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$C^{1,\alpha}$-regularity for p-harmonic functions on SU(3) and semi-simple Lie groups

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Abstract

In this paper, when $1 < p < 2$, we establish the $C^{1,\alpha}_{\text{loc}}$-regularity of weak solutions to the degenerate subelliptic p-Laplacian equation

$$\triangle_{\mathcal{H},p}u(x) = \sum_{i=1}^{6} X_i^*(|\nabla_{\mathcal{H}}u|^{p-2}X_i u) = 0$$

on SU(3) endowed with the horizontal vector fields $X_1, \ldots, X_6$. The result can be extended to a class of compact connected semi-simple Lie group.

Keywords p-Laplacian equation · $C^{1,\alpha}$-regularity · SU(3) · Caccioppoli inequality · De Giorgi · p-harmonic function · Semi-simple Lie group

Mathematics Subject Classification 35H20 · 35B65

1 Introduction

In this research article, we consider the special unitary group of 3×3 complex matrices SU(3) endowed with a horizontal vector field $X_1, X_2, \ldots, X_6$; see Sect. 2 for more geometries and properties of SU(3). Given a domain $\Omega \subset$ SU(3), we consider the quasilinear subelliptic equation

$$\sum_{i=1}^{6} X_i^*(a_i(\nabla_{\mathcal{H}}u)) = 0 \quad \text{in } \Omega. \tag{1.1}$$

Here $\nabla_{\mathcal{H}}u = (X_1u, X_2u, \ldots, X_6u)$ is the horizontal gradient of a function $u \in C^1(\Omega)$; X_i^* is the formal adjoint of X_i; the vector function $a := (a_1, a_2, \ldots, a_6) \in C^2(\mathbb{R}^6, \mathbb{R}^6)$ satisfies

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the following growth and ellipticity conditions:

$$\sum_{i,j=1}^{6} \frac{\partial a_i(\xi)}{\partial \xi_j} \eta_i \eta_j \geq l_0(\delta + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2, \tag{1.2}$$

$$\sum_{i,j=1}^{6} \left| \frac{\partial a_i(\xi)}{\partial \xi_j} \right| \leq L(\delta + |\xi|^2)^{\frac{p-2}{2}}, \tag{1.3}$$

$$|a_i(\xi)| \leq L(\delta + |\xi|^2)^{\frac{p-2}{2}} |\xi| \tag{1.4}$$

for all $\xi, \eta \in \mathbb{R}^6$, where $0 \leq \delta \leq 1$, $1 < p < \infty$ and $0 < l_0 < L$. Note that conditions (1.2) and (1.3) are the same as conditions [5, (2.3) and (2.4)], but the condition (1.4) is stronger than the condition [5, (2.5)]

$$|a_i(\xi)| \leq L(\delta + |\xi|^2)^{\frac{p-1}{2}}. \tag{1.5}$$

We call a function $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ as a weak solution to (1.1) if

$$\sum_{i=1}^{6} \int_\Omega a_i(\nabla_{\mathcal{H}} u) X_i \varphi dx = 0, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.6}$$

Here $W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ is the first order p-th integrable horizontal local Sobolev space, namely, all functions $u \in L^p_{\mathrm{loc}}(\Omega)$ with their distributional horizontal gradients $\nabla_{\mathcal{H}} u \in L^p_{\mathrm{loc}}(\Omega)$. Given the typical example $a(\xi) = (\delta + |\xi|^2)^{\frac{p-2}{2}} \xi$, equation (1.1) becomes the non-degenerate p-Laplacian equation

$$\sum_{i=1}^{6} X_i\big((\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} X_i u\big) = 0 \quad \text{if } \delta > 0, \tag{1.7}$$

and the p-Laplacian equation

$$\sum_{i=1}^{6} X_i\big(|\nabla_{\mathcal{H}} u|^{p-2} X_i u\big) = 0 \quad \text{if } \delta = 0. \tag{1.8}$$

Particularly, we call weak solutions to (1.8) as p-harmonic functions in $\Omega \subset \mathrm{SU}(3)$.

In the linear case $p = 2$, p-harmonic functions in SU(3) are harmonic functions and their C^∞-regularity was established by Hörmander [11]. In the quasilinear case $p \neq 2$, Domokos-Manfredi [5] obtained the local boundedness of horizontal gradient $\nabla_{\mathcal{H}} u$ of p-harmonic functions u in SU(3), that is, $\nabla_{\mathcal{H}} u \in L^\infty_{\mathrm{loc}}(\Omega)$. Moreover, when $2 < p < \infty$, they obtain the Hölder regularity of $\nabla_{\mathcal{H}} u$, that is, $\nabla_{\mathcal{H}} u \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ independent of u. But when $1 < p < 2$, the Hölder regularity of $\nabla_{\mathcal{H}} u$ was unknown.

For the general quasi-linear equation (1.1) in SU(3), Domokos-Manfredi [5] also built up analogue regularity. To be precise, if a satisfies conditions (1.2), (1.3) and (1.5) for some $1 < p < \infty$, weak solutions u are proved to satisfy $\nabla_{\mathcal{H}} u \in L^\infty_{\mathrm{loc}}(\Omega)$. If $2 \leq p < \infty$, they further proved $\nabla_{\mathcal{H}} u \in C^\alpha_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0, 1)$. But when $1 < p < 2$, the Hölder regularity of $\nabla_{\mathcal{H}} u$ was also unavailable.

In this paper, we focus on the case $1 < p < 2$. Moreover, instead of the condition (1.5) assumed by [5] when $2 \leq p < \infty$, we work with the stronger condition (1.4). Indeed, if a satisfies (1.2), (1.3) and (1.4) for some $1 < p < 2$, we obtain an Hölder regularity of horizontal gredient of weak solutions u to (1.1). As a consequence, when $1 < p < 2$, the horizontal gradient of p-harmonic functions in SU(3) has an Hölder regularity. See Theorem 1.1 below for details. We always denote by $B_r(x)$ the Carnot–Carathéodory ball centered at $x \in \mathrm{SU}(3)$ with radius r with respect to the Carnot–Carathéodory distance d_{CC} which is defined in Sect. 2. For convenience we write B_r as $B_r(x)$ and denote by $C(a, b, \dots)$ a positive constant depending on parameters $a, b, \dots$ whose value may change line to line.

Theorem 1.1 *Suppose that $a \in C^2(\mathbb{R}^6, \mathbb{R}^6)$ satisfies the conditions (1.2), (1.3) and (1.4) for some l_0, L and such $1 < p < 2$, $\delta \geq 0$. If $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ is a weak solution to (1.1), then $\nabla_{\mathcal{H}} u \in C^{\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0, 1)$ depending on l_0, L and such p, δ. Moreover, for all $B_{r_0} \subset \Omega$ and any $0 < r \leq r_0$, we have*

$$\max_{1\leq l\leq 6} \operatorname{osc}_{B_r} X_l u \leq C \left(\frac{r}{r_0}\right)^{\alpha} \left(⨍_{B_{r_0}} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}, \tag{1.9}$$

where $0 < \alpha < 1$ depends on p, l_0 and L, and the constant $C > 0$ depends on p, l_0, L and r_0.

Consequently, when $1 < p < 2$, the horizontal gradients of p-harmonic functions on SU(3) *have the Hölder regularity and satisfy* (1.9).

Recall that in the Euclidean space $\mathbb{R}^n$ (corresponding to the vector field $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$), the $C^{0,1}$ and $C^{1,\alpha}$-regualrity of solutions to Eq. (1.1) under conditions (1.2), (1.3) and (1.4) have been established by [8, 14, 21–23]. In the Heisenberg group $\mathbb{H}^n$, the $C^{0,1}$ and $C^{1,\alpha}$-regularity of solutions to Eq. (1.1) under conditions (1.2), (1.3) and (1.4) have been established by [4, 7, 16–18, 20, 25].

We will prove Theorem 1.1 in Sect. 5 by borrowing some ideas from [18] to use De Giorgi's method in [3], and also adapting some apriori estimates in [5] under conditions (1.2), (1.3) and (1.4). Indeed, given any weak solution u to (1.1), to get the Hölder regularity for $\nabla_{\mathcal{H}} u$, the central is to show that $\nabla_{\mathcal{H}} u$ belongs to De Giorgi's class in SU(3), which will be recalled in Sect. 3. To this end, we consider the double truncation of $X_l u$, that is,

$$v = \min(\mu(r)/8, \max(\mu(r)/4 - X_l u, 0)),$$

where $\mu(r) = \max_{1\leq k\leq 6} \sup_{B_r} |X_k u|$ and $l \in \{1, \dots, 6\}$. It then suffices to buil up the following crucial Caccioppoli inequality

$$\int_{B_r} \eta^{\beta+4} v^{\beta+2} |\nabla_{\mathcal{H}} v|^2 dx \leq C(p, L, l_0, \gamma)(\beta+2)^2 \mu(r)^4 \left(1 + \frac{1}{r^2}\right) \times |B_r|^{1-1/\gamma} \left(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta} dx \right)^{1/\gamma} \tag{1.10}$$

for all $\beta \geq 0$, where $\gamma > 1$ is a constant and $\eta \in C_0^{\infty}(B_r)$ is a standard cut-off function in B_r. Indeed, if (1.10) holds true, following [18] line by line and using an interation argument as in [9, Lemma 7.3], we are able to conclude (1.9) and hence Theorem 1.1 holds true. For details see Sect. 5.

To prove (1.10), firstly, under the stronger condition (1.4), by choosing different test functions, we are able to adapt or modify the arguments in [5] so to get several a priori estimates as in Lemma 2.3, which are stronger than the corresponding estimates in [5]. See Sect. 2 for details.

Next, from these apriori estimates in Lemma 2.3, in Sect. 4 we deduce two auxiliary Caccioppoli inequalities for $\nabla_{\mathcal{H}} u$ and $\nabla_{\mathcal{T}} u$ involving v as in Lemmas 4.5 and 4.6, whose proofs are postponed to Sect. 6. Since there is no nilpotent structure in SU(3), we have to deal with all integrals involving $[X_i, X_j]u$ for all $i, j \in \{1, \ldots, 8\}$ according to Table 1 and use a priori estimates in Lemma 2.3 to bound them.

Finally, we choose the function $\eta^{\beta+4}v^{\beta+3}$ to test (4.7) in Lemma 4.4 and obtain (4.14), where $\beta \geq 0$. Then using conditions (1.2), (1.3), (1.4), a priori estimates in Lemma 2.3 and Lemmas 4.5, 4.6, we conclude (1.10) in Lemma 4.1. The proof is postponed to Sect. 4.

To prove Lemma 4.5, first, we choose the function $\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2 X_l u$ to test (4.7) in Lemma 4.4 and obtain (6.1), where $\beta \geq 0$ and $l \in \{1, \ldots, 6\}$. Then using conditions (1.2), (1.3), (1.4), we conclude (4.8) in Lemma 4.5. The proof is postponed to Sect. 6.

The proof of Lemma 4.6 is based on Lemma 4.5. To prove Lemma 4.6, first, we choose the function $\eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}|\nabla_{\mathcal{H}}u|^4 X_l u$ to test (4.7) in Lemma 4.4 and obtain (6.6), where $\beta \geq 0$, $\tau \in (1/2, 1)$ and $l \in \{7, 8\}$. Then using conditions (1.2), (1.3), (1.4), a priori estimates in Lemmas 2.3 and 4.5, we conclude (4.12) in Lemma 4.6. The proof is postponed to Sect. 6.

2 Preliminaries

We identify the group SU(3) with the unitary group of 3×3 complex matrices

$$\{g \in \mathrm{GL}(3, \mathbb{C}) : g \cdot g^* = I, \det g = 1\}.$$

Its Lie algebra is given by

$$\mathrm{su}(3) := \{X \in \mathrm{gl}(3, \mathbb{C}) : X + X^* = 0, \mathrm{tr}X = 0\}$$

with the inner product $\langle X, Y \rangle := -\frac{1}{2}\mathrm{tr}(XY)$.

Noting that the two-dimensional maximal torus

$$\mathbb{T} := \left\{ \begin{pmatrix} e^{ia_1} & 0 & 0 \\ 0 & e^{ia_2} & 0 \\ 0 & 0 & e^{ia_3} \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R}, a_1 + a_2 + a_3 = 0 \right\},$$

we choose its Lie algebra

$$\mathcal{T} := \left\{ \begin{pmatrix} ia_1 & 0 & 0 \\ 0 & ia_2 & 0 \\ 0 & 0 & ia_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R}, a_1 + a_2 + a_3 = 0 \right\}$$

as the Cartan subalgebra.

Recalling the definition of su(3), we use a set of Gell Mann matrices $\mathcal{G}$ to form its orthonormal basis, that is,

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -\frac{i}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{i}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2i}{\sqrt{3}} \end{pmatrix}.$$

Table 1 $[X_i, X_j]$ in SU(3)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$-X_7$	X_5	$-X_6$	$-X_3$	X_4	$4X_2$	$2X_2$
X_2	X_7	0	X_6	X_5	$-X_4$	$-X_3$	$-4X_1$	$-2X_1$
X_3	$-X_5$	$-X_6$	0	$-X_8$	X_1	X_2	$2X_4$	$4X_4$
X_4	X_6	$-X_5$	X_8	0	X_2	$-X_1$	$-2X_3$	$-4X_3$
X_5	X_3	X_4	$-X_1$	$-X_2$	0	$X_8 - X_7$	$2X_6$	$-2X_6$
X_6	$-X_4$	X_3	$-X_2$	X_1	$X_7 - X_8$	0	$-2X_5$	$2X_5$
X_7	$-4X_2$	$4X_1$	$-2X_4$	$2X_3$	$-2X_6$	$2X_5$	0	0
X_8	$-2X_2$	$2X_1$	$-4X_4$	$4X_3$	$2X_6$	$-2X_5$	0	0

Consider the following two vector fields:

$$X_7 = -[X_1, X_2] = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_8 = -[X_3, X_4] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$

Since $T_1 = \frac{1}{2}X_7$ and $T_2 = \frac{1}{2\sqrt{3}}X_7 - \frac{1}{\sqrt{3}}X_8$, we choose the vertical vector field $\nabla_{\mathcal{T}} = \{X_7, X_8\}$ as an orthonormal basis of the Cartan subalgebra $\mathcal{T}$.

The following table is [5, Table 2.1], which gives all $[X_i, X_j]$ for any two vector fields $X_i, X_j \in \{X_1, X_2, \dots, X_8\}$.

Table 1 shows that

$$[X_i, X_j] = \sum_{k=1}^{8} C_{i,j}^k X_k \quad \text{for any } i, j \in \{1, 2, \dots, 8\} \tag{2.1}$$

and that

$$[X_i, X_j] = \sum_{k=1}^{6} C_{i,j}^k X_k \quad \text{for any } i \in \{1, 2, \dots, 8\} \text{ and any } j \in \{7, 8\}, \tag{2.2}$$

where $C_{i,j}^k \in \mathbb{R}$ are constants and are completely determined by Table 1. Note that the orthonormal basis $\nabla_{\mathcal{H}} = \{X_1, X_2, \dots, X_6\}$ generates the horizontal subspace $\mathcal{H}$ in SU(3). Since the matries $\mathcal{G}$ are left-invariant vector fields, the basis $\nabla_{\mathcal{H}}$ is left-invariant. From Table 1, at every point of SU(3) the basis $\nabla_{\mathcal{H}}$ satisfies the Hörmander condition. From this, the horizontal distribution of a sub-Riemannian manifold is generated by $\nabla_{\mathcal{H}}$.

In the following, we define the Carnot–Carathéodory distance d_{CC}. An absolutely continuous function curve $\gamma : [0, T] \to \mathrm{SU}(3)$ is subunitary associated to $\nabla_{\mathcal{H}}$ if there are measurable $\{\alpha_i \in L^\infty[0, T]\}_{1 \le i \le 6}$ such that

$$\gamma'(t) = \sum_{i=1}^{6} \alpha_i(t) X_i(\gamma(t)) \quad \text{and} \quad \sum_{i=1}^{6} \alpha_i^2(t) \le 1 \quad \text{for a.e. } t \in [0, T].$$

According to [2], since $\nabla_{\mathcal{H}}$ satisfies the Hörmander condition, there exist curves γ subunitary associated to $\nabla_{\mathcal{H}}$ connecting any two given points $x, y \in \mathrm{SU}(3)$. The Carnot–Carathéodory distance d_{CC} is then defined as

$$d_{CC}(x, y) = \inf\{T \geq 0 : \text{there exists a subunitary curve } \gamma : [0, T] \to \mathrm{SU}(3) \text{ connecting } x \text{ and } y\}.$$

We denote by dx the bi-invariant Harr-measure, by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathrm{SU}(3)$ and by $⨍_E f dx = \frac{1}{|E|}\int_E f dx$ the average of an integrable function f over set E.

In the rest of this section, we recall some Caccioppoli inequalities established by Domokos-Manfredi [5] and use similar methods to get stronger estimates; see [5, Theorem 2.1, Lemmas 3.1, 3.2, 3.3 and 3.4, Remark 3.2 and Corollary 3.1].

Domokos-Manfredi [5] established the following uniform gradient estimate; see [5, Theorem 2.1].

Proposition 2.1 [5, Theorem 2.1] *Let* $1 < p < \infty$ *and* $0 \leq \delta \leq 1$. *Assume that* $a \in C^2(\mathbb{R}^6, \mathbb{R}^6)$ *satisfies the conditions* (1.2), (1.3) *and* (1.4). *If* $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ *is a weak solution to* (1.1), *then* $\nabla_{\mathcal{H}} u \in L^\infty_{\mathrm{loc}}(\Omega)$. *Moreover, for all ball* $B_r \subset \Omega$, *we have*

$$\sup_{B_{r/2}} |\nabla_{\mathcal{H}} u| \leq C(p, L, l_0) \left(⨍_{B_r} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

Combining Proposition 2.1 and [6, Theorem 1.1], we have the following corollary.

Corollary 2.2 *Let* $1 < p < \infty$ *and* $\delta > 0$. *If* $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ *is a weak solution to* (1.1), *then* $u \in C^\infty(\Omega)$.

The following lemma gives some Caccioppoli estimates which will be used to prove Lemmas 4.5, 4.6 and 4.1; see Sect. 2.1 for its proof. For simplicity, we write $w = (\delta + |\nabla_{\mathcal{H}} u|^2)$. Note the fact that in all the integral terms in the following lemma, only $w^{\frac{p-2}{2}}$ includes δ. This fact is necessary for us to establish the Caccioppoli inequality for v in Sect. 4; see Lemma 4.1.

Lemma 2.3 *Let* $0 < \delta < 1$. *Assume that* $a \in C^2(\mathbb{R}^6, \mathbb{R}^6)$ *satisfies the conditions* (1.2), (1.3) *and* (1.4). *If* $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ *is a weak solution to* (1.1), *then for any* $\eta \in C^\infty_0(\Omega)$ *with* $0 \leq \eta \leq 1$, *the followings hold:*

(i) *If* $\beta \geq 0$, *then*

$$\int_\Omega \eta^2 w^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u|^2 dx \leq C(p, L, l_0) \int_\Omega |\nabla_{\mathcal{H}} \eta|^2 w^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{2\beta+2} dx + C(p, L, l_0)(\beta+1)^2 \int_\Omega \eta^2 w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^2 |\nabla_{\mathcal{T}} u|^{2\beta} dx. \tag{2.3}$$

(ii) *If* $\beta \geq 0$, *then*

$$\int_\Omega \eta^2 w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u|^2 dx \leq C(p, L, l_0)(\beta + 1)^4 \int_\Omega \eta^2 w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta} |\nabla_{\mathcal{T}} u|^2 dx + C(p, L, l_0)(\beta+1)^2 K_\eta \int_{supp(\eta)} w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta+2} dx, \tag{2.4}$$

(iii) *If* $\beta \geq 1$, *then*

$$\int_{\Omega} \eta^{2\beta+2} w^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u|^2 dx \leq C(p, L, l_0)^{\beta} (\beta+1)^{4\beta} \|\nabla_{\mathcal{H}} \eta\|_{L^\infty}^{2\beta} \int_{\Omega} \eta^2 w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u|^2 dx. \tag{2.5}$$

(iv) *If* $\beta \geq 1$, *then*

$$\int_{\Omega} \eta^2 w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u|^2 dx \leq C(p, L, l_0)(\beta+1)^{12} K_\eta \int_{supp(\eta)} w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta+2} dx. \tag{2.6}$$

(v) *If* $\beta \geq 1$, *then*

$$\int_{\Omega} \eta^{2\beta+2} w^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u|^2 dx \leq C(p, L, l_0, \beta) K_\eta^{\beta+1} \int_{supp(\eta)} w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta+2} dx. \tag{2.7}$$

(vi) *If* $\beta \geq 1$, *then*

$$\int_{\Omega} \eta^{2\beta+4} w^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u|^2 dx \leq C(p, L, l_0, \beta) K_\eta^{\beta+2} \int_{supp(\eta)} w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta+2} dx. \tag{2.8}$$

Above

$$K_\eta = 1 + \|\nabla_{\mathcal{H}} \eta\|_{L^\infty(\Omega)}^2 + \|\nabla_{\mathcal{T}} \eta\|_{L^\infty(\Omega)}. \tag{2.9}$$

2.1 Proof of Lemma 2.3

In this subsection, we prove Lemma 2.3. To this end, we consider the Riemannian approximation to (1.1) as in [5]. For any fixed constant $\varepsilon \in (0, 1)$, we write $\nabla^\varepsilon = (\nabla^\varepsilon_{\mathcal{H}}, \nabla^\varepsilon_{\mathcal{T}})$, where $\nabla^\varepsilon_{\mathcal{H}} = \nabla_{\mathcal{H}}$ and $\nabla^\varepsilon_{\mathcal{T}} = \varepsilon \nabla_{\mathcal{T}}$. Consider a Riemannian approximation to (1.1), that is,

$$\sum_{i=1}^{8} X_i^\varepsilon (a_i(\nabla^\varepsilon u)) = 0 \text{ in } \Omega; \tag{2.10}$$

see [5, Section 3] for details. Let u_ε be a weak solution to (2.10) with conditions (1.2), (1.3) and (1.4). By [5, Remark 3.1], we obtain that $u_\varepsilon \in C^\infty(\Omega)$ when $\delta > 0$ and $\varepsilon > 0$.

We have the following series of a priori estimates for u_ε, that is, Lemmas 2.4, 2.5, 2.6 and 2.7. They correspond to Lemmas 3.1, 3.2, 3.3 and 3.4 in [5], where they assume (1.2), (1.3) and (1.5). Here we work with (1.2), (1.3) and the stronger one (1.4). For simplicity, we write $w_\varepsilon = (\delta + |\nabla^\varepsilon u_\varepsilon|^2)$.

Lemma 2.4 *Let* $0 < \delta < 1$. *For any* $\beta \geq 0$ *and any* $\eta \in C_0^\infty(\Omega)$ *with* $0 \leq \eta \leq 1$, *we have*

$$\int_{\Omega} \eta^2 w_\varepsilon^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u_\varepsilon|^{2\beta} |\nabla^\varepsilon \nabla_{\mathcal{T}} u_\varepsilon|^2 dx \le C(p, L, l_0) \int_{\Omega} |\nabla^\varepsilon \eta|^2 w_\varepsilon^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u_\varepsilon|^{2\beta+2} dx + C(p, L, l_0)(\beta+1)^2 \int_{\Omega} \eta^2 w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^2 |\nabla_{\mathcal{T}} u_\varepsilon|^{2\beta} dx. \tag{2.11}$$

Lemma 2.4 is a stronger version of [5, Lemma 3.1], since $w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^2$ in the second term in the right hand side of (2.11) is more accurate than $w_\varepsilon^{\frac{p}{2}}$ in the second term in the right hand side of [5, (3.3)]. To prove Lemma 2.4, we follows the proof of [5, Lemma 3.1] line by line. The only difference is that we use the condition (1.4) instead of the condition (1.5) they used in the proof of [5, Lemma 3.1]. Here we omit the details.

Lemma 2.5 *Let* $0 < \delta < 1$. *For any* $\beta \ge 0$ *and any* $\eta \in C_0^\infty(\Omega)$ *with* $0 \le \eta \le 1$, *we have*

$$\begin{aligned} &\int_{\Omega} \eta^2 w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^{2\beta} |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 dx \\ &\le C(p, L, l_0)(\beta+1)^4 \int_{\Omega} \eta^2 w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^{2\beta} |\nabla_{\mathcal{T}} u_\varepsilon|^2 dx \\ &\quad + C(p, L, l_0)(\beta+1)^2 \int_{\Omega} (\eta^2 + |\nabla^\varepsilon \eta|^2 + \eta |\nabla_{\mathcal{T}} \eta|) w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^{2\beta+2} dx. \end{aligned} \tag{2.12}$$

Lemma 2.5 is similar to [5, Lemma 3.2]. The only difference between the two lemmas is that $w_\varepsilon^{\frac{p-2}{2}+\beta}$ and $w_\varepsilon^{\frac{p}{2}+\beta}$ in [5, (3.8)] are replaced by $w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^{2\beta}$ and $w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^{2\beta+2}$ separately. The proof of Lemma 2.5 follows the same line as that of [5, Lemma 3.2]. To prove [5, Lemma 3.2], they used $\phi = \eta^2 w_\varepsilon^\beta X_i^\varepsilon u_\varepsilon$ to test [5, (3.10)]. Now, to prove Lemma 2.5, we use $\phi = \eta^2 |\nabla^\varepsilon u_\varepsilon|^{2\beta} X_i^\varepsilon u_\varepsilon$ instead of $\phi = \eta^2 w_\varepsilon^\beta X_i^\varepsilon u_\varepsilon$ as the new test function in [5, (3.10)] in the proof of [5, Lemma 3.2]. Then we use the condition (1.4) whenever the condition (1.5) is used in the rest of the proof of [5, Lemma 3.2]. Here we omit the details.

Lemma 2.6 follows from Lemma 2.4.

Lemma 2.6 *Let* $0 < \delta < 1$. *For any* $\beta \ge 1$ *and any* $\eta \in C_0^\infty(\Omega)$ *with* $0 \le \eta \le 1$, *we have*

$$\begin{aligned} &\int_{\Omega} \eta^{2\beta+2} w_\varepsilon^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u_\varepsilon|^{2\beta} |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 dx \\ &\le C(p, L, l_0)(\beta+1)^4 \|\nabla^\varepsilon \eta\|_{L^\infty}^2 \int_{\Omega} \eta^{2\beta} w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^2 |\nabla_{\mathcal{T}} u_\varepsilon|^{2\beta-2} |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 dx. \end{aligned} \tag{2.13}$$

Lemma 2.6 is stronger than [5, Lemma 3.3], since $w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^2$ in the right hand side of (2.13) is more accurate than $w_\varepsilon^{\frac{p}{2}}$ in the right hand side of [5, (3.11)]. To prove Lemma 2.6, we follows the proof of [5, Lemma 3.3] line by line. The only difference is that we use (2.11) with $\eta \to \eta^{\beta+2}$ and the condition (1.4) to replace [5, (3.7)] and the condition (1.5) they used in the proof of [5, Lemma 3.3] separately. Here we omit the details.

By Lemma 2.6, we obtain Lemma 2.7.

Lemma 2.7 *Let* $0 < \delta < 1$. *For any* $\beta \ge 1$ *and* $\eta \in C_0^\infty(\Omega)$ *with* $0 \le \eta \le 1$, *we have*

$$\begin{aligned} &\int_{\Omega} \eta^{2\beta+2} w_\varepsilon^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u_\varepsilon|^{2\beta} |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 dx \\ &\le C(p, L, l_0)^\beta (\beta+1)^{4\beta} \|\nabla^\varepsilon \eta\|_{L^\infty}^{2\beta} \int_{\Omega} \eta^2 w_\varepsilon^{\frac{p-2}{2}} |\nabla^\varepsilon u_\varepsilon|^{2\beta} |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 dx. \end{aligned} \tag{2.14}$$

Lemma 2.7 is a stronger version of [5, Lemma 3.4], since $w_\varepsilon^{\frac{p-2}{2}}|\nabla^\varepsilon u_\varepsilon|^{2\beta}$ in the right hand side of (2.14) is more accurate than $w_\varepsilon^{\frac{p-2}{2}+\beta}$ in the right hand side of [5, (3.13)]. To prove Lemma 2.7, we follows the proof of [5, Lemma 3.4] line by line. The only difference is that we use Lemma 2.6 and the condition (1.4) to replace [5, Lemma 3.3] and the condition (1.5) they used in the proof of [5, Lemma 3.4] separately. Here we omit the details.

Lemma 2.3 follows from Lemmas 2.4, 2.5 and 2.7. Specifically, by Theorem 2.2, letting $\varepsilon \to 0$ in the above estimates, we get some intrinsic Cacciopoli inequalities for weak solutions u to (1.1) from Lemmas 2.4, 2.5 and 2.7, which are stronger than that in [5, Corollary 4.1]; see [5, Section 4] for more details.

Proof of Lemma 2.3 The estimates (2.3), (2.4) and (2.5) follow from Lemmas 2.4, 2.5 and 2.7 respectively in a direct way.

Next we prove (2.6). By Hölder's inequality, we have

$$\begin{aligned} &C(p,L,l_0)(\beta+1)^4\int_\Omega \eta^2 w^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^{2\beta}|\nabla_{\mathcal{T}}u|^2dx \\ &\le C(p,L,l_0)(\beta+1)^4\left(\int_\Omega \eta^{2\beta+2}w^{\frac{p-2}{2}}|\nabla_{\mathcal{T}}u|^{2\beta+2}dx\right)^{\frac{1}{\beta+1}} \\ &\quad \left(\int_{supp(\eta)} w^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^{2\beta+2}dx\right)^{\frac{\beta}{\beta+1}}. \end{aligned} \tag{2.15}$$

Noting that $|\nabla_{\mathcal{T}}u|^2 \le 2|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2$, we use (2.5) to bound the first term in the right hand side of (2.15). Then by Young's inequality, we have

$$\begin{aligned} &C(p,L,l_0)(\beta+1)^4\int_\Omega \eta^2 w^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^{2\beta}|\nabla_{\mathcal{T}}u|^2dx \\ &\le \frac{1}{\beta+1}\int_\Omega \eta^2 w^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^{2\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2dx \\ &\quad + C(p,L,l_0)(\beta+1)^{12}\|\nabla_{\mathcal{H}}\eta\|^2_{L^\infty}\int_{supp(\eta)} w^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^{2\beta+2}dx, \end{aligned}$$

which, together with (2.4), yields (2.6).

Combining (2.5) and (2.6), we conclude (2.7).

Next we prove (2.8). By Hölder's inequality, we have

$$\begin{aligned} &C(p,L,l_0,\beta)\int_\Omega \eta^{2\beta+2}w^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^2|\nabla_{\mathcal{T}}u|^{2\beta}dx \\ &\le C(p,L,l_0,\beta)\left(\int_\Omega \eta^{2\beta+2}w^{\frac{p-2}{2}}|\nabla_{\mathcal{T}}u|^{2\beta+2}dx\right)^{\frac{\beta}{\beta+1}}\left(\int_{supp(\eta)} w^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^{2\beta+2}dx\right)^{\frac{1}{\beta+1}}. \end{aligned} \tag{2.16}$$

By changing η to $\eta^{\beta+2}$ in (2.3), we have

$$\begin{aligned} \int_\Omega \eta^{2\beta+4}w^{\frac{p-2}{2}}|\nabla_{\mathcal{T}}u|^{2\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{T}}u|^2dx &\le C(p,L,l_0,\beta)\int_\Omega |\nabla_{\mathcal{H}}\eta|^2\eta^{2\beta+2}w^{\frac{p-2}{2}}|\nabla_{\mathcal{T}}u|^{2\beta+2}dx \\ &\quad + C(p,L,l_0,\beta)\int_\Omega \eta^{2\beta+4}w^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^2|\nabla_{\mathcal{T}}u|^{2\beta}dx. \end{aligned} \tag{2.17}$$

We combine (2.16) and (2.17). Then by Young's inequality therein, we have

$$\int_{\Omega} \eta^{2\beta+4} w^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u|^2 dx \leq C(p, L, l_0, \beta) K_\eta \int_{\Omega} \eta^{2\beta+2} w^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{2\beta+2} dx + C(p, L, l_0, \beta) \int_{supp(\eta)} w^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{2\beta+2} dx. \tag{2.18}$$

Since $|\nabla_{\mathcal{T}} u|^2 \leq 2|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u|^2$, applying (2.7) to the first term in the right hand side of (2.18), we conclude (2.8). □

2.2 Sobolev inequalities

In this subsection, we recall local Sobolev inequalities and Poincaré inequalities on SU(3). Denote by $Q = 10$ the homogeneous dimension of SU(3). The following lemma follows from [10, Theorem 11.20 and Corollary 9.8] and [1, Proposition 2.1].

Lemma 2.8 *Let $1 \leq p_1 < Q$. For any ball $B_\rho \subset \Omega$, we have*

$$\left(\fint_{B_\rho} |f|^{\frac{Qp_1}{Q-p_1}} dx \right)^{\frac{Q-p_1}{Qp_1}} \leq C(p_1) \rho \left(\fint_{B_\rho} |\nabla_{\mathcal{H}} f|^{p_1} dx \right)^{\frac{1}{p_1}}$$

for any $f \in C_0^\infty(B_\rho)$, and

$$\left(\fint_{B_\rho} |f - f_{B_\rho}|^{\frac{Qp_1}{Q-p_1}} dx \right)^{\frac{Q-p_1}{Qp_1}} \leq C(p_1) \rho \left(\fint_{B_\rho} |\nabla_{\mathcal{H}} f|^{p_1} dx \right)^{\frac{1}{p_1}}$$

for any $f \in C^\infty(B_\rho)$, where $f_{B_\rho} = \fint_{B_\rho} f dx$.

3 De Giorgi's class of functions

In this section, we recall De Giorgi's class of functions defined in SU(3). Let Ω be a domain in SU(3).

Definition 3.1 Let $B_{\rho_0} \subset \Omega$ be a ball. We call a function $u \in W^{2,2}_{\mathcal{H}}(B_{\rho_0}) \cap L^\infty(B_{\rho_0})$ belongs to $DG^+(B_{\rho_0})$ if there exists non-negative constants γ and χ such that for any balls $B_{\rho'}$, B_ρ with the same center as B_{ρ_0} and $0 < \rho' < \rho < \rho_0$, for any $k \in \mathbb{R}$, and for any $q > Q$, the inequality

$$\int_{A^+_{k,\rho'}} |\nabla_{\mathcal{H}} u|^2 dx \leq \frac{\gamma}{(\rho - \rho')^2} \int_{A^+_{k,\rho}} |u - k|^2 dx + \chi^2 |A^+_{k,\rho}|^{1-2/q} \tag{3.1}$$

holds true, where $A^+_{k,\rho}(u) = \{x \in B_\rho : (u(x) - k)^+ = \max(u(x) - k, 0) > 0\}$.

Remark 3.2 By replacing $A^+_{k,\rho}(u)$ with $A^-_{k,\rho}(u)$ in (3.1), we define that $u \in DG^-(B_{\rho_0})$ in a similar way, where $A^-_{k,\rho}(u) = \{x \in B_\rho : (u(x) - k)^- = \min(u(x) - k, 0) < 0\}$.

It is apparent that $u \in DG^-(B_{\rho_0})$ if $-u \in DG^+(B_{\rho_0})$. Denote $DG(B_{\rho_0}) = DG^+(B_{\rho_0}) \cap DG^-(B_{\rho_0})$.

The following lemma is almost the same as [25, Lemma 4.1]; see the "Appendix" for its proof.

Lemma 3.3 *For any $b \in (0, 1)$, there exists a constant $\theta_1 = \theta_1(\gamma, q, b) \in (0, 1)$ such that for any function $u \in DG^+(B_\rho)$ and for any constant k, the following holds:*
If

$$H = \sup_{B_\rho} u(x) - k \geq \chi\rho^{1-Q/q}$$

for any $q > Q$, then

$$\left|A^+_{k,\rho}\right| \leq \theta_1 |B_\rho|$$

implies that

$$\sup_{B_{\rho/2}} u(x) \leq k + bH.$$

The following lemma is almost the same as [25, Lemma 4.2]; see the "Appendix" for the details of its proof.

Lemma 3.4 *For any $\tau \in (0, 1]$, there exists a constant $s = s(\gamma, q, \tau, R) > 0$ such that for any function $u \in DG^+(B_\rho)$ and for any $k \in \mathbb{R}$, the following holds:*
If

$$\left|A^-_{k,\rho/2}\right| \geq \tau |B_{\rho/2}|,$$

then

$$\sup_{B_{\rho/4}} u(x) \leq \sup_{B_\rho} u(x) - 2^{-s} H + \chi\rho^{1-\frac{Q}{q}}, \tag{3.2}$$

where $H = \sup_{B_\rho} u(x) - k$.

Remark 3.5 Changing k to $-k$ and $u(x)$ to $-u(x)$ in Lemma 3.4, for any function $u \in DG^-(B_\rho)$ the following holds:

$$|A^+_{k,\rho/2}| \geq \tau |B_{\rho/2}|$$

implies that

$$\inf_{B_{\rho/4}} u(x) \geq \inf_{B_\rho} u(x) + 2^{-s} H - \chi\rho^{1-\frac{Q}{q}},$$

where $H = -\inf_{B_\rho} u(x) + k$.

Applying Lemma 3.4 and Remark 3.5 with $k = \frac{1}{2}\left(\sup_{B_{\rho/4}} u(x) + \inf_{B_{\rho/4}} u(x)\right)$, we have the following theorem. We omit its proof.

Theorem 3.6 *For any function $u \in DG(B_\rho)$, there exists $s_0 = s_0(\gamma, q) > 0$ such that*

$$\mathrm{osc}_{B_{\rho/4}} u(x) \leq (1 - 2^{-s_0})\mathrm{osc}_{B_\rho} u(x) + 2\chi\rho^{1-Q/q}.$$

4 Crucial Caccioppli inequality

In this section, we give the crucial Caccioppli inequality and its proof. Let $1 < p < \infty$ and $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ be a weak solution to (1.1) with $0 < \delta \le 1$. Fix a ball $B_{r_0} \subset \Omega$. For any ball B_r with the same center as B_{r_0} and $0 < r \le r_0/16$, we denote $\varpi(r) = \max\limits_{1\le k\le 6} \operatorname{osc}_{B_r} X_k u$ and

$$v = \min(\mu(r)/8, \max(\mu(r)/4 - X_l u, 0)),$$

where $\mu(r) = \max_{1\le k\le 6} \sup_{B_r} |X_k u|$ and $l \in \{1, \dots, 6\}$. For any fixed $l \in \{1, \dots, 6\}$, we consider the set

$$E = \{x \in \Omega : \mu(r)/8 < X_l u < \mu(r)/4\}.$$

Note that for any $i \in \{1, \dots, 6, 7, 8\}$,

$$X_i v = -X_i X_l u \quad \text{in } E \quad \text{and} \quad X_i v = 0 \quad \text{in } \Omega \setminus E. \tag{4.1}$$

We have the following lemma, which includes the crucial Caccioppli inequality for v. In this paper, for convenience we write $a_{i,j}(\xi) = \frac{\partial a_i(\xi)}{\partial \xi_j}$ for any $\xi \in \mathbb{R}^6$ and $i, j \in \{1, 2, \dots, 6\}$.

Lemma 4.1 *Let $B_r \subset \Omega$ be a ball. Assume that the cut-off function $\eta \in C_0^\infty(B_r)$ satisfies (4.10) and (4.11). Then for any $\gamma > 1$ and any $\beta \ge 0$, we have*

$$\int_\Omega \eta^{\beta+4} v^{\beta+2} |\nabla_{\mathcal{H}} v|^2 dx \le C(p, L, l_0, \gamma)(\beta+2)^2 \mu(r)^4 \left(1 + \frac{1}{r^2}\right) \times |B_r|^{1-\frac{1}{\gamma}} \left(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta} dx\right)^{\frac{1}{\gamma}}. \tag{4.2}$$

Remark 4.2 For any $l \in \{1, \dots, 6\}$, we use the same method as that of Lemma 4.1 to get (4.2) with

$$v' = \min(\mu(r)/8, \max(\mu(r)/4 + X_l u, 0)).$$

From Lemma 4.1 and Remark 4.2, we get the following lemma; see Sect. 6 for its proof.

Lemma 4.3 *For any ball $B_r \subset \Omega$ with the same center B_R and $0 < r \le R$, there is $\theta = \theta(p, L, l_0, R) > 0$ such that the followings hold:*

(i) *If*

$$|\{x \in B_r : X_k u < \mu(r)/4\}| \le \theta |B_r| \tag{4.3}$$

holds true for an index $k \in \{1, \dots, 6\}$, then

$$\inf_{B_{r/2}} X_k u \ge 3\mu(r)/16. \tag{4.4}$$

(ii) *If*

$$|\{x \in B_r : X_k u > -\mu(r)/4\}| \le \theta |B_r| \tag{4.5}$$

holds true for an index $k \in \{1, \dots, 6\}$, then

$$\sup_{B_{r/2}} X_k u \le -3\mu(r)/16. \tag{4.6}$$

4.1 Proof of Lemma 4.1

Before proving Lemma 4.1, we need the following lemmas. See Sect. 6 for their proofs.

Lemma 4.4 *For any $l \in \{1, \ldots, 6, 7, 8\}$, $X_l u$ is a weak solution to the equation*

$$\sum_{i,j=1}^{6} X_i(a_{i,j}(\nabla_{\mathcal{H}}u)X_jX_lu) + \sum_{i,j=1}^{6} X_i(a_{i,j}(\nabla_{\mathcal{H}}u)[X_l, X_j]u)$$
$$- \sum_{i=1}^{6}[X_i, X_l](a_i(\nabla_{\mathcal{H}}u)) = 0. \tag{4.7}$$

Lemma 4.5 *For any $\beta \geq 0$ and any non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \eta^{\beta+2} v^{\beta+2} \big(\delta + |\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}}u|^2 |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2 dx$$
$$\leq C(p, L, l_0)(\beta+2)^2 \int_\Omega \eta^\beta(\eta^2 + |\nabla_{\mathcal{H}}\eta|^2 + \eta|\nabla_{\mathcal{T}}\eta|)v^{\beta+2}\big(\delta + |\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}}u|^4 dx$$
$$+ C(p, L, l_0)(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^\beta \big(\delta + |\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}}u|^4 |\nabla_{\mathcal{H}}v|^2 dx$$
$$+ C(p, L, l_0) \int_\Omega \eta^{\beta+2} v^{\beta+2} \big(\delta + |\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}}u|^2 |\nabla_{\mathcal{T}}u|^2 dx. \tag{4.8}$$

Before starting the following lemma, we note that if $\delta \geq 0$ and $q \geq 0$ such that $p+q-2 \geq 0$, the following holds:

$$\big(\delta + |\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}}u|^q \leq C(p, q)(\delta + \mu(r)^2)^{\frac{p-2}{2}} \mu(r)^q \text{ in } B_r, \tag{4.9}$$

where $C(p, q) = 6^{(q+p-2)/2}$ when $p \geq 2$ and $C(p, q) = 6^{q/2}$ when $1 < p < 2$. In the rest of this section, for any fixed ball $B_r \subset \Omega$ we consider a cut-off function $\eta \in C_0^\infty(B_r)$ such that

$$0 \leq \eta \leq 1 \quad \text{in } B_r, \quad \eta = 1 \quad \text{in } B_{r/2} \tag{4.10}$$

and

$$|\nabla_{\mathcal{H}}\eta| \leq C/r, \quad |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}\eta| \leq C/r^2, \quad |\nabla_{\mathcal{T}}\eta| \leq C/r^2 \quad \text{in } B_r. \tag{4.11}$$

The following lemma gives a Caccioppoli inequality for $\nabla_{\mathcal{T}}u$ weighted with $|\nabla_{\mathcal{H}}u|^4$ involving v; see Sect. 6 for its proof.

Lemma 4.6 *Let $B_r \subset \Omega$ be a ball. Assume that the cut-off function $\eta \in C_0^\infty(B_r)$ satisfies (4.10) and (4.11). Then for any $\tau \in (1/2, 1)$, any $\gamma \in (1, 2)$ and any $\beta \geq 0$, we have*

$$\int_\Omega \eta^{\tau(\beta+2)+4} v^{\tau(\beta+4)} (\delta + |\nabla_{\mathcal{H}}u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}}u|^4 |\nabla_{\mathcal{H}}\nabla_{\mathcal{T}}u|^2 dx$$
$$\leq C(p, L, l_0, \tau, \gamma)(\beta+2)^{2\tau} \left(1 + \frac{1}{r^{2(2-\tau)}}\right) |B_r|^{1-\tau} (\delta + \mu(r)^2)^{\frac{p-2}{2}} \mu(r)^6 J^\tau, \tag{4.12}$$

where

$$J=\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\nabla_{\mathcal{H}}v|^2dx+\left(1+\frac{1}{r^2}\right)\mu(r)^4|B_r|^{1-\frac{1}{\gamma}}\left(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}dx\right)^{\frac{1}{\gamma}}. \tag{4.13}$$

Now we use Lemmas 4.4, 4.5 and 4.6 to prove Lemma 4.1.

Proof of Lemma 4.1 We consider two cases: $1<\gamma<3/2$ and $\gamma\geq 3/2$. In the case that $\gamma\geq 3/2$, we use Hölder's inequality for the integral in the right hand side of (4.2) when (4.2) holds for some $1<\gamma_0<3/2$. Below we assume $1<\gamma<3/2$. Recall that

$$v=\min(\mu(r)/8,\max(\mu(r)/4-X_lu,0)),$$

where $\mu(r)=\max_{1\leq k\leq 6}\sup_{B_r}|X_ku|$ and $l\in\{1,\dots,6\}$. Let $B_r\subset\Omega$ be a ball. Consider the cut-off function $\eta\in C_0^\infty(\Omega)$ with (4.10) and (4.11). For any $\beta\geq 0$ and any $l\in\{1,\dots,6\}$, letting $\varphi=\eta^{\beta+4}v^{\beta+3}$ be a test function in (4.7), we have

$$\begin{aligned}
&-(\beta+3)\int_\Omega\sum_{i,j=1}^6\eta^{\beta+4}v^{\beta+2}a_{i,j}(\nabla_{\mathcal{H}}u)X_jX_luX_ivdx\\
&=(\beta+4)\int_\Omega\sum_{i,j=1}^6\eta^{\beta+3}v^{\beta+3}a_{i,j}(\nabla_{\mathcal{H}}u)X_jX_luX_i\eta dx\\
&\quad+(\beta+4)\int_\Omega\sum_{i,j=1}^6\eta^{\beta+3}v^{\beta+3}a_{i,j}(\nabla_{\mathcal{H}}u)[X_l,X_j]uX_i\eta dx\\
&\quad+(\beta+3)\int_\Omega\sum_{i,j=1}^6\eta^{\beta+4}v^{\beta+2}a_{i,j}(\nabla_{\mathcal{H}}u)[X_l,X_j]uX_ivdx\\
&\quad+\int_\Omega\sum_{i=1}^6[X_i,X_l](a_i(\nabla_{\mathcal{H}}u))\eta^{\beta+4}v^{\beta+3}dx,
\end{aligned}$$

which, together with $X_lX_ju=X_jX_lu+[X_l,X_j]u$, yields

$$\begin{aligned}
L_l&:=-(\beta+3)\int_\Omega\sum_{i,j=1}^6\eta^{\beta+4}v^{\beta+2}a_{i,j}(\nabla_{\mathcal{H}}u)X_jX_luX_ivdx\\
&=(\beta+4)\int_\Omega\sum_{i=1}^6\eta^{\beta+3}v^{\beta+3}X_l(a_i(\nabla_{\mathcal{H}}u))X_i\eta dx\\
&\quad+(\beta+3)\int_\Omega\sum_{i,j=1}^6\eta^{\beta+4}v^{\beta+2}a_{i,j}(\nabla_{\mathcal{H}}u)[X_l,X_j]uX_ivdx\\
&\quad+\int_\Omega\sum_{i=1}^6[X_i,X_l](a_i(\nabla_{\mathcal{H}}u))\eta^{\beta+4}v^{\beta+3}dx=:I_1+I_2+I_3.
\end{aligned}\tag{4.14}$$

Recalling that

$$E=\{x\in\Omega:\mu(r)/8<X_lu<\mu(r)/4\},$$

and that

$$X_i v = -X_i X_l u \quad \text{in } E \quad \text{and} \quad X_i v = 0 \quad \text{in } \Omega \setminus E,$$

by the condition (1.2), we have

$$\begin{aligned} L_l &\geq l_0(\beta+3)\int_\Omega \eta^{\beta+4} v^{\beta+2}(\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} v|^2 dx \\ &\geq C_0(p, L, l_0)(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}} \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\nabla_{\mathcal{H}} v|^2 dx. \end{aligned}$$

Here the integration domain in the right hand side of the above inequality is the set E.

Next, we bound each item in the right hand of (4.14) in turn. By integration by parts, we have

$$I_1 = -(\beta+4)\int_\Omega \sum_{i=1}^{6} a_i(\nabla_{\mathcal{H}} u) X_l(\eta^{\beta+3} v^{\beta+3} X_i \eta) dx.$$

By the condition (1.4) and (4.9), we have

$$\begin{aligned} |I_1| &\leq C(p, L, l_0)(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^{\beta+3} \big(\delta + |\nabla_{\mathcal{H}} u|^2\big)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|(|\nabla_{\mathcal{H}} \eta|^2 + \eta|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} \eta|) dx \\ &\quad + C(p, L, l_0)(\beta+2)^2 \int_\Omega \eta^{\beta+3} v^{\beta+2} \big(\delta + |\nabla_{\mathcal{H}} u|^2\big)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u||\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}} \eta| dx \\ &\leq \frac{C(p, L, l_0)}{r^2}(\beta+2)^2(\delta+\mu(r)^2)^{\frac{p-2}{2}} \mu(r)^4 \int_{B_r} \eta^\beta v^\beta dx \\ &\quad + \frac{C(p, L, l_0)}{r}(\beta+2)^2(\delta+\mu(r)^2)^{\frac{p-2}{2}} \mu(r)^2 \int_{B_r} \eta^{\beta+2} v^{\beta+1} |\nabla_{\mathcal{H}} v| dx. \end{aligned} \tag{4.15}$$

Applying Young's inequality to the final item in the right hand side of (4.15), we have

$$\begin{aligned} |I_1| &\leq \frac{C_0(p, L, l_0)}{1000}(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}} \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\nabla_{\mathcal{H}} v|^2 dx \\ &\quad + \frac{C(p, L, l_0)}{r^2}(\beta+2)^3(\delta+\mu(r)^2)^{\frac{p-2}{2}} \mu(r)^4 \int_{B_r} \eta^\beta v^\beta dx. \end{aligned}$$

For I_2, by (2.1) and the condition (1.3), we have

$$\begin{aligned} |I_2| &\leq C(p, L, l_0)(\beta+2) \int_\Omega \eta^{\beta+4} v^{\beta+2} \big(\delta + |\nabla_{\mathcal{H}} u|^2\big)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u||\nabla_{\mathcal{H}} v| dx \\ &\quad + C(p, L, l_0)(\beta+2) \int_\Omega \eta^{\beta+4} v^{\beta+2} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{T}} u| dx =: I_{21} + I_{22}. \end{aligned} \tag{4.16}$$

For I_{21}, by (4.9) and Young's inequality, we have

$$\begin{aligned} I_{21} &\leq C(p, L, l_0)(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}} \mu(r)^2 \int_{B_r} \eta^{\beta+2} v^{\beta+1} |\nabla_{\mathcal{H}} v| dx \\ &\leq \frac{C_0(p, L, l_0)}{2000}(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}} \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\nabla_{\mathcal{H}} v|^2 dx \\ &\quad + C(p, L, l_0)(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}} \mu(r)^4 \int_{B_r} \eta^\beta v^\beta dx. \end{aligned} \tag{4.17}$$

For I_{22}, by Hölder's inequality with $q=\frac{2\gamma}{\gamma-1}$, we have

$$I_{22}\le C(p,L,l_0,\gamma)(\beta+2)\left(\int_E \eta^{\beta+4}v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}v|^2dx\right)^{\frac{1}{2}}$$
$$\times\left(\int_E \eta^{\gamma(\beta+2)}v^{\gamma(\beta+2)}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}dx\right)^{\frac{1}{2\gamma}}\left(\int_\Omega \eta^q\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{T}}u|^q dx\right)^{\frac{1}{q}}. \tag{4.18}$$

Here the integration domains in (4.16) and (4.18) are essentially the set E. Applying (2.7) with $\beta=\frac{q-2}{2}$ in Lemma 2.3 to the final integral in the right hand side of (4.18), we have

$$\int_\Omega \eta^q\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{T}}u|^q dx$$
$$\le C(p,L,l_0,\gamma)\left(1+\frac{1}{r^q}\right)\int_{B_r}(\delta+|\nabla_{\mathcal{H}}u|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^q dx. \tag{4.19}$$

Combining (4.18) and (4.19), from (4.9), we have

$$I_{22}\le C(p,L,l_0,\gamma)(\beta+2)\left(1+\frac{1}{r}\right)|B_r|^{\frac{1}{q}}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^2$$
$$\times\left(\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\nabla_{\mathcal{H}}v|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}dx\right)^{\frac{1}{2\gamma}}. \tag{4.20}$$

Combining (4.16), (4.17) and (4.20), we have

$$|I_2|\le\frac{C_0(p,L,l_0)}{1000}(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}}\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\nabla_{\mathcal{H}}v|^2dx$$
$$+C(p,L,l_0,\gamma)(\beta+2)^3\left(1+\frac{1}{r^2}\right)|B_r|^{1-\frac{1}{\gamma}}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^4$$
$$\times\left(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}dx\right)^{\frac{1}{\gamma}}.$$

For I_3, by integration by parts and (2.1), we have

$$I_3=-\int_\Omega\sum_{i=1}^6 a_i(\nabla_{\mathcal{H}}u)[X_i,X_l](\eta^{\beta+4}v^{\beta+3})dx$$
$$=-\sum_{i=1}^6\sum_{k=1}^8 C_{i,l}^k\int_\Omega a_i(\nabla_{\mathcal{H}}u)X_k(\eta^{\beta+4}v^{\beta+3})dx.$$

Since

$$X_kv=-X_kX_lu=-X_lX_ku+[X_l,X_k]u\quad\text{in } E$$

holds for any $k\in\{7,8\}$, by (2.2) and the condition (1.4), we have

$$|I_3|\le C(p,L,l_0)(\beta+2)\int_\Omega\eta^{\beta+3}v^{\beta+3}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u||\nabla_{\mathcal{T}}\eta|dx$$
$$+C(p,L,l_0)(\beta+2)\int_E\eta^{\beta+4}v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u||\nabla_{\mathcal{H}}\nabla_{\mathcal{T}}u|dx$$

$$
\begin{aligned}
&+C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+3}v^{\beta+3}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u||\nabla_{\mathcal H}\eta|dx\\
&+C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+4}v^{\beta+2}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u||\nabla_{\mathcal H}v|dx\\
&+C(p,L,l_0)(\beta+2)\int_E \eta^{\beta+4}v^{\beta+2}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^2dx\\
=:&\ I_{31}+I_{32}+I_{33}+I_{34}+I_{35}.
\end{aligned}
$$

For I_{31}, I_{33} and I_{35}, by (4.9), we have

$$
\begin{aligned}
I_{31}&\le \frac{C(p,L,l_0)(\beta+2)}{r^2}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^4\int_{B_r}\eta^\beta v^\beta dx,\\
I_{33}&\le \frac{C(p,L,l_0)(\beta+2)}{r}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^4\int_{B_r}\eta^\beta v^\beta dx,\\
I_{35}&\le C(p,L,l_0)(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^4\int_{B_r}\eta^\beta v^\beta dx.
\end{aligned}
$$

For I_{34}, by (4.9) and Young's inequality, we have

$$
\begin{aligned}
I_{34}\le &\frac{C_0(p,L,l_0)}{1000}(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}}\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\nabla_{\mathcal H}v|^2dx\\
&+C(p,L,l_0)(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^4\int_{B_r}\eta^\beta v^\beta dx.
\end{aligned}
$$

For I_{32}, Hölder's inequality yields

$$
\begin{aligned}
I_{32}\le &C(p,L,l_0)(\beta+2)\left(\int_E \eta^{\gamma(\beta+2)}v^{\gamma\beta+4(\gamma-1)}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}dx\right)^{\frac12}\\
&\times\left(\int_E \eta^{(2-\gamma)(\beta+2)+4}v^{(2-\gamma)(\beta+4)}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^2|\nabla_{\mathcal H}\nabla_{\mathcal T}u|^2dx\right)^{\frac12}.
\end{aligned}
$$

Noting that the integration domains in the above inequality are the set E, we have

$$
I_{32}\le C(p,L,l_0)(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{4}}\mu(r)^{2(\gamma-1)-1}M^{\frac12}\left(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\right)^{\frac12},\tag{4.21}
$$

where

$$
M:=\int_\Omega \eta^{(2-\gamma)(\beta+2)+4}v^{(2-\gamma)(\beta+4)}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4|\nabla_{\mathcal H}\nabla_{\mathcal T}u|^2dx.
$$

We use Lemma 4.6 with $\tau=2-\gamma$ to estimate M. Thus

$$
M\le C(p,L,l_0,\gamma)(\beta+2)^{2(2-\gamma)}\left(1+\frac{1}{r^{2\gamma}}\right)|B_r|^{\gamma-1}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^6J^{2-\gamma},\tag{4.22}
$$

where J is as in (4.12). Combining (4.21) and (4.22), we have

$$
\begin{aligned}
I_{32}\le &C(p,L,l_0,\gamma)(\beta+2)^{3-\gamma}\left(1+\frac{1}{r^\gamma}\right)|B_r|^{\frac{\gamma-1}{2}}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^{2\gamma}J^{\frac{2-\gamma}{2}}\\
&\times\left(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\right)^{\frac12},
\end{aligned}
$$

where J is as in (4.12). From this, by Young's inequality again, we have

$$
\begin{aligned}
I_{32} \le\ & \frac{C_0(p,L,l_0)}{1000}(\beta+2)(\delta+\mu(r)^2)^{\frac{p-2}{2}} J \\
& + C(p,L,l_0,\gamma)(\beta+2)^{\frac{4}{\gamma}-1}\left(1+\frac{1}{r^2}\right)|B_r|^{1-\frac{1}{\gamma}}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^4 \\
& \times\left(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}dx\right)^{\frac{1}{\gamma}},
\end{aligned}
$$

where J is as in (4.12).

Combining all the above estimates together, we use Hölder's inequality to conclude (4.2). □

5 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by considering two cases: $\delta > 0$ and $\delta = 0$. First, we prove Theorem 1.1 for the case $\delta > 0$.

The following lemma is similar to [5, Lemma 4.1]. It is a preparation for proving Theorem 1.1. We omit its proof.

Lemma 5.1 *Let $1 < p < 2$ and $u \in C^\infty(\Omega)$ be a solution to (1.1) with $\delta > 0$. Let $B_{3r_0/2}$ be a ball in Ω. Assume that for any ball B_r with the same center as $B_{3r_0/2}$ and $0 < r < r_0/2$, there is $\tau > 0$ such that*

$$|\nabla_{\mathcal{H}}u| \ge \tau\mu(r) \quad \text{in } A^+_{k,r}(X_l u) \tag{5.1}$$

holds for an index $l \in \{1, 2, \ldots, 6\}$ and for a constant $k \in \mathbb{R}$. Then for any $q \ge 4$ and any $0 < r'' < r' \le r$, the following holds:

$$
\begin{aligned}
&\int_{B_{r''}}\left(\delta+|\nabla_{\mathcal{H}}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}(X_l u-k)^+|^2dx \\
&\le \frac{C(p,L,l_0,q,\tau,r_0)}{(r'-r'')^2}\int_{B_{r'}}(\delta+|\nabla_{\mathcal{H}}u|^2)^{\frac{p-2}{2}}|(X_l u-k)^+|^2dx \\
&\quad + C(p,L,l_0,q,\tau,r_0)(\delta+\mu(r_0)^2)^{\frac{p}{2}}|A^+_{k,r'}(X_l u)|^{1-\frac{2}{q}}.
\end{aligned}
\tag{5.2}
$$

Here the range of p in Lemma 5.1 is $1 < p < 2$. When $1 < p < 2$, we use the extra assumption (5.1) to replace the condition $2 \le p < \infty$ in the proof of [5, Lemma 4.1]. Since the assumption (5.1) implies that

$$\left(\delta+|\nabla_{\mathcal{H}}u|^2\right)^{\frac{p-2}{2}} \le C(\tau)(\delta+\mu(r)^2)^{\frac{p-2}{2}} \quad \text{in } A^+_{k,r}(X_l u),$$

we can prove Lemma 5.1 by following the proof of [5, Lemma 4.1] line by line. We omit the details.

Remark 5.2 By changing $(X_l u-k)^+$ to $(X_l u-k)^-$ and $A^+_{k,r}(X_l u)$ to $A^-_{k,r}(X_l u)$, we get the same conclusion as (5.2).

By an interation argument (see [9, Lemma 7.3]), Theorem 1.1 follows from the following theorem. For simplicity we write $\varpi(r) = \max_{1\le k\le 6} \operatorname{osc}_{B_r} X_k u$; see Sect. 4 for details.

Theorem 5.3 *Let $1 < p < 2$ and B_{r_0} be a ball in Ω. There is $s = s(p, L, l_0, r_0) \geq 1$ such that for any $0 < r \leq r_0/16$ the following holds:*

$$\varpi(r/2) \leq (1 - 2^{-s})\varpi(8r) + 2^s(\delta + \mu(r_0)^2)^{\frac{1}{2}} \left(\frac{r}{r_0}\right)^{\frac{1}{2}}. \tag{5.3}$$

Proof Let B_{r_0} be a ball in Ω. For any ball B_r with the same center as B_{r_0} and $0 < r \leq r_0/16$, we may assume that

$$\varpi(r/2) \geq (\delta + \mu(r_0)^2)^{\frac{1}{2}} \left(\frac{r}{r_0}\right)^{\frac{1}{2}}, \tag{5.4}$$

otherwise, the inequality (5.3) holds true for $s = 1$. Below we assume that (5.4) holds true. To prove Theorem 5.3, we consider the following two cases.

Case 1. There exists a constant $\theta = \theta(p, L, l_0, r_0) > 0$ as in Corollary 4.3 such that either

$$|\{x \in B_{4r} : X_l u < \mu(4r)/4\}| \leq \theta |B_{4r}| \tag{5.5}$$

or

$$|\{x \in B_{4r} : X_l u > -\mu(4r)/4\}| \leq \theta |B_{4r}| \tag{5.6}$$

holds true for at least one index $l \in \{1, \ldots, 6\}$. Below we assume that (5.5) holds true. We use the same method to deal with the case (5.6). By (i) in Corollary 4.3, we have

$$|X_l u| \geq 3\mu(4r)/16 \quad \text{in } B_{2r}.$$

Thus

$$|\nabla_{\mathcal{H}} u| \geq 3\mu(2r)/16 \quad \text{in } B_{2r}.$$

From this, Lemma 5.1 with $q = 2Q = 20$ implies that

$$\int_{B_{r''}} |\nabla_{\mathcal{H}}(X_i u - k)^+|^2 dx \leq \frac{C(p, L, l_0, r_0)}{(r' - r'')^2} \int_{B_{r'}} |(X_i u - k)^+|^2 dx + C(p, L, l_0, r_0) \times (\delta + \mu(r_0)^2)^{\frac{p}{2}} (\delta + \mu(2r)^2)^{\frac{2-p}{2}} |A^+_{k,r'}(X_i u)|^{1-\frac{1}{Q}}$$

holds true for any $0 < r'' < r' \leq 2r$, any $i \in \{1, \ldots, 6\}$ and any constant k. Hence for each $i \in \{1, \ldots, 6\}$, we have $X_i u \in DG^+(B_{2r})$; see Definition 3.1 for details. On the other hand, Remark 5.2 implies that $X_i u \in DG^-(B_{2r})$ holds true for each $i \in \{1, \ldots, 6\}$. According to Remark 3.2, for each $i \in \{1, \ldots, 6\}$ we have $X_i u \in DG(B_{2r})$. By Theorem 3.6, there exists $s_0 = s_0(p, L, l_0, r_0) > 0$ such that for each $i \in \{1, \ldots, 6\}$, we have

$$\text{osc}_{B_{r/2}} X_i u \leq (1 - 2^{-s_0})\text{osc}_{B_{2r}} X_i u + C(p, L, l_0, r_0)(\delta + \mu(r_0)^2)^{\frac{p}{4}} (\delta + \mu(2r)^2)^{\frac{2-p}{4}} r^{\frac{1}{2}}. \tag{5.7}$$

Since $1 < p < 2$, one has

$$(\delta + \mu(2r)^2)^{\frac{2-p}{4}} \leq (\delta + \mu(r_0)^2)^{\frac{2-p}{4}},$$

which, together with (5.7), yields

$$\text{osc}_{B_{r/2}} X_i u \leq (1 - 2^{-s_0})\text{osc}_{B_{2r}} X_i u + C(p, L, l_0, r_0)(\delta + \mu(r_0)^2)^{\frac{1}{2}} \left(\frac{r}{r_0}\right)^{\frac{1}{2}}.$$

Choosing $s_0 = s_0(p, L, l_0, r_0) \geq 1$ large enough such that $2^{s_0} \geq C(p, L, l_0, r_0)$, we conclude (5.3) in this case.

Case 2. If Case 1 does not happen, then

$$|\{x \in B_{4r} : X_i u < \mu(4r)/4\}| > \theta |B_{4r}| \tag{5.8}$$

and

$$|\{x \in B_{4r} : X_i u > -\mu(4r)/4\}| > \theta |B_{4r}| \tag{5.9}$$

hold true for every $i \in \{1, \ldots, 6\}$, where the constant $\theta = \theta(p, L, l_0, r_0) > 0$ is as in Corollary 4.3. Consider the set $\{x \in B_{8r} : X_i u > \mu(8r)/4\}$. Since

$$|\nabla_{\mathcal{H}} u| \geq \mu(8r)/4 \quad \text{in } A^+_{k,8r}(X_i u)$$

holds true for all $k \geq \mu(8r)/4$, by Lemma 5.1 with $q = 2Q = 20$, for any $0 < r'' < r' \leq 8r$, any $i \in \{1, \ldots, 6\}$ and any $k \geq k_0 = \mu(8r)/4$, we have

$$\begin{aligned}\int_{B_{r''}} |\nabla_{\mathcal{H}}(X_i u - k)^+|^2 dx \leq \frac{C(p, L, l_0, r_0)}{(r' - r'')^2} \int_{B_{r'}} |(X_i u - k)^+|^2 dx + C(p, L, l_0, r_0) \\ \times (\delta + \mu(r_0)^2)^{\frac{p}{2}} (\delta + \mu(8r)^2)^{\frac{2-p}{2}} |A^+_{k,r'}(X_i u)|^{1-\frac{1}{Q}}.\end{aligned}$$

Since (5.8) implies that

$$|\{x \in B_{4r} : X_i u < \mu(8r)/4\}| > \theta |B_{4r}|,$$

by Lemma 3.4, there exists a constant $s_1 = s_1(p, L, l_0, r_0) > 0$ such that

$$\begin{aligned}\sup_{B_{2r}} X_i u \leq \sup_{B_{8r}} X_i u - 2^{-s_1} \left(\sup_{B_{8r}} X_i u - \mu(8r)/4 \right) \\ + C(p, L, l_0, r_0)(\delta + \mu(r_0)^2)^{\frac{p}{4}} (\delta + \mu(8r)^2)^{\frac{2-p}{4}} r^{\frac{1}{2}}.\end{aligned} \tag{5.10}$$

On the other hand, by (5.9) and Remark 3.5, we have

$$\begin{aligned}\inf_{B_{2r}} X_i u \geq \inf_{B_{8r}} X_i u + 2^{-s_1} \left(-\inf_{B_{8r}} X_i u - \mu(8r)/4 \right) \\ - C(p, L, l_0, r_0)(\delta + \mu(r_0)^2)^{\frac{p}{4}} (\delta + \mu(8r)^2)^{\frac{2-p}{4}} r^{\frac{1}{2}}.\end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11), we have

$$\begin{aligned}\varpi(2r) \leq (1 - 2^{-s_1})\varpi(8r) + 2^{-s_1-1}\mu(8r) \\ + C(p, L, l_0, r_0)(\delta + \mu(r_0)^2)^{\frac{p}{4}} (\delta + \mu(8r)^2)^{\frac{2-p}{4}} r^{\frac{1}{2}}.\end{aligned} \tag{5.12}$$

Since $1 < p < 2$, one has

$$(\delta + \mu(8r)^2)^{\frac{2-p}{4}} \leq (\delta + \mu(r_0)^2)^{\frac{2-p}{4}}.$$

Thus (5.12) becomes

$$\varpi(2r) \leq (1 - 2^{-s_1})\varpi(8r) + 2^{-s_1-1}\mu(8r) + C(p, L, l_0, r_0)(\delta + \mu(r_0)^2)^{\frac{1}{2}} r_0^{\frac{1}{2}} \left(\frac{r}{r_0}\right)^{\frac{1}{2}}. \tag{5.13}$$

Note that the conditions (5.8) and (5.9) implies that

$$\varpi(8r) \geq \mu(8r) - \mu(4r)/4 \geq 3\mu(8r)/4,$$

which, together with (5.13), yields

$$\varpi(2r) \leq (1 - 2^{-s_1-2})\varpi(8r) + C(p, L, l_0, r_0)(\delta + \mu(r_0)^2)^{\frac{1}{2}} \left(\frac{r}{r_0}\right)^{\frac{1}{2}}.$$

Choosing $s_0 = s_0(p, L, l_0, r_0) \geq 1$ large enough such that $2^{s_0} \geq C(p, L, l_0, r_0)$, we conclude (5.3) in this case.

Finally, we choose the constant $s = \max\{1, s_0, s_1 + 2, \log_2 C(p, L, l_0, r_0)\}$ to complete the proof of Theorem 5.3. □

In the rest of this section, we prove Theorem 1.1 for the case $\delta = 0$.

Proof of Theorem 1.1 for the case $\delta = 0$. When $1 < p < 2$ and $\delta = 0$, the vector function $a := (a_1, a_2, \ldots, a_6) \in C^2(\mathbb{R}^6, \mathbb{R}^6)$ satisfies conditions (1.2), (1.3) and (1.4) with $\delta = 0$, that is,

$$\sum_{i,j=1}^{6} \frac{\partial a_i(\xi)}{\partial \xi_j}\eta_i\eta_j \geq l_0|\xi|^{p-2}|\eta|^2, \tag{5.14}$$

$$\sum_{i,j=1}^{6} |\frac{\partial a_i(\xi)}{\partial \xi_j}| \leq L|\xi|^{p-2}, \tag{5.15}$$

$$|a_i(\xi)| \leq L|\xi|^{p-1} \tag{5.16}$$

for all $\xi, \eta \in \mathbb{R}^6$, where $0 < l_0 < L$. For any $\delta > 0$ and all $\xi \in \mathbb{R}^6$, we define the new vector function $a^\delta := (a_1^\delta, a_2^\delta, \ldots, a_6^\delta) \in C^2(\mathbb{R}^6, \mathbb{R}^6)$ as

$$a^\delta(\xi) := (1 + \eta_\delta(\xi))a(\xi) + \eta_\delta(\xi)(\delta + |\xi|^2)^{\frac{p-2}{2}}\xi.$$

Here by [15, P343], we choose $\eta_\delta \in C^{0,1}([0, \infty))$ such that a^δ converges to a uniformly on compact subsets of $\mathbb{R}^6$ as $\delta \to 0$ and satisfies the conditions:

$$\sum_{i,j=1}^{6} \frac{\partial a_i^\delta(\xi)}{\partial \xi_j}\eta_i\eta_j \geq \frac{1}{\tilde{L}}(\delta + |\xi|^2)^{\frac{p-2}{2}}|\eta|^2$$

$$\sum_{i,j=1}^{6} |\frac{\partial a_i^\delta(\xi)}{\partial \xi_j}| \leq \tilde{L}(\delta + |\xi|^2)^{\frac{p-2}{2}},$$

$$|a_i^\delta(\xi)| \leq \tilde{L}(\delta + |\xi|^2)^{\frac{p-2}{2}}|\xi|$$

for all $\xi, \eta \in \mathbb{R}^6$, where $\tilde{L} = \tilde{L}(p, L, l_0) \geq 1$. Let Ω be a domain in SU(3). Given an any domain $\Omega' \Subset \Omega$ and a weak solution $u \in W^{1,p}_{\mathcal{H}}(\Omega)$ to (1.1) with the conditions (5.14), (5.15) and (5.16), we let $u^\delta \in W^{1,p}_{\mathcal{H}}(\Omega')$ be the unique weak solution to the following Dirichlet problem

$$\begin{cases} \sum_{i=1}^{6} X_i(a_i^\delta(\nabla_{\mathcal{H}}u')) = 0 \text{ in } \Omega', \\ u' - u \in W^{1,p}_{\mathcal{H},0}(\Omega'). \end{cases}$$

By Theorem 1.1 for the case $\delta \in (0, 1]$, we get the uniform estimate (1.9) for $\nabla_{\mathcal{H}} u^{\delta}$ (see [19, Section 3.4]). Since the constant C in (1.9) is independent of δ, letting $\delta \to 0$, we conclude (1.9) for the case $\delta = 0$. □

6 Proofs of Lemmas 4.4, 4.5, 4.6 and 4.3

In this section, we prove Lemmas 4.4, 4.5, 4.6 and 4.3. Firstly, we prove Lemma 4.4.

Proof of Lemma 4.4 For any function $\phi \in C_0^{\infty}(\Omega)$ and any $l \in \{1, \ldots, 6, 7, 8\}$, letting $X_l \phi$ be a test function in (1.1), we have

$$0 = -\int_{\Omega} \sum_{i=1}^{6} X_i (a_i(\nabla_{\mathcal{H}} u)) X_l \phi dx = \int_{\Omega} \sum_{i=1}^{6} a_i(\nabla_{\mathcal{H}} u) X_i X_l \phi dx.$$

Since $X_i X_l = X_l X_i + [X_i, X_l]$, integration by parts implies that

$$\begin{aligned} 0 &= \int_{\Omega} \sum_{i=1}^{6} a_i(\nabla_{\mathcal{H}} u) X_l X_i \phi dx + \int_{\Omega} \sum_{i=1}^{6} a_i(\nabla_{\mathcal{H}} u)[X_i, X_l]\phi dx \\ &= -\int_{\Omega} \sum_{i,j=1}^{6} a_{i,j}(\nabla_{\mathcal{H}} u) X_l X_j u X_i \phi dx + \int_{\Omega} \sum_{i=1}^{6} a_i(\nabla_{\mathcal{H}} u)[X_i, X_l]\phi dx. \end{aligned}$$

Since $X_l X_j = X_j X_l + [X_l, X_j]$ again, we have

$$\begin{aligned} 0 = &-\int_{\Omega} \sum_{i,j=1}^{6} a_{i,j}(\nabla_{\mathcal{H}} u) X_j X_l u X_i \phi dx - \int_{\Omega} \sum_{i,j=1}^{6} a_{i,j}(\nabla_{\mathcal{H}} u)[X_l, X_j] u X_i \phi dx \\ &+ \int_{\Omega} \sum_{i=1}^{6} a_i(\nabla_{\mathcal{H}} u)[X_i, X_l]\phi dx. \end{aligned}$$

By integration by parts again, we conclude (4.7). □

Secondly, we prove Lemma 4.5.

Proof of Lemma 4.5 Let $\eta \in C_0^{\infty}(\Omega)$ be a non-negative cut-off function. For any $\beta \geq 0$ and any $l \in \{1, \ldots, 6\}$, taking $\varphi = \eta^{\beta+2} v^{\beta+2} |\nabla_{\mathcal{H}} u|^2 X_l u$ as a test function in (4.7), we have

$$\begin{aligned} L^l &:= \int_{\Omega} \sum_{i,j=1}^{6} a_{i,j}(\nabla_{\mathcal{H}} u) X_j X_l u \eta^{\beta+2} v^{\beta+2} X_i (|\nabla_{\mathcal{H}} u|^2 X_l u) dx \\ &= -(\beta+2) \int_{\Omega} \sum_{i,j=1}^{6} a_{i,j}(\nabla_{\mathcal{H}} u) X_j X_l u \eta^{\beta+1} X_i \eta v^{\beta+2} |\nabla_{\mathcal{H}} u|^2 X_l u dx \\ &\quad - (\beta+2) \int_{\Omega} \sum_{i,j=1}^{6} a_{i,j}(\nabla_{\mathcal{H}} u) X_j X_l u \eta^{\beta+2} v^{\beta+1} X_i v |\nabla_{\mathcal{H}} u|^2 X_l u dx \end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \sum_{i,j=1}^{6} a_{i,j}(\nabla_{\mathcal{H}}u)[X_l, X_j]uX_i(\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_lu)dx\\
&+\int_\Omega \sum_{i=1}^{6}[X_i, X_l](a_i(\nabla_{\mathcal{H}}u))\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_ludx =: I_1^l + I_2^l + I_3^l + I_4^l. \qquad (6.1)
\end{aligned}
$$

Noting that

$$
X_i(|\nabla_{\mathcal{H}}u|^2X_lu) = |\nabla_{\mathcal{H}}u|^2X_iX_lu + X_i(|\nabla_{\mathcal{H}}u|^2)X_lu,
$$

by the condition (1.2), we have

$$
\begin{aligned}
&\sum_{i,j,l=1}^{6} a_{i,j}(\nabla_{\mathcal{H}}u)X_jX_luX_i(|\nabla_{\mathcal{H}}u|^2X_lu)\\
&\geq l_0\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^2|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2 + \frac{l_0}{2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}(|\nabla_{\mathcal{H}}u|^2)|^2.
\end{aligned}
$$

From this, summing L^l with respect to l from 1 to 6, we have

$$
\sum_{l=1}^{6} L^l \geq l_0\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^2|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2dx.
$$

Next, we estimate each item in the right hand side of (6.1) in turn. By the condition (1.3), we have

$$
\begin{aligned}
|I_1^l| &\leq C(p, L, l_0)(\beta+2)\int_\Omega \eta^{\beta+1}|\nabla_{\mathcal{H}}\eta|v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|dx,\\
|I_2^l| &\leq C(p, L, l_0)(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u||\nabla_{\mathcal{H}}v|dx.
\end{aligned}
$$

Since

$$
\begin{aligned}
&X_i(\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_lu)\\
&= (\beta+2)\eta^{\beta+1}X_i\eta v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_lu + (\beta+2)\eta^{\beta+2}v^{\beta+1}X_iv|\nabla_{\mathcal{H}}u|^2X_lu\\
&\quad + 2\sum_{k=1}^{6}X_kuX_iX_kuX_lu\eta^{\beta+2}v^{\beta+2} + \eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_iX_lu,
\end{aligned}
$$

one gets

$$
\begin{aligned}
&|X_i(\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_lu)|\\
&\leq 3\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u| + (\beta+2)\eta^{\beta+1}v^{\beta+2}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{H}}\eta|\\
&\quad + (\beta+2)\eta^{\beta+2}v^{\beta+1}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{H}}v|. \qquad (6.2)
\end{aligned}
$$

Note that (2.1) shows

$$
|[X_l, X_j]u| \leq C(|\nabla_{\mathcal{H}}u| + |\nabla_{\mathcal{T}}u|),\ \forall l, j \in \{1, \ldots, 6\}.
$$

From this, then by (6.2) and the condition (1.3), we have

$$
\begin{aligned}
|I_3^l| \le C(p,L,l_0)\int_\Omega \eta^{\beta+2}v^{\beta+2}(\delta+|\nabla_{\mathcal{H}}u|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^2|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u||\nabla_{\mathcal{T}}u|dx\\
+C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+1}|\nabla_{\mathcal{H}}\eta|v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{T}}u|dx\\
+C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{H}}v||\nabla_{\mathcal{T}}u|dx\\
+C(p,L,l_0)\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|dx\\
+C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+1}|\nabla_{\mathcal{H}}\eta|v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^4dx\\
+C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^4|\nabla_{\mathcal{H}}v|dx.
\end{aligned}
$$

By (2.1) again, we have

$$
I_4^l=\int_\Omega\sum_{i=1}^6[X_i,X_l](a_i(\nabla_{\mathcal{H}}u))\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_lu\,dx=I_{41}^l+I_{42}^l,
$$

where

$$
\begin{aligned}
I_{41}^l&:=\sum_{i=1}^6\sum_{k=1}^6C_{i,l}^k\int_\Omega X_k(a_i(\nabla_{\mathcal{H}}u))\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_lu\,dx,\\
I_{42}^l&:=\sum_{i=1}^6\sum_{k=7}^8C_{i,l}^k\int_\Omega X_k(a_i(\nabla_{\mathcal{H}}u))\eta^{\beta+2}v^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_lu\,dx.
\end{aligned}
$$

By the condition (1.3) again, we have

$$
|I_{41}^l|\le C(p,L,l_0)\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|dx. \tag{6.3}
$$

Now we estimate I_{42}^l. We only consider the case that $k=7$ below. In the case that $k=8$, we use the similar method. Since $X_7=-[X_1,X_2]$, integration by parts yields

$$
\begin{aligned}
I_{42}^{l,7}&:=C_{i,l}^7\int_\Omega X_7(a_i(\nabla_{\mathcal{H}}u))\varphi\,dx=-C_{i,l}^7\int_\Omega[X_1,X_2](a_i(\nabla_{\mathcal{H}}u))\varphi\,dx\\
&=C_{i,l}^7\int_\Omega X_2(a_i(\nabla_{\mathcal{H}}u))X_1\varphi\,dx-C_{i,l}^7\int_\Omega X_1(a_i(\nabla_{\mathcal{H}}u))X_2\varphi\,dx.
\end{aligned}\tag{6.4}
$$

Denote $\phi:=\eta^{\beta+2}|\nabla_{\mathcal{H}}u|^2X_lu$. Rewriting $\varphi=v^{\beta+2}\phi$ in (6.4), we have

$$
X_i\varphi=(\beta+2)v^{\beta+1}\phi X_iv+v^{\beta+2}X_i\phi,\ i\in\{1,\dots,6\},
$$

which yields

$$
\begin{aligned}
I_{42}^{l,7}&=C_{i,l}^7(\beta+2)\int_\Omega v^{\beta+1}\phi[X_2(a_i(\nabla_{\mathcal{H}}u))X_1v-X_1(a_i(\nabla_{\mathcal{H}}u))X_2v]dx\\
&\quad+C_{i,l}^7\int_\Omega v^{\beta+2}[X_2(a_i(\nabla_{\mathcal{H}}u))X_1\phi-X_1(a_i(\nabla_{\mathcal{H}}u))X_2\phi]dx=:J^l+K^l.
\end{aligned}
$$

We estimate J^l as below. By the condition (1.3) again, we have

$$
|J^l|\le C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^3|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u||\nabla_{\mathcal{H}}v|dx.
$$

We estimate K^l as below. By the integration by parts again, we have

$$K^l = C^7_{i,l}(\beta+2)\int_\Omega v^{\beta+1} a_i(\nabla_{\mathcal{H}} u)[X_1 v X_2\phi - X_2 v X_1 \phi]dx - C^7_{i,l}\int_\Omega v^{\beta+2} a_i(\nabla_{\mathcal{H}} u) X_7\phi dx$$
$$=: K^l_1 + K^l_2.$$

For K^l_1, the condition (1.4) yields

$$|K^l_1| \le C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+2} v^{\beta+1}\left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^3 |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u||\nabla_{\mathcal{H}} v| dx$$
$$+ C(p,L,l_0)(\beta+2)^2 \int_\Omega \eta^{\beta+1} v^{\beta+1}(\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^4 |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}}\eta| dx.$$

For K^l_2, since

$$X_7\phi = (\beta+2)\eta^{\beta+1}|\nabla_{\mathcal{H}} u|^2 X_l u X_7\eta + \eta^{\beta+2}|\nabla_{\mathcal{H}} u|^2 X_7 X_l u + \sum_{j=1}^6 2\eta^{\beta+2} X_l u X_j u X_7 X_j u,$$
$$X_7 X_j = X_j X_7 + [X_7, X_j],\ \ j \in \{1,\ldots,6\},$$

we have

$$\begin{aligned} K^l_2 = &-C^7_{i,l}(\beta+2)\int_\Omega \eta^{\beta+1} v^{\beta+2} a_i(\nabla_{\mathcal{H}} u)|\nabla_{\mathcal{H}} u|^2 X_l u X_7\eta dx \\ &- C^7_{i,l}\int_\Omega \left[\eta^{\beta+2} v^{\beta+2} a_i(\nabla_{\mathcal{H}} u)|\nabla_{\mathcal{H}} u|^2\right] X_l X_7 u dx \\ &- C^7_{i,l}\int_\Omega \eta^{\beta+2} v^{\beta+2} a_i(\nabla_{\mathcal{H}} u)|\nabla_{\mathcal{H}} u|^2 [X_7, X_l] u dx \\ &- 2C^7_{i,l}\sum_{j=1}^6 \int_\Omega \left[\eta^{\beta+2} v^{\beta+2} a_i(\nabla_{\mathcal{H}} u) X_l u X_j u\right] X_j X_7 u dx \\ &- 2C^7_{i,l}\sum_{j=1}^6 \int_\Omega \eta^{\beta+2} v^{\beta+2} a_i(\nabla_{\mathcal{H}} u) X_l u X_j u [X_7, X_j] u dx. \end{aligned} \tag{6.5}$$

Next we use (2.2) and the condition (1.4) to bound the first, third and fifth terms in the right hand side of (6.5), and use integration by parts and the condition (1.3) to bound the second and fourth terms. Thus

$$\begin{aligned} |K^l_2| \le\ & C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+1} v^{\beta+2}\left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^4 |\nabla_{\mathcal{T}}\eta| dx \\ &+ C(p,L,l_0)\int_\Omega \eta^{\beta+2} v^{\beta+2}(\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^2 |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u||\nabla_{\mathcal{T}} u| dx \\ &+ C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+2} v^{\beta+1}\left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^3 |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{T}} u| dx \\ &+ C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\beta+1} v^{\beta+2}\left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^3 |\nabla_{\mathcal{H}}\eta||\nabla_{\mathcal{T}} u| dx \\ &+ C(p,L,l_0)\int_\Omega \eta^{\beta+2} v^{\beta+2}\left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^4 dx. \end{aligned}$$

Finally, combining all the above estimates together, then by Young's inequality, we conclude (4.8). □

Thirdly, based on Lemma 4.5, we prove Lemma 4.6.

Proof of Lemma 4.6 For simplicity we write the left hand side of (4.12) as

$$M := \int_\Omega \eta^{\tau(\beta+2)+4} v^{\tau(\beta+4)} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^4 |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u|^2 dx,$$

where $\tau \in (1/2, 1)$. Let $B_r \subset \Omega$ be a ball. Consider the cut-off function $\eta \in C_0^\infty(\Omega)$ with (4.10) and (4.11). For any $\beta \geq 0$ and any $l \in \{7, 8\}$, letting $\varphi = \eta^{\tau(\beta+2)+4} v^{\tau(\beta+4)} |\nabla_{\mathcal{H}} u|^4 X_l u$ be a test function in (4.7), we have

$$\begin{aligned}
L_l &= \int_\Omega \sum_{i,j=1}^{6} \eta^{\tau(\beta+2)+4} v^{\tau(\beta+4)} |\nabla_{\mathcal{H}} u|^4 a_{i,j}(\nabla_{\mathcal{H}} u) X_j X_l u X_i X_l u dx \\
&= -(\tau(\beta+2)+4) \int_\Omega \sum_{i,j=1}^{6} \eta^{\tau(\beta+2)+3} v^{\tau(\beta+4)} |\nabla_{\mathcal{H}} u|^4 X_l u a_{i,j}(\nabla_{\mathcal{H}} u) X_j X_l u X_i \eta dx \\
&\quad - \tau(\beta+4) \int_\Omega \sum_{i,j=1}^{6} \eta^{\tau(\beta+2)+4} v^{\tau(\beta+4)-1} |\nabla_{\mathcal{H}} u|^4 X_l u a_{i,j}(\nabla_{\mathcal{H}} u) X_j X_l u X_i v dx \\
&\quad - 4 \int_\Omega \sum_{i,j,k=1}^{6} \eta^{\tau(\beta+2)+4} v^{\tau(\beta+4)} |\nabla_{\mathcal{H}} u|^2 X_k u X_l u a_{i,j}(\nabla_{\mathcal{H}} u) X_j X_7 u X_i X_k u dx \\
&\quad - \int_\Omega \sum_{i,j=1}^{6} a_{i,j}(\nabla_{\mathcal{H}} u)[X_l, X_j] u X_i \varphi dx \\
&\quad - \int_\Omega \sum_{i=1}^{6} [X_i, X_l](a_i(\nabla_{\mathcal{H}} u)) \varphi dx =: K_1^l + K_2^l + K_3^l + K_4^l + K_5^l.
\end{aligned} \tag{6.6}$$

By the condition (1.2), we have

$$\sum_{l=7}^{8} L_l \geq l_0 \int_\Omega \eta^{\tau(\beta+2)+4} v^{\tau(\beta+4)} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^4 |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u|^2 dx = l_0 M.$$

Next we estimate each item in the right hand side of (6.6) in turn. For simplicity we write

$$\tilde{K} := \int_\Omega \eta^{(2\tau-1)(\beta+2)+6} v^{(2\tau-1)(\beta+4)} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^4 |\nabla_{\mathcal{T}} u|^2 |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u|^2 dx.$$

By the condition (1.3), then by Hölder's inequality, we have

$$\begin{aligned}
|K_1^l| &\leq C(p, L, l_0)(\beta+2) \int_\Omega \eta^{\tau(\beta+2)+3} v^{\tau(\beta+4)} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} \\
&\qquad \times |\nabla_{\mathcal{H}} u|^4 |\nabla_{\mathcal{T}} u| |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u| |\nabla_{\mathcal{H}} \eta| dx \\
&\leq C(p, L, l_0)(\beta+2) \tilde{K}^{1/2} \left(\int_\Omega \eta^{\beta+2} v^{\beta+4} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^4 |\nabla_{\mathcal{H}} \eta|^2 dx \right)^{1/2},
\end{aligned} \tag{6.7}$$

$$|K_2^l| \le C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)-1}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}} \times |\nabla_{\mathcal H}u|^4|\nabla_{\mathcal T}u||\nabla_{\mathcal H}\nabla_{\mathcal T}u||\nabla_{\mathcal H}v|dx$$
$$\le C(p,L,l_0)(\beta+2)\tilde K^{1/2}\left(\int_\Omega \eta^{\beta+4}v^{\beta+2}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4|\nabla_{\mathcal H}v|^2dx\right)^{1/2}. \tag{6.8}$$

By the condition (1.3), Hölder's inequality and Lemma 4.5, we have

$$|K_3^l| \le C(p,L,l_0)\int_\Omega \eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}} \times |\nabla_{\mathcal H}u|^3|\nabla_{\mathcal T}u||\nabla_{\mathcal H}\nabla_{\mathcal T}u||\nabla_{\mathcal H}\nabla_{\mathcal H}u|dx$$
$$\le C(p,L,l_0)\tilde K^{1/2}\left(\int_\Omega \eta^{\beta+4}v^{\beta+4}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^2|\nabla_{\mathcal H}\nabla_{\mathcal H}u|^2dx\right)^{1/2}$$
$$\le C(p,L,l_0)\tilde K^{1/2}I^{1/2}, \tag{6.9}$$

where I is denoted as the right hand side of (4.8) in Lemma 4.5, that is,

$$I := C(p,L,l_0)(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta+4}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4(\eta^2+|\nabla_{\mathcal H}\eta|^2+\eta|\nabla_{\mathcal T}\eta|)dx$$
$$+C(p,L,l_0)(\beta+2)^2\int_\Omega \eta^{\beta+4}v^{\beta+2}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4|\nabla_{\mathcal H}v|^2dx$$
$$+C(p,L,l_0)\int_\Omega \eta^{\beta+4}v^{\beta+4}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^2|\nabla_{\mathcal T}u|^2dx. \tag{6.10}$$

Combining (6.7), (6.8) and (6.9), we have

$$\sum_{i=1}^3 |K_i^l| \le C(p,L,l_0)\tilde K^{1/2}I^{1/2}. \tag{6.11}$$

Now we bound $\tilde K$. By Hölder's inequality, we have

$$\tilde K \le \left(\int_\Omega \eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4|\nabla_{\mathcal H}\nabla_{\mathcal T}u|^2dx\right)^{\frac{2\tau-1}{\tau}}$$
$$\times\left(\int_\Omega \eta^{\frac{2\tau}{1-\tau}+4}(\delta+|\nabla_{\mathcal H}u|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4|\nabla_{\mathcal T}u|^{\frac{2\tau}{1-\tau}}|\nabla_{\mathcal H}\nabla_{\mathcal T}u|^2dx\right)^{\frac{1-\tau}{\tau}} =: M^{\frac{2\tau-1}{\tau}}G^{\frac{1-\tau}{\tau}}, \tag{6.12}$$

where

$$G := \int_\Omega \eta^{\frac{2\tau}{1-\tau}+4}(\delta+|\nabla_{\mathcal H}u|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4|\nabla_{\mathcal T}u|^{\frac{2\tau}{1-\tau}}|\nabla_{\mathcal H}\nabla_{\mathcal T}u|^2dx.$$

Let $q=2/(1-\tau)$. By (2.8) with $\beta=(q-2)/2$ in Lemma 2.3 and (4.9), we have

$$G \le C(p,L,l_0)\mu(r)^4\int_\Omega \eta^{q+2}(\delta+|\nabla_{\mathcal H}u|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal T}u|^{q-2}|\nabla_{\mathcal H}\nabla_{\mathcal T}u|^2dx$$
$$\le C(p,L,l_0)\left(1+\frac{1}{r^{q+2}}\right)\mu(r)^4\int_{B_r}\big(\delta+|\nabla_{\mathcal H}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^q dx$$
$$\le C(p,L,l_0)\left(1+\frac{1}{r^{q+2}}\right)|B_r|(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^{q+4}. \tag{6.13}$$

Here we apply Lemma 2.3 to estimate the second inequality in (6.13), and apply (4.9) to estimate the last inequality.

Next we bound I as in (6.10). Fix $1 < \gamma < 2$. For the second item of I, by Hölder's inequality and (4.9), we have

$$\begin{aligned}&\int_\Omega \eta^{\beta+2} v^{\beta+4}(\delta+|\nabla_{\mathcal{H}}u|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^4(\eta^2+|\nabla_{\mathcal{H}}\eta|^2+\eta|\nabla_{\mathcal{T}}\eta|)dx\\ &\quad\le C(p,L,l_0,\gamma)\left(1+\frac{1}{r^2}\right)(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^8|B_r|^{1-1/\gamma}\left(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\right)^{1/\gamma}. \quad (6.14)\end{aligned}$$

For the second item of I, by (4.9), we have

$$\begin{aligned}&\int_\Omega \eta^{\beta+4} v^{\beta+2}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^4|\nabla_{\mathcal{H}}v|^2dx\\ &\quad\le C(p,L,l_0)(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^4\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\nabla_{\mathcal{H}}v|^2dx. \quad (6.15)\end{aligned}$$

For the finally item of I, by Hölder's inequality, (2.7) with $\beta=\frac{1}{\gamma-1}$ in Lemma 2.3 and (4.9), we have

$$\begin{aligned}&\int_\Omega \eta^{\beta+4} v^{\beta+4}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^2|\nabla_{\mathcal{T}}u|^2dx\\ &\le\left(\int_\Omega \eta^{\frac{2\gamma}{\gamma-1}}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^2|\nabla_{\mathcal{T}}u|^{\frac{2\gamma}{\gamma-1}}dx\right)^{1-1/\gamma}\\ &\quad\times\left(\int_\Omega \eta^{\gamma(\beta+2)}v^{\gamma(\beta+4)}\big(\delta+|\nabla_{\mathcal{H}}u|^2\big)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^2dx\right)^{1/\gamma}\\ &\le C(p,L,l_0,\gamma)\left(1+\frac{1}{r^2}\right)(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^8|B_r|^{1-1/\gamma}\left(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\right)^{1/\gamma}. \quad (6.16)\end{aligned}$$

Here we apply Hölder's inequality to estimate the first inequality in (6.16) and apply Lemma 2.3 and (4.9) to estimate the finally inequality. Combining (6.14), (6.15) and (6.16), we have

$$I\le C(p,L,l_0,\gamma)(\beta+2)^2(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^4J, \quad (6.17)$$

where J is as in (4.13). Combining (6.11), (6.12), (6.13) and (6.17), by Young's inequality therein, we have

$$\begin{aligned}\sum_{i=1}^3|K_i^l|&\le\frac{l_0M}{1000}+C(p,L,l_0,\tau,\gamma)G^{1-\tau}I^\tau\le\frac{l_0M}{1000}+C(p,L,l_0,\tau,\gamma)(\beta+2)^{2\tau}\\ &\quad\times\left(1+\frac{1}{r^{2(2-\tau)}}\right)|B_r|^{1-\tau}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^6J^\tau.\end{aligned}$$

Next, we bound K_4^l. Noting that

$$\begin{aligned}X_i\varphi&=(\tau(\beta+2)+4)\eta^{\tau(\beta+2)+3}v^{\tau(\beta+4)}|\nabla_{\mathcal{H}}u|^4X_7uX_i\eta\\ &\quad+\tau(\beta+4)\eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)-1}|\nabla_{\mathcal{H}}u|^4X_7uX_iv\\ &\quad+4\eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}|\nabla_{\mathcal{H}}u|^2X_kuX_7uX_iX_ku\\ &\quad+\eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}|\nabla_{\mathcal{H}}u|^4X_iX_7u,\end{aligned}$$

by (2.2) and the condition (1.3), we have

$$\begin{aligned}|K_4^l| &\le C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\tau(\beta+2)+3}v^{\tau(\beta+4)}\left(\delta+|\nabla_{\mathcal H}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^5|\nabla_{\mathcal T}u||\nabla_{\mathcal H}\eta|dx\\ &\quad+C(p,L,l_0)(\beta+2)\int_\Omega \eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)-1}\left(\delta+|\nabla_{\mathcal H}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^5|\nabla_{\mathcal T}u||\nabla_{\mathcal H}v|dx\\ &\quad+C(p,L,l_0)\int_\Omega \eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}\left(\delta+|\nabla_{\mathcal H}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4|\nabla_{\mathcal T}u||\nabla_{\mathcal H}\nabla_{\mathcal H}u|dx\\ &\quad+C(p,L,l_0)\int_\Omega \eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}\left(\delta+|\nabla_{\mathcal H}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^5|\nabla_{\mathcal H}\nabla_{\mathcal T}u|dx\\ &=K_{41}+K_{42}+K_{43}+K_{44}.\end{aligned}\tag{6.18}$$

Before bounding each item in the right hand side of (6.18), we bound K_5. By (2.2) and the condition (1.3), we have

$$|K_5|\le C(p,L,l_0)\int_\Omega \eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}\left(\delta+|\nabla_{\mathcal H}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^4|\nabla_{\mathcal T}u||\nabla_{\mathcal H}\nabla_{\mathcal H}u|dx\le K_{43}.$$

From this, to bound K_5, we only need to bound K_{43}. Now we bound each item in the right hand side of (6.18). By Hölder's inequality, we have

$$\begin{aligned}K_{41} &\le \frac{C(p,L,l_0,\tau,\gamma)(\beta+2)}{r}\left(\int_\Omega |\nabla_{\mathcal H}u|^4\eta^\beta v^\beta dx\right)^\tau\\ &\quad\times\left(\int_\Omega \eta^{\frac{2\tau+3}{1-\tau}}v^{\frac{4\tau}{1-\tau}}(\delta+|\nabla_{\mathcal H}u|^2)^{\frac{p-2}{2(1-\tau)}}|\nabla_{\mathcal H}u|^{\frac{5-4\tau}{1-\tau}}|\nabla_{\mathcal T}u|^{\frac{1}{1-\tau}}dx\right)^{1-\tau}\\ &\le \frac{C(p,L,l_0,\tau,\gamma)(\beta+2)}{r}(\delta+\mu(r)^2)^{\frac{(p-2)(1+\tau)}{4}}\mu(r)^5J^\tau\\ &\quad\times\left(\int_\Omega \eta^{\frac{1}{1-\tau}}\left(\delta+|\nabla_{\mathcal H}u|^2\right)^{\frac{p-2}{4}}|\nabla_{\mathcal T}u|^{\frac{1}{1-\tau}}dx\right)^{1-\tau}.\end{aligned}\tag{6.19}$$

Applying the fact $|\nabla_{\mathcal T}u|^2\le 2|\nabla_{\mathcal H}\nabla_{\mathcal H}u|^2$, (2.7) with $\beta=\frac{\tau}{1-\tau}$ in Lemma 2.3 and (4.9) to estimate the final item in the right hand side of (6.19), we have

$$\begin{aligned}&\int_\Omega \eta^{\frac{1}{1-\tau}}(\delta+|\nabla_{\mathcal H}u|^2)^{\frac{p-2}{4}}|\nabla_{\mathcal T}u|^{\frac{1}{1-\tau}}dx\\ &\le |B_r|^{1/2}\left(\int_\Omega \eta^{\frac{2}{1-\tau}}\left(\delta+|\nabla_{\mathcal H}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal T}u|^{\frac{2}{1-\tau}}dx\right)^{1/2}\\ &\le C(p,L,l_0,\tau)|B_r|^{1/2}K_\eta^{\frac{1}{2(1-\tau)}}\left(\int_{B_r}\left(\delta+|\nabla_{\mathcal H}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal H}u|^{\frac{2}{1-\tau}}dx\right)^{1/2}\\ &\le C(p,L,l_0,\tau)|B_r|K_\eta^{\frac{1}{2(1-\tau)}}(\delta+\mu(r)^2)^{\frac{p-2}{4}}\mu(r)^{\frac{1}{1-\tau}}.\end{aligned}\tag{6.20}$$

Combining (6.19) and (6.20), we have

$$K_{41}\le \frac{C(p,L,l,\tau,\gamma)(\beta+2)}{r}K_\eta^{\frac12}|B_r|^{1-\tau}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^6J^\tau,$$

where K_η is as in (2.9). Since $\eta\in C_0^\infty(B_r)$ satisfies (4.10) and (4.11), one gets

$$K_\eta\le 1+\frac{C}{r^2}.$$

For K_{42}, by Hölder's inequality, Young's inequality and (4.9), we have

$$
\begin{aligned}
K_{42} \leq & C(p, L, l_0, \tau, \gamma)(\beta+2) \left(\int_\Omega [|\nabla_{\mathcal{H}} u|^2 \eta^{\beta/2} v^{\beta/2}][\eta^{(\beta+4)/2} v^{(\beta+2)/2} |\nabla_{\mathcal{H}} v|] dx \right)^\tau \\
& \times \left(\int_\Omega \eta^{\frac{4}{1-\tau}} v^{\frac{3\tau-1}{1-\tau}} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2(1-\tau)}} |\nabla_{\mathcal{H}} u|^{\frac{5-2\tau}{1-\tau}} |\nabla_{\mathcal{T}} u|^{\frac{1}{1-\tau}} |\nabla_{\mathcal{H}} v| dx \right)^{1-\tau} \\
\leq & C(p, L, l_0, \tau, \gamma)(\beta+2)(\delta + \mu(r)^2)^{\frac{(p-2)(1+\tau)}{4}} \mu(r)^{4+\tau} J^\tau \\
& \times \left(\int_\Omega \eta^{\frac{4}{1-\tau}} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{4}} |\nabla_{\mathcal{T}} u|^{\frac{1}{1-\tau}} |\nabla_{\mathcal{H}} v| dx \right)^{1-\tau}. \qquad (6.21)
\end{aligned}
$$

Applying Hölder's inequality, (2.7) with $\beta = \frac{1}{1-\tau}$ in Lemma 2.3 and (4.9) to estimate the final item in the right hand side of (6.21), we have

$$
\begin{aligned}
& \int_\Omega \eta^{\frac{4}{1-\tau}} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{4}} |\nabla_{\mathcal{T}} u|^{\frac{1}{1-\tau}} |\nabla_{\mathcal{H}} v| dx \\
& \leq |B_r|^{1/2} \left(\int_\Omega \eta^{\frac{8}{1-\tau}} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{\frac{2}{1-\tau}} |\nabla_{\mathcal{H}} v|^2 dx \right)^{1/2} \\
& \leq C(p, L, l_0, \tau) |B_r|^{1/2} K_\eta^{\frac{2-\tau}{2(1-\tau)}} \left(\int_{B_r} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^{\frac{2(2-\tau)}{1-\tau}} dx \right)^{1/2} \\
& \leq C(p, L, l_0, \tau) |B_r| K_\eta^{\frac{2-\tau}{2(1-\tau)}} (\delta + \mu(r)^2)^{\frac{p-2}{4}} \mu(r)^{\frac{2-\tau}{1-\tau}}. \qquad (6.22)
\end{aligned}
$$

Combining (6.21) and (6.22), we have

$$
K_{42} \leq C(p, L, l, \tau, \gamma)(\beta+2) K_\eta^{\frac{2-\tau}{2}} |B_r|^{1-\tau} (\delta + \mu(r)^2)^{\frac{p-2}{2}} \mu(r)^6 J^\tau,
$$

where K_η is as in (2.9). For K_{43}, Hölder's inequality yields

$$
K_{43} \leq C(p, L, l_0) I^{\frac{1}{2}} \left(\int_\Omega \eta^{(2\tau-1)(\beta+2)+6} v^{(2\tau-1)(\beta+4)} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^6 |\nabla_{\mathcal{T}} u|^2 dx \right)^{\frac{1}{2}}, \qquad (6.23)
$$

where I is as in (6.10). Applying Hölder's inequality to estimate the final item in the right hand side of (6.23), we have

$$
\begin{aligned}
& \int_\Omega \eta^{(2\tau-1)(\beta+2)+6} v^{(2\tau-1)(\beta+4)} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^6 |\nabla_{\mathcal{T}} u|^2 dx \\
& \leq \left(\int_\Omega \eta^{\frac{2\tau}{1-\tau}} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^6 |\nabla_{\mathcal{T}} u|^{\frac{2\tau}{1-\tau}} dx \right)^{\frac{1-\tau}{\tau}} \\
& \quad \times \left(\int_\Omega \eta^{\tau\beta + \frac{2\tau(2\tau+1)}{2\tau-1}} v^{\tau(\beta+4)} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^6 dx \right)^{\frac{2\tau-1}{\tau}}. \qquad (6.24)
\end{aligned}
$$

Applying Hölder's inequality, (2.7) with $\beta = \frac{3\tau-1}{1-\tau}$ in Lemma 2.3 and (4.9) to estimate the first item in the right hand side of (6.24), we have

$$
\begin{aligned}
& \int_\Omega \eta^{\frac{2\tau}{1-\tau}} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} u|^6 |\nabla_{\mathcal{T}} u|^{\frac{2\tau}{1-\tau}} dx \\
& \leq |B_r|^{1/2} (\delta + \mu(r)^2)^{\frac{p-2}{4}} \mu(r)^6 \left(\int_\Omega \eta^{\frac{4\tau}{1-\tau}} \left(\delta + |\nabla_{\mathcal{H}} u|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u|^{\frac{4\tau}{1-\tau}} dx \right)^{1/2}
\end{aligned}
$$

$$\leq C(p,L,l_0,\tau)K_\eta^{\frac{\tau}{1-\tau}}|B_r|^{1/2}(\delta+\mu(r)^2)^{\frac{p-2}{4}}\mu(r)^6$$
$$\times\left(\int_{B_r}\left(\delta+|\nabla_{\mathcal{H}}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^{\frac{4\tau}{1-\tau}}dx\right)^{1/2}$$
$$\leq C(p,L,l_0,\tau)K_\eta^{\frac{\tau}{1-\tau}}|B_r|(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^{6+\frac{2\tau}{1-\tau}}. \tag{6.25}$$

Applying (4.9) to estimate the final item in the right hand side of (6.24), we have

$$\int_\Omega \eta^{\tau\beta+\frac{2\tau(2\tau+1)}{2\tau-1}}v^{\tau(\beta+4)}\left(\delta+|\nabla_{\mathcal{H}}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^6dx$$
$$\leq(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^{4\tau+6}\int_\Omega\eta^{\tau\beta}v^{\tau\beta}dx\leq|B_r|^{1-\tau}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^6J^\tau. \tag{6.26}$$

Combining (6.23), (6.24), (6.25) and (6.26), we have

$$K_{43}\leq C(p,L,l_0,\tau,\gamma)K_\eta|B_r|^{1-\tau}(\delta+\mu(r)^2)^{\frac{p-2}{4}}\mu(r)^4J^{\tau-1/2}I^{1/2},$$

where K_η is as in (2.9). For K_{44}, by Young's inequality, Hölder's inequality and (4.9), we have

$$K_{44}\leq\frac{l_0M}{1000}+C(p,L,l_0,\tau,\gamma)\int_\Omega\eta^{\tau(\beta+2)+4}v^{\tau(\beta+4)}\left(\delta+|\nabla_{\mathcal{H}}u|^2\right)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}u|^6dx$$
$$+C(p,L,l_0,\tau,\gamma)\left(\int_\Omega|\nabla_{\mathcal{H}}u|^4\eta^\beta v^\beta\right)^\tau$$
$$\times\left(\int_{B_r}\eta^{\frac{2\tau+4}{1-\tau}}v^{\frac{4\tau}{1-\tau}}\left(\delta+|\nabla_{\mathcal{H}}u|^2\right)^{\frac{p-2}{2(1-\tau)}}|\nabla_{\mathcal{H}}u|^{\frac{6-4\tau}{1-\tau}}dx\right)^{1-\tau}$$
$$\leq\frac{l_0M}{1000}+C(p,L,l_0,\tau,\gamma)|B_r|^{1-\tau}(\delta+\mu(r)^2)^{\frac{p-2}{2}}\mu(r)^6J^\tau.$$

Finally, combining all the above estimates together, we conclude (4.12). □

Finally, we use Lemma 4.1 and Remark 4.2 to prove Lemma 4.3.

Proof of Lemma 4.3 We consider two cases: (i) that (4.3) holds true for an index $k\in\{1,\ldots,6\}$ and (ii) that (4.5) holds true for an index $k\in\{1,\ldots,6\}$. In the case (i), we use Lemma 4.1 with v to conclude (4.4). Similarly, in the case (ii), we use Remark 4.2 with v' to conclude (4.6). Below we assume that (4.3) holds true for an index $k\in\{1,\ldots,6\}$.

Let B_R be a ball in Ω. For any ball $B_r\subset\Omega$ with the same center B_R and $0<r\leq R$, we consider the cut-off function $\eta\in C_0^\infty(B_r)$ with (4.10) and (4.11). Recall that

$$v=\min(\mu(r)/8,\max(\mu(r)/4-X_ku,0)).$$

For any $\beta\geq0$, we write $\psi=\eta^{\beta/2+2}v^{\beta/2+2}$. Note that

$$X_i\psi=(\beta/2+2)\eta^{\beta/2+1}v^{\beta/2+2}X_i\eta+(\beta/2+2)\eta^{\beta/2+2}v^{\beta/2+1}X_iv.$$

For $\gamma>1$, by Hölder's inequality and Lemma 4.1, we have

$$
\begin{aligned}
\int_{B_r} |\nabla_{\mathcal{H}}\psi|^2 dx \le C(p,L,l_0,\gamma)(\beta+2)^2 \Bigg[\int_{B_r} \eta^{\beta+2} v^{\beta+4} |\nabla_{\mathcal{H}}\eta|^2 dx \\
+ \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\nabla_{\mathcal{H}} v|^2 dx \Bigg] \\
\le C(p,L,l_0,\gamma)(\beta+2)^4 \left(1+\frac{1}{r^2}\right) \mu(r)^4 |B_r|^{1-\frac{1}{\gamma}} \left(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta} dx \right)^{\frac{1}{\gamma}},
\end{aligned} \tag{6.27}
$$

Here we use Hölder's inequality and Lemma 4.1 to get the second inequality in (6.27). By Lemma 2.8 with $p_1 = 2$, we have

$$
\left(⨍_{B_r} |\psi|^{\frac{2Q}{Q-2}} dx \right)^{\frac{Q-2}{Q}} \le C r^2 ⨍_{B_r} |\nabla_{\mathcal{H}}\psi|^2 dx, \tag{6.28}
$$

where $Q = 10$ is the homogeneous dimension of SU(3). Here for any ball B_r we use [24, Theorem V.4.1 on P69] to control its volume, that is, $C_1 r^Q \le |B_r| \le C_2 r^Q$. Combining (6.27) and (6.28), we have

$$
\left(⨍_{B_r} (\eta v)^{\frac{Q(\beta+4)}{Q-2}} dx \right)^{\frac{Q-2}{Q}} \le C(p,L,l_0,\gamma)(\beta+2)^4 (1+r^2) \mu(r)^4 \left(⨍_{B_r} (\eta v)^{\gamma\beta} dx \right)^{\frac{1}{\gamma}}. \tag{6.29}
$$

Now we choose $\gamma = \frac{Q-1}{Q-2} = \frac{9}{8}$ such that $1 < \gamma < \frac{Q}{Q-2} = \frac{5}{4}$. For simplicity we write

$$
\beta_i = 4Q\left(\left(\frac{Q}{Q-1} \right)^{i+1} - 1 \right), \ i = 0, 1, 2, \dots.
$$

Since $\gamma\beta_{i+1} = \frac{Q}{Q-2}(\beta_i + 4)$, by (6.29) with $\beta = \beta_i$, we have

$$
\begin{aligned}
\left(⨍_{B_r} (\eta v)^{\gamma\beta_{i+1}} dx \right)^{\frac{Q-2}{Q}} \le C(p,L,l,\gamma)(\beta_i+2)^4 (1+r^2) \mu(r)^4 \\
\times \left(⨍_{B_r} (\eta v)^{\gamma\beta_i} dx \right)^{\frac{1}{\gamma}}, \ i = 0, 1, 2, \dots.
\end{aligned} \tag{6.30}
$$

Denote

$$
M_i = \left(⨍_{B_r} \left(\frac{\eta v}{(1+r^2)^{\frac{1}{4}} \mu(r)} \right)^{\gamma\beta_i} dx \right)^{\frac{1}{\gamma\beta_i}}, \ i = 0, 1, 2, \dots.
$$

Thus (6.30) becomes

$$
M_{i+1} \le C_i M_i^{\frac{Q\beta_i}{(Q-2)\gamma\beta_{i+1}}}, \ i = 0, 1, 2, \dots, \tag{6.31}
$$

where

$$
C_i = C(p,L,l_0)^{\frac{Q}{(Q-1)\beta_{i+1}}} (\beta_i + 2)^{\frac{4Q}{(Q-1)\beta_{i+1}}}.
$$

For any natural number $N \ge 1$, we iterate (6.31) with respect to i from 0 to $N-1$. Thus

$$
M_N \le C(p,L,l_0) M_0^{\left(\frac{Q}{Q-1}\right)^N \frac{\beta_0}{\beta_N}}.
$$

Letting $N \to \infty$, one gets

$$\sup_{B_r}\left(\frac{\eta v}{\mu(r)}\right) \le C(p, L, l_0)(1+r^2)^{\frac{Q-1}{4Q}}\left(\fint_{B_r}\left(\frac{\eta v}{\mu(r)}\right)^{\frac{4Q}{Q-2}}dx\right)^{\frac{Q-2}{4Q^2}}.$$

Recall that the cut-off function $\eta \in C_0^\infty(B_r)$ satisfies (4.10) and (4.11). Then the assumption (4.3) implies that

$$\sup_{B_{r/2}} v \le C(p, L, l_0)(1+r^2)^{\frac{Q-1}{4Q}}\theta^{\frac{Q-2}{4Q^2}}\mu(r). \tag{6.32}$$

We choose $\theta = \theta(p, L, l_0, R) > 0$ small enough such that

$$\theta^{\frac{Q-2}{4Q^2}} \le C(p, L, l_0)^{-1}(1+R^2)^{\frac{1-Q}{4Q}}/16.$$

Then (6.32) becomes

$$\sup_{B_{r/2}} v \le \mu(r)/16,$$

which yields that $X_k u \ge 3\mu(r)/16$ in $B_{r/2}$. We conclude (4.4). □

7 A class of compact connected semi-simple Lie group

In this section, we consider a class of compact connected semi-simple Lie group $\mathbb{LG}$, which was first proposed by Domokos-Manfredi [5]. The semi-simple Lie group $\mathbb{LG}$ is connected and compact. We notate $\mathcal{LG}$ as its Lie algebra. The inner product on $\mathcal{LG}$ satisfies properties

$$\langle gXg^{-1}, gYg^{-1}\rangle = \langle X, Y\rangle, \quad \forall\, g \in \mathbb{LG}, \text{ and } X, Y \in \mathcal{LG},$$

and

$$\langle [X, Y], Z\rangle = -\langle Y, [X, Z]\rangle, \quad \forall\, X, Y, Z \in \mathcal{LG}.$$

Let $\mathbb{LS}$ be the maximal torus of $\mathbb{LG}$. We notate $\mathcal{LS}$ as its Lie algebra. Owing to that $\mathcal{LS}$ is a maximal commutative subalgebra of $\mathcal{LG}$, we call it as the Cartan subalgebra. Denote by $\mathcal{R}$ the set of all roots, where we say that $R \in \mathcal{LS}$ is a root if $R \ne 0$ with the root space $\mathcal{LG}_R \ne \{0\}$. Here $\mathcal{LG}_R = \{Z \in \mathcal{LG}_{\mathbb{C}} : [S, Z] = i\langle R, S\rangle Z, \quad \forall S \in \mathcal{LS}\}$.

According to [5, Section 5], we can define the orthogonal complement of $\mathcal{LS}$ denoted by $\mathcal{H}$, and we can choose its orthonormal basis satisfying Property 7.1. We notate $\mathcal{B}_{\mathcal{H}} = \{X_1, X_2, \ldots, X_{2n}\}$ as the orthonormal basis of $\mathcal{H}$.

Proposition 7.1 (i) $\forall 1 \le k \le n,\ \exists R_k \in \mathcal{R}^+\ s.t.\ \mathrm{span}\{X_{2k-1}, X_{2k}\} = \mathcal{H}_{R_k}$.
(ii) $[X_{2k-1}, X_{2k}] = -R_k,\quad [X_{2k}, R_k] = -\|R_k\|^2 X_{2k-1},\quad [R_k, X_{2k-1}] = \|R_k\|^2 X_{2k}$.
(iii) $[X_l, X_m] \in \mathcal{H}$ *when* $(l, m) \ne (2k-1, 2k)$.
(iv) $\{[X_{2k-1}, S], [X_{2k}, S]\} \subset \mathcal{H}_{R_k}$ *when* $S \in \mathcal{LS}$.

Based on properties of $\mathcal{B}_{\mathcal{H}}$, a basis of $\mathcal{LS}$ can be selected, that is, $\{R_1, R_2, \ldots, R_\upsilon\}$. For any function v, we denote by

$$\nabla_{\mathcal{H}} v = (X_1 v, X_2 v, \ldots, X_{2n}),\quad \nabla_{\mathcal{T}} v = (R_1 v, R_2 v, \ldots, R_\upsilon v)$$

the horizontal and vertical gradients. Moreover, from Property 7.1, we draw the conclusion

$$[X_i, X_j] = \sum_{k=1}^{2n} \lambda_{i,j}^{(k)} X_k + \sum_{l=1}^{\upsilon} \theta_{i,j}^{(l)} R_l, \quad [X_i, R_j] = \sum_{k=1}^{2n} \vartheta_{i,j}^{(k)} X_k. \tag{7.1}$$

Here $\lambda_{i,j}^{(k)}$, $\theta_{i,j}^{(l)}$ and $\vartheta_{i,j}^{(k)}$ are constants.

Given a domain $\Omega \subset \mathbb{LG}$, we consider the equation

$$\sum_{i=1}^{2n} X_i^*(a_i(\nabla_{\mathcal{H}} u)) = 0 \quad \text{in } \Omega. \tag{7.2}$$

Here $\nabla_{\mathcal{H}} u = (X_1 u, \ldots, X_{2n} u)$ is the horizontal gradient of u, and the vector function $a = (a_1, \ldots, a_{2n})$ satisfies the conditions:

$$\sum_{i,j=1}^{2n} \frac{\partial a_i(\xi)}{\partial \xi_j} \eta_i \eta_j \geq l_0(\delta + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2, \tag{7.3}$$

$$\sum_{i,j=1}^{2n} \left| \frac{\partial a_i(\xi)}{\partial \xi_j} \right| \leq L(\delta + |\xi|^2)^{\frac{p-2}{2}}, \tag{7.4}$$

$$|a_i(\xi)| \leq L(\delta + |\xi|^2)^{\frac{p-2}{2}} |\xi| \tag{7.5}$$

for all $\xi, \eta \in \mathbb{R}^{2n}$, where $0 \leq \delta \leq 1$, $1 < p < \infty$ and $0 < l_0 < L$. Based on the conclusion (7.1) and above conditions, our method can be expended to the semi-simple Lie group $\mathbb{LG}$. We list our main results for $\mathbb{LG}$ and omit the proof.

Theorem 7.2 *Suppose that the conditions* (7.3)*,* (7.4) *and* (7.5) *hold for some* l_0*,* L *and such* $1 < p < 2$*,* $\delta \geq 0$*. If* $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ *is a weak solution to* (7.2)*, then* $\nabla_{\mathcal{H}} u \in C^{\alpha}_{\mathrm{loc}}(\Omega)$ *for some* $\alpha \in (0, 1)$ *depending on* l_0*,* L *and such* p*,* δ*. Moreover, for all* $B_{r_0} \subset \Omega$ *and any* $0 < r \leq r_0$*, we have*

$$\max_{1 \leq l \leq 2n} \operatorname{osc}_{B_r} X_l u \leq C \left(\frac{r}{r_0}\right)^{\alpha} \left(⨍_{B_{r_0}} (\delta + |\nabla_{\mathcal{H}} u|^2)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}, \tag{7.6}$$

where $0 < \alpha < 1$ *depends on* p*,* l_0 *and* L*, and the constant* $C > 0$ *depends on* p*,* l_0*,* L *and* r_0*.*

Consequently, when $1 < p < 2$*, the horizontal gradients of* p*-harmonic functions on* $\mathbb{LG}$ *have the Hölder regularity and satisfy* (7.6)*.*

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Appendix

The following lemma is [13, Lemma 4.7], which will be used to prove Lemma 3.3.

Lemma 8.3 *For a non-negative sequence* $\{y_l\}_{l=0,1,2,\ldots}$*,* $y_{l+1} \leq c_0 b_0^l y_l^{1+\varepsilon}$ *implies that*

$$y_l \leq c_0^{\frac{(1+\varepsilon)^l - 1}{\varepsilon}} b_0^{\frac{(1+\varepsilon)^l - 1}{\varepsilon^2} - \frac{l}{\varepsilon}} y_0^{(1+\varepsilon)^l},$$

where $c_0 > 0$, $\varepsilon > 0$ and $b_0 > 1$.

Moreover, if $y_0 \le \theta = c_0^{-\frac{1}{\varepsilon}} b_0^{-\frac{1}{\varepsilon^2}}$, then $y_l \le \theta b_0^{-\frac{l}{\varepsilon}}$. Consequently, $y_l \to 0$ as $l \to \infty$.

Proof of Lemma 3.3 For any $b \in (0, 1)$, letting $\rho_h = \frac{\rho}{2} + \frac{\rho}{2^{h+1}}$, $\rho = \rho_h$, $\rho' = \rho_{h+1}$ and $k_h = k + bH - b^{h+1}H$ in (3.1), we have

$$\begin{aligned}\int_{A^+_{k_h,\rho_{h+1}}} |\nabla_{\mathcal{H}} u|^2 dx &\le \frac{\gamma}{(\rho_h - \rho_{h+1})^2} \int_{A^+_{k_h,\rho_h}} |u-k|^2 dx + \chi^2 |A^+_{k_h,\rho_h}|^{1-2/q} \\ &\le \frac{2^{2h+4}\gamma}{\rho^2} |A^+_{k_h,\rho_h}| H^2 + \chi^2 |A^+_{k_h,\rho_h}|^{1-2/q}, \end{aligned} \tag{8.7}$$

where $h = 0, 1, 2, \dots$ and $q > Q = 10$.

The following inequality comes from [12, Lemma 2.3]

$$(l-k)|A^+_{l,\rho}|^{1-\frac{1}{Q}} \le \frac{C\rho |B_\rho|^{1-1/Q}}{|B_\rho \setminus A^+_{k,\rho}|} \int_{A^+_{k,\rho} \setminus A^+_{l,\rho}} |\nabla_{\mathcal{H}} u| dx. \tag{8.8}$$

Letting $l = k_{h+1}$, $k = k_h$ and $\rho = \rho_{h+1}$ in (8.8), we have

$$(k_{h+1} - k_h)|A^+_{k_{h+1},\rho_{h+1}}|^{1-\frac{1}{Q}} \le \frac{C\rho_{h+1} |B_{\rho_{h+1}}|^{1-1/Q}}{|B_{\rho_{h+1}} \setminus A^+_{k_h,\rho_{h+1}}|} \int_{A^+_{k_h,\rho_{h+1}} \setminus A^+_{k_{h+1},\rho_{h+1}}} |\nabla_{\mathcal{H}} u| dx. \tag{8.9}$$

The condition that $|A^+_{k,\rho}| \le \theta_1 |B_\rho|$ shows that

$$|A^+_{k_h,\rho_{h+1}}| \le |A^+_{k,\rho}| \le C 2^Q \theta_1 |B_{\rho_{h+1}}|.$$

From this, we choose θ_1 small enough such that $C2^Q\theta_1 \le 1/2$. Then (8.9) becomes

$$(b^{h+1} - b^{h+2}) H |A^+_{k_{h+1},\rho_{h+1}}|^{1-\frac{1}{Q}} \le \frac{C\rho_{h+1}}{|B_{\rho_{h+1}}|^{1/Q}} \int_{A^+_{k_h,\rho_{h+1}}} |\nabla_{\mathcal{H}} u| dx. \tag{8.10}$$

Applying Hölder's inequality to (8.10), then combining (8.7), from the assumption that $H = \sup_{B_\rho} u(x) - k \ge \chi \rho^{1-Q/q}$, we have

$$\begin{aligned}(b^{h+1} - b^{h+2})^2 H^2 |A^+_{k_{h+1},\rho_{h+1}}|^{2-\frac{2}{Q}} &\le C |A^+_{k_h,\rho_{h+1}}| \int_{A^+_{k_h,\rho_{h+1}}} |\nabla_{\mathcal{H}} u|^2 dx \\ &\le C |A^+_{k_h,\rho_h}|^{2-\frac{2}{q}} \left(\frac{2^{2h}\gamma}{\rho^2} |A^+_{k_h,\rho_h}|^{\frac{2}{q}} + \frac{\chi^2}{H^2} \right) H^2 \\ &\le C 2^{2h} (1+\gamma) H^2 \rho^{\frac{2Q}{q} - 2} |A^+_{k_h,\rho_h}|^{2-\frac{2}{q}}. \end{aligned}$$

Here for any ball B_ρ we use [24, Theorem V.4.1 on P69] to control its volume, that is, $C_1 \rho^Q \le |B_\rho| \le C_2 \rho^Q$. Thus

$$\left(\frac{|A^+_{k_{h+1},\rho_{h+1}}|}{\rho^Q} \right)^{1-\frac{1}{Q}} \le C 2^h (1-b)^{-1} b^{-h-1} (1+\gamma)^{\frac{1}{2}} \left(\frac{|A^+_{k_h,\rho_h}|}{\rho^Q} \right)^{1-\frac{1}{q}}. \tag{8.11}$$

Denote

$$\mu_h = \frac{|A^+_{k_h,\rho_h}|}{\rho^Q} \text{ and } \mu_0 = \frac{|A^+_{k,\rho}|}{\rho^Q} \le C\theta_1.$$

From (8.11), by Lemma 8.3 with $y_l = \mu_h$, there exists $\theta_1 = \theta_1(\gamma, q, b) \in (0, 1)$ such that $\lim_{h\to\infty} \mu_h = 0$, that is, $|A^+_{k+bH,\rho/2}| = 0$. □

Proof of Lemma 3.4 The following inequality comes from [12, Lemma 2.3]

$$(l-k)|A^+_{l,\rho}|^{1-\frac{1}{Q}} \le \frac{C\rho|B_\rho|^{1-1/Q}}{|B_\rho \setminus A^+_{k,\rho}|} \int_{A^+_{k,\rho}\setminus A^+_{l,\rho}} |\nabla_{\mathcal{H}} u|dx. \tag{8.12}$$

Denote

$$\mu_1 = \sup_{B_\rho} u(x) \ge k,\ \ w_1 = \mu_1 - k \text{ and } D_t = A^+_{\mu_1-\frac{w_1}{2^t},\rho/2} \setminus A^+_{\mu_1-\frac{w_1}{2^{t+1}},\rho/2},\ t = 1, 2, \dots.$$

Letting $l = \mu_1 - \frac{w_1}{2^{t+1}}$, $k = \mu_1 - \frac{w_1}{2^t}$ and $\rho \to \rho/2$ in (8.12), we have

$$\frac{w_1}{2^{t+1}} \left|A^+_{\mu_1-\frac{w_1}{2^{t+1}},\rho/2}\right|^{1-1/Q} \le \frac{C\rho|B_{\rho/2}|^{1-1/Q}}{|B_{\rho/2} \setminus A^+_{\mu_1-\frac{w_1}{2^t},\rho/2}|} \int_{D_t} |\nabla_{\mathcal{H}} u|dx,\ \ t = 1, 2, \dots. \tag{8.13}$$

The assumption that $|A^-_{k,\rho/2}| \ge \tau|B_{\rho/2}|$ implies that

$$\left|A^+_{\mu_1-\frac{w_1}{2^t},\rho/2}\right| \le \left|A^+_{\mu_1-\frac{w_1}{2},\rho/2}\right| = \left|A^+_{\frac{\mu_1}{2}+\frac{k}{2},\rho/2}\right| \le \left|A^+_{k,\rho/2}\right| \le (1-\tau)|B_{\rho/2}|,$$

which, together with Hölder's inequality to (8.13), yields

$$\left(\frac{w_1}{2^{t+1}}\right)^2 \left|A^+_{\mu_1-\frac{w_1}{2^{t+1}},\rho/2}\right|^{2-2/Q} \le \frac{C}{\tau^2}|D_t| \int_{D_t} |\nabla_{\mathcal{H}} u|^2 dx. \tag{8.14}$$

Letting $k = \mu_1 - \frac{w_1}{2^t}$, $\rho' \to \rho/2$ and $\rho \to \rho$ in (3.1), we have

$$\begin{aligned}
\int_{A^+_{\mu_1-\frac{w_1}{2^t},\rho/2}} |\nabla_{\mathcal{H}} u|^2 dx &\le \frac{4\gamma}{(\rho)^2} \int_{A^+_{\mu_1-\frac{w_1}{2^t},\rho}} \left|u - \left(\mu_1 - \frac{w_1}{2^t}\right)\right|^2 dx + \chi^2 \left|A^+_{\mu_1-\frac{w_1}{2^t},\rho}\right|^{1-2/q} \\
&\le C\left[\frac{\gamma}{\rho^2}\left|A^+_{\mu_1-\frac{w_1}{2^t},\rho}\right|^{2/q} \sup_{B_\rho}\left|u - \left(\mu_1 - \frac{w_1}{2^t}\right)\right|^2 + \chi^2\right] \left|A^+_{\mu_1-\frac{w_1}{2^t},\rho}\right|^{1-2/q} \\
&\le C\left[\gamma\rho^{2Q/q-2}\left(\frac{w_1}{2^t}\right)^2 + \chi^2\right] \left|A^+_{\mu_1-\frac{w_1}{2^t},\rho}\right|^{1-2/q}.
\end{aligned} \tag{8.15}$$

Below we may assume that

$$\chi\rho^{1-Q/q} \le w_1/2^t = (\mu_1 - k)/2^t, \tag{8.16}$$

otherwise we have

$$2^{-t_0}\mu_1 - 2^{-t_0}k < \chi\rho^{1-Q/q},$$

that is,

$$0 < 2^{-t_0}\left(k - \sup_{B_\rho} u(x)\right) + \chi\rho^{1-Q/q}.$$

Thus (3.2) holds. Combining (8.16), (8.15) and (8.14), we have

$$\left|A^+_{\mu_1-\frac{w_1}{2^{s-2}},\rho/2}\right|^{2-2/Q} \leq \left|A^+_{\mu_1-\frac{w_1}{2^{t+1}},\rho/2}\right|^{2-2/Q} \leq \frac{C(1+\gamma)}{\tau^2}|D_t|\rho^{2Q/q-2}. \tag{8.17}$$

Summing (8.17) with respect to t from 1 to $s-3$, we have

$$(s-2)\left|A^+_{\mu_1-\frac{w_1}{2^{s-2}},\rho/2}\right|^{2-2/Q} \leq \frac{C(1+\gamma)}{\tau^2}|B_{\rho/2}|^{2-2/Q}.$$

Here for any ball B_ρ we use [24, Theorem V.4.1 on P69] to control its volume, that is, $C_1\rho^Q \leq |B_\rho| \leq C_2\rho^Q$. Thus

$$\left|A^+_{\mu_1-\frac{w_1}{2^{s-2}},\rho/2}\right| \leq \left(\frac{C(1+\gamma)}{\tau^2(s-2)}\right)^{Q/(2Q-2)}|B_{\rho/2}|.$$

Denote

$$\theta_1 = \left(\frac{C(1+\gamma)}{\tau^2(s-2)}\right)^{Q/(2Q-2)}.$$

By (8.16) and Lemma 3.3 with $b = 1/2$ and $k = \mu_1 - \frac{w_1}{2^{s-2}}$, there exists $s = s(\gamma, q, \tau) > 0$ such that

$$\sup_{B_{\rho/4}} u(x) \leq \mu_1 - 2^{-s+1}w_1 = \sup_{B_\rho} u(x) - 2^{-s+1}\left(\sup_{B_\rho} u(x) - k\right).$$

Thus (3.2) holds true. □

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