrm{new}}, z)$$

where $\Gamma = \overline{\Gamma}_0(\mathfrak{f}\mathfrak{n}) \cap \Gamma_0(\mathfrak{n}) = \overline{\Gamma}_0(\mathfrak{f}\mathfrak{n})$ when $d \equiv 1 \bmod 4$ or $\Gamma = \overline{\Gamma}_0(\mathfrak{f}\mathfrak{n}\mathfrak{q}_2) \cap \Gamma_0(\mathfrak{n}) = \overline{\Gamma}_0(\mathfrak{f}\mathfrak{n}\mathfrak{q}_2)$ when $d \equiv 2, 3 \bmod 4$. Moreover, the Fourier coefficients are given as periods of

$\eta_{\mathcal{F}}$ over certain special cycles C_β in $\Gamma\backslash D$ attached to positive definite $\beta \in \mathrm{Sym}_2(\mathbb{Q})$, i.e.,

$$\Theta(\eta_{\mathcal{F}})(g') = \sum_{\beta>0} a_\beta(\eta_{\mathcal{F}}) e^{2\pi i\, \mathrm{tr}(\beta g')}$$

By Lemma 4.3, 4.6, 4.9 and 4.14, we can determine that it has level

$$\mathcal{L}_{m,\mathfrak{n}} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) : A \in \begin{pmatrix} \mathbb{Z} & n_1 N(\mathfrak{n})\mathbb{Z} \\ n_1 N(\mathfrak{n})\mathbb{Z} & \mathbb{Z} \end{pmatrix}, \right.$$
$$\left. B \in M_2(n_2 m|d|N(\mathfrak{n})\mathbb{Z}),\, C \in M_2(m^3 d^2 N(\mathfrak{n})\mathbb{Z}) \right\}$$

with

$$\begin{cases} n_1 = n_2 = 1, & \text{if } d \equiv 1 \bmod 4 \\ n_1 = 2, n_2 = 2^4, & \text{if } d \equiv 2, 3 \bmod 4. \end{cases}$$

Recall

$$\Omega_\beta = \left\{ (\mathbf{x}_1, \mathbf{x}_2) \in V(\mathbb{Q})^2 : \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & (\mathbf{x}_1, \mathbf{x}_2) \\ (\mathbf{x}_1, \mathbf{x}_2) & (\mathbf{x}_2, \mathbf{x}_2) \end{pmatrix} = \beta \right\}.$$

By [3, Theorem 9], the Fourier coefficient of the theta lifting $\Theta_\varphi(\eta_{\mathcal{F}})$ at $\beta > 0$ is given by

$$\begin{aligned} C_{\Theta_\varphi(\eta_{\mathcal{F}}),\beta} &= \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma\backslash\Omega_\beta} \varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_{\mathcal{F}} \\ &= \sum_{[\kappa_i]\in\Gamma\backslash\mathbb{P}^1(F)} \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_{\mathcal{F}} \end{aligned} \tag{25}$$

where the second equality is the consequence of Proposition 3.4. For simplicity we will denote $C_{\Theta_{\varphi^{\mathrm{new}}}(\eta_{\mathcal{F}}),\beta} := I = \sum_{[\kappa_i]\in\Gamma\backslash\mathbb{P}^1(F)} I_{\kappa_i}$ where

$$I_{\kappa_i} = \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_{\mathcal{F}}. \tag{26}$$

We will first express I_∞ in terms of the twisted L-value $L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)$ in Subsect. 5.1 and then use Atkin-Lehner operators to calculate I_{κ_i} for $\kappa_i \neq \infty$ in Subsect. 5.2.

Remark 5.2 We will describe how to choose the Gram matrix β for which we will show that $C_{\Theta_\varphi(\eta_{\mathcal{F}}),\beta}$ is non-vanishing.

- Let $\det \beta \in -d(\mathbb{Q}^\times)^2$. Then for $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_\beta$, $U(\mathbf{x}_1, \mathbf{x}_2)^\perp$ is split over $\mathbb{Q}$ due to Proposition 3.3 and $U(\mathbf{x}_1, \mathbf{x}_2)^\perp$ has signature (1,1). The stabilizer $\Gamma_U \subset \Gamma$ of $U = U(\mathbf{x}_1, \mathbf{x}_2)$ is trivial if $U^\perp$ is split over $\mathbb{Q}$, see [11, Lemma 4.2].
- For $(\mathbf{x}_1, \mathbf{x}_2) = \left(\begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{pmatrix} \right) \in \Omega_{\beta,\infty,+}$, we have

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} b_1\bar{b}_1 & \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) \\ \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) & b_2\bar{b}_2 \end{pmatrix}.$$

  We want this pair to satisfy the condition † as in Remark 3.6. This will allow us to apply Lemma 3.5 to deduce that the corresponding cusp $z_{U(\mathbf{x}_1,\mathbf{x}_2)}$ runs through all the representatives in $\mathfrak{f}^{-1}/\mathcal{O}$. For the non-vanishing of $\varphi_q^{\chi_{\mathfrak{m}}}$ at split or inert q dividing $\mathfrak{f}$, we only count $(\mathbf{x}_1, \mathbf{x}_2)$ such that b_i $(i = 1, 2)$ is divisible by m. Via imposing conditions on β itself, we can achieve that for any pair $(\mathbf{x}_1, \mathbf{x}_2)$ in $\Omega_{\beta,\infty,+}$ the assumption † holds.

Explicit examples of β will be given at the end of the Subsect. 5.1, see from Example 5.6 to 5.11.

Assume that β is given as in above Remark 5.2. For $(\mathbf{x}_1, \mathbf{x}_2)$ in $\Omega_{\beta,\infty,+}$, we have

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix} \text{ if } d \equiv 1 \tag{27}$$

or

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix} \text{ if } d \equiv 2, 3 \tag{28}$$

with $x, y, z, w \in \mathbb{Z}$ and $xw - yz = \pm 1$. We want to find out if there is another pair $(\mathbf{y}_1, \mathbf{y}_2)$ in $\Omega_{\beta,\infty,+}$ such that it gives rise to the same cycle D_U as that generated by $(\mathbf{x}_1, \mathbf{x}_2)$.

Assume that such a pair $(\mathbf{y}_1, \mathbf{y}_2)$ exists in $\Omega_{\beta,\infty,+}$. For $U(\mathbf{x}_1, \mathbf{x}_2) = U(\mathbf{y}_1, \mathbf{y}_2)$ we consider an element $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}. \tag{29}$$

To make $\langle u_\infty, \mathbf{x}_1, \mathbf{x}_2, u_{\kappa_j} \rangle$ and $\langle u_\infty, \mathbf{y}_1, \mathbf{y}_2, u_{\kappa_j} \rangle$ represent the same orientation, we need $\sigma \in \mathrm{GL}_2^+(\mathbb{Q})$. Additionally the Gram matrix β's corresponding to $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ must be identical.

Expressing $(\mathbf{y}_i, \mathbf{y}_j)$ in terms of $(\mathbf{x}_i, \mathbf{x}_j)$ and using bilinearity, we have

$$\begin{aligned}
(\mathbf{y}_1, \mathbf{y}_1) &= a^2(\mathbf{x}_1, \mathbf{x}_1) + 2ab(\mathbf{x}_1, \mathbf{x}_2) + b^2(\mathbf{x}_2, \mathbf{x}_2), \\
(\mathbf{y}_2, \mathbf{y}_2) &= c^2(\mathbf{x}_1, \mathbf{x}_1) + 2cd(\mathbf{x}_1, \mathbf{x}_2) + d^2(\mathbf{x}_2, \mathbf{x}_2), \\
(\mathbf{y}_1, \mathbf{y}_2) &= ac(\mathbf{x}_1, \mathbf{x}_1) + (ad + bc)(\mathbf{x}_1, \mathbf{x}_2) + bd(\mathbf{x}_2, \mathbf{x}_2).
\end{aligned}$$

Consider that $\det \beta$ is preserved; more explicitly,

$$\begin{aligned}
\det \beta =& \det((\mathbf{y}_i, \mathbf{y}_j)) = (\mathbf{y}_1, \mathbf{y}_1)(\mathbf{y}_2, \mathbf{y}_2) - (\mathbf{y}_1, \mathbf{y}_2)^2 \\
=& a^2c^2(\mathbf{x}_1, \mathbf{x}_1)^2 + b^2d^2(\mathbf{x}_2, \mathbf{x}_2)^2 + 4abcd(\mathbf{x}_1, \mathbf{x}_2)^2 + (a^2d^2 + b^2c^2)(\mathbf{x}_1, \mathbf{x}_1)(\mathbf{x}_2, \mathbf{x}_2) \\
&+ 2ac(ad + bc)(\mathbf{x}_1, \mathbf{x}_1)(\mathbf{x}_1, \mathbf{x}_2) + 2bd(ad + bc)(\mathbf{x}_1, \mathbf{x}_2)(\mathbf{x}_2, \mathbf{x}_2) \\
&- a^2c^2(\mathbf{x}_1, \mathbf{x}_1)^2 - b^2d^2(\mathbf{x}_2, \mathbf{x}_2)^2 - (ad + bc)^2(\mathbf{x}_1, \mathbf{x}_2)^2 - 2abcd(\mathbf{x}_1, \mathbf{x}_1)(\mathbf{x}_2, \mathbf{x}_2) \\
&- 2ac(ad + bc)(\mathbf{x}_1, \mathbf{x}_1)(\mathbf{x}_1, \mathbf{x}_2) - 2bd(ad + bc)(\mathbf{x}_1, \mathbf{x}_2)(\mathbf{x}_2, \mathbf{x}_2) \\
=& (\det \sigma)^2 \det((\mathbf{x}_i, \mathbf{x}_j)) = \det((\mathbf{x}_i, \mathbf{x}_j)).
\end{aligned}$$

Since σ has positive determinant we know that $\sigma \in \mathrm{SL}_2(\mathbb{Q})$.

Moreover, to preserve β the following identities must hold:

$$(\mathbf{x}_1, \mathbf{x}_1) = a^2(\mathbf{x}_1, \mathbf{x}_1) + 2ab(\mathbf{x}_1, \mathbf{x}_2) + b^2(\mathbf{x}_2, \mathbf{x}_2), \tag{30}$$

$$(\mathbf{x}_2, \mathbf{x}_2) = c^2(\mathbf{x}_1, \mathbf{x}_1) + 2cd(\mathbf{x}_1, \mathbf{x}_2) + d^2(\mathbf{x}_2, \mathbf{x}_2), \tag{31}$$

$$(\mathbf{x}_1, \mathbf{x}_2) = ac(\mathbf{x}_1, \mathbf{x}_1) + (ad + bc)(\mathbf{x}_1, \mathbf{x}_2) + bd(\mathbf{x}_2, \mathbf{x}_2). \tag{32}$$

As $\det \sigma = ad - bc = 1$, we can rewrite (32) as

$$ac(\mathbf{x}_1, \mathbf{x}_1) + 2bc(\mathbf{x}_1, \mathbf{x}_2) + bd(\mathbf{x}_2, \mathbf{x}_2) = 0. \tag{33}$$

We will describe σ in different cases in the following.

(I) Let b be 0. From (30) we know that $a^2 = 1$, and from (32) we have that $(\mathbf{x}_1, \mathbf{x}_2) = ac(\mathbf{x}_1, \mathbf{x}_1) + (\mathbf{x}_1, \mathbf{x}_2)$ which implies $c = 0$. So $\sigma = \pm\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. In the same way, if $c = 0$ then $\sigma = \pm\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$.

(II) Assume that $bc \neq 0$. Substituting $(\mathbf{x}_1, \mathbf{x}_2)$ in (30), we have

$$(\mathbf{x}_1, \mathbf{x}_1) = a^2(\mathbf{x}_1, \mathbf{x}_1) + b^2(\mathbf{x}_2, \mathbf{x}_2) - \frac{a}{c}(ac(\mathbf{x}_1, \mathbf{x}_1) + bd(\mathbf{x}_2, \mathbf{x}_2))$$

which is simplified to be

$$c(\mathbf{x}_1, \mathbf{x}_1) + b(\mathbf{x}_2, \mathbf{x}_2) = 0. \tag{34}$$

Combining (33) and (34) we have

$$ac(\mathbf{x}_1, \mathbf{x}_1) + 2bc(\mathbf{x}_1, \mathbf{x}_2) - cd(\mathbf{x}_1, \mathbf{x}_1) = 0$$

and then $d = a + 2b\frac{(\mathbf{x}_1,\mathbf{x}_2)}{(\mathbf{x}_1,\mathbf{x}_1)}$. As $ad - bd = 1$, we have

$$a^2 + 2ab\frac{(\mathbf{x}_1, \mathbf{x}_2)}{(\mathbf{x}_1, \mathbf{x}_1)} + b^2\frac{(\mathbf{x}_2, \mathbf{x}_2)}{(\mathbf{x}_1, \mathbf{x}_1)} = 1. \tag{35}$$

Set $\mathbf{x}_1 = \begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & d_1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & d_2 \end{pmatrix}$, $\mathbf{y}_1 = \begin{pmatrix} a_1' & b_1' \\ \bar{b}_1' & d_1' \end{pmatrix}$ and $\mathbf{y}_2 = \begin{pmatrix} a_2' & b_2' \\ \bar{b}_2' & d_2' \end{pmatrix}$. Combining (27) and (29), we can rewrite

$$b_1' = ab_1 + bb_2 = m((ax + bz) + (ay + bw)\omega)$$

or combining (28) and (29)

$$b_1' = ab_1 + bb_2 = \frac{1}{2}((ax + bz) + (ay + bw)\omega).$$

Then we need $ax + bz = \mu \in \mathbb{Z}$ and $ay + bw = \nu \in \mathbb{Z}$. Solving these two equations we get

$$a = \frac{\mu\delta - \nu\gamma}{\alpha\delta - \beta\gamma} \in \mathbb{Z} \quad \text{and} \quad b = \frac{\nu\alpha - \mu\beta}{\alpha\delta - \beta\gamma} \in \mathbb{Z}.$$

Similarly, we can deduce that $c, d \in \mathbb{Z}$ when treating b_2'. Therefore, the linear transform $\sigma \in \mathrm{SL}_2(\mathbb{Q})$ on $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_{\beta,\kappa_i,+}$ generates the same cycle, but the Schwartz function on $(\mathbf{y}_1, \mathbf{y}_2)$ vanishes if $\sigma \notin \mathrm{SL}_2(\mathbb{Z})$. For particular choices of β we get limited possibilities of above σ. We can rewrite (35) as

$$\left(a + b\frac{(\mathbf{x}_1, \mathbf{x}_2)}{(\mathbf{x}_1, \mathbf{x}_1)}\right)^2 + b^2\frac{(\mathbf{x}_1, \mathbf{x}_1)(\mathbf{x}_2, \mathbf{x}_2) - (\mathbf{x}_1, \mathbf{x}_2)^2}{(\mathbf{x}_1, \mathbf{x}_1)^2} = 1. \tag{36}$$

(II.1) If $(\mathbf{x}_1, \mathbf{x}_2) = 0$ and $(\mathbf{x}_1, \mathbf{x}_1) = (\mathbf{x}_2, \mathbf{x}_2)$, then $a^2 = 0$ as $bc \neq 0$ by our assumption. So in this case $\sigma = \pm\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$.

(II.2) If $(\mathbf{x}_1, \mathbf{x}_2) = 0$ and $(\mathbf{x}_1, \mathbf{x}_1) < (\mathbf{x}_2, \mathbf{x}_2)$, i.e. $\frac{\det\beta}{(\mathbf{x}_1,\mathbf{x}_1)^2} > 1$, then there is no such a σ that $bc \neq 0$.

(II.3) If $(\mathbf{x}_1, \mathbf{x}_2) \neq 0$ and $\frac{(\mathbf{x}_1,\mathbf{x}_1)(\mathbf{x}_2,\mathbf{x}_2)-(\mathbf{x}_1,\mathbf{x}_2)^2}{(\mathbf{x}_1,\mathbf{x}_1)^2} > 1$, then b has to be 0 which is a contradiction to $bc \neq 0$.

Remark 5.3 The possibilities of σ in (29) will determine the constant μ_β in Proposition 5.5. After the whole treatment of this section, we will see that this μ_β does not effect the non-vanishing of our theta liftings since it appears in the Fourier coefficient as a non-zero multiplier. In Example from 5.6 to 5.11, we will show how to get the exact values of μ_β.

5.1 On cycles through ∞

We will first calculate the part I_∞ corresponding to the cusp ∞ as in (26). We pick a fundamental domain for $\Gamma_\infty\backslash D_U$ and integrate with respect to the cycle. Since we are integrating along a vertical path with z-coordinate constant, we can ignore dz and $d\bar{z}$. We obtain

$$I_\infty = \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}_1,\mathbf{x}_2)\int_{(z,r)\in C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \frac{1}{2}\mathcal{F}_1(z,r)dr. \tag{37}$$

Lemma 5.4 *For* $(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}$, *we have*

$$\varphi_f^{\mathrm{new}}(\mathbf{x}_1,\mathbf{x}_2) = \lambda_{\mathfrak{m},\mathfrak{n}}\varphi_f(\mathbf{x}_1,\mathbf{x}_2)$$

where

$$\lambda_{\mathfrak{m},\mathfrak{n}} = \prod_{q_1|m|d|}\left[\overline{\Gamma}_0(q_1):\overline{\Gamma}(q_1)\right]\prod_{\mathfrak{q}_2 \text{ above ramified } 2}\left[\overline{\Gamma}_0(\mathfrak{q}_2):\overline{\Gamma}(\mathfrak{q}_2)\right]\prod_{\mathfrak{q}_3|\mathfrak{n}}\left[\overline{\Gamma}_0(\mathfrak{q}_3):\overline{\Gamma}(\mathfrak{q}_3)\right].$$

Proof Note that any pair $(\mathbf{x}_1,\mathbf{x}_2)$ in $\Omega_{\beta,\infty}$ is of form $\left(\begin{pmatrix} a_1 & b_1\\ \bar{b}_1 & 0\end{pmatrix},\begin{pmatrix} a_2 & b_2\\ \bar{b}_2 & 0\end{pmatrix}\right)$. Recall from [26, Sect. 1.3], for $a\in\mathcal{O}^\times$ satisfying $((a),\mathfrak{f})=1$ we have

$$\prod_{v|\mathfrak{f}}\widetilde{\chi}_v(a_v) = \prod_{v\nmid\mathfrak{f}}\widetilde{\chi}_v^{-1}(a_v) = \chi^{-1}((a)).$$

Then, for our choice of finite Schwartz function φ_f, we have

$$\begin{aligned}\varphi_f(\mathbf{x}_1,\mathbf{x}_2) &= \prod_{v|\mathfrak{f}}\widetilde{\chi}_{\mathfrak{m},v}\left(\frac{a_2b_1-a_1b_2}{m}\right)\\ &= \prod_{v\nmid\mathfrak{f}}\widetilde{\chi}_{\mathfrak{m},v}^{-1}\left(\frac{a_2b_1-a_1b_2}{m}\right) = \chi^{-1}\left(\left(\frac{a_2b_1-a_1b_2}{m}\right)\right)\end{aligned}$$

or

$$\varphi_f(\mathbf{x}_1,\mathbf{x}_2) = \chi^{-1}\left(\left(\frac{2(a_2b_1-a_1b_2)}{m}\right)\right).$$

Let q be the split prime dividing m. Consider the representative

$$\gamma = (\gamma_1,\gamma_2) = \left(\begin{pmatrix} x_{\gamma,1} & y_{\gamma,1}\\ 0 & x_{\gamma,1}^{-1}\end{pmatrix},\begin{pmatrix} x_{\gamma,2} & y_{\gamma,2}\\ 0 & x_{\gamma,2}^{-1}\end{pmatrix}\right)$$

for $\overline{\Gamma}_0(q)/\overline{\Gamma}(q)$ with $[x_{\gamma,j}]\in(\mathcal{O}/(q))^\times$ and $[y_{\gamma,j}]\in\mathcal{O}/(q)$. By the computation in the proof of Lemma 4.4, we can observe that

$$\begin{aligned}\varphi_q^{\chi_\mathfrak{m},\mathrm{new}}(\mathbf{x}_1,\mathbf{x}_2) &= \sum_{[\gamma]\in\overline{\Gamma}_0(q)/\overline{\Gamma}(q)}(\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})(x_{\gamma,1}^{-2})(\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})\left(\frac{a_2b_1-a_1b_2}{m}\right)\\ &= \sum_{[\gamma]\in\overline{\Gamma}_0(q)/\overline{\Gamma}(q)}(\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})\left(\frac{a_2b_1-a_1b_2}{m}\right)\\ &= [\overline{\Gamma}_0(q):\overline{\Gamma}(q)]\varphi_q^{\chi_\mathfrak{m}}(\mathbf{x}_1,\mathbf{x}_2)\end{aligned}$$

Similarly we have for inert $q|m$

$$\varphi_q^{\chi_{\mathfrak{m}},\mathrm{new}}(\mathbf{x}_1,\mathbf{x}_2)=[\overline{\Gamma}_0(q):\overline{\Gamma}(q)]\varphi_q^{\chi_{\mathfrak{m}}}(\mathbf{x}_1,\mathbf{x}_2),$$

and for ramified prime q with $(q)=\mathfrak{q}^2$

$$\varphi_q^{\chi_{\mathfrak{m}},\mathrm{new}}(\mathbf{x}_1,\mathbf{x}_2)=[\overline{\Gamma}_0(\mathfrak{q}):\overline{\Gamma}(\mathfrak{q})]\varphi_q^{\chi_{\mathfrak{m}}}(\mathbf{x}_1,\mathbf{x}_2).$$

At the place $q|N(\mathfrak{n})$ we will show that

$$\varphi_q^{\mathfrak{n},\mathrm{new}}=[\overline{\Gamma}_0(q):\overline{\Gamma}(q)]\varphi_q^{\mathfrak{n}}.$$

For q split, set $\mathbf{x}_i=\begin{pmatrix}a_i & b_i\\ c_i & 0\end{pmatrix}$ and compute

$$w(1,\gamma)\cdot(\mathbf{x}_1,\mathbf{x}_2)=\left(\begin{pmatrix}* & x_{\gamma_1}^{-1}x_{\gamma_2}^{-1}b_1\\ x_{\gamma_1}x_{\gamma_2}c_1 & 0\end{pmatrix},\begin{pmatrix}* & x_{\gamma_1}^{-1}x_{\gamma_2}^{-1}b_2\\ x_{\gamma_1}x_{\gamma_2}c_2 & 0\end{pmatrix}\right).$$

It is not hard to observe that the condition on $b_1c_2+b_2c_1$ is preserved as $x_{\gamma,j}\in\mathbb{Z}_q^{\times}$. It follow that

$$w(1,\gamma)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)$$

which implies the assertion. For q inert, set $\mathbf{x}_i=\begin{pmatrix}b_i & a_i\sqrt{d}\\ 0 & \bar{b}_i\end{pmatrix}$ and compute

$$\gamma^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)=\left(\begin{pmatrix}x_\gamma^{-1}\bar{x}_\gamma b_1 & *\\ 0 & x_\gamma\bar{x}_\gamma^{-1}\bar{b}_1\end{pmatrix},\begin{pmatrix}x_\gamma^{-1}\bar{x}_\gamma b_2 & *\\ 0 & x_\gamma\bar{x}_\gamma^{-1}\bar{b}_2\end{pmatrix}\right).$$

Again the condition on $b_1\bar{b}_2+\bar{b}_1b_2$ is preserved and so the assertion follows. The case at ramified 2 can be treated similarly.

Now we have proven the lemma. □

Proposition 5.5 *Assume that the Gram matrix β is chosen so that the condition † in Lemma 3.5 is satisfied. Then we can calculate*

$$I_\infty=\frac{\mu_\beta\lambda_{\mathfrak{m},\mathfrak{n}}L(\mathcal{F},\chi_{\mathfrak{m}},1)}{2A(1,1,\chi_{\mathfrak{m}},1)}\tag{38}$$

where μ_β is a non-zero integer depending on β as stated in Remark 5.3 and $A(1,1,\chi_{\mathfrak{m}},1)$ is given explicitly in [24, Theorem 1.8].

Proof By the above lemma, we can express

$$\begin{aligned}I_\infty&=\sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}}\varphi_f^{\mathrm{new}}(\mathbf{x}_1,\mathbf{x}_2)\int_{(z,r)\in C_{U(\mathbf{x}_1,\mathbf{x}_2)}}\frac{1}{2}\mathcal{F}_1(z,r)dr\\ &=\lambda_{\mathfrak{m},\mathfrak{n}}\sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}}\varphi_f(\mathbf{x}_1,\mathbf{x}_2)\int_{(z,r)\in C_{U(\mathbf{x}_1,\mathbf{x}_2)}}\frac{1}{2}\mathcal{F}_1(z,r)dr.\end{aligned}$$

Under our assumption on β, by Lemma 3.5 we have

$$I_\infty=\mu_\beta\lambda_{\mathfrak{m},\mathfrak{n}}\sum_{[z_U]\in\mathfrak{f}^{-1}/\mathcal{O},(z_U\mathfrak{f},\mathfrak{f})=1}\chi_{\mathfrak{m}}^{-1}(z_U\mathfrak{f})\int_0^\infty\frac{1}{2}\mathcal{F}_1(z,r)dr$$

where μ_β is a non-zero integer depending on the possibilities of σ as discussed in Remark 5.3. At last, by [24, Theorem 1.8] with $n = 1$, we can compute

$$I_\infty = \mu_\beta \lambda_{\mathfrak{m},\mathfrak{n}} \sum_{[z_U]\in\mathfrak{f}^{-1}/\mathcal{O},(z_U\mathfrak{f},\mathfrak{f})=1} \chi_{\mathfrak{m}}^{-1}(z_U\mathfrak{f}) \int_0^\infty \frac{1}{2}\mathcal{F}_1(z,r)dr = \frac{\mu_\beta\lambda_{\mathfrak{m},\mathfrak{n}}L(\mathcal{F},\chi_{\mathfrak{m}},1)}{2A(1,1,\chi_{\mathfrak{m}},1)}.$$

□

For a diagonal Gram matrix β, the pair $(\mathbf{x}_1,\mathbf{x}_2)\in\Omega_{\beta,\infty,+}$ has $b_1\bar{b}_2+\bar{b}_1b_2=0$. It follows that $\varphi_q^{\mathfrak{n}}$ is vanishing on such a pair $(\mathbf{x}_1,\mathbf{x}_2)$. So, for the non-vanishing of I_∞, the Gram matrix β being diagonal is ruled out of our consideration. In the following we give some examples of β satisfying the condition † (as promised in Remark 5.2) for which I_∞ can be expressed in terms of $L(\mathcal{F},\chi_{\mathfrak{m}},1)$.

Example 5.6 Let $F=\mathbb{Q}(\sqrt{-3})$ with $d_F=-3$ and $\mathcal{O}=\mathbb{Z}[\omega]$ with $\omega=\frac{1+\sqrt{-3}}{2}$. Suppose that

$$\beta = \begin{pmatrix} b_1\bar{b}_1 & \frac{1}{2}(b_1\bar{b}_2+\bar{b}_1b_2) \\ \frac{1}{2}(b_1\bar{b}_2+\bar{b}_1b_2) & b_2\bar{b}_2 \end{pmatrix} = \begin{pmatrix} m^2 & \frac{1}{2}m^2 \\ \frac{1}{2}m^2 & m^2 \end{pmatrix}.$$

We have

$$I_\infty = \frac{4\lambda_{\mathfrak{m},\mathfrak{n}}L(\mathcal{F},\chi_{\mathfrak{m}},1)}{A(1,1,\chi_{\mathfrak{m}},1)}.$$

Proof For the non-vanishing of $\varphi_q^{\chi_{\mathfrak{m}}}$, we need $m|b_i$. Solving $b_i\bar{b}_i=m^2$, we must take $b_i=\pm m, \pm m\omega$ or $\pm m\bar{\omega}$. Observing

$$(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{2}(b_1\bar{b}_2+\bar{b}_1b_2) = \frac{1}{2}m^2,$$

we can determine b_i with the condition † as in Proposition 3.5 satisfied:

$$\begin{cases} b_1 = m \\ b_2 = m\omega \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \omega \end{pmatrix}$$
$$\begin{cases} b_1 = m \\ b_2 = m\bar{\omega} \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \omega \end{pmatrix}$$
$$\begin{cases} b_1 = -m \\ b_2 = -m\omega \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \omega \end{pmatrix}$$
$$\begin{cases} b_1 = -m \\ b_2 = -m\bar{\omega} \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \omega \end{pmatrix}$$
$$\begin{cases} b_1 = m\omega \text{ or } m\bar{\omega} \\ b_2 = m, \end{cases} \quad \begin{cases} b_1 = -m\omega \text{ or } -m\bar{\omega} \\ b_2 = -m. \end{cases}$$

We have seen in Lemma 3.7 that the sign of $\mathrm{Im}(b_1\bar{b}_2)$ determines the orientation ε of $T_Z(D_U)$ via $\varepsilon\mathrm{Im}(b_1\bar{b}_2)>0$. If the cycle D_U integrated over is directed from the cusp on the complex plane to the cusp ∞, we need $\varepsilon<0$ which implies $\mathrm{Im}(b_1\bar{b}_2)<0$. We will list all pairs in $\Gamma_\infty\backslash\Omega_{\beta,\infty,+}$ with $\mathrm{Im}(b_1\bar{b}_2)<0$.

First we consider one pair

$$(\mathbf{x}_{1,1},\mathbf{x}_{1,2}) = \left(\begin{pmatrix} a_{1,1} & m \\ m & 0 \end{pmatrix}, \begin{pmatrix} a_{1,2} & m\omega \\ m\bar{\omega} & 0 \end{pmatrix}\right)$$

with $a_{1,1}, a_{1,2} \in \mathbb{Z}$ which gives rise to the cycle $D_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$ directed from the cusp $z_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = \frac{a_{1,1}\omega - a_{1,2}}{m\sqrt{-3}} \in F$ to the cusp ∞. Rewriting (35) as $a^2 - ab + b^2 = 1$, we have either $a^2 = 1, b^2 = 0$ or $a^2 = 0, b^2 = 1$ and then $\sigma = \pm\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ or $\pm\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. So the following four pairs give rise to the same cycle $D_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$:

$$(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}), (-\mathbf{x}_{1,1}, -\mathbf{x}_{1,2}), (\mathbf{x}_{1,2}, -\mathbf{x}_{1,1}), (-\mathbf{x}_{1,2}, \mathbf{x}_{1,1}). \tag{39}$$

Lemma 3.5 tells us that for $(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}) \in \Gamma_\infty \backslash \Omega_{\beta,\infty,+}$, $z_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$ ranges over $\mathfrak{f}^{-1}/\mathcal{O}$ with $\mathfrak{f} = \sqrt{-3}\mathfrak{m}$.

Suppose that

$$(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) = \left(\begin{pmatrix} a_{2,1} & -m \\ -m & 0 \end{pmatrix}, \begin{pmatrix} a_{2,2} & -m\bar{\omega} \\ -m\omega & 0 \end{pmatrix}\right)$$

with $a_{2,1}, a_{2,2} \in \mathbb{Z}$ is another pair in $\Omega_{\beta,\infty,+}$ which gives rise to the cycle $D_{U(\mathbf{x}_{2,1},\mathbf{x}_{2,2})}$ directed from the cusp $z_{U(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} = \frac{a_{2,1}\bar{\omega} - a_{2,2}}{m\sqrt{-3}} \in F$ to the cusp ∞. Similarly we have following pairs

$$(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}), (-\mathbf{x}_{2,1}, -\mathbf{x}_{2,2}), (\mathbf{x}_{2,2}, -\mathbf{x}_{2,1}), (-\mathbf{x}_{2,2}, \mathbf{x}_{2,1}) \tag{40}$$

giving rise to the same cycle $D_{U(\mathbf{x}_{2,1},\mathbf{x}_{2,2})}$. Also for $(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) \in \Gamma_\infty \backslash \Omega_{\beta,\infty,+}$ we have $z_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$ running through $\mathfrak{f}^{-1}/\mathcal{O}$ with $\mathfrak{f} = \sqrt{-3}\mathfrak{m}$.

It is obvious that the eight pairs in (39) and (40) are not Γ_∞-equivalent since the Γ_∞-action on the pair preserves off-diagonal entries of each component of the pair. Then we can split I_∞ as

$$\begin{aligned} I_\infty = & I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} + I_{(-\mathbf{x}_{1,1},-\mathbf{x}_{1,2})} + I_{(\mathbf{x}_{1,2},-\mathbf{x}_{1,1})} + I_{(-\mathbf{x}_{1,2},\mathbf{x}_{1,1})} \\ & + I_{(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} + I_{(-\mathbf{x}_{2,1},-\mathbf{x}_{2,2})} + I_{(\mathbf{x}_{2,2},-\mathbf{x}_{2,1})} + I_{(-\mathbf{x}_{2,2},\mathbf{x}_{2,1})}, \end{aligned}$$

where the subscript $(-,-)$ indicates the sum as in (37) over $[z_{U(-,-)}] \in \mathfrak{f}^{-1}/\mathcal{O}$. By [24, Theorem 1.8] with $n = 1$, we can calculate

$$I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = I_{(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} = \frac{\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{2A(1, 1, \chi_{\mathfrak{m}}, 1)}.$$

So, in this case we have $\mu_\beta = 8$ and then we can deduce that

$$I_\infty = 8 \cdot I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = \frac{4\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, 1, \chi_{\mathfrak{m}}, 1)}.$$

□

Detailed calculations in the following examples can be found in [26, Sect. 4.4.1].

Example 5.7 Let $F = \mathbb{Q}(\sqrt{d})$ with $d \equiv 1 \pmod 4$ and $d \neq -3$ in which case $d_F = d$ and $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{d}}{2}$. Suppose that

$$\beta = \begin{pmatrix} m^2 & \frac{1}{2}m^2 \\ \frac{1}{2}m^2 & \frac{1-d}{4}m^2 \end{pmatrix}.$$

We have

$$I_\infty = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, 1, \chi_{\mathfrak{m}}, 1)}.$$

Example 5.8 Let $F = \mathbb{Q}(\sqrt{d})$ with $d \equiv 2, 3 \pmod 4$ and $d \neq -1$ in which case $d_F = 4d$ and $\mathcal{O} = \mathbb{Z}[d]$. Let

$$\beta = \begin{pmatrix} \frac{1}{4}m^2 & \frac{1}{4}nm^2 \\ \frac{1}{4}nm^2 & \frac{1}{4}(n^2 - d)m^2 \end{pmatrix}$$

with $n \in \mathbb{Z}$ coprime to $2\mathfrak{n}$ (for the non-vanishing of $\varphi_q^{\mathfrak{n}}$ and φ_2). We have

$$I_\infty = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, 1, \chi_{\mathfrak{m}}, 1)}.$$

Example 5.9 Let $F = \mathbb{Q}(i)$ with $d_F = -4$ and $\mathcal{O} = \mathbb{Z}[i]$. Set

$$\beta = \begin{pmatrix} \frac{1}{4}m^2 & \frac{1}{4}nm^2 \\ \frac{1}{4}nm^2 & \frac{1}{4}(n^2 + 1)m^2 \end{pmatrix}$$

with $1 < n \in \mathbb{Z}$ coprime to $2\mathfrak{n}$ (for the non-vanishing of $\varphi_q^{\mathfrak{n}}$ and φ_2 at ramified 2). We have

$$I_\infty = \frac{4\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, 1, \chi_{\mathfrak{m}}, 1)}.$$

Example 5.10 Let $F = \mathbb{Q}(\sqrt{d})$ with $d \equiv 1 \pmod 4$ and $d \neq -3$ in which case $d_F = d$ and $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{d}}{2}$. Set

$$\beta = \begin{pmatrix} m^2 & \frac{nm^2}{2} \\ \frac{nm^2}{2} & \frac{n^2-d}{4}m^2 \end{pmatrix}$$

with $1 < n \in \mathbb{Z}$ coprime to $q_{\mathfrak{n}}$. We have

$$I_\infty = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, 1, \chi_{\mathfrak{m}}, 1)}.$$

Example 5.11 Let $F = \mathbb{Q}(\sqrt{-3})$ with $d_F = d = -3$ and $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{-3}}{2}$. Set

$$\beta = \begin{pmatrix} m^2 & \frac{nm^2}{2} \\ \frac{nm^2}{2} & \frac{n^2-d}{4}m^2 \end{pmatrix}$$

with odd $n \in \mathbb{Z}$ greater than 1 and coprime to $\mathfrak{n}$. We have

$$I_\infty = \frac{3\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, 1, 1, \chi_{\mathfrak{m}}, 1)}.$$

Remark We can swap the diagonal entries of each β in Example from 5.6 to 5.11 and obtain same results.

5.2 On other cycles

We introduce the Atkin-Lehner operator as defined in Lingham's thesis [15, Sect. 5.3]. Lingham developed this for all odd class numbers while we shall only use results for class number 1 since then we can follow Asai's treatment of cusps (see [1, Sect. 1.1]) in the case of principal ideal domain. For $\mathfrak{m}$ in $\mathcal{O}$ dividing $\mathfrak{n}$ such that $\mathfrak{m}$ and $\frac{\mathfrak{n}}{\mathfrak{m}}$ are coprime, take

$$W_{\mathfrak{m}} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \tag{41}$$

where $x \in \mathfrak{m}$, $y \in \mathcal{O}$, $z \in \mathfrak{n}$,$w \in \mathfrak{m}$ and $(xw - yz)\mathcal{O} = \mathfrak{m}$.

Proposition 5.12 *(1) For any ideal* $\mathfrak{m}$ *dividing* $\mathfrak{n}$ *such that* $\mathfrak{m}$ *and* $\frac{\mathfrak{n}}{\mathfrak{m}}$ *are coprime, we can find a matrix of the form (41).*

(2) $W_{\mathfrak{m}}$ *is an involution (i.e.* $W_{\mathfrak{m}}^2$ *(modulo scalars) lies in* $\Gamma_0(\mathfrak{n})$*), normalizes* $\Gamma_0(\mathfrak{n})$ *and is independent of the particular choice of* x, y, z, w.

Proof See [15, Lemma 5.3.1 and Lemma 5.3.2]. □

In particular if we take $\mathfrak{m} = \mathcal{O}$ we get an element of $\Gamma_0(\mathfrak{n})$ and if we take $\mathfrak{m} = \mathfrak{n}$ we get the analogue of the classical Fricke involution. One can check that the Fricke involution can be formed as a product of Atkin-Lehner involutions, where $\mathfrak{m}$ runs over prime power divisors of $\mathfrak{n}$.

Lemma 5.13 *Let* $\alpha_1 = \frac{p_1}{q_1}$, $\alpha_2 = \frac{p_2}{q_2}$ *be two cusps such that* $\langle p_1, q_1\rangle = \langle p_2, q_2\rangle = \mathcal{O}$*. Then the following are equivalent*

(1) $\alpha_2 = M\alpha_1$ *for some* $M \in \Gamma_0(\mathfrak{n})$*;*

(2) $q_2 s_1 - q_1 s_2 \in q_1 q_2 \mathcal{O} + \mathfrak{n}$*, where* s_i *satisfies* $p_i s_i \equiv 1$ *(mod* q_i*).*

Proof See [15, Lemma 1.5.1] for a more general version holding over any number field. □

It follows that two cusps are equivalent relative to $\Gamma_0(\mathfrak{n})$ if and only if the ideals generated by the denominators have the same 'greatest common divisor' with $\mathfrak{n}$, so each equivalence class of cusps is in one-to-one correspondence with each ordered decomposition $\mathfrak{n} = \mathfrak{M}\mathfrak{L}$. Following Asai's treatment (see [1, Sect. 1.1]) we say a cusp κ_2/κ_1 belongs to $\mathfrak{L}$-class if $\gcd(\kappa_1\mathcal{O}, \mathfrak{n}) = \mathfrak{L}$. For each decomposition $\mathfrak{n} = \mathfrak{M}\mathfrak{L}$ with $\mathfrak{M} = M\mathcal{O}$ and any cusp $\kappa = \kappa_2/\kappa_1$ of $\mathfrak{L}$-class, we can take a typical matrix W_κ which transforms κ to ∞:

$$W_\kappa = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \alpha_\kappa \quad \text{with} \quad \alpha_\kappa = \begin{pmatrix} M\lambda_1 & \lambda_2 \\ -\kappa_1 & \kappa_2 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}). \tag{42}$$

As $\langle \kappa_1, \kappa_2\rangle = \langle \kappa_1, M\rangle = \mathcal{O}$ there exist $b, c \in \mathcal{O}$ such that $b\kappa_2 \equiv 1 \pmod{\kappa_1}$ and $cM \equiv 1 \pmod{\kappa_1}$. Taking $\lambda_1 = bc \in \mathcal{O}$ we observe that $\lambda_2 = \frac{1 - M\lambda_1\kappa_2}{\kappa_1}$ belongs to $\mathcal{O}$. So W_κ is well-defined. It is not difficult to see that W_κ is of type of Atkin-Lehner operator as defined in (41).

Fix a representative $\kappa_i = \kappa_{i,2}/\kappa_{i,1} \in \mathbb{P}^1(F)/\Gamma$ of each equivalence class of cusps corresponding to the ordered decomposition $m\sqrt{d_F}\mathfrak{n} = \mathfrak{M}_i\mathfrak{L}_i$ with $\mathfrak{M}_i$ generated by M_i and $\mathfrak{L}_i$ by L_i. Write as defined in (42)

$$W_{\kappa_i} = \begin{pmatrix} 1 & 0 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} M_i\lambda_1 & \lambda_2 \\ -\kappa_{i,1} & \kappa_{i,2} \end{pmatrix}$$

which transforms κ_i to ∞.

It is well known that the fractional linear transformation on the extended upper half space is composition of an even number of inversions (see e.g. [2, Sect. 2.3]). So the action of $\mathrm{GL}_2(\mathbb{C})$ on the subspace U preserves the orientation. By Proposition 3.2 we know that if $U(\mathbf{x}_1, \mathbf{x}_2) \perp \nu(\infty)$ then $U(W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)) \perp \nu(W_{\kappa_i}^{-1} \cdot \infty)$. We have proven that the bilinear form on a pair of vectors is preserved under the action of $\mathrm{GL}_2(\mathbb{C})$ in (3) and hence so is the Gram matrix β. Thus for $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_{\beta,\infty,+}$ we have $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2) \in \Omega_{\beta,\kappa_i,+}$. Then we obtain

$$I_{\kappa_i} = \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_{\mathcal{F}}$$

$$= \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}} \varphi_f^{\mathrm{new}}(|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)) \int_{C_{U(|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))}} \eta_{\mathcal{F}}$$

$$= \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}} \varphi_f^{\mathrm{new}}(|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)) \int_{C_{U(W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))}} \eta_{\mathcal{F}}$$

where the last equality is the consequence of $U(|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)) = U(W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))$.

Remark 5.14 We introduce the factor $|\det(W_{\kappa_i})|$ to make sure that for $(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}$, $|\det(W_{\kappa_i})|^{-1}\cdot W_{\kappa_i}\cdot(\mathbf{x}_1,\mathbf{x}_2)$ lies in $V(\mathbb{Q})^2$ (so in $\Gamma_\infty\backslash\Omega_{\beta,\infty,+}$) but not just in $V(\mathbb{R})^2$.

Next we will analyze $\varphi_f^{\mathrm{new}}(|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))$ for $(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}$. For simplicity we write $\chi=\chi_{\mathfrak{m}}$.

We begin the calculation in a slightly more general setting. Given $g=\begin{pmatrix} x & y\\ z & w\end{pmatrix}$ and $(\mathbf{x}_1,\mathbf{x}_2)=\left(\begin{pmatrix} a_1 & b_1\\ \bar{b}_1 & 0\end{pmatrix},\begin{pmatrix} a_2 & b_2\\ \bar{b}_2 & 0\end{pmatrix}\right)$, we compute

$$\begin{aligned}&\left(\begin{pmatrix} a_1' & b_1'\\ \bar{b}_1' & d_1'\end{pmatrix},\begin{pmatrix} a_2' & b_2'\\ \bar{b}_2' & d_2'\end{pmatrix}\right) := |\det g|^{-1}\cdot g\cdot\left(\begin{pmatrix} a_1 & b_1\\ \bar{b}_1 & 0\end{pmatrix},\begin{pmatrix} a_2 & b_2\\ \bar{b}_2 & 0\end{pmatrix}\right)\\ &=|\det(g)|^{-2}\left(\begin{pmatrix} a_1x\bar{x}+\bar{b}_1\bar{x}y+b_1x\bar{y} & a_1x\bar{z}+\bar{b}_1y\bar{z}+b_1x\bar{w}\\ a_1\bar{x}z+\bar{b}_1\bar{x}w+b_1\bar{y}z & a_1z\bar{z}+\bar{b}_1\bar{z}w+b_1z\bar{w}\end{pmatrix},\begin{pmatrix} a_2x\bar{x}+\bar{b}_2\bar{x}y+b_2x\bar{y} & a_2x\bar{z}+\bar{b}_2y\bar{z}+b_2x\bar{w}\\ a_2\bar{x}z+\bar{b}_2\bar{x}w+b_2\bar{y}z & a_2z\bar{z}+\bar{b}_2\bar{z}w+b_2z\bar{w}\end{pmatrix}\right)\end{aligned} \tag{43}$$

and then

$$\begin{aligned}(a_2'b_1'-a_1'b_2') =&|\det(g)|^{-4}[(a_2x\bar{x}+\bar{b}_2\bar{x}y+b_2x\bar{y})(a_1x\bar{z}+\bar{b}_1y\bar{z}+b_1x\bar{w})\\ &-(a_1x\bar{x}+\bar{b}_1\bar{x}y+b_1x\bar{y})(a_2x\bar{z}+\bar{b}_2y\bar{z}+b_2x\bar{w})]\\ =&|\det(g)|^{-4}[a_2b_1(x\bar{x}x\bar{w}-x\bar{y}x\bar{z})-a_1b_2(x\bar{x}x\bar{w}-x\bar{y}x\bar{z})\\ &+b_1\bar{b}_2(x\bar{x}y\bar{w}-x\bar{y}x\bar{z})+\bar{b}_1b_2(x\bar{y}y\bar{z}-\bar{x}yx\bar{w})]\\ =&\det(g)^{-2}\det(\bar{g})^{-1}[(a_2b_1-a_1b_2)x^2+(b_1\bar{b}_2-\bar{b}_1b_2)xy],\end{aligned} \tag{44}$$

$$\begin{aligned}(\bar{b}_2'd_1'-\bar{b}_1'd_2') =&|\det(g)|^{-4}[(a_2\bar{x}z+\bar{b}_2\bar{x}w+b_2\bar{y}z)(a_1z\bar{z}+\bar{b}_1\bar{z}w+b_1z\bar{w})\\ &-(a_1\bar{x}z+\bar{b}_1\bar{x}w+b_1\bar{y}z)(a_2z\bar{z}+\bar{b}_2\bar{z}w+b_2z\bar{w})]\\ =&|\det(g)|^{-4}[a_2b_1(\bar{x}zz\bar{w}-\bar{y}zz\bar{z})-a_1b_2(\bar{x}zz\bar{w}-\bar{y}zz\bar{z})]\\ &+b_1\bar{b}_2(\bar{x}wz\bar{w}-\bar{y}z\bar{z}w)-\bar{b}_1b_2(\bar{x}wz\bar{w}-\bar{y}z\bar{z}w)\\ =&\det(g)^{-2}\det(\bar{g})^{-1}[(a_2b_1-a_1b_2)z^2+(b_1\bar{b}_2-\bar{b}_1b_2)wz].\end{aligned} \tag{45}$$

Remark 5.15 With our choice of β, the pair $(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}$ satisfies the condition † as in Remark 3.6. It means that $b_i\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}\times\bar{\mathfrak{q}}\mathcal{O}_{\bar{\mathfrak{q}}}$ (i.e. $q|b_i$) for split $q|m$ with $(q)=\mathfrak{q}\bar{\mathfrak{q}}$, and $b_i\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ for inert $q|m$ with $(q)=\mathfrak{q}$. So $b_1\bar{b}_2-\bar{b}_1b_2$ appearing in (44) and (45) turns out to be divisible by q^2 for each prime $q|m$.

Recall the ordered decomposition $\mathfrak{f}\mathfrak{n}=\mathfrak{M}_i\mathfrak{L}_i$ (or $\mathfrak{f}\mathfrak{n}\mathfrak{q}_2=\mathfrak{M}_i\mathfrak{L}_i$ with $\mathfrak{q}_2$ above 2 when $d\equiv 2,3 \bmod 4$) and its corresponding representative $\kappa_i=\frac{\kappa_{i,2}}{\kappa_{i,1}}$ of equivalence class of cusps with $\kappa_{i,1}$ and $\kappa_{i,2}$ coprime.

Lemma 5.16 *For* $(\mathbf{x}_1, \mathbf{x}_2) \in \Gamma_\infty \backslash \Omega_{\beta,\infty,+}$ *and non-trivial* $\mathfrak{M}_i$ *dividing* $\mathfrak{fn}$ *when* $d \equiv 1 \bmod 4$ *(or* $\mathfrak{fn}\mathfrak{q}_2$ *with* $\mathfrak{q}_2$ *above 2 when* $d \equiv 2, 3 \bmod 4$*), we have that* φ_f^{new} *is vanishing on* $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$.

Proof Write

$$W_{\kappa_i}^{-1} = \begin{pmatrix} \kappa_{i,2} & -\frac{\lambda_2}{M_i} \\ \kappa_{i,1} & \lambda_1 \end{pmatrix} =: \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad \text{with } \det(W_{\kappa_i}^{-1}) = \frac{1}{M_i}$$

and for $(\mathbf{x}_1, \mathbf{x}_2) = \left(\begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{pmatrix} \right) \in \Gamma_\infty \backslash \Omega_{\beta,\infty,+}$, set

$$(\mathbf{x}_1^{'}, \mathbf{x}_2^{'}) = \left(\begin{pmatrix} a_1^{'} & b_1^{'} \\ \bar{b}_1^{'} & d_1^{'} \end{pmatrix}, \begin{pmatrix} a_2^{'} & b_2^{'} \\ \bar{b}_2^{'} & d_2^{'} \end{pmatrix} \right) := |\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot \left(\begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}, \begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix} \right).$$

By (43), we have for $j = 1, 2$

$$\begin{aligned}
a_j^{'} &= |M_i|^2 (a_j x\bar{x} + \bar{b}_j \bar{x} y + b_j x \bar{y}) = |M_i|^2 a_j \kappa_{i,2}\bar{\kappa}_{i,2} - \overline{M}_i \bar{b}_j \bar{\kappa}_{i,2}\lambda_2 - M_i b_j \kappa_{i,2}\bar{\lambda}_2, \\
b_j^{'} &= |M_i|^2 (a_j x\bar{z} + \bar{b}_j y\bar{z} + b_j x\bar{w}) = |M_i|^2 a_j \kappa_{i,2}\bar{\kappa}_{i,1} - \overline{M}_i \bar{b}_j \lambda_2 \bar{\kappa}_{i,1} + |M_i|^2 b_j \kappa_{i,2}\bar{\lambda}_1, \\
\bar{b}_j^{'} &= |M_i|^2 (a_j \bar{x} z + b_j \bar{y} z + \bar{b}_j \bar{x} w) = |M_i|^2 a_j \bar{\kappa}_{i,2}\kappa_{i,1} - M_i b_j \bar{\lambda}_2 \kappa_{i,1} + |M_i|^2 \bar{b}_j \bar{\kappa}_{i,2}\lambda_1, \\
d_j^{'} &= |M_i|^2 (a_j z\bar{z} + \bar{b}_j \bar{z} w + b_j z\bar{w}) = |M_i|^2 (a_j \kappa_{i,1}\bar{\kappa}_{i,1} + \bar{b}_j \bar{\kappa}_{i,1}\lambda_1 + b_j \kappa_{i,1}\bar{\lambda}_1).
\end{aligned}$$

(I) Let q a prime dividing $m|d_F|$ which is split, inert or ramified. We will only treat in details the case when q is split with $(q) = \mathfrak{q}\bar{\mathfrak{q}}$ and other cases can be treated similarly. We want to show that if

$$(\mathfrak{M}_i, \mathfrak{m}) = \mathfrak{q}, \ (\mathfrak{M}_i, (m)) = \bar{\mathfrak{q}} \text{ or } (\mathfrak{M}_i, \mathfrak{m}) = (q)$$

then $\omega(1, \gamma)\varphi_q^\chi$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$ for $[\gamma] \in \overline{\Gamma}_0(q)/\overline{\Gamma}(q)$.

(I.1) Let $(\mathfrak{M}_i, \mathfrak{m}) = \mathfrak{q}$ and then we have

$$\mathfrak{q}|\mathfrak{M}_i, \ \bar{\mathfrak{q}}|\overline{\mathfrak{M}}_i, \ \bar{\mathfrak{q}}|(\kappa_{i,1}), \ \mathfrak{q}|(\bar{\kappa}_{i,1}), \ \bar{\mathfrak{q}} \nmid (\kappa_{i,2}), \ \mathfrak{q} \nmid (\bar{\kappa}_{i,2})$$

By Remark 5.15, there is no need to discuss the integrality of b_j but we care for that of a_j.

Suppose that $a_j \in \mathbb{Z}_q$. It is easy to observe that $a_j^{'}, b_j^{'}, \bar{b}_j^{'}, d_j^{'} \in \mathfrak{q}\mathcal{O}_\mathfrak{q}$. Set

$$\gamma = (\gamma_1, \gamma_2) = \left(\begin{pmatrix} u_1 & v_1 \\ 0 & u_1^{-1} \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & u_2^{-1} \end{pmatrix} \right)$$

with $[u_1], [u_2] \in (\mathcal{O}/(q))^\times$ and $[v_1], [v_2] \in \mathcal{O}/(q)$. We write

$$(\mathbf{x}_1^{''}, \mathbf{x}_2^{''}) = \left(\begin{pmatrix} a_1^{''} & b_1^{''} \\ c_1^{''} & d_1^{''} \end{pmatrix}, \begin{pmatrix} a_2^{''} & b_2^{''} \\ c_2^{''} & d_2^{''} \end{pmatrix} \right) := (\gamma_1^{-1} \mathbf{x}_1^{'t} (\gamma_2^{-1})^*, \gamma_1^{-1} \mathbf{x}_2^{'t} (\gamma_2^{-1})^*)$$

and compute

$$\begin{aligned}
\begin{pmatrix} a_j^{''} & b_j^{''} \\ c_j^{''} & d_j^{''} \end{pmatrix} &= \begin{pmatrix} u_1^{-1} & -v_1 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} a_j^{'} & b_j^{'} \\ \bar{b}_j^{'} & d_j^{'} \end{pmatrix} \begin{pmatrix} u_2 & 0 \\ v_2 & u_2^{-1} \end{pmatrix} \\
&= \begin{pmatrix} u_1^{-1} u_2 a_j^{'} - v_1 u_2 \bar{b}_j^{'} + u_1^{-1} v_2 b_j^{'} - v_1 v_2 d_j^{'} & u_1^{-1} u_2^{-1} b_j^{'} - v_1 u_2^{-1} d_j^{'} \\ u_1 u_2 \bar{b}_j^{'} + u_1 v_2 d_j^{'} & u_1 u_2^{-1} d_j^{'} \end{pmatrix}.
\end{aligned}$$

Then, as $a_j^{'}, b_j^{'}, \bar{b}_j^{'}, d_j^{'} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, we can observe that $a_j^{''}, b_j^{''}, c_j^{''}, d_j^{''} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ as well which implies that

$$\frac{a_2^{''}b_1^{''} - a_1^{''}b_2^{''}}{m} + \frac{c_2^{''}d_1^{'} - c_1^{''}d_2^{'}}{m} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}.$$

It immediately follows that $\omega(1,\gamma)\varphi_q^{\chi,\mathrm{new}}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$. Suppose that $a_j \notin \mathbb{Z}_q$ and set $l_q = \min\{\mathrm{ord}_q(a_j)\} \leq -1$. Assume that $\omega(1,\gamma)\varphi_q^{\chi,\mathrm{new}}$ is non-vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$ which requires that $a_j^{''}, d_j^{''} \in \mathbb{Z}_q$ and $b_j^{''}, c_j^{''} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$.

- We first consider $v_2 \in (\mathcal{O}/(q))^{\times}$. Observing

  $$c_j^{''} = u_1 u_2 \bar{b}_j^{'} + u_1 v_2 d_j^{'} \quad \text{and} \quad d_j^{''} = u_1 u_2^{-1} d_j^{'},$$

  we know that for $c_j^{''} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ and $d_j^{''} \in \mathbb{Z}_q$ we need $d_j^{'} \in \mathbb{Z}_q$ and at least $\bar{b}_j^{'} \in \mathcal{O}_{\mathfrak{q}}$. As

  $$\bar{b}_j^{'} = |M_i|^2 a_j \bar{\kappa}_{i,2}\kappa_{i,1} - M_i b_j \bar{\lambda}_2 \kappa_{i,1} + |M_i|^2 \bar{b}_j \bar{\kappa}_{i,2}\lambda_1$$

  with $\bar{\kappa}_{i,2}, \kappa_{i,1} \in \mathcal{O}_{\mathfrak{q}}^{\times}$, we then need $M_i \in \mathfrak{q}^{-l_q}$ which makes

  $$d_j^{'} = |M_i|^2 (a_j \kappa_{i,1}\bar{\kappa}_{i,1} + \bar{b}_j \bar{\kappa}_{i,1}\lambda_1 + b_j \kappa_{i,1}\bar{\lambda}_1) \quad (q|\kappa_{i,1}\bar{\kappa}_{i,1})$$

  lie in $q\mathbb{Z}_q$. Looking back to $c_j^{''} = u_1 u_2 \bar{b}_j^{'} + u_1 v_2 d_j^{'} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, that $d_j^{'} \in q\mathbb{Z}_q$ makes $\bar{b}_j^{'} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$. It follows that we need $M_i \in \mathfrak{q}^{-l_q+1}\mathcal{O}_{\mathfrak{q}}$, a contradiction to that $\mathfrak{M}_i$ is square-free.
- Let $v_2 = 0$. Then we have

  $$\begin{pmatrix} a_j^{''} & b_j^{''} \\ c_j^{''} & d_j^{''} \end{pmatrix} = \begin{pmatrix} * & * \\ u_1 u_2 \bar{b}_j^{'} & u_1 u_2^{-1} d_j^{'} \end{pmatrix}.$$

  For $c_j^{''} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, we need

  $$\bar{b}_j^{'} = |M_i|^2 a_j \bar{\kappa}_{i,2}\kappa_{i,1} - M_i b_j \bar{\lambda}_2 \kappa_{i,1} + |M_i|^2 \bar{b}_j \bar{\kappa}_{i,2}\lambda_1 \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}} \quad (\bar{\kappa}_{i,2}, \kappa_{i,1} \in \mathcal{O}_{\mathfrak{q}}^{\times})$$

  which requires $M_i \in \mathfrak{q}^{-l_q+1}\mathcal{O}_{\mathfrak{q}}$ contradicting to that $\mathfrak{M}_i$ is square-free.

Therefore, when $(\mathfrak{M}_i, \mathfrak{m}) = \mathfrak{q}$, we can deduce that $\omega(1,\gamma)\varphi_q^{\chi}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$.

(I.2) When $(\mathfrak{M}_i, \mathfrak{m}) = \bar{\mathfrak{q}}$, we can prove it in the same way.

(I.3) Let $(\mathfrak{M}_i, \mathfrak{m}) = q$ and then we have $q|M_i$ and $\kappa_{i,1}, \bar{\kappa}_{i,1} \in \mathcal{O}_{\mathfrak{q}}^{\times}$. It is clear for $a_j \in \mathbb{Z}_q$. Suppose that $a_j \notin \mathbb{Z}_q$ and set $l_q = \min\{\mathrm{ord}_q(a_j)\} \leq -1$. First assume v_1 is a unit. Let $l_q = -1$. Then we have that all $a_j^{'}, d_j^{'}, b_j^{'}, \bar{b}_j^{'}$ are divisible by q and so are $a_j^{''}, d_j^{''}, b_j^{''}, c_j^{''}$. Let $l_q = -2$. It is clear that $d_j^{'} \in \mathbb{Z}_q$. Then, for $b_j^{''}, \bar{b}_j^{''} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, we need $b_j^{'}, \bar{b}_j^{'} \in \mathcal{O}_{\mathfrak{q}}$. Expand

$$\begin{aligned} a_2^{''}b_1^{''} - a_1^{''}b_2^{''} = & u_1^{-2}(a_2^{'}b_1^{'} - a_1^{'}b_2^{'}) + u_1^{-1}v_1(a_2^{'}d_1^{'} - a_1^{'}d_2^{'}) \\ & + u_1^{-1}v_1(b_1^{'}\bar{b}_2^{'} - \bar{b}_1^{'}b_2^{'}) + v_1^2(\bar{b}_2^{'}d_1^{'} - \bar{b}_1^{'}d_2^{'}) \end{aligned}$$

and

$$c_2^{''}d_1^{''} - c_1^{''}d_2^{''} = u_1^2(\bar{b}_2^{'}d_1^{'} - \bar{b}_1^{'}d_2^{'}).$$

It is not hard to observe that $a_2^{'}d_1^{'} - a_1^{'}d_2^{'} \in q^2\mathbb{Z}_q$ and $b_1^{'}\bar{b}_2^{'} - \bar{b}_1^{'}b_2^{'} \in \mathfrak{q}^2\mathcal{O}_{\mathfrak{q}}$. By (44) and (45), we have

$$a_2^{'}d_1^{'} - a_1^{'}d_2^{'} = M_i|M_i|^2((a_2b_1 - a_1b_2)\kappa_{i,2}^2 + (b_1^{'}\bar{b}_2^{'} - \bar{b}_1^{'}b_2^{'})\kappa_{i,2}^2(-\lambda/M_i))$$

and

$$\bar{b}_2^{'}d_1^{'} - \bar{b}_1^{'}d_2^{'} = M_i|M_i|^2((a_2b_1 - a_1b_2)\kappa_{i,1}^2 + (b_1^{'}\bar{b}_2^{'} - \bar{b}_1^{'}b_2^{'})\kappa_{i,1}\lambda_1),$$

both of which lie in $\mathfrak{q}^2\mathcal{O}_{\mathfrak{q}}$. It follows again that

$$\frac{a_2^{''}b_1^{''} - a_1^{''}b_2^{''}}{m} + \frac{c_2^{''}d_1^{'} - c_1^{''}d_2^{'}}{m} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}.$$

For $l_q < -2$ there is no chance for $d_j^{''} \in \mathbb{Z}_q$ as $M_{i,1}$ is square-free. It is clear for $v_1 = 0$. Therefore $\omega(1, \gamma)\varphi_q^{\chi}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$ in this case.

(II) Next we consider $\varphi_q^{\mathfrak{n}}$ for split $q|N(\mathfrak{n})$ and omit details for inert q. As discussed in Subsect. 4.3.4, we can have $(\mathfrak{n}, (q)) = \mathfrak{q}$, $(\mathfrak{n}, (q)) = \bar{\mathfrak{q}}$ or $(\mathfrak{n}, (q)) = (q)$. Again we want to show that if $(\mathfrak{n}, (q))|\mathfrak{M}_i$ then $\omega(1, \gamma)\varphi_q^{\mathfrak{n}}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$ for $[\gamma] \in \overline{\Gamma}_0(\mathfrak{q})/\overline{\Gamma}(\mathfrak{q})$, $\overline{\Gamma}_0(\bar{\mathfrak{q}})/\overline{\Gamma}(\bar{\mathfrak{q}})$ or $\overline{\Gamma}_0(q)/\overline{\Gamma}(q)$ respectively.
Let $a_j \in \mathbb{Z}_q$. Assume that $(\mathfrak{n}, (q)) = \mathfrak{q}$ and $\mathfrak{q}|\mathfrak{M}_i$. Then it is clear that $\bar{b}_j^{'} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ and $d_j^{'} \in q\mathbb{Z}_q$. Expanding

$$\begin{aligned} b_1^{''}c_2^{''} + b_2^{''}c_1^{''} =&(u_1^{-1}u_2^{-1}b_1^{'} - v_1u_2^{-1}d_1^{'})(u_1u_2\bar{b}_2^{'} + u_1v_2d_2^{'}) \\ &+ (u_1^{-1}u_2^{-1}b_2^{'} - v_1u_2^{-1}d_2^{'})(u_1u_2\bar{b}_1^{'} + u_1v_2d_1^{'}), \end{aligned}$$

we see it is in $\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$. So $\omega(1, \gamma)\varphi_q^{\mathfrak{n}}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$. Also it is clear for $\bar{\mathfrak{q}}|\mathfrak{M}_i$ or $(q)|\mathfrak{M}_i$.
Let $a_j \notin \mathbb{Z}_q$ and set $l_q = \min\{\mathrm{ord}_q(a_j)\} \le -1$. Assume that $(\mathfrak{n}, (q)) = \mathfrak{q}$ and $\mathfrak{q}|\mathfrak{M}_i$. Then we have $q \nmid \kappa_{i,1}$. Look at

$$d_j^{'} = |M_i|^2(a_j\kappa_{i,1}\bar{\kappa}_{i,1} + \bar{b}_j\bar{\kappa}_{i,1}\lambda_1 + b_j\kappa_{i,1}\bar{\lambda}_1).$$

Then there is no chance for $d_j^{''} = d_j^{'}$ to be in $q\mathbb{Z}_q$ as M_i is square-free. So $\omega(1, \gamma)\varphi_q^{\mathfrak{n}}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$. This also occurs in the case that $(\mathfrak{n}, (q)) = \bar{\mathfrak{q}}$ and $\bar{\mathfrak{q}}|\mathfrak{M}_i$. Now assume that $(\mathfrak{n}, (q)) = (q)$ and $(q)|\mathfrak{M}_i$. If $l_q = -1$, then we have $b_j^{'}, \bar{b}_j^{'} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ which implies that $b_1^{''}c_2^{''} + b_2^{''}c_1^{''} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$. If $l_q \le -2$, we can observe that there is no room for $d_j^{''} = u_1u_2^{-1}d_j^{'} \in q\mathbb{Z}$ for square-free M_i.
Let $q^{'}$ be another prime dividing $N(\mathfrak{n})$ with $q^{'} = \mathfrak{q}^{'}\bar{\mathfrak{q}}^{'}$. Similarly, if $\mathfrak{q}^{'}|\mathfrak{M}_i$, $\bar{\mathfrak{q}}^{'}|\mathfrak{M}_i$ or $(q)|\mathfrak{M}_i$, we can show that $\omega(1, \gamma)\varphi_{q^{'}}^{\mathfrak{n}}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$.

(III) To finish our proof we consider φ_2 if 2 is ramified with $(2) = \mathfrak{q}_2^2$. Set $\gamma = \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}$ with $[u] \in (\mathcal{O}/\mathfrak{q}_2)^{\times}$ and $[v] \in \mathcal{O}/\mathfrak{q}_2$. We write

$$(\mathbf{x}_1^{''}, \mathbf{x}_2^{''}) = \left(\begin{pmatrix} \bar{b}_1^{''} & a_1^{''}\sqrt{d} \\ c_1^{''}\sqrt{d} & \bar{b}_1^{''} \end{pmatrix}, \begin{pmatrix} \bar{b}_2^{''} & a_2^{''}\sqrt{d} \\ c_2^{''}\sqrt{d} & \bar{b}_2^{''} \end{pmatrix}\right) := (\gamma\mathbf{x}_1^{'}\bar{\gamma}^*, \gamma\mathbf{x}_2^{'}\bar{\gamma}^*).$$

Suppose that $\mathfrak{q}_2|\mathfrak{M}_i$. If $l_q = \min\{\mathrm{ord}_q(a_j)\} \ge -1$, then we have $b_j^{'} \in \frac{1}{\mathfrak{q}_2}\mathcal{O}_{\mathfrak{q}_2}$ and then $b_j^{''} \in \frac{1}{\mathfrak{q}_2}\mathcal{O}_{\mathfrak{q}_2}$ as well. So $b_1^{''}\bar{b}_2^{''} \in \frac{1}{2}\mathcal{O}_{\mathfrak{q}_2}$ and then $b_1^{''}\bar{b}_2^{''} + \bar{b}_1^{''}b_2^{''} = 2\mathrm{Re}(b_1^{''}\bar{b}_2^{''}) \in \mathbb{Z}_2$ which makes φ_2 vanish on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$. If $l_q = \min\{\mathrm{ord}_q(a_j)\} \leq -2$, then there is no chance for $d_j' \in \mathbb{Z}_2$ as $\mathfrak{M}_i$ is square-free, and so for c_j''. So again we have $\omega(1,\gamma)\varphi_2$ vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$.

□

If follows that I_{κ_i} is vanishing for $\kappa_i \neq \infty$. So we have proven our main theorem:

Theorem 5.17 *Suppose that $F = \mathbb{Q}(\sqrt{d})$ is an imaginary quadratic field of class number 1 with the discriminant d_F and denote its ring of integers by $\mathcal{O}$. Let m be a square-free product of inert or split primes, and put $\mathfrak{m} = m\mathcal{O}$ and $\mathfrak{f} = \sqrt{d}\mathfrak{m}$. Choose a quadratic Hecke character $\chi_{\mathfrak{m}}$ of conductor $\mathfrak{f}$. Given a square-free ideal $\mathfrak{n}$ coprime to $(m|d_F|)$, let $\mathcal{F}$ be a weight 2 Bianchi cusp form of level $\Gamma_0(\mathfrak{n})$. Choose the Schwartz function as in Remark 5.1 and β as in Remark 5.2. Then the Fourier coefficient of the theta lift at β as in (25) is*

$$I_\infty = \frac{\mu_\beta \lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{2A(1, 1, \chi_{\mathfrak{m}}, 1)}.$$

So, if $L(\mathcal{F}, \chi_{\mathfrak{m}}, 1) \neq 0$, we can deduce the non-vanishing of our theta lifting as above.

6 Non-vanishing of theta lifting

Recall from [6] that a new form in $S_2(\Gamma_0(\mathfrak{n}))$ is an eigenform for all the Hecke operators $T_{\mathfrak{p}}$ for $\mathfrak{p}$ not dividing $\mathfrak{n}$, which is not induced from in $S_2(\Gamma_0(\mathfrak{m}))$ for any level $\mathfrak{m}$ properly dividing $\mathfrak{n}$. There is an involution J induced by the action on $\mathbb{H}_3$ of the matrix $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where ϵ generates the unit group of $\mathcal{O}$. The effect of J on Fourier coefficients is $c(\alpha) \to c(\epsilon\alpha)$; the involution commutes with the Hecke operators, and splits $S_2(\Gamma_0(\mathfrak{n}))$ into two eigenspaces,

$$S_2(\Gamma_0(\mathfrak{n})) = S_2^+(\Gamma_0(\mathfrak{n})) \oplus S_2^-(\Gamma_0(\mathfrak{n})).$$

Newforms in $S_2^+(\Gamma_0(\mathfrak{n}))$ were called plusforms, and their Fourier coefficients satisfy the additional condition $c(\epsilon\alpha) = c(\alpha)$ for all $\alpha \in \mathcal{O}$. Denote by $S_2^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ the space of newforms in $S_2(\Gamma_0(\mathfrak{n}))$ and by $S_2^{\mathrm{new},+}(\Gamma_0(\mathfrak{n}))$ the space of plusforms in $S_2^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$. More discussions on newforms and plusforms of weight 2 Bianchi modular forms can be found in [6].

We can choose

$$\beta = \left(-\frac{dz}{r}, \frac{dr}{r}, \frac{d\bar{z}}{r}\right) \quad \text{for } (z, r) \in \mathbb{H}_3$$

as a basis for the left-invariant differential forms on $\mathbb{H}_3$. Let $\mathcal{F} \in S_2^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ and recall its Mellin transform from [6, Sect. 2.5]

$$\Lambda(\mathcal{F}, s) = \frac{(4\pi)^2}{|d_F|} \cdot \int_0^\infty t^{2s-2} \mathcal{F} \cdot \beta \tag{46}$$

for $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$.

Proposition 6.1 *[6, Proposition 2.1] Let $\mathcal{F} \in S_2^{\mathrm{new},+}(\Gamma_0(\mathfrak{n}))$. Then*

(1) For $Re(s) > 3/2$ we have

$$\Lambda(\mathcal{F}, s) = (2\pi)^{2-2s} |d_F|^{s-1} \Gamma(s)^2 L(\mathcal{F}, s). \tag{47}$$

(2) Assume that $\mathcal{F}$ is an eigenform for the Fricke involution $\omega_{\mathfrak{n}} = \begin{pmatrix} 0 & -1 \\ \mathfrak{n} & 0 \end{pmatrix}$, *i.e.,* $\mathcal{F}|\omega_{\mathfrak{n}} = \varepsilon_{\mathfrak{n}}\mathcal{F}$ *with* $\varepsilon_{\mathfrak{n}} = \pm 1$. *Then* $\Lambda(\mathcal{F}, s)$ *satisfies the functional equation*

$$\Lambda(\mathcal{F}, s) = -\varepsilon_{\mathfrak{n}} N(\mathfrak{n})^{1-s}\Lambda(\mathcal{F}, 2-s). \tag{48}$$

Put $\alpha(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\omega_N = \omega(N) = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Let ψ be a character of $(\mathcal{O}/\mathfrak{m}_\psi)^\times$ with conductor $\mathfrak{m}_\psi$. Similar to the twisted Hilbert modular forms [22, Sect. 5], the twist of $\mathcal{F}$ by ψ can be defined as, for $m \in \mathfrak{m}_\psi$,

$$\mathcal{F}_\psi = G(\psi^{-1}, 1/m)^{-1} \sum_{u \in (\mathcal{O}/\mathfrak{m}_\psi)^\times} \psi^{-1}(u)\mathcal{F}|_2\alpha(u/m)$$

where $G(\psi^{-1}, 1/m)^{-1}$ is the Gauss sum of ψ^{-1}.

Lemma 6.2 *Let* $\mathcal{F} \in S_2(\Gamma_0(\mathfrak{n}))$, ψ *a character of* $(\mathcal{O}/\mathfrak{m}_\psi)^\times$, *and* $\mathfrak{M}$ *the least common multiple of* $\mathfrak{n}$, $\mathfrak{m}_\psi^2$, *and* $\mathfrak{m}_\psi$. *Then* $\mathcal{F}_\psi \in S_2(\Gamma_0(\mathfrak{M}), \psi^2)$.

Proof We will apply Miyake's treatment in [16, Lemma 4.3.10] to our case without any new techniques.

Let $\gamma = \begin{pmatrix} a & b \\ cM & d \end{pmatrix} \in \Gamma_0(\mathfrak{M})$ where $M \in \mathfrak{M}$ and put

$$\gamma' = \alpha(u/m)\gamma\alpha(d^2u/m)^{-1},$$

then $\gamma' \in \Gamma_0(\mathfrak{M}) \subset \Gamma_0(\mathfrak{n})$. Writing $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, we have

$$\mathcal{F}|_2\alpha(u/m)\gamma = \mathcal{F}|_2\gamma'\alpha(d^2u/m) = \mathcal{F}|_2\alpha(d^2u/m).$$

Therefore

$$\begin{aligned}
\mathcal{F}_\psi|_2\gamma &= G(\psi^{-1}, 1/m)^{-1} \sum_{u \in (\mathcal{O}/\mathfrak{m}_\psi)^*} \psi^{-1}(u)\mathcal{F}|_2\alpha(u/m)\gamma \\
&= G(\psi^{-1}, 1/m)^{-1} \sum_{u \in (\mathcal{O}/\mathfrak{m}_\psi)^*} \psi^{-1}(u)\mathcal{F}|_2\alpha(d^2u/m) \\
&= \psi(d^2)G(\psi^{-1}, 1/m)^{-1} \sum_{u \in (\mathcal{O}/\mathfrak{m}_\psi)^*} \psi^{-1}(d^2u)\mathcal{F}|_2\alpha(d^2u/m) \\
&= \psi^2(d)\mathcal{F}_\psi
\end{aligned}$$

which implies that $f_\psi \in S_2(\Gamma_0(\mathfrak{M}), \psi^2)$. □

Lemma 6.3 *Let* $\mathcal{F} \in S_2(\Gamma_0(\mathfrak{n}))$ *and* ψ *a character of* $(\mathcal{O}/\mathfrak{m}_\psi)^\times$. *If* $(\mathfrak{n}, \mathfrak{m}_\psi) = 1$, *then*

$$\mathcal{F}_\psi|_2\omega(\mathfrak{n}\mathfrak{m}_\psi^2) = C_\psi\mathcal{G}_{\psi^{-1}}$$

where $\mathcal{G} = \mathcal{F}|_2\omega_{\mathfrak{n}}$ *and*

$$C_\psi = C_{\psi,\mathfrak{n}} = \psi(\mathfrak{n})G(\psi)/G(\psi^{-1}).$$

Proof We will apply Miyake's treatment in [16, Lemma 4.3.11] to our case without any new techniques.

For $u \in \mathcal{O}$ prime to $m \in \mathfrak{m}_\psi$, take $n, v \in \mathcal{O}$ and $N \in \mathfrak{n}$ so that $nm - Nuv = 1$. Then

$$\alpha(u/m)\omega(Nm^2) = m \cdot \omega(N) \begin{pmatrix} m & -v \\ -uN & n \end{pmatrix} \alpha(v/m). \tag{49}$$

Since $\mathcal{G} = \mathcal{F}|_2\omega_\mathfrak{n}$ belongs to $S_2(\Gamma_0(\mathfrak{n}))$, (49) implies

$$\mathcal{F}|_2\alpha(u/m)\omega(Nm^2) = \mathcal{G}|_2\alpha(v/m),$$

so that

$$\begin{aligned} G(\psi^{-1})\mathcal{F}_\psi|_2\omega(Nm^2) &= \sum_{u\in(\mathcal{O}/\mathfrak{m}_\psi)^\times} \psi^{-1}(u)\mathcal{F}|_2\alpha(u/m)\omega(Nm^2) \\ &= \sum_{v\in(\mathcal{O}/\mathfrak{m}_\psi)^\times} \psi(-Nv)\mathcal{G}|_2\alpha(v/m) \\ &= \psi(-N) \sum_{v\in(\mathcal{O}/\mathfrak{m}_\psi)^\times} \psi(v)\mathcal{G}|_2\alpha(v/m). \end{aligned}$$

Then the assertion follows immediately. □

Combining Lemma 6.3 and Proposition 6.1, for the central value at $s = 1$ we obtain:

Proposition 6.4 *For $\mathcal{F} \in S_2^{\text{new},+}(\Gamma_0(\mathfrak{n}))$ and ψ a quadratic Hecke character, we have*

$$L(\mathcal{F}_\psi, 1) = -\varepsilon_\mathfrak{n}\psi(\mathfrak{n})L(\mathcal{F}_\psi, 1).$$

Let $\mathfrak{n}$, $\chi_\mathfrak{m}$ and $\mathfrak{m}$ be as in Theorem 5.17. For $\mathcal{F} \in S_2^{\text{new},+}(\Gamma_0(\mathfrak{n}))$, it follows that for the non-vanishing of $L(\mathcal{F}, \chi_\mathfrak{m}, 1) = L(\mathcal{F}_{\chi_\mathfrak{m}}, 1)$, we need at least $\varepsilon_\mathfrak{n}\chi_\mathfrak{m}(\mathfrak{n}) = -1$.

Lemma 6.5 *Given a Bianchi modular form $\mathcal{F} \in S_2^{\text{new},+}(\Gamma_0(\mathfrak{n}))$, there always exists a quadratic Hecke character $\chi_\mathfrak{m}$ of conductor $\mathfrak{m}$ such that $\varepsilon_\mathfrak{n}\chi_\mathfrak{m}(\mathfrak{n}) = -1$.*

Proof Assume that

$$\varepsilon_\mathfrak{n}\chi_\mathfrak{m}(\mathfrak{n}) = \prod_{\text{prime } \mathfrak{q}_i|\mathfrak{n}} \varepsilon_{\mathfrak{q}_i}\chi_\mathfrak{m}(\mathfrak{q}_i) = -1.$$

We denote, for each prime $\mathfrak{q}_i$ dividing $\mathfrak{n}$,

$$\lambda_{\mathfrak{q}_i} := \chi_\mathfrak{m}(\mathfrak{q}_i)\varepsilon_{\mathfrak{q}_i} \in \{\pm 1\}. \tag{50}$$

Recall the Chinese Remainder Theorem in the following. Let $N = \prod_i n_i$ with the n_i being pairwise coprime. Given any integer a_i there exists an integer x such that $x \equiv a_i \pmod{n_i}$ for every i. To solve the system of congruences consider $N_i = N/n_i$ and then there exists integers M_i such that $N_iM_i \equiv 1 \pmod{n_i}$. A solution of the system of congruences is $x = \sum_i a_iN_iM_i$. The way for computing the solution can also be applied into principal ideal domains.

Recall the quadratic residue symbol from [18, Chapter V]. The quadratic residue symbol for $\mathcal{O}$ is defined by, for a prime ideal $\mathfrak{p} \subset \mathcal{O}$,

$$\left(\frac{\alpha}{\mathfrak{p}}\right) = \alpha^{\frac{N\mathfrak{p}-1}{2}} \bmod \mathfrak{p}.$$

It has properties completely analogous to those of classical Legendre symbol

$$\left(\frac{\alpha}{\mathfrak{p}}\right) = \begin{cases} 0, \ \alpha \in \mathfrak{p}, \\ 1, \ \alpha \notin \mathfrak{p} \text{ and } \exists \eta \in \mathcal{O} : \alpha \equiv \eta^2 \bmod \mathfrak{p}, \\ -1, \ \alpha \notin \mathfrak{p} \text{ and there is no such } \eta. \end{cases}$$

The quadratic residue symbol can be extended to take non-prime ideals or non-zero elements as its denominator, in the same way that the Jacobi symbol extends the Legendre symbol. For $0 \neq \beta \in \mathcal{O}$ then we define $\left(\frac{\alpha}{\beta}\right) := \left(\frac{\alpha}{(\beta)}\right)$ where (β) is the principal ideal generated by β. Analogous to the Jacobi symbol, this symbol is multiplicative in the top and bottom parameters.

We are interested in the quadratic reciprocity law in the case of the imaginary quadratic field $F = \mathbb{Q}(\sqrt{d})$ with class number one (see [12, Chapter VIII]. For any $\alpha \in \mathcal{O}$ with odd norm we define elements $t_\alpha, t_\alpha^{'} \in \mathbb{Z}/2\mathbb{Z}$ by

$$\alpha \equiv \sqrt{d}^{t_\alpha}(1 + 2\sqrt{d})^{t'_\alpha}\xi^2 \bmod 4 \quad \text{for } \xi \in \mathcal{O}.$$

Then the quadratic reciprocity law for coprime elements of odd norm is given by

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = (-1)^T$$

where

$$T \equiv \begin{cases} t_\alpha t_\beta^{'} + t_\alpha^{'} t_\beta + t_\alpha t_\beta \bmod 2, & \text{if } d \equiv 1, 2 \bmod 4 \\ t_\alpha t_\beta^{'} + t_\alpha^{'} t_\beta \bmod 2, & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

In particular, if $\alpha \equiv 1 \bmod 4$, we can observe that $t_\alpha = t_\alpha^{'} = 0$ which implies that $T \equiv 0 \bmod 2$. It follows that

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = 1 \quad \text{for } \alpha \equiv 1 \bmod 4. \tag{51}$$

We want to find a quadratic character defined by the quadratic residue symbol, $\chi_{\mathfrak{m}} = \left(\frac{\cdot}{m\sqrt{d}}\right)$, such that $\chi_{\mathfrak{m}}(\mathfrak{q}_i) = \varepsilon_{\mathfrak{q}_i}\lambda_{\mathfrak{q}_i}$ for $\lambda_{\mathfrak{q}_i}$ given in (50). By our assumption m is the product of inert or split primes. We can impose that $m \equiv 1 \bmod 4$ to get $\left(\frac{\cdot}{m}\right) = \left(\frac{m}{\cdot}\right)$ by the above quadratic reciprocity law (51). To achieve $\chi_{\mathfrak{m}}(\mathfrak{q}_i) = \left(\frac{m}{\mathfrak{q}_i}\right)\left(\frac{\mathfrak{q}_i}{\sqrt{d}}\right) = \varepsilon_{\mathfrak{q}_i}\lambda_{\mathfrak{q}_i}$, we need $\left(\frac{m}{\mathfrak{q}_i}\right) = \left(\frac{\mathfrak{q}_i}{\sqrt{d}}\right)\varepsilon_{\mathfrak{q}_i}\lambda_{\mathfrak{q}_i}$ which can be done via imposing congruence conditions $(*)$ on m modulo $\mathfrak{q}_i$. Therefore, by the Chinese remainder theorem, there exists a m satisfying

$$\begin{cases} m \equiv 1 \bmod 4 \\ \text{congruence conditions } (*) \text{ on } m \bmod \mathfrak{q}_i \text{ for each prime } \mathfrak{q}_i | \mathfrak{n}. \end{cases} \tag{52}$$

Now we have proven this lemma. □

Write $S := \{\text{place } v : v \mid 2|d|\mathfrak{n}\}$. Let ξ be a quadratic idelic Hecke character of conductor $M_\xi\mathcal{O}$ such that $M_\xi \equiv 1 \bmod 4$, $M_\xi \equiv m \bmod \mathfrak{q}_i$ for each $\mathfrak{q}_i|\mathfrak{n}$ and at v dividing $\sqrt{d}\mathcal{O}$ the local component ξ_v is ramified with square-free conductor. Note that its conductor is coprime to $2\mathfrak{n}$ and divisible by $\sqrt{d}\mathcal{O}$, and so is its induced character χ_ξ of $(\mathcal{O}/M_\xi\mathcal{O})^\times$. Also we can observe that M_ξ satisfies the conditions in (52). So, by the preceding lemma there exists a χ_ξ attached to ξ such that $\varepsilon_\mathfrak{n}\chi_\xi(\mathfrak{n}) = -1$. Let $\Psi(S; \xi)$ denote the set of quadratic characters χ_ξ such that $\widetilde{\chi}_{\xi,v} = \xi_v$ for all $v \in S$. Recall from [8, Theorem B(1)]

Proposition 6.6 *Suppose π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ which is self-contragredient. Suppose that for some quadratic character $\chi \in \Psi(S;\xi)$ one has root number $\epsilon(\pi\otimes\chi)=1$. Then there exist infinitely many quadratic characters $\chi' \in \Psi(S;\xi)$ such that $L(\pi\otimes\chi',1)\neq 0$.*

In Sect. 1.6 we have discussed the automorphic representation π on the space of weight 2 Bianchi modular forms. Also, we have shown that there exists a $\chi_\xi \in \Psi(S;\xi)$ such that $\varepsilon_{\mathfrak{n}}\chi_\xi(\mathfrak{n}) = -1$, i.e., $\epsilon(\pi\otimes\chi_\xi)=1$. So we can apply the above proposition to deduce that, for $\mathcal{F}\in S_2^{\mathrm{new},+}$, there are infinitely many quadratic characters $\chi\in\Psi(S;\xi)$ such that $L(\mathcal{F},\chi,1)$ is non-vanishing.

We will explain that these infinitely many quadratic characters always include a quadratic character with square-free conductor. This is necessary since the quadratic character $\chi_{\mathfrak{m}}$ as in Theorem 5.17 has the square-free conductor $\mathfrak{m}$. Suppose that $\Psi(S;\xi)\ni\chi_{\mathfrak{M}}:(\mathcal{O}/\mathfrak{M})^\times\to\mathbb{C}^\times$ is a quadratic Hecke character. Set $\mathfrak{M}=\prod_{\text{prime }\mathfrak{p}_i|\mathfrak{M}}\mathfrak{p}_i^{r_i}$ with $r_i\geq 1$. By the Chinese Remainder Theorem, we have $(\mathcal{O}/\mathfrak{M})^\times\simeq\prod_{\mathfrak{p}_i|\mathfrak{M}}(\mathcal{O}/\mathfrak{p}_i^{r_i})^\times$ and then can write $\chi_{\mathfrak{M}}=\prod\chi_{\mathfrak{M},\mathfrak{p}_i}$ with $\chi_{\mathfrak{M},\mathfrak{p}_i}$ defined on $(\mathcal{O}/\mathfrak{p}_i^{r_i})^\times$. It is known that $(\mathcal{O}/\mathfrak{p}^r)^\times$ has cyclic order of either $p^r(p-1)$ for $\mathfrak{p}$ above split prime p or $p^{2(r-1)}(p^2-1)$ for $\mathfrak{p}$ above inert prime p. So $\chi_{\mathfrak{M},\mathfrak{p}_i}$ is induced from a character defined on $(\mathcal{O}/\mathfrak{p})^\times$ which implies that $\chi_{\mathfrak{M}}$ is induced from a primitive character $\chi_{\mathfrak{m}_0}$ of square-free conductor $\mathfrak{m}_0$.

We will show that the non-vanishing of $L(\mathcal{F},\chi_{\mathfrak{M}},1)$ is equivalent to that of $L(\mathcal{F},\chi_{\mathfrak{m}_0},1)$. Write $\mathfrak{M}=\mathfrak{m}_0\mathfrak{n}_0^2$. It is a fact that

$$L(\mathcal{F},\chi_{\mathfrak{M}},s)=L(\mathcal{F},\chi_{\mathfrak{m}_0},s)\prod_{v|\mathfrak{n}_0}(1-a_{\mathcal{F}}(\mathfrak{p}_v)\chi_{\mathfrak{m}_0}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s}+N(\mathfrak{p}_v)^{1-2s})$$

where $a_{\mathcal{F}}$ denotes the Fourier coefficient of $\mathcal{F}$. It suffices to show the non-vanishing of

$$1-a_{\mathcal{F}}(\mathfrak{p}_v)\chi_{\mathfrak{m}_0}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s}+N(\mathfrak{p}_v)^{1-2s}\quad\text{at } s=1$$

which can be rewritten as the Hecke polynomial

$$(1-\alpha_{\mathcal{F}}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s})(1-\beta_{\mathcal{F}}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s}).$$

As $|\alpha_{\mathcal{F}}(\mathfrak{p}_v)|<N(\mathfrak{p}_v)$ and $|\beta_{\mathcal{F}}(\mathfrak{p}_v)|<N(\mathfrak{p}_v)$, we can deduce the non-vanishing of

$$(1-\alpha_{\mathcal{F}}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-1})(1-\beta_{\mathcal{F}}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-1}).$$

As $\mathfrak{m}_0$ is square-free, divisible by $\sqrt{d}$ and coprime to $\mathfrak{n}$ such that $L(\mathcal{F},\chi_{\mathfrak{m}_0},1)$ is non-vanishing, following Theorem 5.17 we can deduce that

Theorem 6.7 *Given a Bianchi modular form $\mathcal{F}\in S_2^{\mathrm{new},+}$ with $\mathfrak{n}$ coprime to $d_F\mathcal{O}$, there always exists a quadratic Hecke character such that the theta lifting as in Theorem 5.17 is non-vanishing.*

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