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The distribution of the multiplicative index of algebraic numbers over residue classes

Pieter Moree[1] · **Antonella Perucca**[2] · **Pietro Sgobba**[3]

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Abstract
Let K be a number field and G a finitely generated torsion-free subgroup of $K^\times$. Given a prime $\mathfrak{p}$ of K we denote by $\mathrm{ind}_\mathfrak{p}(G)$ the index of the subgroup ($G \bmod \mathfrak{p}$) of the multiplicative group of the residue field at $\mathfrak{p}$. Under the Generalized Riemann Hypothesis we determine the natural density of primes of K for which this index is in a prescribed set S and has prescribed Frobenius in a finite Galois extension F of K. We study in detail the natural density in case S is an arithmetic progression, in particular its positivity.

Keywords Reductions of algebraic numbers · Multiplicative index and order · Primes in arithmetic progression · Natural density

Mathematics Subject Classification Primary: 11R45 · Secondary: 11A07, 11R44

1 Introduction

The distribution of the multiplicative index of an integer seems to have been first studied by Pappalardi [17] in 1995. Under the Generalized Riemann Hypothesis (GRH) he provided asymptotic formulae for $\sum_{p\leq x} f(\mathrm{ind}_p(g))$, for f satisfying fairly mild restrictions (here and in the sequel we denote the rational primes by p). This line of investigation was continued in 2012 by Felix and Murty [5] and later by Felix for higher rank in [3]. Given a set of integers

✉ Pietro Sgobba
pietro.sgobba@xjtlu.edu.cn

Pieter Moree
moree@mpim-bonn.mpg.de

Antonella Perucca
antonella.perucca@uni.lu

1 Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

2 Department of Mathematics, University of Luxembourg, 6 Avenue de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg

3 Department of Pure Mathematics, Xi'an Jiaotong–Liverpool University, 111 Ren'ai Road, Suzhou 215123, China

S and a natural number g, in [5] it was proven that

$$\pi_{g,S}(x) := \left|\{p \le x : \mathrm{ind}_g(p) \in S\}\right| = c_{g,S}\,\mathrm{Li}(x) + O\left(\frac{x}{(\log x)^{2-\epsilon}}\right), \tag{1}$$

where $c_{g,S}$ is a constant defined by a series whose terms depend on the set S, $\mathrm{Li}(x) := \int_2^x dt/\log t$ denotes the logarithmic integral and $\epsilon > 0$ is arbitrary. It is a difficult problem to determine whether $c_{g,S}$ is positive or not, cf. Felix [4]. The special case where S is an arithmetic progression was already considered by Moree [12, Thm. 5] in 2005. For example, he proved Theorem 8 below in case $G = \langle g \rangle$, $F = K = \mathbb{Q}$.

In this paper we consider the behavior of $\pi_{g,S}(x)$, with $\mathbb{Q}$ replaced by a number field K and g by a finitely generated torsion-free subgroup G of $K^\times$. Instead of over rational primes, we sum now over primes $\mathfrak{p}$ of norm $\le x$. Under GRH we establish in Theorem 1, see Sect. 2, an asymptotics similar to (1), but with a weaker error term depending on the rank of G. Notice that our result relies on variations for number fields of Hooley's proof of Artin's primitive root conjecture under the assumption of GRH [8]. In Sect. 3 we then restrict to the case where S consists of integers in an arithmetic progression a mod d. In Theorem 8 we show that in this case the natural density can be expressed as a linear combination of at most $\varphi(d') - 1$ Artin-type constants, with $d' = d/(a, d)$. The positivity of the density is studied in Sect. 4, the numerical evaluation of the Artin-type constants in Sect. 5. In the final section we demonstrate our results by determining the density for two examples and compare the outcome with an experimental approximation.

We take G to be fixed, but one can also ask what happens for a "typical" G. Ambrose [1] considered the average index of the group generated by a finite number of elements in the residue field at a prime of a number field and provided asymptotic formulae for the average order of this quantity.

Likewise we can wonder about the above questions, but for the multiplicative order, rather than the index. As far as the authors know, these were first studied by Chinen and Murata [2] for $d = 4$, and a little later by Moree by a simpler method. Both Chinen and Murata, and independently Moree, went on to write various further papers (he surveyed his results in [15]). Under an appropriate generalization of the Riemann Hypothesis it turns out that the natural density of primes $p \le x$ such that the multiplicative order of g modulo p is congruent to a mod d exists. Denote it by $\delta_g(a, d)$ and the associated counting function by $N_g(a, d)(x)$. The proof of the existence of $\delta_g(a, d)$ by Moree is based on the identity

$$N_g(a, d)(x) = \sum_{t=1}^{\infty} \left|\{p \le x : \mathrm{ind}_p(g) = t,\ p \equiv 1 + ta \bmod dt\}\right|.$$

The average density of elements of order congruent to a mod d in a field of prime characteristic also exists, but is a much simpler quantity, see Moree [11]. It has very similar features to $\delta_g(a, d)$.

In the special case where d divides a, we are just asking for the density of primes p such that d divides the multiplicative order of g modulo p. This density is much easier to deal with and turns out to be a rational number. This can be proven unconditionally, see for example [14, 22, 23].

Ziegler [24], using the approach of Moree, was the first to study the order in arithmetic progression problem in the setting of number fields. His work was generalized by Perucca and Sgobba in [19, 20], who obtained in particular uniformity results for the distribution of the order. It is expected that, likewise, some uniformity also holds for the distribution of the index into suitably related congruence classes, however at the moment it is not clear how to obtain such a result. For example, it does not follow from [19, Cor. 5.2], in spite of the fact that congruence conditions on both the order and the size of the multiplicative group lead to congruence conditions on the index. We leave this as a research direction and as an open problem to the reader.

It is also still unknown whether this kind of results can be proven unconditionally. Although the results of this paper mostly rely on GRH, there are fundamental papers on Artin's conjecture for primitive roots providing unconditional results, see for example [6] by Gupta and Murty, and [7] by Heath-Brown. However, the contrast between what can be proven conditionally versus unconditionally in this area is quite dramatic. We note though that the infinitude of primes p in a prescribed arithmetic progression with $\mathrm{ind}_p(g) \neq t$, with t prescribed, *can* be unconditionally determined (however, not its density) [16]. Last but not least, Pappalardi obtained quantitative results without relying on GRH under certain convergence conditions, see for example [17, Thm. 1], from which one can determine the density for $\mathrm{ind}_p(g)$ being squarefree (and, more generally, k-free with $k \geq 2$).

2 The existence of the density of primes with prescribed index and Frobenius

Let K be a number field, and F/K a finite Galois extension. Let G be a finitely generated and torsion-free subgroup of $K^\times$ having positive rank r. Our goal is to determine the density of the set P of primes $\mathfrak{p}$ of K (defined in the next theorem) with prescribed index and Frobenius. The notation F, K, G and r will be maintained throughout. We also set $K_{m,n} := K(\zeta_m, G^{1/n})$ for $m \mid n$, and similarly for $F_{m,n}$. Further we make use of the following usual notation: ζ_n denotes an n-th primitive root of unity, μ the Möbius function, and φ Euler's totient function. We write $\log^a x$ as shorthand for $(\log x)^a$, and (a, b) for $\gcd(a, b)$.

We recall that Landau's prime ideal theorem states that

$$|\{\mathfrak{p} : \mathrm{N}\,\mathfrak{p} \leq x\}| = \mathrm{Li}(x) + O_K(xe^{-c_K\sqrt{\log x}}), \tag{2}$$

where $c_K > 0$ is a constant depending on K.

Theorem 1 (under GRH). *Let K be a number field, and let G be a finitely generated and torsion-free subgroup of $K^\times$ of positive rank r. Let F/K be a finite Galois extension, and let C be a union of conjugacy classes in* $\mathrm{Gal}(F/K)$. *Let S be a non-empty set of positive integers. Define*

$$P := \{\mathfrak{p} : \mathrm{ind}_\mathfrak{p}(G) \in S,\ \mathrm{Frob}_{F/K}(\mathfrak{p}) \in C\},$$

where $\mathfrak{p}$ ranges over the primes of K unramified in F and for which $\mathrm{ind}_\mathfrak{p}(G)$ *is well-defined. We let $P(x)$ be the number of prime ideals in P of norm $\leq x$. We have the asymptotic estimate*

$$P(x) = \frac{x}{\log x} \sum_{t \in S} \sum_{v=1}^{\infty} \frac{\mu(v)c(vt)}{[F_{vt,vt} : K]} + O\left(\frac{x}{\log^{2-\frac{1}{r+1}} x}\right),$$

where

$$c(n) = \left|\{\sigma \in \mathrm{Gal}(F_{n,n}/K) : \sigma|_{K_{n,n}} = \mathrm{id},\ \sigma|_F \in C\}\right|.$$

The implicit constant in the O-term depends only on K, F and G.

This result in combination with the prime ideal theorem leads to the following corollary.

Corollary 2 (under GRH). *Let S be a non-empty set of positive integers. The natural density of the primes $\mathfrak{p}$ of K such that $\mathrm{ind}_{\mathfrak{p}}(G) \in S$ and $\mathrm{Frob}_{F/K}(\mathfrak{p}) \in C$ exists and is given by*

$$\sum_{t \in S} \sum_{v=1}^{\infty} \frac{\mu(v)c(vt)}{[F_{vt,vt} : K]}.$$

We will now formulate some preliminaries required for the proof of Theorem 1. Our starting point is [19, Prop. 5.1], which was established for rank 1 in [24, Prop. 1].

Theorem 3 (under GRH). *For $x \geq t^3$, the number $R_t(x)$ of primes $\mathfrak{p}$ with norm up to x, unramified in F, and such that $\mathrm{ind}_{\mathfrak{p}}(G) = t$ and $\mathrm{Frob}_{F/K}(\mathfrak{p}) \in C$ satisfies*

$$R_t(x) = \mathrm{Li}(x) \sum_{v=1}^{\infty} \frac{\mu(v)c(vt)}{[F_{vt,vt} : K]} + O\left(\frac{x}{\log^2 x}\right) + O\left(\frac{x \log\log x}{\varphi(t) \log^2 x}\right).$$

The implicit constant in the O-term depends only on K, F and G.

The following lemma is a straightforward generalization of [12, Lem. 6], taking into account that for every natural number n, the ratio

$$C(n) := \frac{\varphi(n)n^r}{[K_{n,n} : K]} \tag{3}$$

is bounded above by some constant D, depending only on K and G (see [19, Thm. 1.1]).

Lemma 4 *For every real number $y \geq 1$ we have*

$$\sum_{t \leq y} \sum_{n=1}^{\infty} \frac{\mu(n)}{[K_{nt,nt} : K]} = 1 + O\left(\frac{D}{y^r}\right),$$

where the implicit constant is absolute.

Proof We claim that

$$\sum_{n > y} \frac{1}{n^r \varphi(n)} = O\Big(\frac{1}{y^r}\Big).$$

For $r = 1$ this is due to Landau [10], who first proved that

$$\sum_{n \leq x} \frac{1}{\varphi(n)} = A \log x + B + O\Big(\frac{\log x}{x}\Big), \tag{4}$$

with A and B explicit constants, and then applied partial integration. The proof for arbitrary r is completely analogous. Since $\varphi(nt) \geq \varphi(n)\varphi(t)$, we obtain

$$\sum_{t > y} \sum_{n=1}^{\infty} \frac{1}{[K_{nt,nt} : K]} \leq D \sum_{t > y} \frac{1}{t^r \varphi(t)} \sum_{n=1}^{\infty} \frac{1}{n^r \varphi(n)} \ll \sum_{t > y} \frac{D}{t^r \varphi(t)} \ll \frac{D}{y^r}, \tag{5}$$

where we used that the fourth sum is bounded above by a constant not depending on r. The estimate (5) shows that the double sum in the statement of Lemma 4 is absolutely convergent for all y. Thus, we may rearrange the double sum as follows:

$$\sum_{t=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(n)}{[K_{nt,nt}:K]}=\sum_{m=1}^{\infty}\sum_{s|m}\frac{\mu(m/s)}{[K_{m,m}:K]}=\sum_{m=1}^{\infty}\sum_{d|m}\frac{\mu(d)}{[K_{m,m}:K]}$$
$$=\sum_{m=1}^{\infty}\frac{1}{[K_{m,m}:K]}\sum_{d|m}\mu(d)=\frac{1}{[K_{1,1}:K]}=1,$$

completing the proof. □

The following is a generalization of Ziegler [24, Lem. 13].

Lemma 5 (under GRH). *We have*

$$\left|\left\{\mathfrak{p}:\mathrm{N}\,\mathfrak{p}\le x,\ \mathrm{ind}_{\mathfrak{p}}(G)>(\log x)^{\frac{1}{r+1}}\right\}\right|=O\left(\frac{x}{\log^{2-\frac{1}{r+1}}x}\right),$$

where the primes $\mathfrak{p}$ *of* K *are restricted to those for which* $\mathrm{ind}_{\mathfrak{p}}(G)$ *is well-defined. The implicit constant in the* O*-term depends only on* K *and* G*.*

Proof The number of primes with ramification index or residue class degree at least 2 is of order $O(\sum_{p\le\sqrt{x}}1)=O\left(\sqrt{x}/\log x\right)$. We make use of the functions $R_t(x)$ from Theorem 3 with $F=K$. For any real number $y\ge 1$, let $\mathcal{E}_y(x)$ be the number of primes $\mathfrak{p}$ with $\mathrm{N}\,\mathfrak{p}\le x$ and such that $\mathrm{ind}_{\mathfrak{p}}(G)>y$. Notice that

$$\mathcal{E}_y(x)=|\{\mathfrak{p}:\mathrm{N}\,\mathfrak{p}\le x\}|-\sum_{t\le y}R_t(x)+O\left(\frac{\sqrt{x}}{\log x}\right).$$

Landau's prime ideal theorem (2) implies the (much) weaker estimate

$$|\{\mathfrak{p}:\mathrm{N}\,\mathfrak{p}\le x\}|=\mathrm{Li}(x)+O\left(\frac{x}{\log^2 x}\right)=\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right), \tag{6}$$

which is all we need. By Theorem 3 and Lemma 4 we obtain

$$\sum_{t\le y}R_t(x)=\mathrm{Li}(x)\sum_{t\le y}\sum_{n=1}^{\infty}\frac{\mu(n)}{[K_{nt,nt}:K]}+O\left(\frac{xy}{\log^2 x}\right)$$
$$+O\left(\frac{x\log\log x}{\log^2 x}\sum_{t\le y}\frac{1}{\varphi(t)}\right)$$
$$=\mathrm{Li}(x)+O\left(\frac{x}{y^r\log x}\right)+O\left(\frac{xy}{\log^2 x}\right)$$
$$+O\left(\frac{x\log\log x}{\log^2 x}\sum_{t\le y}\frac{1}{\varphi(t)}\right).$$

On taking $y=(\log x)^{1/(r+1)}$, we now obtain on invoking (6) and (4), the estimate

$$\mathcal{E}_y(x)=O\left(\frac{x}{y^r\log x}\right)+O\left(\frac{xy}{\log^2 x}\right)+O\left(\frac{x(\log\log x)^2}{\log^2 x}\right)=O\left(\frac{x}{\log^{2-\frac{1}{r+1}}x}\right),$$

completing the proof. □

Proof of Theorem 1 Set $\rho = 2 - \frac{1}{r+1}$. Lemma 5 with $y = (\log x)^{\frac{1}{r+1}}$ yields

$$P(x) = \sum_{\substack{t \le y \\ t \in S}} R_t(x) + O\left(\frac{x}{\log^\rho x}\right).$$

Estimating the sum as in the proof of Lemma 5, we obtain

$$P(x) = \mathrm{Li}(x) \sum_{\substack{t \le y \\ t \in S}} \sum_{v=1}^{\infty} \frac{\mu(v)c(vt)}{[F_{vt,vt} : K]} + O\left(\frac{x}{\log^\rho x}\right).$$

Now we focus on the main term. We have

$$\left| \sum_{t \in S} \sum_{v=1}^{\infty} \frac{\mu(v)c(vt)}{[F_{vt,vt} : K]} - \sum_{\substack{t \le y \\ t \in S}} \sum_{v=1}^{\infty} \frac{\mu(v)c(vt)}{[F_{vt,vt} : K]} \right| \le \sum_{t > y} \sum_{v=1}^{\infty} \frac{1}{[F_{vt,vt} : F]}$$

By (5) the right-hand side is bounded by $\ll y^{-r}$. Using this estimate the proof is easily completed. □

3 The distribution of the index over residue classes

Let a, d be integers with $d \ge 2$. We study the density $\mathrm{dens}_G(a, d)$ of primes $\mathfrak{p}$ of K such that $\mathrm{ind}_{\mathfrak{p}}(G) \equiv a \bmod d$. Under GRH, by Theorem 1 this density exists and we have

$$\mathrm{dens}_G(a, d) = \sum_{t \equiv a \bmod d} \sum_{v \ge 1} \frac{\mu(v)}{[K_{vt,vt} : K]}. \tag{7}$$

The goal of this section is to prove Theorem 8, which expresses $\mathrm{dens}_G(a, d)$ as a finite sum of terms depending on Dirichlet characters χ of modulus d. These terms involve Artin-type constants $B_\chi(r)$ that can be evaluated with multi-precision using Theorem 16, thus allowing one to evaluate $\mathrm{dens}_G(a, d)$ with multi-precision.

We start by explaining our notation. Given an integer $n \ge 1$ we let G_n be the group of characters defined on $(\mathbb{Z}/n\mathbb{Z})^\times$, so that $G_n \cong (\mathbb{Z}/n\mathbb{Z})^\times$. For a Dirichlet character χ we denote by h_χ the (Dirichlet) convolution $\mu * \chi$ of the Möbius function μ with χ, that is $\mu * \chi(n) = \sum_{d|n} \mu(d)\chi(n/d)$. Recall that the Dirichlet convolution of two multiplicative functions is again a multiplicative function.

Put $w = \gcd(a, d)$, $a' = a/w$ and $d' = d/w$. The integers $t \equiv a \bmod d$ are of the form wt' with $t' \equiv a' \bmod d'$. Thus we can rewrite $\mathrm{dens}_G(a, d)$ as

$$\mathrm{dens}_G(a, d) = \sum_{t \equiv a' \bmod d'} \sum_{v \ge 1} \frac{\mu(v)}{[K_{vwt,vwt} : K]}.$$

This expression on its turn can be rewritten as

$$\begin{aligned}
\mathrm{dens}_G(a,d) &= \sum_{t\equiv a' \bmod d'} \sum_{\substack{v_1\geq 1\\ t\mid v_1}} \frac{\mu(v_1/t)}{[K_{v_1w,v_1w}:K]}\\
&= \sum_{v_1\geq 1} \sum_{\substack{t\equiv a' \bmod d'\\ t\mid v_1}} \frac{\mu(v_1/t)}{[K_{v_1w,v_1w}:K]}\\
&= \sum_{v_1\geq 1}\left(\frac{1}{\varphi(d')}\sum_{\chi\in G_{d'}}\overline{\chi(a')}h_\chi(v_1)\right)\frac{1}{[K_{v_1w,v_1w}:K]}\\
&= \frac{1}{\varphi(d')}\sum_{\chi\in G_{d'}}\overline{\chi(a')}\sum_{v_1\geq 1}\frac{h_\chi(v_1)}{[K_{v_1w,v_1w}:K]}\,. \qquad (8)
\end{aligned}$$

In the second step we used that the double series is absolutely convergent (see the proof of Lemma 4). In the third step we used [12, Lem. 9], where χ runs over the Dirichlet characters modulo d'.

We now focus on the final sum in (8). Recall the definition (3) of $C(n)$. By Perucca et al. [21, Thm. 1.1] there exists an integer n_0 (depending only on G and K) such that

$$C(n) = C(\gcd(n,n_0))\,. \qquad (9)$$

One can easily show that for $m \mid n$, one has $C(m) \mid C(n)$, and hence n_0 can be taken to be the minimal integer satisfying

$$C(n_0) = \max_{n\geq 1}\frac{\varphi(n)n^r}{[K_{n,n}:K]}.$$

By (9) we have

$$\frac{1}{[K_{n,n}:K]} = \frac{C(\gcd(n,n_0))}{\varphi(n)n^r},$$

and therefore

$$\sum_{n\geq 1}\frac{1}{[K_{n,n}:K]} = \sum_{g\mid n_0}\sum_{\substack{n\geq 1\\ (n,n_0)=g}}\frac{C(g)}{\varphi(n)n^r}.$$

In our case,

$$\sum_{v\geq 1}\frac{h_\chi(v)}{[K_{vw,vw}:K]} = \sum_{g\mid n_0}\sum_{\substack{v\geq 1\\ (vw,n_0)=g}}\frac{C(g)h_\chi(v)}{\varphi(vw)v^r w^r}\,. \qquad (10)$$

If $\sum_{v\geq 1} f(v)$ is some absolute convergent series, we have

$$\begin{aligned}
\sum_{\substack{v\geq 1\\ (vw,n_0)=g}} f(v) &= \sum_{\substack{v\geq 1\\ (\frac{vw}{g},\frac{n_0}{g})=1}} f(v) = \sum_{v\geq 1} f(v)\sum_{n\mid \frac{n_0}{g},\, n\mid\frac{vw}{g}}\mu(n)\\
&= \sum_{n\mid\frac{n_0}{g}}\mu(n)\sum_{\substack{v\geq 1\\ n\mid\frac{vw}{g}}} f(v) = \sum_{n\mid\frac{n_0}{g}}\mu(n)\sum_{\substack{v\geq 1\\ \frac{gn}{(gn,w)}\mid v}} f(v),
\end{aligned}$$

where we used that n divides the integer vw/g if and only if $gn/(gn, w)$ divides v. Thus, in particular,

$$\sum_{\substack{v\geq 1\\ (vw,n_0)=g}} \frac{C(g)h_\chi(v)}{\varphi(vw)v^r w^r} = \frac{C(g)}{w^r}\sum_{n|\frac{n_0}{g}}\mu(n)\sum_{\substack{v\geq 1\\ \frac{gn}{(gn,w)}|v}}\frac{h_\chi(v)}{\varphi(vw)v^r}.$$

Inserting the right-hand side into (10) and inserting the resulting expression into (8) yields

$$\mathrm{dens}_G(a,d) = \frac{1}{\varphi(d')}\sum_{\chi\in G_{d'}}\overline{\chi(a')}\sum_{g|n_0}\frac{C(g)}{w^r}\sum_{n|\frac{n_0}{g}}\mu(n)\sum_{\substack{v\geq 1\\ \frac{gn}{(gn,w)}|v}}\frac{h_\chi(v)}{\varphi(vw)v^r}.$$

Denoting

$$C_\chi(N,w,r) = \sum_{\substack{v\geq 1\\ N|v}}\frac{h_\chi(v)}{\varphi(vw)v^r},$$

we can write this as

$$\mathrm{dens}_G(a,d) = \frac{1}{\varphi(d')}\sum_{\chi\in G_{d'}}\overline{\chi(a')}\sum_{g|n_0}\frac{C(g)}{w^r}\sum_{n|\frac{n_0}{g}}\mu(n)\,C_\chi\left(\frac{gn}{(gn,w)},w,r\right). \tag{11}$$

Let $\kappa(n) = \prod_{p|n} p$ denote the squarefree kernel of n. Recall that $h_\chi = \mu * \chi$. The following result is a special case of [12, Lem. 10] and expresses $C_\chi(N,w,r)$ as an Euler product.

Lemma 6 *We have*

$$C_\chi(N,w,r) = c_\chi(N,w,r)B_\chi(r),$$

where

$$c_\chi(N,w,r) = \frac{h_\chi(N)\kappa(Nw)}{N^{r+1}w}\prod_{p|N}\frac{p^{r+1}}{p^{r+2}-p^{r+1}-p+\chi(p)}\prod_{\substack{p\nmid N\\ p|w}}\frac{p^{r+1}-1}{p^{r+2}-p^{r+1}-p+\chi(p)},$$

and

$$B_\chi(r) = \prod_p\left(1+\frac{p(\chi(p)-1)}{(p-1)(p^{r+1}-\chi(p))}\right), \tag{12}$$

where p runs over all rational prime numbers.

Corollary 7 *We have $C_\chi(1,1,r) = \sum_{v\geq 1}\frac{h_\chi(v)}{v\varphi(v)} = B_\chi(r)$.*

Proof of Lemma 6 We distinguish two cases:
a) The case where $h_\chi(N) = 0$.

We have to verify that $C_\chi(N,w,r) = 0$. Since h_χ is multiplicative and we have $h_\chi(p^k) = \chi(p)^{k-1}(\chi(p)-1)$, it follows that if $h_\chi(N) = 0$, then there is a prime divisor p of N with $\chi(p) = 1$. Hence, $h_\chi(v) = 0$ for all v that are divisible by N and so $C_\chi(N,w,r) = 0$.
b) The case where $h_\chi(N) \neq 0$.

We rewrite $C_\chi(N, w, r)$ as

$$C_\chi(N, w, r) = \frac{h_\chi(N)}{\varphi(Nw)N^r} \sum_{v \geq 1} \frac{h_\chi(Nv)\varphi(Nw)}{h_\chi(N)\varphi(Nvw)v^r}, \tag{13}$$

and note that the argument is a multiplicative function in v. We apply the Euler product identity to evaluate the sum and obtain

$$\prod_{p|N} \frac{p^{r+1}}{p^{r+1} - \chi(p)} \prod_{\substack{p\nmid N \\ p|w}} \left(1 + \frac{(\chi(p)-1)}{p^{r+1} - \chi(p)}\right) \prod_{p\nmid Nw} \left(1 + \frac{p(\chi(p)-1)}{(p-1)(p^{r+1} - \chi(p))}\right),$$

which can be rewritten as

$$B_\chi(r) \prod_{p|N} \frac{p^{r+1}(p-1)}{p^{r+2} - p^{r+1} - p + \chi(p)} \prod_{\substack{p\nmid N \\ p|w}} \frac{(p-1)(p^{r+1}-1)}{p^{r+2} - p^{r+1} - p + \chi(p)}.$$

On inserting this in (13) and noting that

$$\varphi(\kappa(Nw)) = \prod_{p|N}(p-1) \prod_{\substack{p\nmid N \\ p|w}}(p-1), \quad \frac{\varphi(\kappa(Nw))}{\varphi(Nw)} = \frac{\kappa(Nw)}{Nw},$$

the proof is completed. □

The density $\text{dens}_G(a, d)$ can be expressed as a finite linear combination involving the constants $B_\chi(r)$. Our result generalizes [12, Thm. 5] by Moree, who dealt with the case $F = K = \mathbb{Q}$ and G of rank 1.

Theorem 8 (under GRH). *Let a and d be two natural numbers. Put $d' = d/(a, d)$. Assuming that the function $C(n)$, defined in (3), is explicitly given, we can write*

$$\text{dens}_G(a, d) = \sum_{\chi \in G_{d'}} d_\chi B_\chi(r),$$

with the d_χ's explicit complex numbers (they can be determined using (11) and Lemma 6).

The equality of the series given in (7) and the linear combination of Theorem 8 is unconditional: it is establishing that these two quantities are densities that requires assuming GRH.

Proof In the identity (11) for $\text{dens}_G(a, d)$ we make the substitution

$$C_\chi\left(\frac{gn}{(gn, w)}, w, r\right) = c_\chi\left(\frac{gn}{(gn, w)}, w, r\right) B_\chi(r)$$

(which is allowed by Lemma 6). The constants d_χ are obtained by factoring out the terms $B_\chi(r)$, so that for each $\chi \in G_{d'}$ we have

$$d_\chi = \frac{\overline{\chi(a')}}{\varphi(d')} \sum_{g|n_0} \frac{C(g)}{w^r} \sum_{n|\frac{n_0}{g}} \mu(n) c_\chi\left(\frac{gn}{(gn, w)}, w, r\right).$$

3.1 Generic aspects of the behaviour of $\mathbf{dens}_G(a, d)$

Generically the degree $[K_{vt,vt} : K]$ equals $vt\varphi(vt)$ if G has rank 1. If every degree occurring in (7) satisfied this, then we would obtain

$$\rho(a, d) := \sum_{t \equiv a \bmod d} \sum_{v \geq 1} \frac{\mu(v)}{vt\varphi(vt)}.$$

The inner sum is easily seen to equal $A \cdot r(t)$, with

$$r(t) = \frac{1}{t^2} \prod_{p|t} \frac{p^2 - 1}{p^2 - p - 1}.$$

Thus we can alternatively write

$$\rho(a, d) = A_1 \sum_{t \equiv a \bmod d} r(t),$$

with

$$A_r := \prod_p \left(1 - \frac{1}{p^r(p-1)}\right) \tag{14}$$

the *rank r Artin constant*. The "incomplete" rank r Artin constant, defined by restricting to p odd, appears also in other works, such as in Pappalardi [18]. For every $B > 0$ we have, see [11, Thm. 4],

$$\sum_{p \leq x} \rho(p; a, d) = \rho(a, d)\,\mathrm{Li}(x) + O\Big(\frac{x}{\log^B x}\Big),$$

with $\rho(p; a, d)$ the density of elements of $\mathbb{F}_p^\times$ having index congruent to a mod d. Thus on average a finite field of prime order has $\rho(a, d)$ elements having index congruent to a mod d. Two cases are particularly easy.

Proposition 9 ([11, Prop. 4]). *One has*

$$\rho(0, d) = \frac{1}{d\varphi(d)} \quad \text{and} \quad \rho(d, 2d) = \begin{cases} \rho(0, 2d) & \text{if } d \text{ is odd;} \\ 3\rho(0, 2d) & \text{if } d \text{ is even.} \end{cases}$$

In the remaining cases it is not difficult to express $\rho(a, d)$ in terms of the $B_\chi(1)$'s, see [11, Prop. 6]. When $(a, d) = 1$, this expression takes a particularly simple form, namely

$$\rho(a, d) = \frac{1}{\varphi(d)} \sum_{\chi \bmod d} \overline{\chi(a)} B_\chi(1)\,. \tag{15}$$

In the examples in Sect. 6 we will meet $\rho(a, d)$ again.

4 The positivity of $\mathbf{dens}_G(a, d)$

As in the previous section we consider a number field K, a finitely generated and torsion-free subgroup G of $K^\times$, and the natural density $\mathrm{dens}_G(a, d)$ of the primes $\mathfrak{p}$ of K such that $\mathrm{ind}_{\mathfrak{p}}(G) \equiv a \bmod d$. We are interested in characterizing when this density is positive.

Example 10 Recall that for a prime $\mathfrak{p}$ of K of degree 1 such that $\mathrm{ind}_{\mathfrak{p}}(G)$ is well-defined, we have $d \mid \mathrm{ind}_{\mathfrak{p}}(G)$ if and only if $\mathfrak{p}$ splits completely in $K_{d,d}$ (cf. [24, Lem. 2]). So by Chebotarev's density theorem we have (without relying on GRH)

$$\mathrm{dens}_G(0, d) = \frac{1}{[K_{d,d} : K]} > 0, \tag{16}$$

In view of the above example, we may suppose in the following that $0 < a < d$.

We denote by $\mathrm{dens}_G(h)$ the density of primes $\mathfrak{p}$ such that $\mathrm{ind}_{\mathfrak{p}}(G) = h$, with h a prescribed integer, and by n_0 an integer satisfying $C(n) = C(\gcd(n, n_0))$ for all $n \geq 1$, where $C(n)$ was defined in (3). With this notation we are ready to recall the following result by Järviniemi and Perucca:

Theorem 11 ([9, Main Thm. and Rem. 4.2], under GRH). *The density* $\mathrm{dens}_G(h)$ *is well-defined for all* $h \geq 1$*, and we have* $\mathrm{dens}_G(h) > 0$ *if and only if* $\mathrm{dens}_G(\gcd(h, n_0)) > 0$*. For any set* S *of positive integers the following holds: if the density of primes* $\mathfrak{p}$ *of* K *such that* $\mathrm{ind}_{\mathfrak{p}}(G) \in S$ *is positive, then there is some* $h \in S$ *such that* $\mathrm{dens}_G(h) > 0$.

Proposition 12 (under GRH). *If* $d \geq 2$ *is coprime to* n_0*, then* $\mathrm{dens}_G(a, d) > 0$.

Proof By Theorem 11 (taking S to be the set of all positive integers) we know that there is some $h \geq 1$ such that $\mathrm{dens}_G(h) > 0$. Moreover, we deduce that there is an integer $h_0 \mid n_0$ such that for every integer t coprime to n_0 we have $\mathrm{dens}_G(th_0) > 0$. We conclude by taking $t \equiv 1 \bmod n_0$ and $t \equiv ah_0^{-1} \bmod d$. □

The following result tells us in particular that for every prime number ℓ and for every $e \gg 0$ there is a positive density of primes $\mathfrak{p}$ of K such that $v_\ell(\mathrm{ind}_{\mathfrak{p}}(G)) = e$.

Proposition 13 *For every prime number* ℓ *there is some non-negative integer* e_ℓ *(and we can take* $e_\ell = 0$ *for all but finitely many* ℓ*) such that for every* $e \geq e_\ell$ *we have*

$$\mathrm{dens}_G(0, \ell^e) > \mathrm{dens}_G(0, \ell^{e+1}).$$

Under GRH, for every $n > 0$ *and for every integer* z*, we have* $\mathrm{dens}_G(z\ell^{e_\ell}, \ell^n) > 0$.

Proof By Chebotarev's density theorem the primes $\mathfrak{p}$ of K such that $v_\ell(\mathrm{ind}_{\mathfrak{p}}(G)) = e$ have density $1/[K_{\ell^e,\ell^e} : K] - 1/[K_{\ell^{e+1},\ell^{e+1}} : K]$, so the first assertion follows from the eventual maximal growth of the Kummer degrees, see [19, Lem. 3.2]. By the first assertion (and by applying Theorem 11 to the set S of positive integers having ℓ-adic valuation equal to e) for every $e \geq e_\ell$ there is some b coprime to ℓ such that $\mathrm{dens}_G(b\ell^e) > 0$. Then, for every prime q coprime to n_0 we have $\mathrm{dens}_G(qb\ell^e) > 0$, so we may conclude by selecting $q \equiv b^{-1}z\ell^{-v_\ell(z)} \bmod \ell^n$, which is possible by Dirichlet's theorem on primes in arithmetic progressions. □

If x, y are positive integers, then we use the notation $\gcd(x, y^\infty)$ to denote the positive integer obtained from x by removing the prime factors that do not divide y.

Theorem 14 (under GRH). *We have* $\mathrm{dens}_G(a, d) > 0$ *if and only if*

$$\mathrm{dens}_G(a, \gcd(d, n_0^\infty)) > 0.$$

Proof Set $d_0 = \gcd(d, n_0^\infty)$. The former inequality in the statement clearly implies the latter because the integers congruent to $a \bmod d$ are also congruent to $a \bmod d_0$. Now suppose that there is a positive density of primes $\mathfrak{p}$ of K such that $\mathrm{ind}_\mathfrak{p}(G) \equiv a \bmod d_0$. From Theorem 11 we deduce that there exists $h \geq 1$ such that $h \equiv a \bmod d_0$ and $\mathrm{dens}_G(h) > 0$. For every pair of positive integers t, s coprime to n_0 such that $s \mid th$, we have $\mathrm{dens}_G(th/s) > 0$. If we choose $t \equiv s \bmod d_0$, then $th/s \equiv a \bmod d_0$. We claim that we may also choose t, s so that $th/s \equiv a \bmod d/d_0$. Because of the Chinese remainder theorem it will be possible to simultaneously ensure that the two conditions hold, and hence $\mathrm{dens}_G(th/s) > 0$ implies $\mathrm{dens}_G(a, d) > 0$. To prove the claim, we first choose t, s so that $\gcd(a, d/d_0) = \gcd(th/s, d/d_0)$, and then multiply t by an integer invertible modulo d/d_0 to obtain the requested congruence. □

Theorem 15 (under GRH). *The following conditions are equivalent:*

(1) *the density* $\mathrm{dens}_G(a, d)$ *is positive;*
(2) *there is an integer* A *such that* $A \equiv a \bmod d$ *and* $\mathrm{dens}_G(\gcd(A, n_0))$ *is positive.*

Proof Write $D := \mathrm{lcm}(d, n_0)$. By Theorem 11 the density $\mathrm{dens}_G(a, d)$ is positive if and only if there is an integer $A \equiv a \bmod d$ for which $\mathrm{dens}_G(A, D) > 0$. The latter holds if and only if there is an index h such that $h \equiv A \bmod D$ and $\mathrm{dens}_G(h) > 0$. Since $n_0 \mid D$, we have $\gcd(h, n_0) = \gcd(A, n_0)$, and hence by Theorem 11 we have that $\mathrm{dens}_G(h)$ is positive if and only if $\mathrm{dens}_G(\gcd(A, n_0))$ is positive. □

Since the properties in (2) only depend on A modulo $\mathrm{lcm}(d, n_0)$, we see that it actually suffices to consider A modulo $\mathrm{lcm}(d, n_0)$.

5 The Artin-type constants $B_\chi(r)$

Let $r \geq 1$ be an integer. Recall the Euler product definition (12) of $B_\chi(r)$. For $r = 1$ this was introduced in [11, Sec. 6] and denoted by B_χ, along with a variant A_χ, where p is restricted to those primes for which $\chi(p) \neq 0$. We have

$$B_\chi(1) = A_\chi \prod_{p \mid d} \left(1 - \frac{1}{p(p-1)}\right),$$

where d is the modulus of the character. Note that $A_\chi = 1$ in case χ is the principal character.

If χ_0 is the principal character, then $B_{\chi_0}(r)$ is a rational number. This leaves at most $\varphi(d') - 1$ linearly independent Artin-type constants, with $d' = d/(a, d)$. For example, in case $d' = 3$ and $d' = 4$ only one Artin-type constant is involved. They are real numbers. As an illustration we point out the result that the average density of elements of multiplicative order $\pm 1 \bmod 3$ equals $\frac{5}{16} \pm \frac{3}{10} B_{\chi_3}(1)$, where χ_3 is the non-principal character modulo 3 and $B_{\chi_3}(1) = \frac{5}{6} A_{\chi_3} = 0.1449809353580\ldots$, see [11].

Approximating the numerical value of $B_\chi(r)$ by computing partial Euler products, gives a quite poor accuracy. The following result allows us to do rather better and generalizes [11, Thm. 6] to arbitrary r. It involves special values of Dirichlet L-series. Recall that for $\Re(s) > 1$ and χ a Dirichlet character, we have

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Theorem 16 *Let $p_1(=2), p_2, \ldots$ denote the sequence of consecutive primes and χ be any Dirichlet character. Put*

$$\Lambda_r = A_r L(r+1,\chi)L(r+2,\chi)L(r+3,\chi).$$

Then

$$B_\chi(r) = E_{r,n}\,\Lambda_r \prod_{k=1}^{n}\left(1+\frac{\chi(p_k)}{p_k(p_k^{r+1}-p_k^r-1)}\right)\left(1-\frac{\chi(p_k)}{p_k^{r+2}}\right)\left(1-\frac{\chi(p_k)}{p_k^{r+3}}\right)$$

with

$$1-\frac{1}{p_{n+1}^{r+2}} \le |E_{r,n}| \le 1+\frac{1}{p_{n+1}^{r+2}},$$

provided that $r=1$ and $p_{n+1}\ge 5$, or $r=2$ and $p_{n+1}\ge 3$.

Proof Recall the definition (14) of A_r. Noting that

$$(1-yt^{r+1})\left(\frac{1+\frac{(y-1)t^{r+1}}{(1-yt^{r+1})(1-t)}}{1-\frac{t^{r+1}}{1-t}}\right) = 1+\frac{yt^{r+2}}{1-t-t^{r+1}},$$

we obtain

$$B_\chi(r) = A_r L(r+1,\chi)\prod_{k=1}^{\infty}\left(1+\frac{\chi(p_k)}{p_k(p_k^{r+1}-p_k^r-1)}\right), \tag{17}$$

on setting $y=\chi(p_k)$ and $t=\frac{1}{p_k}$. We rewrite the infinite product as

$$L(r+2,\chi)L(r+3,\chi)\prod_{k=1}^{\infty}\left(1+\frac{\chi(p_k)}{p_k(p_k^{r+1}-p_k^r-1)}\right)\left(1-\frac{\chi(p_k)}{p_k^{r+2}}\right)\left(1-\frac{\chi(p_k)}{p_k^{r+3}}\right)$$

in order to improve its convergence. Denoting the k-th term in the infinite product by $P_{r,k}$, we see that (17) holds with $E_{r,n}=\prod_{k\ge n+1}P_{r,k}$.

It remains to estimate the relative error $E_{r,n}$ (which in general is a complex number). Multiplying out

$$(1-t-t^{r+1}+yt^{r+2})(1-yt^{r+2})(1-yt^{r+3})/(1-t-t^{r+1})$$

gives

$$1+\frac{yt^{r+4}}{1-t-t^{r+1}}\Big(1+t^{r-1}+(1-y)t^r-yt^{r+2}-yt^{2r+2}+y^2t^{2r+3}\Big),$$

and leads to the estimate

$$|P_{r,k}| \le 1+t^{r+3}G_r(t),$$

with

$$G_r(t) = \frac{t(1+t^{r-1}+2t^r+t^{r+2}+t^{2r+2}+t^{2r+3})}{1-t-t^{r+1}}$$

and $t=\frac{1}{p_k}$. Note that $G_r(t)$ is increasing in t and decreasing in r in the region $0<t<1$ and $r\ge 1$. Thus

$$|P_{r,k}| \le 1+p_k^{-r-3}G_r(p_k^{-1}) \le 1+p_k^{-r-3}G_r(p_{n+1}^{-1}) \quad \text{for every } k\ge n+1.$$

Table 1 Examples of densities $\mathrm{dens}_G(a, 5)$ with $K = \mathbb{Q}(\sqrt{5})$

G	$\mathrm{dens}_G(0, 5)$ $\frac{P_{0,5}(10^6)}{\pi_K(10^6)}$	$\mathrm{dens}_G(1, 5)$ $\frac{P_{1,5}(10^6)}{\pi_K(10^6)}$	$\mathrm{dens}_G(2, 5)$ $\frac{P_{2,5}(10^6)}{\pi_K(10^6)}$	$\mathrm{dens}_G(3, 5)$ $\frac{P_{3,5}(10^6)}{\pi_K(10^6)}$	$\mathrm{dens}_G(4, 5)$ $\frac{P_{4,5}(10^6)}{\pi_K(10^6)}$
$\langle \frac{1+\sqrt{5}}{2} \rangle$	0.100000	0.418205	0.296724	0.0950872	0.0899840
	0.100093	0.419351	0.296954	0.0947177	0.0888838
$\langle -\frac{5+\sqrt{5}}{2} \rangle$	0.100000	0.451872	0.266393	0.0995570	0.0821785
	0.099787	0.450979	0.267518	0.0996599	0.0820564

As t tends to zero, $G_r(t)$ tends to zero, and so we can choose n so large that $G_r(p_{n+1}^{-1}) \le 1$. Now

$$|E_{r,n}| = \prod_{k \ge n+1} |P_{r,k}| < \prod_{p > p_n} \left(1 + \frac{1}{p^{r+3}}\right) < 1 + \sum_{m > p_n} \frac{1}{m^{r+3}}.$$

Comparing the sum with an integral leads to the final estimate

$$|E_{r,n}| \le 1 + \frac{1}{p_{n+1}^{r+3}} + \int_{p_{n+1}}^{\infty} \frac{dz}{z^{r+3}} \le 1 + \frac{1}{p_{n+1}^{r+2}},$$

where the sum is over the integers $m > p_n$. Similarly,

$$|E_{r,n}| > \prod_{p > p_n} \left(1 - \frac{1}{p^{r+3}}\right) > 1 - \sum_{m > p_n} \frac{1}{m^{r+3}} > 1 - \frac{1}{p_{n+1}^{r+2}}.$$

Some calculus shows that $G_r(\frac{1}{p}) \le 1$ if and only if $r = 1$ and $p \ge 5$ or $r \ge 2$ and $p \ge 3$. The proof is now completed on invoking the if-part of this statement. □

Remark 17 In the proof of [11, Thm. 6] there are a few typos:
For "$2 + 2t + t^3 + t^5$" read "$2 + 2t + t^3 + t^4 + t^5$".
For "$t \ge 127$" read "$t \le 1/127$".
For "$p_{n+!}$" read "p_{n+1}".

6 Two examples

In this section we demonstrate our results by two relatively easy, but illustrative, examples for $K = \mathbb{Q}(\sqrt{5})$, $r = 1$ and $d = 5$. Some examples for the same r and d values, but with $K = \mathbb{Q}$ are given in Moree [13, Table 2].

In Table 1 we denote by $P_{a,d}(x)$ the number of primes $\mathfrak{p}$ of K of norm up to x such that $\mathrm{ind}_{\mathfrak{p}}(G) \equiv a \bmod d$, and by $\pi_K(x)$ the number of primes $\mathfrak{p}$ of K with norm up to x. The top row gives the theoretical density, the second row an experimental approximation (both with rounding of the final decimal).

We will now treat these two examples without using the machinery of Sect. 3 (however, with complicated enough examples this becomes unavoidable). Our approach requires some

Table 2 The constants $B_\chi(1)$ for $d=5$

χ	$B_\chi(1)$
ψ	$0.34645514515465\ldots + i \cdot 0.21283903970350\ldots$
ψ^2	$0.12284254160167\ldots$
ψ^3	$0.34645514515465\ldots - i \cdot 0.21283903970350\ldots$
ψ^4	0.95

further notation. Given a divisor δ of an integer d_1, we put

$$\rho_{\delta,d_1}(a,d) := \sum_{t\equiv a \bmod d} \sum_{\substack{v\geq 1\\(v,d_1)=\delta}} \frac{\mu(v)}{vt\varphi(vt)}.$$

6.1 First example

Proposition 18 *Set $K=\mathbb{Q}(\sqrt{5})$ and $G=\langle\frac{1+\sqrt{5}}{2}\rangle$. We have*

$$\mathrm{dens}_G(0,5)=\frac{1}{10}=2\rho(0,5)$$

and, for $1\leq a\leq 4$, assuming GRH,

$$\mathrm{dens}_G(a,5)=\frac{18}{19}\rho(a,5)=\frac{9}{38}\left(\frac{19}{20}+\overline{\psi(a)}B_\psi(1)+\psi(a)B_{\psi^3}(1)+\psi^2(a)B_{\psi^2}(1)\right).$$

Proof The first claim follows from (16) and Proposition 9. Next assume that $1\leq a\leq 4$. We have $\rho(a,5)=\rho_{1,5}(a,5)+\rho_{5,5}(a,5)$. If $5\nmid t$, then

$$\sum_{5|v}\frac{\mu(v)}{vt\varphi(vt)}=\sum_{w}\frac{\mu(5w)}{5wt\varphi(5wt)}=-\frac{1}{20}\sum_{5\nmid w}\frac{\mu(w)}{wt\varphi(wt)}.$$

We conclude that $\rho_{5,5}(a,5)=-\frac{1}{20}\rho_{1,5}(a,5)$. It thus follows that $\rho_{1,5}(a,5)=\frac{20}{19}\rho(a,5)$ and $\rho_{5,5}(a,5)=-\frac{1}{19}\rho(a,5)$. Since the degree $[K_{n,n}:K]$ equals $\varphi(n)n$ if $5\nmid n$ and $\frac{1}{2}n\varphi(n)$ otherwise, we infer that $\mathrm{dens}_G(a,5)=\rho_{1,5}(a,5)+2\rho_{5,5}(a,5)=\frac{18}{19}\rho(a,5)$. The proof is completed on invoking (15) and noting that $\psi^2(a)$ is real and $\overline{\psi^3(a)}=\psi(a)$. □

Approximations to $B_\chi(1)$ can be found in Table 2, where ψ denotes the character modulo 5 determined uniquely by $\psi(2)=i$.

The character group has ψ,ψ^2,ψ^3 and ψ^4 as elements, with ψ^4 being the principal character. The table is taken from [11, Table 3], where for $d\leq 12$ further approximations can be found. It was kindly verified by Alessandro Languasco using Theorem 16 with $n=10^6$.

6.2 Second example

Proposition 19 *Set $K=\mathbb{Q}(\sqrt{5})$ and $G=\langle-\frac{5+\sqrt{5}}{2}\rangle$. Let $1\leq a\leq 4$. One of a and $a+5$ is even. Denoting this number by a_1, assuming GRH, we have*

$$\mathrm{dens}_G(a,5)=\frac{20}{19}\rho(a,5)-\frac{4}{19}\rho(a_1,10).$$

Furthermore, $\mathrm{dens}_G(0, 5) = \frac{1}{10}$.

Proof Using (16) we see that $\mathrm{dens}_G(0, 5) = \frac{1}{10}$. We will determine $\mathrm{dens}_G(a, 10)$ in case $5 \nmid a$. The result then follows on adding $\mathrm{dens}_G(a, 10)$ and $\mathrm{dens}_G(a + 5, 10)$.

Since $\mathbb{Q}\Big(\sqrt{-\frac{5+\sqrt{5}}{2}}\Big) = \mathbb{Q}(\zeta_5)$, the degree $[K_{n,n} : K]$ equals $\varphi(n)n$ if $5 \nmid n$, it equals $\frac{1}{2}n\varphi(n)$ if $(n, 10) = 5$, and it equals $\frac{1}{4}n\varphi(n)$ if $10 \mid n$. These degree considerations lead to

$$\mathrm{dens}_G(a, 10) = \begin{cases} \rho_{1,5}(a, 10) + 4\rho_{5,5}(a, 10) & \text{if } 2 \mid a; \\ \rho_{1,5}(a, 10) + 2\rho_{5,10}(a, 10) + 4\rho_{10,10}(a, 10) & \text{if } 2 \nmid a. \end{cases}$$

Reasoning as in the proof of Proposition 18 we deduce that $\rho_{5,5}(a, 10) = -\frac{1}{20}\rho_{1,5}(a, 10)$ and $\rho_{1,5}(a, 10) = \frac{20}{19}\rho(a, 10)$. It follows that $\mathrm{dens}_G(a, 10) = \frac{4}{5}\rho_{1,5}(a, 10) = \frac{16}{19}\rho(a, 10)$ in case a is even.

If a is odd, then so are the integers $t \equiv a \bmod 10$ and so $\rho_{10,10}(a, 10) = -\frac{1}{2}\rho_{5,10}(a, 10)$, leading to $\mathrm{dens}_G(a, 10) = \rho_{1,5}(a, 10)$. Reasoning as in the proof of Proposition 18, we then deduce that $\mathrm{dens}_G(a, 10) = \frac{20}{19}\rho(a, 10)$. □

For reasons of space we refrain here from explicitly writing out $\mathrm{dens}_G(a, 5)$ as a linear sum in the B_χ's, but we will indicate how this is done. For $\rho(a, 5)$ we use (15). For a with $5 \nmid a$ we have by [11, Prop. 6] with $w = 5$ and $\delta = 2$,

$$\rho(2a, 10) = \frac{3}{8} \sum_{\chi \bmod 5} \overline{\chi(a)} \frac{B_\chi(1)}{2 + \chi(2)}.$$

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Declarations

Conflicts of interest The authors have no conflicts of interest to disclose.

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