Abh. Math. Semin. Univ. Hambg. (2023) 93:133–147
https://doi.org/10.1007/s12188-023-00271-z

Towards generic base-point-freeness for hyperkähler manifolds of generalized Kummer type

Mauro Varesco[1]

Received: 6 February 2023 / Accepted: 17 October 2023 / Published online: 16 November 2023

Abstract
We study base-point-freeness for big and nef line bundles on hyperkähler manifolds of generalized Kummer type: For $n \in \{2, 3, 4\}$, we show that, generically in all but a finite number of irreducible components of the moduli space of polarized Kum^n-type varieties, the polarization is base-point-free. We also prove generic base-point-freeness in the moduli space in all dimensions if the polarization has divisibility one.

Keywords Hyperkähler manifolds · Generalized Kummer varieties · Base locus · Fujita's conjecture

Mathematics Subject Classification 14M99

Introduction

The starting point for the analysis presented in this article is the following observation in the context of Fujita's conjecture: Given a K3 surface X and an ample line bundle $H \in \mathrm{Pic}(X)$, Mayer proved in [14] that the line bundle $2H$ is base-point-free. This result is stronger than what one gets when applying Fujita's conjecture to the case of K3 surfaces. Indeed, the conjecture predicts that the line bundle $3H$ is base-point-free, see e.g. [12, Conj. 10.4.1]. This suggests that it might be interesting to study questions related to big and nef line bundles on hyperkähler manifolds.

Let $\mathscr{M}^n_{d,t}$ be the moduli space of hyperkähler manifolds of Kum^n-type with a polarization of square $2d$ and divisibility t. In this article, we study base-point-freeness for big and nef line bundles on hyperkähler manifolds of Kum^n-type: In low dimenision, that is $n \in \{2, 3, 4\}$, we prove that, for all but a finite number of choices of d and t, the polarization of the generic element of $\mathscr{M}^n_{d,t}$ is base-point-free:

Communicated by Daniel Greb.

✉ Mauro Varesco
varesco@math.uni-bonn.de

1 Mathematisches Institut, Universitat Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Theorem (Theorem 5.8) *Let $n \in \{2, 3, 4\}$, and let d and t be positive integers such that the moduli space $\mathscr{M}^n_{d,t}$ is non-empty. Let A be the set of triples*

$$A := \{(2, 1, 2), (3, 4, 2), (3, 28, 8), (3, 92, 8), (4, 3, 2), (4, 20, 5), (4, 55, 10)\}.$$

Then, if $(n, d, t) \notin A$, the polarization of a general pair in $\mathscr{M}^n_{d,t}$ is base-point-free.

In the case where the divisibility of the polarization is one, we prove generic base-point-freeness in all dimensions:

Theorem (Theorem 3.1) *The polarization of the general pair of $\mathscr{M}^n_{d,1}$ is base-point-free for all $d > 0$ and $n > 1$.*

In the article [17] by Rieß , a similar statement is proven for hyperkähler manifolds which are deformation equivalent to the Hilbert scheme of points on a K3 surface: There, the author proves generic base point freeness in the moduli space $\mathscr{F}^2_{d,t}$ parametrizing pairs (X, L) where X is a hyperkähler manifold of $K3^{[2]}$-type and L is a polarization on X of square $2d$ and divisibility t. This result has then been extended by Debarre [6]: There, the author studies in all dimensions the case where the divisibility of the polarization is one or two, showing that, if $t = 1$ and $d \geq n-1$ or if $t = 2$ and $d \geq n+3$, the polarization of a generic pair $(X, L) \in \mathscr{F}^n_{d,t}$ is base-point-free. In dimension six, generic base-point-freeness for $K3^{[3]}$-type varieties is proven by Agostini and Rieß in the draft [1]. For a description of the divisorial component of the base locus of big and nef line bundles on hyperkähler manifolds of $K3^{[n]}$-type and of Kum^n-type see the article [16] by Rieß . There the author provides in particular a criterion for big and nef line bundles to have base divisor.

We are the first, to our knowledge, to study generic base-point-freeness for the moduli space of polarized Kum^n-type varieties. Nevertheless, most ideas and techniques that we have used in this paper are inspired by the work of the aforementioned authors. The structure of this article is the following:

In Sect. 1, we recall the definition of *hyperkähler manifold*, and we collect some results on the lattice structure of the integral second cohomology group of hyperkähler manifolds of Kum^n-type.

In Sect. 2, we deduce from a result in [15] that the moduli space $\mathscr{M}^n_{d,t}$ is connected when non-empty for $n \in \{2, 3, 4\}$. Since it is known that this space is always smooth, we conclude that it is irreducible. We then recall the proof of the fact that base-point-freeness is an open property in families of polarized hyperkähler manifolds. This will lead us to a useful criterion for generic base-point-freeness in the moduli space.

Section 3 is devoted to study the case where the divisibility of the polarization is one. We deal with it separately since the technique that we use differs from the ones we use in the other cases. In particular, we prove that any effective line bundle L on an abelian surface T induces a base-point-free line bundle on $\mathrm{Kum}^n(T)$ for all $n > 1$. We will use this to prove generic base-point-freeness in the moduli space $\mathscr{M}^n_{d,1}$ for all $n > 1$ and $d > 0$.

In Sect. 4, we recall the notions of tautological bundles on the Hilbert scheme of points and of k-very ampleness. These are key ingredients for the proof of generic base-point-freeness in the remaining cases.

Finally, in Sect. 5, we deal with polarizations with higher divisibility. We prove that, for all but a finite number of cases, generic base-point-freeness holds for $\mathscr{M}^n_{d,t}$ for $n \in \{2, 3, 4\}$. This will conclude the proof of the main result of this article.

1 Hyperkähler manifolds

Let us begin by recalling the definition of hyperkähler manifold. For an introduction to the subject see [7].

Definition 1.1 A *hyperkähler manifold* is a simply-connected compact Kähler manifold X, such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form.

A particular family of examples of hyperkähler manifolds can be constructed as follows: Let T be an abelian surface and let $T^{[n+1]}$ be the Hilbert scheme of $n+1$ points on T. Then, the fibre over 0 of the natural morphism $T^{[n+1]} \to T$ induced by the sum is a hyperkähler manifold of dimension $2n$. It is called *generalized Kummer variety*, and it is denoted by $\mathrm{Kum}^n(T)$. In this paper, we focus on varieties which are deformation equivalent to some generalized Kummer variety. These varieties are still hyperkähler manifolds and are called Kum^n-type varieties.

Recall that the second cohomology group $H^2(X, \mathbb{Z})$ of a hyperkähler manifold X has a natural lattice structure induced by the Beauville–Bogomolov quadratic form q. Talking about the lattice $(H^2(X, \mathbb{Z}), q)$, we will use the following definitions:

Definition 1.2 Let (Λ, q) be a lattice and let α be an element in Λ, then:

(i) The element α is called *primitive*, if it is not a non-trivial multiple of another element, i.e., $\alpha = k \cdot \alpha'$ for some $k \in \mathbb{Z}$ and $\alpha' \in \Lambda$ implies $k = \pm 1$.
(ii) The *divisibility of* α, $\mathrm{div}(\alpha)$, is the multiplicity of the element $q(-, \alpha) \in \Lambda^\vee$, i.e., $\mathrm{div}(\alpha) = m$ if m spans the ideal $\{q(B, \alpha) | \ B \in \Lambda\} \subseteq \mathbb{Z}$.
(iii) Let $\{e_i\}$ be a $\mathbb{Z}$-basis of the lattice Λ. The *discriminant* of the lattice (Λ, q) is defined as the determinant of the matrix $(q(e_i, e_j))_{i,j}$.

As the discriminant of a lattice does not depend on the choice of the basis, we see that the divisibility of a primitive element divides the discriminant of the lattice.

Remark 1.3 If X is a hyperkähler manifold, then $H^1(X, \mathscr{O}_X) = 0$. Therefore, using the exponential sequence, one identifies the Picard group of X with a subgroup of the lattice $H^2(X, \mathbb{Z})$. Given an element $L \in \mathrm{Pic}(X)$, we will consider its divisibility with respect to the lattice $(H^2(X, \mathbb{Z}), q)$. As a consequence, the divisibility of any element of $\mathrm{Pic}(X)$ is invariant under deformation of X.

In the case of hyperkähler manifolds of Kum^n-type, the lattice structure on the second cohomology has been described:

Proposition 1.4 [3, Prop. 8] *Let* $X := \mathrm{Kum}^n(T)$ *for some abelian surface* T. *Then:*

(i) The lattice $(H^2(X, \mathbb{Z}), q)$ *has the following orthogonal decomposition:*

$$(H^2(X, \mathbb{Z}), q) = (H^2(T, \mathbb{Z}), \cup) \oplus (\mathbb{Z}\delta),$$

where $\cup$ *is the cup product on* $H^2(T, \mathbb{Z})$ *and* δ *is the restriction to* X *of the divisor* $\frac{1}{2}E$, *where* E *is the exceptional divisor on* $T^{[n+1]}$ *of the Hilbert–Chow morphism. In particular,* $q(\delta) = -2n - 2$.
(ii) Let Y *be a hyperkähler manifold of* Kum^n*-type which is deformation equivalent to* X. *Then, the lattice* $(H^2(Y, \mathbb{Z}), q)$ *has the following orthogonal decomposition:*

$$(H^2(Y, \mathbb{Z}), q) \simeq (H^2(X, \mathbb{Z}), q) = (H^2(T, \mathbb{Z}), \cup) \oplus (\mathbb{Z}\delta).$$

Remark 1.5 By abuse of notation, we will use δ to denote both the divisor $\frac{1}{2}E$ on the Hilbert scheme and its restriction on the generalized Kummer variety as in Proposition 1.4. Similarly, given a line bundle L on an abelian surface T, we will denote by L_n both the induced divisor on $T^{[n+1]}$ and its restriction to $\mathrm{Kum}^n(T)$. It will be clear from the context which of the two we are referring to.

Remark 1.6 Let $X = \mathrm{Kum}^n(T)$ for some abelian surface T. Proposition 1.4 implies that any element $\alpha \in H^2(X, \mathbb{Z})$ can be expressed as $\alpha = a\lambda_n + b\delta$ for a primitive element $\lambda \in H^2(T, \mathbb{Z})$ and $a, b \in \mathbb{Z}$. Since $(H^2(T, \mathbb{Z}), \cup)$ is a unimodular lattice, the divisibility of α is

$$\mathrm{div}(\alpha) = \gcd(a, 2(n+1)b).$$

2 The moduli space

Let $\mathscr{M}^n_{d,t}$ be the moduli space which parametrizes pairs (X, L) where X is a hyperkähler manifold of Kum^n-type and L is a primitive and ample line bundle on X with $q(L) = 2d$ and $\mathrm{div}(L) = t$.

Remark 2.1 Note that, by Remark 1.6, the moduli space $\mathscr{M}^n_{d,t}$ is empty if $t \nmid \gcd(2d, 2n+2)$.

In [15], the author characterizes the number of connected components of the moduli space $\mathscr{M}^n_{d,t}$. Let us fix the necessary notation to recall that result. Let $n, d, t > 0$ be integers such that $t \mid \gcd(2d, 2n+2)$ and set

$$d_1 := \frac{2d}{\gcd(2d, 2n+2)}, \quad n_1 := \frac{2n+2}{\gcd(2d, 2n+2)}, \quad g := \frac{\gcd(2d, 2n+2)}{t},$$

$$w := \gcd(g, t), \qquad g_1 := \frac{g}{w}, \qquad t_1 := \frac{t}{w}.$$

Let ϕ be the Euler function, i.e., the function that associates to a positive integer l the number of integers in $\{1, \dots, l\}$ which are coprime with l. Moreover, let $\rho(l)$ be the number of primes in the factorization of l. For w and t_1 as above, write $w = w_+(t_1)w_-(t_1)$, where $w_+(t_1)$ is the product of all powers of primes dividing $\gcd(w, t_1)$. Finally, denote by $|\mathscr{M}^n_{d,t}|$ the number of connected components of $\mathscr{M}^n_{d,t}$.

Theorem 2.2 [15, Thm. 5.3] *With the above notation we have:*

(1) $|\mathscr{M}^n_{d,t}| = w_+(t_1)\phi(w_-(t_1))2^{\rho(t_1)-1}$ *if* $t > 2$ *and one of the following holds:*

- *(a)* g_1 *is even,* $\gcd(d_1, t_1) = 1 = \gcd(n_1, t_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* t_1*;*
- *(b)* $g_1, t_1,$ *and* d_1 *are odd,* $\gcd(d_1, t_1) = 1 = \gcd(n_1, 2t_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* $2t_1$*;*
- *(c)* $g_1, t_1,$ *and* w *are odd,* d_1 *is even,* $\gcd(d_1, t_1) = 1 = \gcd(n_1, 2t_1)$ *and* $-d_1/4n_1$ *is a quadratic residue mod* t_1.

(2) $|\mathscr{M}^n_{d,t}| = w_+(t_1)\phi(w_-(t_1))2^{\rho(t_1/2)-1}$ *if* $t > 2$*,* g_1 *is odd,* t_1 *is even,* $\gcd(d_1, t_1) = 1 = \gcd(n_1, 2t_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* $2t_1$.

(3) $|\mathscr{M}^n_{d,t}| = 1$ *if* $t \le 2$ *and one of the following holds:*

- *(a)* g_1 *is even,* $\gcd(d_1, t_1) = 1 = \gcd(n_1, t_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* t_1*;*

(b) *g_1, t_1, and d_1 are odd,* $\gcd(d_1, t_1) = 1 = \gcd(n_1, 2t_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* $2t_1$*;*
(c) *g_1, t_1, and w are odd,* $\gcd(d_1, t_1) = 1 = \gcd(n_1, 2t_1)$ *and* $-d_1/4n_1$ *is a quadratic residue mod* t_1*;*
(d) *g_1 is odd,* $\gcd(d_1, t_1) = 1 = \gcd(n_1, 2t_1)$ *and* $-d_1/n_1$ *is a quadratic residue mod* $2t_1$.

(4) $|\mathscr{M}^n_{d,t}| = 0$ *otherwise.*

In the case $n \in \{2, 3, 4\}$, Theorem 2.2 implies the connectedness of $\mathscr{M}^n_{d,t}$:

Corollary 2.3 *Let $n \in \{2, 3, 4\}$ and let $t, d > 0$ be integers such that $\mathscr{M}^n_{d,t}$ is non-empty. Then, $\mathscr{M}^n_{d,t}$ is connected.*

Proof We do the computation only for $n = 2$, the other cases can be checked in the same way. By Remark 2.1, the moduli space $\mathscr{M}^n_{d,t}$ is non-empty by assumption only if t divides $\gcd(2d, 2n + 2)$. For $n = 2$, we then have to consider only the cases where $t \in \{1, 2, 3, 6\}$.

If $t = 1, 2$, the moduli space $\mathscr{M}^2_{d,t}$ is connected by the case (3) of Theorem 2.2.

If $t = 3$, then $g = \frac{\gcd(2d,6)}{3} = 2$, $w = \gcd(2, 3) = 1$, $g_1 = 2$, and $t_1 = 3$. Therefore, since g_1 is even and since we are assuming that $\mathscr{M}^2_{d,t}$ is non-empty, we are in the case $(1.a)$ of Theorem 2.2. From $w_+(t_1) = w_-(t_1) = 1$, we then deduce that

$$|\mathscr{M}^2_{d,3}| = 1\phi(1)2^{\rho(3)-1} = 1.$$

If $t = 6$, then $g = \frac{\gcd(2d,6)}{6} = 1$, $w = \gcd(1, 6) = 1$, $g_1 = 1$, and $t_1 = 6$. Therefore, since g_1 is odd and t_1 is even and since we are assuming that $\mathscr{M}^2_{d,t}$ is non-empty, we are in the case (2) of Theorem 2.2. As $w_+(t_1) = w_-(t_1) = 1$, we deduce that

$$|\mathscr{M}^2_{d,6}| = 1\phi(1)2^{\rho(6/2)-1} = 1.$$

□

It is a standard fact that the moduli space $\mathscr{M}^n_{d,t}$ is smooth:

Proposition 2.4 [9, Sec. 1.14] *For every $n > 1$ and every positive integers d and t, the moduli space $\mathscr{M}^n_{d,t}$ is smooth.*

Proof Let (X, L) be a point of $\mathscr{M}^n_{d,t}$, and let $\mathrm{Def}(X, L)$ be the universal deformation of the pair (X, L) as in [9, Sec. 1.14]. Locally around (X, L), the moduli space $\mathscr{M}^n_{d,t}$ is isomorphic to the open subset of $\mathrm{Def}(X, L)$ where the line bundle L remains ample. The statement then follows from the fact that $\mathrm{Def}(X, L)$ is a smooth hypersurface of the universal deformation space $\mathrm{Def}(X)$. □

As an immediate corollary we have the following:

Corollary 2.5 *Every connected component of $\mathscr{M}^n_{d,t}$ is irreducible. In particular, for $n \in \{2, 3, 4\}$, the moduli space $\mathscr{M}^n_{d,t}$ is irreducible whenever it is non-empty.*

In the following proposition, we recall the standard fact that base-point-freeness is an open property in families of hyperkähler manifolds:

Proposition 2.6 *Let $\zeta : \mathscr{X} \to S$ be a family of hyperkähler manifolds over a connected base S. Let $\mathscr{L}$ be a line bundle on $\mathscr{X}$ such that $q(\mathscr{L}_s) > 0$ for all $s \in S$, where $\mathscr{L}_s$ denotes the restriction of $\mathscr{L}$ to the fibre $\mathscr{X}_s$. Then, if $\mathscr{L}_0$ is base-point-free for some $0 \in S$, then $\mathscr{L}_s$ is base-point-free for all s in an open neighbourhood of $0 \in S$.*

Proof Let us begin by reducing to the case where $\mathscr{L}$ is a line bundle on $\mathscr{X}$ whose push-forward $\zeta_*\mathscr{L}$ is a free sheaf of rank $h^0(\mathscr{L}_0)$: By assumption, the line bundle $\mathscr{L}_0$ is base-point-free and satisfies $q(\mathscr{L}_0) > 0$. This implies that $\mathscr{L}_0$ is big and nef by the Beauville–Fujiki relation [7, Prop. 23.14]. Therefore, $h^i(\mathscr{L}_0) = 0$ for all $i > 0$ by the Kodaira vanishing theorem. The semicontinuity theorem then implies that $h^i(\mathscr{L}_s) = 0$ for all $i > 0$ and all s in a neighbourhood U of $0 \in S$, see [8, Thm. III.12.8]. Since the Euler characteristic $\chi(\mathscr{L}_s)$ does not depend on $s \in S$ by flatness, $h^0(\mathscr{L}_s)$ is constant in the neighbourhood U. Therefore, the push forward $(\zeta|_{\zeta^{-1}U})_*(\mathscr{L}|_{\zeta^{-1}U})$ is a locally free sheaf of rank $m = h^0(\mathscr{L}_0)$ by Grauert theorem, see [8, Cor. III.12.9]. After substituting S with a suitable open neighbourhood of 0 and $\mathscr{X}$ with the preimage via ζ of this neighbourhood, we may assume that $\mathscr{L}$ is a line bundle on $\mathscr{X}$ such that

$$\zeta_*\mathscr{L} \simeq \mathscr{O}_S^{\oplus m}.$$

Let $\{e_1, \ldots, e_m\}$ be the standard basis of the $H^0(\mathscr{O}_S)$-module $H^0(\mathscr{O}_S^{\oplus m})$. Since $H^0(\mathscr{L}) \simeq H^0(\mathscr{O}_S^{\oplus m})$, we may view $\{e_1, \ldots, e_m\}$ as a basis of $H^0(\mathscr{L})$. The base locus of $|\mathscr{L}|$ is then

$$\mathrm{BL}(|\mathscr{L}|) = \bigcap_{i=1}^{m} V(e_i) \subseteq \mathscr{X}.$$

The set $\zeta(\mathrm{BL}(|\mathscr{L}|)) \subseteq S$ is closed since ζ is a proper map and $\mathrm{BL}(|\mathscr{L}|)$ is closed. By assumption, the line bundle $\mathscr{L}_0$ is base-point-free. Therefore, the element $0 \in S$ does not belong to $\zeta(\mathrm{BL}(|\mathscr{L}|)$ since $H^0(\mathscr{L})$ surjects onto $H^0(\mathscr{L}_0)$ via restriction. This allows us to conclude that $W := S \backslash \zeta(\mathrm{BL}(|\mathscr{L}|)$ is an open neighbourhood of $0 \in S$ such that for all $s \in W$ the line bundle $\mathscr{L}_s$ is base-point-free. □

Let us deduce from Proposition 2.6 the following criterion for generic base-point-freeness in the moduli space $\mathscr{M}^n_{d,t}$:

Proposition 2.7 *Let $n > 1$, and let d and t be positive integers such that the moduli space $\mathscr{M}^n_{d,t}$ is irreducible. Assume that there exists a pair (X, L) where X is a* Kum^n*-type variety and L is a base-point-free line bundle on X with $q(L) = 2d$ and $\mathrm{div}(L) = t$. Then, the polarization of a generic pair in $\mathscr{M}^n_{d,t}$ is base-point-free.*

Proof Proposition 2.6 shows that base-point-freeness is an open property in families of polarized hyperkähler manifolds. Therefore, as the moduli space $\mathscr{M}^n_{d,t}$ is irreducible by assumption, we just need to find an element $(\tilde{X}, \tilde{L}) \in \mathscr{M}^n_{d,t}$ for which the line bundle $\tilde{L}$ is base-point-free. Note that L is not assumed to be ample, so the pair (X, L) does not necessarily belong to $\mathscr{M}^n_{d,t}$. Let $\mathscr{X} \to S$ be a family of hyperkähler manifolds of generic Picard rank one such that $\mathscr{X}_0 \simeq X$ for some $0 \in S$, and for which there exists a line bundle $\mathscr{L} \in \mathrm{Pic}(\mathscr{X})$ such that $\mathscr{L}|_{\mathscr{X}_0} = L$. By the projectivity criterion (see [9, Thm. 3.11] and [11, Thm. 2]), the variety $\mathscr{X}_s$ is projective for all $s \in S$. From Proposition 2.6, we conclude that the line bundle $\mathscr{L}_s$ is base-point-free and ample for general $s \in S$. For this last step, we use the fact that the Picard rank of $\mathscr{X}_s$ is one for general $s \in S$, and the fact that $-\mathscr{L}_s$ cannot be ample since $\mathscr{L}_s$ has sections. In particular, for general $s \in S$, the pair $(\mathscr{X}_s, \mathscr{L}_s)$ belongs to $\mathscr{M}^n_{d,t}$ and $\mathscr{L}_s$ is base-point-free. This concludes the proof. □

3 Divisibility one

In this section, we prove generic base-point-freeness in the case where the divisibility of the polarization is one. The results of this section hold in all dimensions.

Theorem 3.1 *The moduli space $\mathscr{M}^n_{d,1}$ is non-empty and irreducible for all $n > 1$ and $d > 0$, and the polarization of a generic pair in $\mathscr{M}^n_{d,1}$ is base-point-free.*

Before diving into the proof of Theorem 3.1, let us recall some results on abelian varieties that we will use throughout this section, see [4] for all the necessary definitions and notations. A line bundle on an abelian variety is determined by its first Chern class and its semicharacter by the Appell–Humbert theorem [4, Thm. 1.2.3]. Denote by $\mathscr{L}(H, \chi)$ the unique line bundle whose first Chern class is H and whose semicharacter is χ. Then, the following results hold:

Lemma 3.2 [4, Lem. 2.3.2] *Let $T := V/\Lambda$ be an abelian variety and let $\mathscr{L}(H, \chi) \in \mathrm{Pic}(T)$ be a line bundle on T. Then, for every $\overline{v} \in T$ with representative $v \in V$ the following holds*

$$t^*_{\overline{v}}\mathscr{L}(H, \chi) = \mathscr{L}(H, \chi\exp(2\pi i \mathrm{Im} H(v, \cdot))),$$

where $t_{\overline{v}}$ is the translation on T induced by the translation t_v on V which sends a vector $x \in V$ to the vector $v + x$. □

Lemma 3.3 [4, Lem. 2.3.4] *Let $f : T \to T'$ be a morphism of abelian varieties with analytic representation $F : V \to V'$. Then, for any $\mathscr{L}(H, \chi) \in \mathrm{Pic}(X')$,*

$$f^*\mathscr{L}(H, \chi) = \mathscr{L}(F^*H, F^*\chi).$$

□

Let us start the proof of Theorem 3.1 by proving that $\mathscr{M}^n_{d,1}$ is always non-empty:

Lemma 3.4 *The space $\mathscr{M}^n_{d,1}$ is non-empty for every $n > 1$ and $d > 0$.*

Proof Let T be an abelian surface which admits a polarization L with $L^2 = 2d$ and let L_n be the induced line bundle on the generalized Kummer variety $\mathrm{Kum}^n(T)$. This line bundle is not necessarily ample, but satisfies $\mathrm{div}(L_n) = 1$ and $q(L_n) = 2d$. By deforming the pair $(\mathrm{Kum}^n(T), L_n)$ and using the projectivity criterion (see [9, Thm. 3.11] and [11, Thm. 2]) as in the proof of Proposition 2.7, we find a hyperkähler manifold belonging to $\mathscr{M}^n_{d,1}$. Thus, this moduli space is non-empty. □

Note that Corollary 2.5 shows that $\mathscr{M}^n_{d,1}$ is irreducible for every n since by Theorem 2.2 the moduli space $\mathscr{M}^n_{d,t}$ is connected for every n if $t = 1$. Therefore, we are in the situation of Proposition 2.7, and, to prove generic base-point-freeness in $\mathscr{M}^n_{d,1}$, it suffices to find a pair (X, L) where X is a Kum^n-type variety and L is a base-point-free line bundle on X with $q(L) = 2d$ and $\mathrm{div}(L) = 1$. We do this by showing that, given any effective line bundle L on an abelian surface T the line bundle L_n induced on the generalized Kummer variety $\mathrm{Kum}^n(T)$ is base-point-free, see Theorem 3.10.

Let us consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Kum}^n(T) & \hookrightarrow & T^{[n+1]} & \xrightarrow{\sigma} & T \\
\downarrow & & \downarrow & \nearrow & \\
\overline{K^n} & \hookrightarrow & T^{(n+1)} & \Sigma & \\
\uparrow & & \uparrow & \nearrow & \\
K^n_T & \hookrightarrow & T^{n+1} & &
\end{array}, \tag{1}$$

where the first column is the base change by 0 of the second column and σ and Σ are the natural maps induced by the summation map on T. By construction, $K^n_T = \ker(\Sigma)$ and the following holds:

Lemma 3.5 *Let T be an abelian surface and let K_T^n as in (1). Then, the map*

$$\varphi\colon T^n \to K_T^n, \qquad (a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, -a_1 - \dots - a_n)$$

is an isomorphism. □

Remark 3.6 Note that K_T^n is a subtorus of T^{n+1} since it is the kernel of the morphism Σ. Moreover, the canonical action of the symmetric group $\mathfrak{S}_{n+1}$ on T^{n+1} preserves K_T^n and induces an action on it. Identifying the torus T^n with K_T^n via the map φ, we see that the induced action of $\mathfrak{S}_{n+1}$ on T^n is generated by the elements

$$[(a_1, \dots, a_n) \mapsto (a_i, a_2, \dots, a_{i-1}, a_1, a_{i+1}, \dots a_n)] \quad \text{for } i = 1, \dots, n, \quad \text{and}$$

$$[(a_1, \dots, a_n) \mapsto (-a_1 - \dots - a_n, a_2, \dots, a_n)].$$

Let k be a positive integer. Given a line bundle L on T, denote by $L^{\boxtimes k} := L \boxtimes \dots \boxtimes L$ the induced line bundle on T^k. Note that, via the map φ, the line bundle $L^{\boxtimes (n+1)}|_{K_T^n}$ is sent to

$$M := (L^{\boxtimes n}) \otimes (-1)^* \mu^* L,$$

where $\mu\colon T^n \to T$ is the addition map and $(-1)\colon T^n \to T^n$ is minus identity.

In the next proposition, we find line bundles M_x on T^n which are isomorphic to M:

Proposition 3.7 *Let L be a line bundle on an abelian surface T, and let M be as above. Then, for all $x \in T$, the line bundle*

$$M_x := t^*_{(nx,\dots,nx)}(L^{\boxtimes n}) \otimes t^*_{(-x,\dots,-x)}(-1)^* \mu^* L$$

is isomorphic to the line bundle M.

Proof By the Appell–Humbert theorem [4, Thm. 1.2.3], it suffices to check that the two line bundles M and M_x have the same first Chern class and the same semicharacter. Let us assume that $L = \mathscr{L}(H, \chi)$ and denote by $\mathrm{pr}_i\colon T^n \to T$ the projection onto the i-th factor. By Lemma 3.3, we have the following equalities:

$$L^{\boxtimes n} = \mathscr{L}\left(\sum_{i=1}^n \mathrm{pr}_i^* H, \prod_{i=1}^n \mathrm{pr}_i^* \chi\right),$$
$$(-1)^* \mu^* L = \mathscr{L}\left(\mu^* H, \frac{1}{\mu^* \chi}\right).$$

The second equality follows from Lemma 3.3 noting that $(-1)^* H = H$ and $(-1)^* \chi = \frac{1}{\chi}$. We then deduce that

$$M = \mathscr{L}\left(\sum_{i=1}^n \mathrm{pr}_i^* H + \mu^* H, \frac{\prod_{i=1}^n \mathrm{pr}_i^* \chi}{\mu^* \chi}\right).$$

On the other hand, applying Lemma 3.2, we see that

$$t^*_{(nx,\dots,nx)}(L^{\boxtimes n}) = \mathscr{L}\left(\sum_{i=1}^n \mathrm{pr}_i^* H, \prod_{i=1}^n \mathrm{pr}_i^* \chi \cdot e^{2\pi i \mathrm{Im}((\sum_{i=1}^n \mathrm{pr}_i^* H)((nx,\dots,nx),\cdot))}\right),$$

$$t^*_{(-x,\dots,-x)}(-1)^*\mu^*L = \mathscr{L}\left(\mu^*H, \frac{e^{2\pi i \mathrm{Im}(\mu^*H((-x,\dots,-x),\cdot))}}{\mu^*\chi}\right).$$

Hence, the Hermitian forms of M and M_x coincide. To see that also their semicharacters coincide just note that, for all $(a_1,\dots,a_n) \in T^n$, the following holds:

$$\left(\sum_{i=1}^n \mathrm{pr}_i^* H\right)((nx,\dots,nx),(a_1,\dots,a_n)) + \mu^*H((-x,\dots,-x),(a_1,\dots,a_n)) =$$

$$= \sum_{i=1}^n H(nx,a_i) + H\left(-nx, \sum_{i=1}^n a_i\right) = 0.$$

□

In the case where L is effective, we aim to show that the line bundle M on T^n is generated by global sections which are symmetric with respect to the $\mathfrak{S}_{n+1}$-action on T^n.

Lemma 3.8 *Let L be an effective line bundle on an abelian surface T, and let $s \in H^0(L)$ be a non-zero section. For $x \in T$, consider the section s_x of M_x defined as follows:*

$$s_x := t^*_{(nx,\dots,nx)}\left(\bigotimes_{i=1}^n \mathrm{pr}_i^* s\right) \otimes t^*_{(-x,\dots,-x)}(-1)^*\mu^* s.$$

Then, s_x is symmetric with respect to the $\mathfrak{S}_{n+1}$-action on T^n.

Proof Let us compute $s_x(a_1,\dots,a_n)$ for $(a_1,\dots,a_n) \in T^n$:

$$\begin{aligned} s_x(a_1,\dots,a_n) &= \left(t^*_{(nx,\dots,nx)}\left(\bigotimes_{i=1}^n \mathrm{pr}_i^* s\right) \otimes t^*_{(-x,\dots,-x)}(-1)^*\mu^* s\right)(a_1,\dots,a_n) \\ &= \left(\bigotimes_{i=1}^n \mathrm{pr}_i^* s\right)(a_1+nx,\dots,a_n+nx)\cdot(-1)^*\mu^*s(a_1-x,\dots,a_n-x) \\ &= s(a_1+nx)\cdot\ldots\cdot s(a_n+nx)\cdot s(-a_1-\ldots-a_n+nx). \end{aligned}$$

Applying Remark 3.6, we deduce from this expression of $s_x(a_1,\dots,a_n)$ that s_x is symmetric with respect to the $\mathfrak{S}_{n+1}$-action on T^n. □

Proposition 3.9 *Let L be an effective line bundle on an abelian surface T. Then, the induced line bundle M on $T^n \simeq K_T^n$ is globally generated by sections which are symmetric with respect to the $\mathfrak{S}_{n+1}$-action inherited from T^{n+1}.*

Proof Let $s \in H^0(L)$ and let D_T be the divisor on T cut out by s. For every $x \in T$, let D_x be the divisor on $T^n \simeq K_T^n$ cut out by the section s_x of the line bundle

$$M_x := t^*_{(nx,\dots,nx)}(L^{\boxtimes n}) \otimes t^*_{(-x,\dots,-x)}(-1)^*\mu^*L.$$

In Proposition 3.7, we proved that the line bundle M_x is isomorphic to M. Therefore, to prove base-point-freeness of M, it suffices to show that for every $(a_1,\dots,a_n) \in T^n$ there exists an element $x \in T$ such that $(a_1,\dots,a_n) \notin D_x$. Note that the following equivalences hold:

$$(a_1,\dots,a_n) \in t^*_{(nx,\dots,nx)}(D^{\boxtimes n}) \iff nx \in t^*_{a_i} D \text{ for some } i,$$

$$(a_1,\dots,a_n) \in t^*_{(-x,\dots,-x)}(-1)^*\mu^* D \iff nx \in t^*_{-a_1-\ldots-a_n} D.$$

Therefore, $(a_1, \dots, a_n) \notin D_x$ for any $x \in T$ such that $nx \notin \sum_i t_{a_i}^* D + t_{-a_1 \dots -a_n}^* D$. As the sections s_x are symmetric with respect to the $\mathfrak{S}_{n+1}$-action on T^n by Lemma 3.8, we conclude that M is generated by symmetric global sections. □

We are now able to prove that the induced line bundle on the generalized Kummer variety is base-point-free:

Theorem 3.10 *Let $n > 1$, and let L be an effective line bundle on an abelian surface T. Then, the induced line bundle L_n on* $\mathrm{Kum}^n(T)$ *is base-point-free.*

Proof Since the line bundle $L^{\boxtimes(n+1)}$ on T^{n+1} is symmetric with respect to the natural $\mathfrak{S}_{n+1}$-action, it descends to a line bundle $L_{(n+1)}$ on the symmetric product $T^{(n+1)}$. By Proposition 3.9, there are symmetric global sections of the line bundle $L^{\boxtimes(n+1)}|_{K_T^n} \simeq M$ which generate it. These sections descend to sections of $L_{(n+1)}|_{\overline{K^n}}$ and they generate $L_{(n+1)}|_{\overline{K^n}}$. Therefore, the line bundle $L_{(n+1)}|_{\overline{K^n}}$ is base-point-free. Finally, since the line bundle L_n on $\mathrm{Kum}^n(T)$ induced by the line bundle L coincides with the pullback of the line bundle $L_{(n+1)}|_{\overline{K^n}}$ via the map $\mathrm{Kum}^n(T) \to \overline{K^n}$, we conclude that L_n is base-point-free. □

We can now conclude the proof of Theorem 3.1:

Proof of Theorem 3.1 For any $n > 1$ and $d > 0$, the moduli space $\mathscr{M}_{d,1}^n$ is non-empty by Lemma 3.4. Let (T, L) be an abelian surface such that $q(L) = 2d$. Since L is a polarization on T, the vector space $H^0(L)$ is non-empty. We can then apply Theorem 3.10 to conclude that the line bundle L_n on $\mathrm{Kum}^n(T)$ is base-point-free. This, together with Proposition 2.7, proves generic base-point-freeness in the moduli space $\mathscr{M}_{d,1}^n$. □

4 Tautological bundles

In this section, we recall the definition of tautological bundles on the Hilbert scheme of points and the notion of k-very ampleness for a line bundle on an abelian surface. These will be used in Corollary 4.5 to deduce a criterion for base-point-freeness of the line bundles of the form $kL_n - \delta$ on $\mathrm{Kum}^n(T)$.

Let T be an abelian surface and consider the universal family for $T^{[n+1]}$:

$$\Xi^{n+1} := \{(P, \xi) \in T \times T^{[n+1]} \mid P \in \xi\}.$$

Denote by $p_T : \Xi^{n+1} \to T$ and by $p_{T^{[n+1]}} : \Xi^{n+1} \to T^{[n+1]}$ the two projections. By construction, the fibre of $p_{T^{[n+1]}}$ over $\xi \in T^{[n+1]}$ is the subscheme $\xi \subseteq T$.

Definition 4.1 Let L be a line bundle on an abelian surface T. The *tautological bundle* associated to L on $T^{[n+1]}$ is

$$L^{[n+1]} := (p_{T^{[n+1]}})_* p_T^*(L).$$

Since $p_{T^{[n+1]}}$ is a finite and flat morphism, $L^{[n+1]}$ is a vector bundle of rank $n + 1$. Denoting by $\det L^{[n+1]} := \bigwedge^{n+1} L^{[n+1]}$ its determinant, the following equality holds:

$$\det L^{[n+1]} = L_n - \delta, \tag{2}$$

where, as in Remark 1.5, L_n is the line bundle that L induces on $T^{[n+1]}$ and δ is the line bundle $\frac{1}{2}E$, see [13, Lem. 3.7 & Thm. 4.6]. The next lemma describes the global sections of the line bundles that we have introduced. For all the details, see [18].

Lemma 4.2 *With the previous notation, we have the following canonical isomorphisms:*

(i) $H^0(T^{[n+1]}, L_n) \simeq \mathrm{Sym}^{n+1} H^0(T, L)$;
(ii) $H^0(T^{[n+1]}, L^{[n+1]}) \simeq H^0(T, L)$;
(iii) $H^0(T^{[n+1]}, L_n - \delta) \simeq \bigwedge^{n+1} H^0(T, L)$.

Definition 4.3 A line bundle L on an abelian surface T is *k-very ample* for some non-negative integer k if, for every $\xi \in T^{[k+1]}$, the evaluation map

$$\mathrm{ev}_{L,\xi} \colon H^0(T, L) \twoheadrightarrow H^0(T, L \otimes \mathscr{O}_\xi)$$

is surjective.

This notion is relevant for us due to the following result:

Proposition 4.4 *Let T be an abelian surface and let L be a line bundle on T. With the previous notation, the following are equivalent:*

(i) The line bundle $L_n - \delta$ on $T^{[n+1]}$ is globally generated.
(ii) The tautological bundle $L^{[n+1]}$ on $T^{[n+1]}$ is globally generated.
(iii) The line bundle L is n-very ample.

Proof Since $H^0(T^{[n+1]}, L^{[n+1]}) \simeq H^0(T, L)$ by Lemma 4.2, the equivalence of (ii) and (iii) is an immediate consequence of the fact that $p^*_{T^{[n+1]}}(\xi) \cong \xi$.

Let us prove the equivalence of (i) and (ii). It is straightforward to see that the vector bundle $L^{[n+1]}$ is globally generated if and only if the map

$$\left(\bigwedge^{n+1} H^0(T^{[n+1]}, L^{[n+1]})\right) \otimes \mathscr{O}_{T^{[n+1]}} \to \bigwedge^{n+1} L^{[n+1]}$$

is surjective. Applying Lemma 4.2 and using (2), this map is the same as the map

$$H^0(T^{[n+1]}, L_n - \delta) \otimes \mathscr{O}_{T^{[n+1]}} \to L_n - \delta.$$

Finally, this last map is surjective if and only if the line bundle $L_n - \delta$ is globally generated. ◻

As an immediate consequence of Proposition 4.4, we have the following corollary:

Corollary 4.5 *Let T be an abelian surface and let L be a line bundle on T. If the line bundle L is n-very ample, then the line bundle $L_n - \delta$ on $\mathrm{Kum}^n(T)$ is globally generated.*

Let us end this section by recalling a criterion for k-very ampleness. Let T be an abelian surface and denote by $\mathrm{Amp}(T)$ its ample cone. Let

$$f \colon \mathrm{Amp}(T) \to \mathbb{Z}_{\geq -1}$$

be the map that associates to an ample line bundle L on T the greatest integer k such that L is k-very ample; we set by definition $f(L) = -1$ if L is not generated by global sections. Then, the following result holds.

Proposition 4.6 [2, Thm. 4.3] *Let (T, L) be an abelian surface such that $\mathrm{NS}(T) = \langle c_1(L)\rangle$ and let $q(L) = 2d$. Then, for all $m \geq 2$,*

$$f(mL) = 2(m-1)d - 2.$$

5 Other divisibilities

We have now all the tools needed to study generic base-point-freeness in the moduli space $\mathscr{M}^n_{d,t}$ for $n \in \{2, 3, 4\}$ when the divisibility is greater than one.

We begin by finding, for all possible values of $t > 1$ and d such that $\mathscr{M}^n_{d,t}$ is non-empty, a generalized Kummer variety together with a line bundle of divisibility t and square $2d$. In the case $n = 2$, the following holds:

Proposition 5.1 *Let $t > 1$ and $d > 0$ be integers such that $\mathscr{M}^2_{d,t}$ is non-empty. Then, there exists an abelian surface (T, L) such that the line bundle $tL_2 - \delta$ on the hyperkähler manifold* $\mathrm{Kum}^2(T)$ *satisfies $q(tL_2 - \delta) = 2d$ and* $\mathrm{div}(tL_2 - \delta) = t$.

Proof Since $n = 2$, we have to consider only $t \in \{2, 3, 6\}$ by Remark 1.6. Given any primitive line bundle L on an abelian surface T, the line bundle $tL_2 - \delta$ on $\mathrm{Kum}^2(T)$ satisfies the condition $\mathrm{div}(tL_2 - \delta) = t$. Therefore, for each given pair of integers (d, t) such that $\mathscr{M}^2_{d,t}$ is non-empty, we just need to find an abelian surface such that $q(tL_2 - \delta) = 2d$.

Let $t = 2$, and let $(X, H) \in \mathscr{M}^2_{d,2}$. By Proposition 1.4, there exists an abelian surface $(\tilde{T}, \tilde{L})$ such that $H^2(X, \mathbb{Z}) \cong H^2(\mathrm{Kum}^2(\tilde{T}), \mathbb{Z})$ and, via this isomorphism,

$$H = a\tilde{L}_2 + b\delta,$$

for some integers a and b. Note that a and b are coprime since H is primitive. Therefore, we deduce from $\mathrm{div}(H) = 2$ that a is even and b is odd. Write $a = 2k$ for some integer k, and let $q(\tilde{L}) = 2\tilde{d}$. Define the following number:

$$\hat{d} := k^2\tilde{d} - \frac{3(b^2 - 1)}{4}. \tag{3}$$

Note that $\hat{d}$ is an integer since b is odd. Let T be an abelian surface which admits a polarization L with $q(L) = 2\hat{d}$. In particular, $q(L_2) = 2\hat{d}$, and

$$q(2L_2 - \delta) = 4(2\hat{d}) - 6 = (2k)^2(2\tilde{d}) - 6b^2 = q(a\tilde{L} + b\delta) = 2d.$$

This allows us to conclude that $(\mathrm{Kum}^2(T), 2L_2 - \delta)$ is a generalized Kummer variety with the prescribed invariants.

In the case of divisibility $t = 3$ (resp., $t = 6$), a similar argument works: Given an integer d such that $\mathscr{M}^2_{d,3}$ (resp., $\mathscr{M}^2_{d,6}$) is non-empty, one sets a, b, and $\tilde{d}$ as in the previous case and checks that the number $\hat{d} := k^2\tilde{d} - \frac{(b^2-1)}{3}$ (resp., $\hat{d} := k^2\tilde{d} - \frac{(b^2-1)}{12}$) is an integer for any possible value of b, k and $\tilde{d}$. This allows us to conclude as in the previous case. □

Similarly, in the case $n = 3$, we have the following:

Proposition 5.2 *Let $t > 1$ and $d > 0$ be integers such that the moduli space $\mathscr{M}^3_{d,t}$ is non-empty. Then, there exists an abelian surface (T, L) such that one of the line bundles $tL_3 - \delta$ and $8L_3 - 3\delta$ on the hyperkähler manifold* $\mathrm{Kum}^3(T)$ *has divisibility t and square $2d$.*

Proof Since $n = 3$, δ has square $2n + 2 = 8$. By Remark 1.6, the only possible values for t are $t = 1, 2, 4, 8$. As in the proof of Proposition 5.1, it suffices to find the correct value for the square $2\hat{d}$ of the line bundle L, for any fixed value of d and t. Given d and t, let a, b, and $\tilde{d}$ be the integers as in the proof of Proposition 5.1.

When $t = 2$ (resp., $t = 4$), set $\hat{d} := \frac{a^2\tilde{d}}{2} - b^2 + 1$ (resp., $\hat{d} = \frac{a^2\tilde{d}}{16} - \frac{b^2-1}{4}$). Then, $\hat{d}$ is an integer, and, given an abelian surface (T, L) such that $q(L) = 2\hat{d}$, the line bundle $2L_3 - \delta$

(resp., $3L_3-\delta$) on the generalized Kummer variety $\mathrm{Kum}^3(T)$ has square $2d$ and divisibility 2 (resp., 3).

For $t=8$, we have that $\gcd(a,8b)=8$. In particular, a is even, and, as a and b are coprime, b is odd. The square b^2 modulo 16 is then either 1 or 9. If $b^2\equiv 1$ modulo 16, set $\hat{d}=\frac{a^2\tilde{d}}{64}-\frac{b^2-1}{16}$. Given an abelian surface (T,L) such that $q(L)=2\hat{d}$, the line bundle $8L_3-\delta$ on $\mathrm{Kum}^3(T)$ has then square $2d$ and divisibility 2. If $b^2\equiv 9$ modulo 16, set $\hat{d}=\frac{a^2\tilde{d}}{64}-\frac{b^2-9}{16}$. Then, given an abelian surface (T,L) such that $q(L)=2\hat{d}$, the line bundle $8L_3-3\delta$ on $\mathrm{Kum}^3(T)$ has square $2d$ and divisibility 8. □

Remark 5.3 Let $t>1$ and let $d>0$ be integers such that the moduli space $\mathscr{M}^3_{d,t}$ is non-empty. From the proof of Proposition 5.2, we can also deduce the following: When $t=8$ and $d=64k-36$ for some positive integer k, it is possible to find an abelian surface (T,L) for which $8L-3\delta$ has square $2d$, but it is not possible to find an abelian surface $(\tilde{T},\tilde{L})$ with $q(8\tilde{L}_3-\delta)=2d$. In all the other cases, a representative of the form $(tL_3-\delta)$ can be found.

Finally, in the case $n=4$, one proves the following result using the same techniques as in Proposition 5.2.

Proposition 5.4 *Let $t>1$, and let $d>0$ be integers such that the moduli space $\mathscr{M}^4_{d,t}$ is non-empty. Then, there exists an abelian surface (T,L) such that one of the following line bundles $tL_4-\delta$, $5L_4-2\delta$, and $10L_4-3\delta$ on the hyperkähler manifold $\mathrm{Kum}^4(T)$ has divisibility t and square $2d$.* □

Proposition 5.1 allows us to prove generic base-point-freeness in $\mathscr{M}^2_{d,t}$ if $(d,t)\neq(1,2)$:

Theorem 5.5 *Let t and d be positive integers such that $\mathscr{M}^2_{d,t}$ is non-empty. Assume that $(d,t)\neq(1,2)$. Then, for a generic pair $(X,L)\in\mathscr{M}^2_{d,t}$ the line bundle L is base-point-free.*

Proof Let (t,d) be a pair of integers such that $\mathscr{M}^2_{d,t}$ is non-empty. By Proposition 2.7, to prove generic base-point-freeness in $\mathscr{M}^2_{d,t}$, it suffices to find a pair (X,L) where X is a hyperkähler manifold of Kum^2-type and L is a base-point-free line bundle (not necessarily ample) on X such that $q(L)=2d$ and $\mathrm{div}(L)=t$. By Remark 1.6, the only possible values for the divisibility t are $t=1,2,3$, and 6.

As the case $t=1$ follows from Theorem 3.1, let t be equal to either 2, 3, or 6, and let d be a positive integer such that $\mathscr{M}^2_{d,t}$ is non-empty. Let (T,L) be an abelian surface such that $q(tL_2-\delta)=2d$. The existence of such an abelian surface is guaranteed by Proposition 5.1. Recall that a generic abelian surface has Picard number 1 (this can be deduced from the analysis presented in [5, Sec. 2.7]). Therefore, by deforming (T,L), we may assume $\mathrm{NS}(T)=\langle c_1(L)\rangle$. Write $2\hat{d}=q(L)$, and let f be the function of Proposition 4.6:

$$f(tL)=2(t-1)\hat{d}-2.$$

Since we are assuming $t>1$, we deduce that $f(tL)\geq 2$ for all the cases that we are considering: Note that, in the case $t=2$, we have $\hat{d}\geq 2$ since we are assuming $(d,t)\neq(1,2)$. By definition of f, this implies that the line bundle tL is 2-very ample. Applying Corollary 4.5, we deduce that the line bundle $tL_2-\delta$ on $\mathrm{Kum}^2(T)$ is base-point-free. This provides the required element and concludes the proof in the case $n=2$. □

Similarly, using Proposition 5.3, it is possible to prove generic base-point-freeness in $\mathscr{M}^3_{d,t}$ in all but three cases:

Theorem 5.6 *Let t and d be positive integers such that $\mathscr{M}^3_{d,t}$ is non-empty. Assume that $(d, t) \neq (4, 2), (28, 8), (92, 8)$. Then, the polarization of a generic pair in $\mathscr{M}^3_{d,t}$ is base-point-free.*

Proof as the case $t = 1$ follows from Theorem 3.1, let $t > 1$, and let d be a positive integer such that $\mathscr{M}^3_{d,t}$ is non-empty. Write $2\hat{d} = q(L)$, and let f be the function of Proposition 4.6:

$$f(tL) = 2(t-1)\hat{d} - 2. \tag{4}$$

Note that $f(tL) \geq 3$ for all the cases we are considering: This is immediate when $t = 4$ and $t = 8$. When $t = 2$ and $\hat{d} = 1$, then $2d = q(2L_3 - \delta) = 0$, which is not possible since d is a positive integer, and when $\hat{d} = 2$, then $2d = q(2L_3 - \delta) = 8$ and this is the case we excluded. In particular, in the cases we are considering, the line bundle L is 3-very ample. Therefore, the line bundle $tL_3 - \delta$ on $\mathrm{Kum}^3(T)$ is base-point-free. In this case, generic base-point-freeness follows from Proposition 2.7.

To conclude the proof, we need to study the cases for which there exists no abelian surface (T, L) for which $tL_3 - \delta$ on $\mathrm{Kum}^3(T)$ has square $2d$. By Proposition 5.2 and Remark 5.3, this happens only when $t = 8$ and $d = 64k - 36$ for some positive integer k, and in this case, there exists an abelian variety (T, L) such that the line bundle $8L_3 - 3\delta$ on $\mathrm{Kum}^3(T)$ has square $2d$ and divisibility 8. In particular, $q(8L_3 - 3\delta) = 128k - 72$, then $q(L) = 2k$. Write $8L_3 - 3\delta = (4L_3 - \delta) + 2(2L_3 - \delta)$. The line bundle $4L_3 - \delta$ is base-point-free by the proof of the case of divisibility 4. On the other hand, the line bundle $2L_3 - \delta$ has square $8k - 8$. Since we are assuming $(d, t) \neq (28, 8), (92, 8)$, we deduce that $k > 2$. Therefore, $8k - 8 > 8$. By using the proof in the case of divisibility 2, we conclude that $2L_3 - \delta$ is base-point-free, since we are not in the case $(d, t) = (4, 2)$. Therefore, the line bundle $8L_3 - 3\delta$ is base-point-free being the sum of line bundles which are base-point-free. This concludes the proof. □

The same technique can be applied to prove generic base-point-freeness in the moduli space $\mathscr{M}^4_{d,t}$ for almost all pairs (d, t):

Theorem 5.7 *Let t and d be positive integers such that $\mathscr{M}^4_{d,t}$ is non-empty. Assume that $(d, t) \neq (3, 2), (20, 5), (55, 10)$. Then, the polarization of a generic pair in $\mathscr{M}^4_{d,t}$ is base-point-free.*

The results of this section can be summarized as follows:

Theorem 5.8 *Let $n \in \{2, 3, 4\}$, and let d and t be positive integers such that the moduli space $\mathscr{M}^n_{d,t}$ is non-empty. Let A be the set of triples*

$$A := \{(2, 1, 2), (3, 4, 2), (3, 28, 8), (3, 92, 8), (4, 3, 2), (4, 20, 5), (4, 55, 10)\}.$$

Then, if $(n, d, t) \notin A$, the polarization of a general pair in $\mathscr{M}^n_{d,t}$ is base-point-free. □

Acknowledgements This work is part of my Master's thesis at ETH Zürich. I would like to thank Ulrike Rieß for suggesting me this topic, for supervising my thesis, and for her continuous advice and support. I am also grateful for the financial support provided by ETH Foundation during my master's, and by ERC Synergy Grant HyperK (Grant agreement No. 854361) during the completion and editing of the article. In particular, I express my gratitude to Daniele Agostini, Fabrizio Anella, and Daniel Huybrechts for reading a preliminary version of this paper.

References

1. Agostini, D., Rieß, U.: Generic base-point-freeness for polarized $K3^{[3]}$-type varieties. Unpublished preprint (2020)

2. Alagal, W., Maciocia, A.: Critical k-very ampleness for abelian surfaces. Kyoto J. Math. **56**(1), 33–47 (2016)
3. Beauville, A.: Variétés Kähleriennes dont la premiere classe de Chern est nulle. J. Differ. Geom. **18**(4), 755–782 (1983)
4. Birkenhake, C., Lange, H.: Complex abelian varieties. Springer Science & Business Media (2013)
5. Birkenhake, C., Lange, H.: Complex tori. Vol. 302. Springer Science & Business Media, New York (1999)
6. Debarre, O.: Hyper-Kähler manifolds. Milan J. Math. **90**(2), 305–387 (2022)
7. Gross, M., Huybrechts, D., Joyce, D.: Calabi-Yau Manifolds and Related Geometries: Lectures at a Summer School in Nordfjordeid, Norway, June 2001. Springer Science & Business Media (2012)
8. Hartshorne, R.: Algebraic Geometry. Springer, Berlin (1977)
9. Huybrechts, D.: Compact hyperkähler manifolds: basic results. Invent. Math. **135**, 63–113 (1999)
10. Huybrechts, D.: Complex Geometry: An Introduction. Springer Science & Business Media (2006)
11. Huybrechts, D.: Erratum to the paper: Compact hyperkähler manifolds: basic results. Invent. Math. **152**, 209–212 (2003)
12. Lazarsfeld, R.: *Positivity in Algebraic Geometry. II, volume 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer (2004)
13. Lehn, M.: Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. Invent. Math. **136**(1), 157–207 (1999)
14. Mayer, A.L.: Families of K3 surfaces. Nagoya Math. J. **48**, 1–17 (1972)
15. Onorati, C.: Connected components of moduli spaces of irreducible holomorphic symplectic manifolds of Kummer type. Rend. Mat. Appl. **7**(43), 251–266 (2022)
16. Rieß, U.: Base divisors of big and nef line bundles on irreducible symplectic varieties. In Annales de l'Institut Fourier, 1–25 (2023)
17. Rieß, U.: On the non-divisorial base locus of big and nef line bundles of $K3^{[2]}$-type varieties. Porc. R. Soc. Edinb. A **151**(1), 52–78 (2021)
18. Scala, L.: Cohomology of the Hilbert scheme of points on a surface with values in representations of tautological bundles. Duke Math. J. **150**(2), 211–267 (2009)