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Biconservative surfaces with constant mean curvature in Lorentzian space forms

Aykut Kayhan[1] · **Nurettin Cenk Turgay**[2]

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Abstract
In this paper, we consider biconservative and biharmonic isometric immersions into the 4-dimensional Lorentzian space form $\mathbb{L}^4(\delta)$ with constant sectional curvature δ. We obtain some local classifications of biconservative CMC surfaces in $\mathbb{L}^4(\delta)$. Further, we get complete classification of biharmonic CMC surfaces in the de Sitter 4-space. We also proved that there is no biharmonic CMC surface in the anti-de Sitter 4-space. Further, we get the classification of biconservative, quasi-minimal surfaces in Minkowski-4 space.

Keywords Biconservative surfaces · Constant mean curvature · Lorentzian space forms · Quasi-minimal surfaces · de Sitter space

Mathematics Subject Classification 53C42 (Primary); 53B25

1 Introduction

After Eells and Lemaire defined k-harmonic maps between Riemannian manifolds for $k = 2, 3, \ldots$ as a natural extension of harmonic maps in [6], the particular case of $k = 2$ has taken the attention of many geometers in the last three decades, [2, 3, 9, 11, 16]. Namely, a map $\psi : (\Omega, g) \to (N, \tilde{g})$ between two semi-Riemannian manifolds is said to be biharmonic if it is a critical point of the bi-energy functional defined by

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✉ Nurettin Cenk Turgay
turgayn@itu.edu.tr

Aykut Kayhan
aykutkayhan@maltepe.edu.tr

1 Maltepe University, Maltepe, Istanbul, Turkey

2 Faculty of Science and Letters, Department of Mathematics, Istanbul Technical University, 34469 Maslak, Istanbul, Turkey

$$E_2(\psi) = \frac{1}{2}\int_{\Omega} |\tau(\psi)|^2 v_g,$$

where v_g is the volume element of g and $\tau(\psi) = -\mathrm{tr}\nabla d\psi$ is the tension field of ψ.

In [12, 13] Jiang obtained the first and second variational formulas of E_2 and concluded that ψ is biharmonic if and only if the corresponding Euler–Lagrange equation

$$\tau_2(\psi) := \Delta\tau(\psi) - \mathrm{tr}\tilde{R}(d\psi, \tau(\psi))d\psi = 0 \tag{1.1}$$

is satisfied, where τ_2 is called the bitension field. We want to note that a harmonic map is trivially biharmonic because it is well-known that a map ϕ is harmonic if and only if $\tau(\phi) = 0$, [7]. So, it is natural and interesting to investigate non-harmonic biharmonic maps, which are called *proper* biharmonic maps, [16].

On the other hand, a map $\psi : (\Omega, g) \to (N, \tilde{g})$ satisfying the condition

$$\langle \tau_2(\psi), d\psi \rangle = 0, \tag{1.2}$$

that is weaker than (1.1), is said to be biconservative. Note that an isometric immersion $\psi = x$ is biconservative if and only if the tangential part of $\tau_2(x)$ vanishes identically, that is

$$(\tau_2(x))^T = 0. \tag{1.3}$$

In the last decade, some authors have obtained results on biconservative submanifold to understand geometrical properties of biharmonic maps, [1, 8, 9, 11, 15, 17, 18]. For example, a classification of quasi-minimal biconservative surfaces in 4-dimesional semi-Riemannian space forms of index 2 was obtained in [18]. Further, in [15], Montaldo et. al. studied biconservative isometric immersions into 4-dimensional Riemannian space-forms where they considered constant mean curvature (CMC) surfaces. They proved the non-existence of *proper* biconservative CMC surfaces when the space form is not flat before they show that a biconservative CMC surface in the Euclidean 4-space must necessarily be a right cylinder with appropriately chosen base curve. We would like to notice that throughthout this paper we use the notion of '**proper**' biconservative submanifolds for submanifolds which has no open part with parallel mean curvature vector.

In this paper, we prove some theorems which shows that the situation is very different when the space form is assumed to be Lorentzian. In Sect. 3, we obtain complete classification of biconservative CMC surfaces in the Minkowski 4-space. In Sect. 4, we consider such surfaces in non-flat Lorentzian space forms. We also obtain a class of biharmonic surfaces in 4-dimensional de-Sitter space.

2 Preliminaries

Let $\mathbb{E}^n_s$ denote the semi-Euclidean n-space with index s whose metric tensor is given by

$$\tilde{g} = \langle\cdot,\cdot\rangle = -\sum_{i=1}^{s} dx_i \otimes dx_i + \sum_{j=s+1}^{n} dx_j \otimes dx_j,$$

where $(x_1, x_2, \ldots, x_n)$ is a Cartesian coordinate system in $\mathbb{R}^n$. We denote the n-dimensional Lorentzian space form with constant sectional curvature $\delta \in \{-1, 0, 1\}$ by $\mathbb{L}^n(\delta)$. In fact, we have

$$\mathbb{L}^n(\delta) = \begin{cases} \mathbb{S}^n_1 & \text{if } \delta = 1, \\ \mathbb{E}^n_1 & \text{if } \delta = 0, \\ \mathbb{H}^n_1 & \text{if } \delta = -1, \end{cases}$$

where $\mathbb{S}^n_1$ and $\mathbb{H}^n_1$ stand for the n-dimensional de Sitter and anti-de Sitter spaces, respectively.

2.1 Submanifolds of Lorentzian space forms

Let M be an m-dimensional semi-Riemannian submanifold of $\mathbb{L}^n(\delta)$. We put ∇ and $\tilde{\nabla}$ for the Levi-Civita connection of M and $\mathbb{L}^n(\delta)$, respectively. Then, Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

and

$$\tilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi, \tag{2.2}$$

respectively, for any vector fields X, Y tangent to M and ξ normal to M, where h is the second fundamental form, A is the shape operator and $\nabla^\perp$ is the normal connection. Denote the curvature tensor of M and $\mathbb{L}^n(\delta)$ with R and $\tilde{R}$, respectively, and let $R^\perp$ stand for the normal curvature tensor of M (in $\mathbb{L}^n(\delta)$). Then, the integrability conditions, called Gauss, Ricci and Codazzi equations,

$$R(X, Y)Z = \delta(\langle Y, Z\rangle X - \langle X, Z\rangle Y) + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.3}$$

$$R^\perp(X, Y)\xi = h(X, A_\xi Y) - h(A_\xi X, Y), \tag{2.4}$$

$$(\bar{\nabla}_Y h)(X, Z) = (\bar{\nabla}_X h)(Y, Z) \tag{2.5}$$

are satisfied, where the covariant derivative $\bar{\nabla}h$ of h is defined by

$$(\bar{\nabla}_X h)(Y, Z, \eta) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Moreover, the second fundamental form and the shape operators are related by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle. \tag{2.6}$$

On the other hand, the mean curvature vector H of M is defined by

$$H = \frac{1}{m}\mathrm{tr}h$$

and its norm $\|H\| = |\langle H, H\rangle|^{1/2}$ is called the mean curvature of M. Note that if $\|H\| = 0$ and $H \neq 0$, then M is said to be quasi-minimal.

In the remaining part of this subsection, we consider the case $m = 2$ and $n = 4$. In this case, if M has index 1, then there exists a semi-geodesic local coordinate system as following:

Proposition 2.1 ([5]) *Let M be a Lorentzian surface with the metric tensor g. Then, there exists a local coordinate system (s, t) such that*

$$g = -(dt \otimes ds + ds \otimes dt) + 2f ds \otimes ds. \tag{2.7}$$

Furthermore, the Levi-Civita connection of M satisfies

$$\begin{aligned} \nabla_{\partial_t}\partial_t &= 0,\\ \nabla_{\partial_t}\partial_s &= \nabla_{\partial_s}\partial_t = -f_t\partial_t,\\ \nabla_{\partial_s}\partial_s &= f_t\partial_s + (2ff_t - f_s)\partial_t \end{aligned} \tag{2.8}$$

We also want to state the following well-known lemma (See, for example, [14]):

Lemma 2.2 *Let M be a Lorentzian surface, $p \in M$ and A be a symmetric endomorphism of T_pM. Then, by choosing an appropriated base for T_pM, A can put into one of the following three canonical forms:*

$$\text{Case (i).} \quad A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad \text{with respect to } \{v_1, v_2\},$$

$$\text{Case (ii).} \quad A = \begin{bmatrix} a_1 & 1 \\ 0 & a_1 \end{bmatrix} \quad \text{with respect to } \{u_1, u_2\},$$

$$\text{Case (iii).} \quad A = \begin{bmatrix} a_1 & b \\ -b & a_1 \end{bmatrix} \quad \text{with respect to } \{v_1, v_2\}$$

Note that $b \neq 0$ and $\{v_1, v_2\}$ is an orthonormal base while $\{u_1, u_2\}$ stands for a pseudo-orthonormal base, that is,

$$\langle u_1, u_2\rangle = -1, \qquad \langle u_1, u_1\rangle = \langle u_2, u_2\rangle = 0.$$

On the other hand, if M is a surface in $\mathbb{L}^4(\delta)$, $\delta = \pm 1$, we are going to put $\hat{\nabla}$ for the Levi-Civita connection of $\mathbb{E}^5_\beta$, where $\beta = \frac{3-\delta}{2}$. Consider an isometric immersion $x : (\Omega, g) \hookrightarrow \mathbb{L}^4(\delta)$ with $x(\Omega) \subset M$. Let $i : \mathbb{L}^4(\delta) \subset \mathbb{R}^5$ be the inclusion and put $\hat{x} = i \circ x$. Then, we have

$$\hat{h}(X, Y) = i_*(h(X, Y)) - \delta g(X, Y)\hat{x}, \tag{2.9}$$

where $\hat{h}$ denotes the second fundamental form of M in $\mathbb{E}^5_\beta$.

2.2 Biconservative submanifolds

In this subsection, we give a summary of well-know facts about biconservative submanifolds in semi-Riemannian space-forms.

Let $x : (\Omega, g) \to (N, \tilde{g})$ be an isometric immersion between semi-Riemannian manifolds and put $M = x(\Omega)$. By splitting $\tau_2(x)$ into its tangential and normal components and considering (1.1), one can obtain the following proposition, (See, for example, [15]).

Proposition 2.3 [15] *x is biharmonic if and only if the equations*

$$m\,\text{grad}\,\|H\|^2 + 4\text{tr}A_{\nabla^\perp_\cdot H}(\cdot) + 4\text{tr}(\tilde{R}(\cdot, H)\cdot)^T = 0 \tag{2.10}$$

and

$$-\Delta^\perp H + \text{tr}h(A_H(\cdot), \cdot) + \text{tr}(\tilde{R}(\cdot, H)\cdot)^\perp = 0 \tag{2.11}$$

are satisfied, where m is the dimension of M and $\Delta^\perp$ is the Laplacian associated with $\nabla^\perp$.

Note that (1.3) implies

Proposition 2.4 [15] *x is biconservative if and only if the equation (2.10) is satisfied.*

Now, we consider the case $(N, \tilde{g}) = \mathbb{L}^n(\delta)$ and assume that M is a CMC surface. In this case, we have $\|H\| = \text{const.}$ and

$$\tilde{R}(X, H)Y = -\delta\langle X, Y\rangle H$$

whenever X, Y are tangent to M. Therefore, one can conclude that M is biconservative if and only if

$$\text{tr}A_{\nabla^\perp_\cdot H}(\cdot) = 0. \tag{2.12}$$

Moreover, the equation (2.11) turns into

$$-\Delta^\perp H + \text{tr}h(A_H(\cdot), \cdot) - \delta m H = 0 \tag{2.13}$$

Remark 2.5 If M is a submanifold of $\mathbb{L}^n(\delta)$ with parallel mean curvature vector, then the equation (2.10) is trivially satisfied. We would like to note that surfaces in $\mathbb{L}^n(\delta)$ with parallel mean curvature vector are classified in [10] (See also [4]).

Before we present our main results in the next sections, we would like to give the following characterization of biconservative surfaces with non-zero CMC.

Lemma 2.6 *Let M be a proper biconservative surface in $\mathbb{L}^4(\delta)$ with non-zero CMC and consider the orthonormal frame field $\{v_3, v_4\}$ of its normal bundle such that $H = cv_3$. Then, we have two cases:*

Case 1: The shape operator A_{v_4} has the matrix representation

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.14}$$

with respect to an appropriately chosen pseudo-orthonormal frame field $\{u_1, u_2\}$ of the tangent bundle of M.

Case 2: A_{v_4} satisfies $A_{v_4}X = 0$ whenever X is tangent to M.

Proof Let M be a proper biconservative surface with CMC. Define the 1-form ω_{34} by

$$\omega_{34}(X) = \langle \tilde{\nabla}_X e_3, e_4 \rangle \tag{2.15}$$

Then, (2.12) takes the form

$$\mathrm{tr}(\langle \nabla^{\perp}_{\cdot} v_3, v_4 \rangle A_{v_4}(\cdot)) = 0 \tag{2.16}$$

Since A_{v_4} is symmetric, it can be put into one of three forms given in case (i), (ii) and (iii) of Lemma 2.2. We are going to consider these cases separately. Note that in each of these cases we have

$$\mathrm{tr} A_{v_4} = 0$$

because $H = cv_3$.

Case (i). In this case, there is an orthonormal frame field $\{v_1, v_2\}$ such that

$$A_{v_4} = \begin{bmatrix} k_1 & 0 \\ 0 & -k_1 \end{bmatrix}$$

for a smooth function k_1. Thus, (2.16) turns into

$$\epsilon \omega_{34}(v_1) k_1 v_1 - \omega_{34}(v_2) k_1 v_2 = 0,$$

where we put $\epsilon = \langle v_1, v_1 \rangle = \pm 1$. Therefore, since M is proper biconservative, the open subset $\mathcal{O} = \{p \in M | k_1(p) \neq 0\}$ must be empty. Consequently, we have *Case 2* of the Lemma.

Case (ii). In this case, there is a pseudo-orthonormal frame field such that

$$A_{v_4} = \begin{bmatrix} k_1 & 1 \\ 0 & k_1 \end{bmatrix}$$

for a smooth function k_1. By considering $\mathrm{tr} A_4 = 0$, we obtain *Case 1* of the lemma.

Case (iii). There is an orthonormal frame field $\{v_1, v_2\}$ so that

$$A_{v_4} = \begin{bmatrix} 0 & -\gamma \\ \gamma & 0 \end{bmatrix}$$

for a smooth non-vanishing function γ because $tr A_4 = 0$. In this case, (2.16) becomes

$$\epsilon\omega_{34}(v_1)\gamma v_1 + \omega_{34}(v_2)\gamma v_2 = 0$$

which gives $\omega_{34}(v_1) = \omega_{34}(v_2) = 0$ because $\gamma \neq 0$. However, this is not possible unless H is parallel. □

3 CMC surfaces in $\mathbb{E}^4_1$

In this section, we obtain the complete local classification of biconservative CMC surfaces in the Minkowski 4-space $\mathbb{E}^4_1$.

3.1 Examples of biconservative surfaces

First, we obtain the following family of biconservative surfaces in $\mathbb{E}^4_1$ which has no counter part in the Euclidean 4-space.

Proposition 3.1 *The ruled surface*

$$x(s,t) = \alpha(s) + t\beta(s) \tag{3.1}$$

has constant curvature c and it is proper biconservative if α, β *satisfy*

$$\langle \beta, \beta \rangle = 0, \quad \langle \beta', \beta' \rangle = c^2, \quad \langle \alpha', \beta \rangle = -1, \tag{3.2}$$

where α *is a curve and* β *is a vector valued function.*

Proof Let M be a surface given by (3.1) and α, β satisfy (3.2). We define functions a_1, a_2, a_3, a_4 by

$$a_1 = \langle \alpha', \alpha' \rangle, \qquad a_2 = \langle \alpha', \beta' \rangle, \qquad a_3 = \langle \alpha', \beta'' \rangle, \qquad a_4 = \langle \beta'', \beta'' \rangle.$$

Then, because of (3.2), $u_1 = \partial_t$ and $u_2 = \partial_s + f\partial_t$ form a pseudo orthonormal frame field for the tangent bundle of M, where f is the smooth function given by

$$f(s,t) = \frac{1}{2}\left(c^2t^2 + 2ta_2(s) + a_1(s)\right). \tag{3.3}$$

Also, we consider the orthonormal frame field $\{v_3, v_4\}$ of the normal bundle of M, where

$$v_3 = -\frac{1}{c}(\beta' + f_t\beta).$$

By a direct computation we get

$$\tilde{\nabla}_{u_1} v_3 = c\partial_t, \tag{3.4}$$
$$h(u_1,u_1) = 0, \qquad H = -h(u_1,u_2) = cv_3, \tag{3.5}$$
$$A_4(u_1) = 0, \tag{3.6}$$

where (3.6) is obtained by combining (3.5) with (2.6). Note that the second equation in (3.5) yields that M has CMC. On the other hand, (3.4) and (3.6) implies

$$\operatorname{tr} A_{\nabla^\perp_{\cdot} H}(\cdot) = -A_{\nabla^\perp_{u_1} H}(u_2) - A_{\nabla^\perp_{u_2} H}(u_1) = 0$$

which yields that M is also biconservative because (2.12) is satisfied. By a further computation, we get

$$\tilde{\nabla}_{\partial_s} H = \left(\frac{2a_2^2 - a_1 + 2a_3 - 2a_2'}{2} + a_2\left(2c^2 - 1\right) t - \frac{1}{2}c^2\left(1 - 2c^2\right) t^2 \right) u_1 - u_2$$
$$+ \sqrt{a_2^2\left(c^2 - 2\right) + a_1 - 2a_3 + a_4 + 2a_2\left(c^2 - 1\right)^2 t + \left(c^3 - c\right)^2 t^2}\, v_4$$

which yields that H is not parallel. □

Before we continue, we want to present an explicit example.

Example 3.2 The vector valued functions

$$\beta(s) = \frac{1}{\sqrt{2}}(\cosh(bs), \sinh(bs), \cos(as), \sin(as))$$
$$\alpha(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{b}\sinh(bs), \frac{1}{b}\cosh(bs), -\frac{1}{a}\sin(as), \frac{1}{a}\cos(as)\right)$$

satisfies the conditions given in (3.2) for $c = \sqrt{(a^2 + b^2)/2}$. Therefore, the CMC surface given by

$$x(s,t) = \frac{1}{\sqrt{2}}\left(t\cosh(bs) + \frac{1}{b}\sinh(bs), t\sinh(bs) + \frac{1}{b}\cosh(bs),\right.$$
$$\left. t\cos(as) - \frac{1}{a}\sin(as), t\sin(as) + \frac{1}{a}\cos(as)\right)$$

is biconservative because of Proposition 3.1.

In the next two propositions, we obtain two families of biconservative cylinders in $\mathbb{E}^4_1$. Note that there exists a similar biconservative surface family in the Euclidean 4-space (See [15, Proposition 5.2]).

Proposition 3.3 *Let M be the cylinder in $\mathbb{E}^4_1$ given by*

$$x(s,t) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), t)$$

for an arc-length parametrized curve $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{E}^3_1$ with the non-null normal vector field. Then M is proper biconservative and CMC if its curvature is constant and torsion is non-vanishing.

Proof By the hypothesis, the vector fields $v_3 = (n_1(s), n_2(s), n_3(s), 0)$ and $v_4 = (b_1(s), b_2(s), b_3(s), 0)$ form a local orthonormal frame field for the normal bundle of M, where $n = (n_1, n_2, n_3)$ and $b = (b_1, b_2, b_3)$ denote unit normal and binormal vector fields of α, respectively. By a direct computation, we obtain

$$H = \epsilon_1 \frac{\kappa}{2} v_3, \qquad A_4 = 0, \qquad \nabla^{\perp}_{\partial_s} v_3 = \epsilon_2 \tau v_4$$

for some $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ depending on the causality of n and b, respectively, where κ and τ are the curvature and torsion of α, respectively. Now, if κ is constant then M is CMC. In this case, $A_4 = 0$ implies (2.12). □

By a similar way, we have

Proposition 3.4 *Let M be the cylinder in $\mathbb{E}^4_1$ given by*

$$x(s,t) = (t, \alpha_1(s), \alpha_2(s), \alpha_3(s))$$

for an arc-length parametrized curve $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{E}^3$. Then M is proper biconservative and CMC if its curvature is constant and torsion is non-vanishing.

We also want to give the following example of quasi-minimal biconservative surface in $\mathbb{E}^4_1$.

Example 3.5 [2] Consider the surface in $\mathbb{E}^4_1$ given by

$$(u, v) = (\psi(u, v), u, v, \psi(u, v)) \tag{3.7}$$

for a smooth function ψ. A direct computation yields that its mean curvature vector is

$$H = \frac{\psi_{uu} + \psi_{vv}}{2}(1, 0, 0, 1)$$

and it satisfies $A_H = 0$. A further computation shows that (2.12) is satisfied and H is not parallel if $\psi_{uu} + \psi_{vv}$ is not a constant.

3.2 Local classification theorem

In this subsection, first we consider two cases given in Lemma 2.6 separately in order to obtain the complete classification biconservative CMC surfaces in the Minkowski 4-space.

Proposition 3.6 *Let M be a proper biconservative surface in $\mathbb{E}^4_1$ satisfying the Case 1 of Lemma 2.6. Then, it is locally congruent to the surface described in Proposition 3.1.*

Proof Assume that M satisfy the condition given in the Case 1 of Lemma 2.6 for the frame field $\{u_1, u_2, v_3, v_4\}$ and let ω_{34} be defined as (2.15). We consider a local coordinate system (s, t) defined on the open set $\mathcal{O} \subset M$ satisfying the conditions given in Proposition 2.1 such that u_1 is proportional to ∂_t. Let $x(s, t)$ be a local parametrization of $\mathcal{O} \subset M$. Then, we have

$$A_{v_4}(\tilde{u}_1) = 0 \text{ and } A_{v_4}(\tilde{u}_2) = \gamma \tilde{u}_1$$

for a non-vanishing smooth function γ, where we define $\tilde{u}_1, \tilde{u}_2$ by

$$\tilde{u}_1 = \partial_t, \tilde{u}_2 = \partial_s + f\partial_t. \tag{3.8}$$

Note that (2.12) implies

$$\omega_{34}(u_1) = 0. \tag{3.9a}$$

and (2.6) gives

$$h(\tilde{u}_1, \tilde{u}_1) = h^3_{11} v_3, \tag{3.9b}$$

$$h(\tilde{u}_1, \tilde{u}_2) = c v_3, \tag{3.9c}$$

$$h(\tilde{u}_2, \tilde{u}_2) = h^3_{22} v_3 - \gamma v_4 \tag{3.9d}$$

for some functions h^3_{11} and h^3_{22}. We combine (3.9) with the Codazzi equation (2.5) for $X = u_2, Y = Z = u_1$ to get

$$u_2(h^3_{11}) = -2\langle \nabla_{u_2} u_1, u_2 \rangle h^3_{11}, \tag{3.10}$$

$$h^3_{11} \omega_{34}(u_2) = 0. \tag{3.11}$$

Since H is not parallel, (3.9a) and (3.11) imply $h^3_{11} = 0$. Consequently, (3.9b) and (2.8) give

$$\tilde{\nabla}_{\partial_t}\partial_t = 0$$

which yields $x_{tt} = 0$. Therefore, we have (3.1) for some α, β. By considering (2.7), we get the first and the third equations in (3.2). On the other hand, by a direct computation, we obtain

$$H = -(\beta' + f_t\beta)$$

which yields the second equation of (3.2). Hence, $\mathcal{O}$ is congruent to the ruled surface given in Proposition 3.1. □

Proposition 3.7 *Let M be a proper biconservative surface in $\mathbb{E}^4_1$ satisfying the Case 2 of Lemma 2.6. Then, it is locally congruent to one of two cylinders described in Propositions 3.3 and 3.4.*

Proof Assume that M satisfy the condition given in the Case 2 of Lemma 2.6 for the frame field $\{v_3, v_4\}$ of the normal bundle of M, $p \in M$ and let ω_{34} be the 1-form defined as (2.15). Since M is proper biconservative, we have $\omega_{34} \neq 0$ outside of a subset of M with empty interior. Note that $H = 2cv_3$ implies $\mathrm{tr}A_{v_3} = 2c$. We are going to consider three canonical forms of A_{v_3} given in Lemma 2.2 separately.
Case (i). There is an orthonormal frame field $\{v_1, v_2\}$ such that

$$A_{v_3} = \begin{bmatrix} k_1 & 0 \\ 0 & 2c - k_1 \end{bmatrix}$$

for a smooth function k_1. We assume $\langle v_2, v_2\rangle = 1$ and put $\epsilon = \langle v_1, v_1\rangle \in \{-1, 1\}$. In this case, by a direct computation using the Codazzi equation (2.5) for $X = v_1$, $Y = Z = v_2$ and $X = v_2$, $Y = Z = v_1$, we obtain

$$(2c - k_1)\omega_{34}(v_1) = 0, \tag{3.12a}$$
$$\epsilon k_1\omega_{34}(v_2) = 0, \tag{3.12b}$$
$$v_1(k_1) = 2c\phi_2, \tag{3.12c}$$
$$v_2(k_1) = -2c\phi_1, \tag{3.12d}$$

where we define ϕ_i by $\nabla_{v_i} v_1 = \phi_i v_2$.

First assume $\omega_{34}(v_1) = 0$ on M. Then, $\omega_{34}(v_2) \neq 0$ and (3.12b) implies $k_1 = 0$. On the other hand, if $\omega_{34}(v_1) \neq 0$ on an open subset $\mathcal{O}$ of M, then (3.12a) and (3.12b) imply $k_1 = c$ and $\omega_{34}(v_2) = 0$, separately. In both cases, (3.12c) and (3.12d) yields that $\phi_1 = \phi_2 = 0$ on M. Therefore, we have $\nabla_{v_i} v_j = 0$, $i, j = 1, 2$ which implies the existence of a local coordinate system (s_1, s_2) such that $v_1 = \partial_{s_1}$, $v_2 = \partial_{s_2}$ defined in a neighborhood $\mathcal{N}_p$ of p. Let $x = x(s,t)$ be a local parametrization of $\mathcal{N}_p$. We put $s_1 = s$, $s_2 = t$ if $\psi_2 = 0$ and $s_1 = t$, $s_2 = s$ if $\psi_1 = 0$. In both cases, the Gauss formula turns into

$$\widetilde{\nabla}_{\partial_t}\partial_t = 0, \qquad \widetilde{\nabla}_{\partial_t}\partial_s = 0$$

which gives $x_{tt} = x_{ts} = 0$. Therefore, we have

$$x(s,t) = \alpha(s) + t\beta_0 \tag{3.13}$$

for a $\mathbb{R}^4$-valued function α and constant vector $\beta_0 \in \mathbb{E}^4_1$. By considering that $\{\partial_s, \partial_t\}$ an orthonormal frame field, we obtain that $\mathcal{N}_p$ is congruent to one of two cylinders given in Propositions 3.3 and 3.4.

Case (ii). Assume that there is a pseudo-orthonormal frame field $\{u_1, u_2\}$ such that

$$A_{v_3} = \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}. \tag{3.14}$$

In this case, by combining $A_{v_4} = 0$ and (3.14) with (2.6), we get

$$h(u_1, u_1) = 0, \ h(u_1, u_2) = -cv_3, h(u_2, u_2) = -v_3.$$

By considering the Codazzi equation (2.5) for we obtain $\omega_{34} = 0$ which is not possible.
Case (iii). Assume that there is an orthonormal frame field $\{v_1, v_2\}$ such that

$$A_{v_3} = \begin{bmatrix} c & \gamma \\ -\gamma & c \end{bmatrix}$$

and $\langle v_1, v_1\rangle = -1$, where γ is a smooth non-vanishing function. Note that we have

$$h(v_1, v_1) = -cv_3, \ h(v_1, v_2) = -\gamma v_4, h(v_2, v_2) = cv_3.$$

In this case, we use the Codazzi equation (2.5) to get

$$c\omega_{34}(v_1) + \gamma\omega_{34}(v_2) = \gamma\omega_{34}(v_1) - c\omega_{34}(v_2) = 0$$

However, since γ is non-vanishing, these equations give $\omega_{34} = 0$ which yields a contradiction. □

By combining Propositions 3.6 and 3.7, we obtain the following classification theorem.

Theorem 3.8 *A surface M in $\mathbb{E}^4_1$ has non-zero CMC and it is biconservative if and only if it is locally congruent to one of the following four types of surfaces.*

(i) *A surface with parallel mean curvature vector,*
(ii) *A ruled surface described in Proposition 3.1,*
(iii) *A cylinder described in Proposition 3.3,*
(iv) *A cylinder described in Proposition 3.4.*

Remark 3.9 The surfaces given in the case (ii) and case (iv) of Theorem 3.8 is not proper biharmonic. On the other hand, if M is a cylinder given in the case (iii) of Theorem 3.8, then it is biharmonic if and only if its profile curve is appropriately chosen (See [2, Theorem 5.1] and [3, Theorem 5.1]).

Now, let M be a quasi-minimal surface in $\mathbb{E}^4_1$ and consider the pseudo-orthonormal frame field $\{u_3, u_4\}$ of its normal bundle such that

$$H = u_3.$$

Note that this equation implies $\mathrm{tr} A_{u_3} = 0$ because of (2.12). Therefore, since M is Riemannian, we can choose orthonormal tangent vector fields v_1, v_2 so that

$$A_{u_3} = \begin{bmatrix} k_1 & 0 \\ 0 & -k_1 \end{bmatrix}.$$

for some smooth functions k_1. Consequently, the biconservativity equation (2.12) implies

$$0 = \mathrm{tr} A_{\nabla^\perp_\cdot H}(\cdot) = \psi_1 k_1 v_1 - \psi_2 k_2 v_2$$

where we define ψ_1, ψ_2 by

$$\nabla^\perp_{v_i} u_3 = \psi_i u_3.$$

Therefore, if M is proper biconservative and quasi-minimal if and only if

$$A_H = 0.$$

By using the exactly same method in [2, Sect. 6], we observe that M is locally congruent to the surface given in Example 3.5. Therefore, we have

Proposition 3.10 *A quasi-minimal surface M in $\mathbb{E}^4_1$ is CMC and proper biconservative if and only if it is locally congruent to the surface given in Example 3.5 for a smooth function ψ such that $\psi_{uu} + \psi_{vv}$ is not a constant.*

4 CMC surfaces in $\mathbb{S}^4_1$ and $\mathbb{H}^4_1$

In this section, we consider CMC surfaces in non-flat Lorentzian space forms. First, we obtain the following classification theorem.

Theorem 4.1 *Let M be a surface in $\mathbb{L}^4(\delta)$, $\delta = \pm 1$. Then, M has non-zero CMC and it is proper biconservative if and only if it is locally congruent to the ruled surface parametrized by (3.1) for some α, β satisfying*

$$\langle \alpha, \alpha \rangle = \delta, \quad \langle \alpha, \beta \rangle = 0, \quad \langle \beta, \beta \rangle = 0, \tag{4.1a}$$

$$\langle \beta, \beta \rangle = 0, \quad \langle \beta', \beta' \rangle = 1 + c^2, \quad \langle \alpha', \beta \rangle = -1, \tag{4.1b}$$

where c is the mean curvature of M.

Proof In order to prove the necessary condition, we assume that M is a proper biconservative CMC surface. First, we consider the subset

$$\mathcal{F} = \{p \in M | A_{v_4}(X) = 0 \text{ whenever } X \in T_pM\}$$

of M and assume that its interior $\tilde{\mathcal{O}}$ is not empty. In this case, similar to the proof of Proposition 3.7, we obtain that A_{v_3} has the matrix representation

$$A_{v_3} = \begin{bmatrix} 2c & 0 \\ 0 & 0 \end{bmatrix},$$

with respect to an orthonormal frame field $\{v_1, v_2\}$ on $\tilde{\mathcal{O}}$, where c is the mean curvature of M. By using the Codazzi equation, we obtain $\nabla_{v_i} v_j = 0,\ i, j = 1, 2$ which yields that $\tilde{\mathcal{O}}$ is flat, i.e., $R = 0$. Then, we consider the Gauss equation (2.3) for $X = Z = v_1,\ Y = v_2$ to get $v_2 = 0$ on $\mathcal{O}$ which is not possible. Therefore, the interior of $\mathcal{F}$ is empty and Lemma 2.6 implies that A_{v_4} has the matrix representation given in (2.14) with respect to an appropriately chosen pseudo-orthonormal frame field $\{u_1, u_2\}$ of the tangent bundle of M.

We consider a local coordinate system (s, t) defined on the open set $\mathcal{O} \subset M$ satisfying the conditions given in Proposition 2.1 and define $\tilde{u}_1, \tilde{u}_2$ as given in (3.8). Consequently, by using the Codazzi equation, we get

$$h(\tilde{u}_1, \tilde{u}_1) = 0.$$

By using this equation, (2.8) and (2.9) we obtain

$$\hat{\nabla}_{\partial_t} \partial_t = 0$$

which give $x_{tt} = 0$, where $x = x(s, t)$ is the local parametrization of $\mathcal{O}$. Therefore, $\mathcal{O}$ is congruent to a ruled surface (3.1) for some α, β. Note that $\langle x, x \rangle = \delta$, implies (4.1a) and

(2.7) gives the first and the third equations in (4.1b). By considering (2.9), (3.1) and (3.8) we obtain

$$-H + \delta x = \hat{\nabla}_{\tilde{u}_1}\tilde{u}_2 = \beta' + f_t\beta$$

from which we get the second equation in (4.1b). Hence, $\mathcal{O}$ is congruent to the ruled surface given in the theorem.

The proof of the sufficient condition follows from a direct computation similar to the proof of Proposition 3.1. □

Remark 4.2 In [15, Theorem 5.1], it is proved that there exists no CMC proper biconservative surface in the non-flat Riemannian space forms $\mathbb{S}^4$ and $\mathbb{H}^4$.

Next, we consider (2.13) for the surface given in Theorem 4.1 to obtain the classification of biharmonic CMC surfaces.

Let M be the proper biconservative CMC surface in $\mathbb{L}^4(\delta)$, $\delta = \pm 1$ parametrized by (3.1) for some vector valued functions α, β satisfying (4.1). We define $\tilde{u}_1, \tilde{u}_2$ as given in (3.8) to get (3.4), (3.5) and

$$H = \delta\alpha + (\delta t - f)\beta - \beta' \tag{4.2}$$

By combining (3.4) and (3.5) with the Ricci equation (2.4), we obtain

$$-\Delta^{\perp} H = 0. \tag{4.3}$$

By using (2.6) and (4.2), we get

$$\mathrm{tr} h(A_H(\cdot), \cdot) = 2c^2 H. \tag{4.4}$$

By considering (4.3) and (4.4), we conclude that (2.13) is equivalent to

$$2(c^2 - \delta)H = 0.$$

Hence, we have the following results.

Theorem 4.3 *Let M be a proper biconservative surface in the de Sitter space $\mathbb{S}^4_1$ with the constant mean curvature $c \neq 0$. Then, M is biharmonic if and only if $c = 1$.*

Theorem 4.4 *There exists no proper biharmonic surface in the anti-de Sitter space $\mathbb{H}^4_1$ with non-zero constant mean curvature.*

Next, we want to present an explicit example:

Example 4.5 The vector valued functions

$$\beta(s) = \frac{1}{\sqrt{2}}(\cosh(bs), \sinh(bs), \cos(as), \sin(as), 0)$$
$$\alpha(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{b}\sinh(bs), \frac{1}{b}\cosh(bs), -\frac{1}{a}\sin(as), \frac{1}{a}\cos(as), 2 - \frac{1}{a^2} - \frac{1}{b^2}\right)$$

satisfies the conditions given in (4.1) for $\delta = 1$ and c satisfying $a^2 + b^2 = 2(1 + c^2)$. Therefore, the ruled surface

$$x(s,t) = \frac{1}{\sqrt{2}}\left(\frac{1}{b}\sinh(bs) + t\cosh(bs), \frac{1}{b}\cosh(bs) + \sinh(bs), -\frac{1}{a}\sin(as) + \cos(as), \frac{1}{a}\cos(as) + \sin(as), 2 - \frac{1}{a^2} - \frac{1}{b^2}\right)$$

is a proper biconservative surface in $\mathbb{S}^4_1$ with the constant curvature c because of Proposition 3.1. Furthermore, this surface is biharmonic in $\mathbb{S}^4_1$ if $a^2 + b^2 = 4$.

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