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CORRECTION

Correction to: Isotropicity of surfaces in Lorentzian 4-manifolds with zero mean curvature vector

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Abstract
In this paper, we see that the hypersurfaces $\mathcal{L}_{\pm}$ in Ando (Abh Math Semin Univ Hambg 92:105–123, 2022, Proposition 1) are neutral but not flat. Nonetheless, we find parallel almost complex structures $\mathcal{I}_{\pm}$ suitable for Ando (Abh Math Semin Univ Hambg 92:105–123, 2022, Theorem 1) and parallel almost paracomplex structures $\mathcal{J}_{\pm}$ suitable for Ando (Abh Math Semin Univ Hambg 92:105–123, 2022, Theorem 2).

Keywords Light cone · $SO(3, 1)$-orbit · Complex structure · Paracomplex structure

Mathematics Subject Classification 53B25 · 53C42 · 53C50

1 Introduction

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Let E^4_1 be the Minkowski 4-space and $\bigwedge^2 E^4_1$ the 2-fold exterior power of E^4_1. Then $\bigwedge^2 E^4_1$ is of dimension 6 and the Minkowski metric of E^4_1 induces an indefinite metric of $\bigwedge^2 E^4_1$ with signature (3, 3). The $SO(3, 1)$-action on E^4_1 yields an $SO(3, 1)$-action on $\bigwedge^2 E^4_1$. In addition, each element of $SO(3, 1)$ gives an isometry of $\bigwedge^2 E^4_1$. In particular, we have an $SO(3, 1)$-action on the light cone $\mathcal{L}$ of $\bigwedge^2 E^4_1$. In the paragraph just before [2, Proposition 1], two hypersurfaces $\mathcal{L}_{\pm}$ of $\mathcal{L}$ are given. These are $SO(3, 1)$-orbits in $\mathcal{L}$. In this proposition, it was asserted that $\mathcal{L}_{\pm}$ are neutral, that is, they have neutral metrics. This assertion has no problems. However, we will see in this paper that $\mathcal{L}_{\pm}$ are not flat, although it was asserted that $\mathcal{L}_{\pm}$ are flat in [2, Proposition 1]. By the equation of Gauss for submanifolds $\mathcal{L}_{\pm}$ of $\bigwedge^2 E^4_1$, we can explicitly represent the curvature tensors of $\mathcal{L}_{\pm}$, and we will see that they

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do not vanish. In [2, Proposition 1], it was also asserted that $\mathcal{L}_\pm$ are neutral hyperKähler. However, according to the proof of [2, Proposition 1], this assertion is based on the flatness. Therefore the assertion that $\mathcal{L}_\pm$ are neutral hyperKähler must be cancelled. Hence we see that Proposition 1 of [2] should be stated as follows:

The 4-*submanifolds* $\mathcal{L}_\pm$ *are neutral and not flat.*

In this paper, we will find one parallel almost complex structure and one parallel almost paracomplex structure on each of $\mathcal{L}_\pm$. In addition, we will see that they are suitable for Theorems 1 and 2 in [2]. Therefore these theorems have no problems.

2 The curvature tensors

As was used in the proof of [2, Proposition 1], let $\tilde{\nabla}^+$ be the Levi-Civita connection of the metric of $\mathcal{L}_+$ induced by the metric $\hat{h}$ of $\bigwedge^2 E_1^4$ and S a surface in $\mathcal{L}_+$ given by $S = \{\tilde{T}_{P_{3,1}} \circ \tilde{T}_{P_{3,2}}(E_{+,1}) \mid \theta, t \in \boldsymbol{R}\}$, where $E_{\pm,i}$ $(i = 1, 2, 3)$ are given in the second paragraph of [2, Section 2] and $\tilde{T}_{P_{k,l}}$ $(k = 1, 2, 3, l = 1, 2)$ are given in the proof of [2, Proposition 1]. Then vector fields $E'_{\pm,2}, E'_{\pm,3}$ along S given in the proof of [2, Proposition 1] are parallel with respect to $\tilde{\nabla}^+$. Let $\hat{\nabla}$ be the Levi-Civita connection of $\hat{h}$. Then $E'_{\pm,3}$ are parallel with respect to $\hat{\nabla}$, while $E'_{\pm,2}$ are not parallel with respect to $\hat{\nabla}$. Let ω_{ij} be as in the second paragraph of [2, Section 2]. Then $\omega_{13}, \omega_{42}, \omega_{23}, \omega_{14}$ form a pseudo-orthonormal basis of the tangent space of $\mathcal{L}_+$ at a point $E_{+,1}$. In addition, ω_{13}, ω_{42} form a pseudo-orthonormal basis of the tangent plane of S at the same point. Let ω'_{ij} be vector fields along S given by $\omega'_{ij} = \tilde{T}_{P_{3,1}} \circ \tilde{T}_{P_{3,2}}(\omega_{ij})$. Then using

$$\tilde{T}_{P_{3,1}}(E_{\pm,2}) = -\sin\theta E_{\pm,1} + \cos\theta E_{\pm,2},$$
$$\tilde{T}_{P_{3,2}}(E_{\mp,2}) = \mp\sinh t E_{\pm,1} + \cosh t E_{\mp,2},$$

which were already obtained in the proof of [2, Proposition 1], we obtain

$$\begin{aligned} \hat{\nabla}_{\omega_{13}}\omega'_{13} &= -\hat{\nabla}_{\omega_{42}}\omega'_{42} = -\frac{1}{\sqrt{2}}(\omega_{12} - \omega_{34}), \\ \hat{\nabla}_{\omega_{13}}\omega'_{42} &= \hat{\nabla}_{\omega_{42}}\omega'_{13} = -\frac{1}{\sqrt{2}}(\omega_{12} + \omega_{34}). \end{aligned} \tag{1}$$

Referring to the previous paragraph, we have an analogous study along a surface $S^\perp$ in $\mathcal{L}_+$ given by $S^\perp = \{\tilde{T}_{P_{2,1}} \circ \tilde{T}_{P_{2,2}}(E_{+,1}) \mid \theta, t \in \boldsymbol{R}\}$. Then ω_{23}, ω_{14} form a pseudo-orthonormal basis of the tangent plane of $S^\perp$ at $E_{+,1}$. Let ω''_{ij} be vector fields along $S^\perp$ given by $\omega''_{ij} = \tilde{T}_{P_{2,1}} \circ \tilde{T}_{P_{2,2}}(\omega_{ij})$. Then we obtain

$$\begin{aligned} \hat{\nabla}_{\omega_{23}}\omega''_{23} &= -\hat{\nabla}_{\omega_{14}}\omega''_{14} = -\frac{1}{\sqrt{2}}(\omega_{12} - \omega_{34}), \\ \hat{\nabla}_{\omega_{14}}\omega''_{23} &= \hat{\nabla}_{\omega_{23}}\omega''_{14} = -\frac{1}{\sqrt{2}}(\omega_{12} + \omega_{34}). \end{aligned} \tag{2}$$

Let $\tilde{R}^+$, $\hat{R}$ be the curvature tensors of $\tilde{\nabla}^+$, $\hat{\nabla}$ respectively. Then using (1), (2) and the equation of Gauss for $\mathcal{L}_+$:

$$\begin{aligned}0 &= \hat{h}(\hat{R}(X,Y)Z,W)\\ &= \hat{h}(\tilde{R}^{+}(X,Y)Z,W) + \hat{h}(\sigma(X,Z),\sigma(Y,W)) - \hat{h}(\sigma(X,W),\sigma(Y,Z))\end{aligned}$$

(σ is the second fundamental form of $\mathcal{L}_+$ in $\bigwedge^2 E_1^4$), we obtain

Proposition 1 *If we set*

$$A = \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix},$$

then the following hold:

$$\begin{aligned}
&\tilde{R}^{+}(\omega_{13},\omega_{42}) = 0,\\
&\tilde{R}^{+}(\omega_{23},\omega_{14}) = 0,\\
&(\tilde{R}^{+}(\omega_{13},\omega_{23})\omega_{13}\ \tilde{R}^{+}(\omega_{13},\omega_{23})\omega_{42}\ \tilde{R}^{+}(\omega_{13},\omega_{23})\omega_{23}\ \tilde{R}^{+}(\omega_{13},\omega_{23})\omega_{14})\\
&\quad = -(\tilde{R}^{+}(\omega_{42},\omega_{14})\omega_{13}\ \tilde{R}^{+}(\omega_{42},\omega_{14})\omega_{42}\ \tilde{R}^{+}(\omega_{42},\omega_{14})\omega_{23}\ \tilde{R}^{+}(\omega_{42},\omega_{14})\omega_{14})\\
&\quad = (\omega_{13}\ \omega_{42}\ \omega_{23}\ \omega_{14})A,\\
&(\tilde{R}^{+}(\omega_{13},\omega_{14})\omega_{13}\ \tilde{R}^{+}(\omega_{13},\omega_{14})\omega_{42}\ \tilde{R}^{+}(\omega_{13},\omega_{14})\omega_{23}\ \tilde{R}^{+}(\omega_{13},\omega_{14})\omega_{14})\\
&\quad = (\tilde{R}^{+}(\omega_{42},\omega_{23})\omega_{13}\ \tilde{R}^{+}(\omega_{42},\omega_{23})\omega_{42}\ \tilde{R}^{+}(\omega_{42},\omega_{23})\omega_{23}\ \tilde{R}^{+}(\omega_{42},\omega_{23})\omega_{14})\\
&\quad = (\omega_{13}\ \omega_{42}\ \omega_{23}\ \omega_{14})B.
\end{aligned}$$

From Proposition 1, we see that $\mathcal{L}_+$ is not flat. Similarly, we see that $\mathcal{L}_-$ is not flat.

3 Complex structures and paracomplex structures

Let $\mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$ denote the tangent space of $\mathcal{L}_+$ at a point $E_{+,1}$. Let $\hat{\wedge}$ denote the exterior product of the exterior algebra of $\mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$. Then we denote by $\hat{\bigwedge}^2 \mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$ the 2-fold exterior power of $\mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$. We set

$$X_1 := \omega_{23}, \quad X_2 := \omega_{14}, \quad Y_1 := \omega_{13}, \quad Y_2 := \omega_{42}.$$

Then $\hat{\bigwedge}^2 \mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$ is decomposed into

$$\hat{\bigwedge}^2 \mathcal{T}_{E_{+,1}}(\mathcal{L}_+) = \hat{\bigwedge}^2_+ \mathcal{T}_{E_{+,1}}(\mathcal{L}_+) \oplus \hat{\bigwedge}^2_- \mathcal{T}_{E_{+,1}}(\mathcal{L}_+),$$

where

(i) $\hat{\bigwedge}^2_+ \mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$ is generated by

$$\frac{1}{\sqrt{2}}(X_1\hat{\wedge}Y_1 - X_2\hat{\wedge}Y_2), \quad \frac{1}{\sqrt{2}}(X_1\hat{\wedge}X_2 + Y_2\hat{\wedge}Y_1), \quad \frac{1}{\sqrt{2}}(X_1\hat{\wedge}Y_2 + Y_1\hat{\wedge}X_2),$$

(ii) $\hat{\bigwedge}^2_- \mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$ is generated by

$$\frac{1}{\sqrt{2}}(X_1\hat{\wedge}Y_1 + X_2\hat{\wedge}Y_2), \quad \frac{1}{\sqrt{2}}(X_1\hat{\wedge}X_2 - Y_2\hat{\wedge}Y_1), \quad \frac{1}{\sqrt{2}}(X_1\hat{\wedge}Y_2 - Y_1\hat{\wedge}X_2).$$

The stabilizer $G(E_{+,1})$ of $SO(3,1)$ at $E_{+,1}$ is generated by $P_{1,1}$, $\pm P_{1,2}$ ($\theta, t \in \boldsymbol{R}$). Then $G(E_{+,1})$ acts on $\mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$. Therefore $G(E_{+,1})$ acts on $\hat{\bigwedge}^2 \mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$.

We see that $(1/\sqrt{2})(X_1 \hat{\wedge} Y_1 - X_2 \hat{\wedge} Y_2)$ is an invariant element of $\hat{\bigwedge}^2_+ \mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$ by the $G(E_{+,1})$-action, which is unique up to a constant, and $(1/\sqrt{2})(X_1 \hat{\wedge} Y_1 - X_2 \hat{\wedge} Y_2)$ defines an almost complex structure $\mathcal{I}_+$ on $\mathcal{L}_+$ by the $SO(3,1)$-action. Using (1) and (2), and referring to [1], we see that $\mathcal{I}_+$ is parallel with respect to $\tilde{\nabla}^+$.

We see that $(1/\sqrt{2})(X_1 \hat{\wedge} Y_2 - Y_1 \hat{\wedge} X_2)$ is an invariant element of $\hat{\bigwedge}^2_- \mathcal{T}_{E_{+,1}}(\mathcal{L}_+)$ by the $G(E_{+,1})$-action, which is unique up to a constant, and $-(1/\sqrt{2})(X_1 \hat{\wedge} Y_2 - Y_1 \hat{\wedge} X_2)$ defines an almost paracomplex structure $\mathcal{J}_+$ on $\mathcal{L}_+$ by the $SO(3,1)$-action. Using (1) and (2), and referring to [1], we see that $\mathcal{J}_+$ is parallel with respect to $\tilde{\nabla}^+$.

We have similar discussions for $\mathcal{L}_-$ and we obtain an almost complex structure $\mathcal{I}_-$ and an almost paracomplex structure $\mathcal{J}_-$ on $\mathcal{L}_-$, which are parallel with respect to the Levi-Civita connection $\tilde{\nabla}^-$ of the metric of $\mathcal{L}_-$ induced by $\hat{h}$. Hence we obtain

Proposition 2 *For $\varepsilon \in \{+, -\}$, $\mathcal{L}_\varepsilon$ has just two almost complex structures $\pm\mathcal{I}_\varepsilon$ and just two almost paracomplex structures $\pm\mathcal{J}_\varepsilon$ by the $SO(3,1)$-action and these are parallel with respect to $\tilde{\nabla}^\varepsilon$.*

We see that $\mathcal{I}_\pm$, $\mathcal{J}_\pm$ satisfy (5), (6) in the proof of [2, Proposition 1] respectively. Therefore Theorems 1 and 2 have no problems.

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