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An example of Tateno disproving conjectures of Bonato–Tardif, Thomasse, and Tyomkyn

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Abstract
In his 2008 thesis [16] , Tateno claimed a counterexample to the Bonato–Tardif conjecture regarding the number of equimorphy classes of trees. In this paper we revisit Tateno's unpublished ideas to provide a rigorous exposition, constructing locally finite trees having an arbitrary finite number of equimorphy classes; an adaptation provides partial orders with a similar conclusion. At the same time these examples also disprove conjectures by Thomassé and Tyomkyn.

Keywords Graph · Tree · Poset · Siblings · Equimorphy

Mathematics Subject Classification 05C05 · 05C60 · 05C75 · 05C99 · 06A06 · 06A07

1 Introduction

Two structures R and S are *equimorphic*, denoted by $R \approx S$, when each embeds in the other; we may also say that one is a *sibling* of the other. If R is finite, there is just one sibling (up to isomorphy). The famous Cantor–Bernstein–Schroeder Theorem states that this is also the

Dedicated to our parents.

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case for structures in a language with pure equality: if there is an injection from one set to another and vice-versa, then there is a bijection between these two sets. The same situation occurs in other structures such as vectors spaces, where embeddings are linear injective maps. But generally one cannot expect equimorphic structures to be necessarily isomorphic: the rational numbers, considered as a linear order, has up to isomorphism continuum many siblings. It is thus a natural problem to understand the siblings of a given structure, and as a first approach to count those siblings (up to isomorphy).

Thus, let $sib(R)$ be the number of siblings of R, these siblings being counted up to isomorphism. Thomassé conjectured that $sib(R) = 1$, $\aleph_0$ or $2^{\aleph_0}$ for countable relational structures made of at most countably many relations (Conjecture 2 in [17]). There is a special case of interest, namely whether $sib(R) = 1$ or infinite for a relational structure of any cardinality. This was unsettled even in the case of locally finite trees, and is connected to the Bonato–Tardif conjecture which asserts that either all trees equimorphic to a given arbitrary tree T are isomorphic, or else there are infinitely many pairwise non-isomoprohic trees equimorphic to T, also called the *Tree Alternative Conjecture* (see [2, 3, 19]). Note that, as a binary relational structure, a ray has infinitely many siblings (add an arbitrary finite disconnected path), but a ray has no non-isomorphic sibling in the category of trees. The subtle connection between these conjectures is through the following observation by Hahn et al. [6]: every sibling of a tree T (as a binary relational structure, or graph) is a tree if and only if $T \oplus 1$ (the graph obtained by adding an isolated vertex to T) is not a sibling of T (more generally, note that every sibling of a connected graph is connected, just in case $G \oplus 1$ is not a sibling). Hence, for a tree T not equimorphic to $T \oplus 1$, the Bonato–Tardif conjecture (in the category of trees) and the special case of Thomassé's conjecture (in the category of relational structures) are equivalent.

Bonato and Tardif [2] proved their conjecture for rayless trees, and this was extended to rayless graphs by Bonato, Bruhn, Diestel and Sprüssel [3]. It was also verified for the case of rooted trees by Tyomkyn [19], and in addition made some progress towards the conjecture for locally finite trees. Tyomkyn made a first conjecture that if there exists a non-surjective embedding of a locally finite tree T, then $sib(T)$ is infinite unless T is a ray, a conjecture which immediately implies the Bonato–Tardif conjecture for locally finite trees. Tyomkyn further conjectured an apparently weaker version that if there exists a non-surjective embedding of a locally finite tree T, then T has at least one non-isomorphic sibling unless T is a ray. Laflamme et al. [13] later proved the Bonato–Tardif conjecture for scattered trees, that is those trees not containing a subdivision of the binary tree. In fact they proved the result under the slightly more general notion of a stable tree. This is based on extensions of results of Polat and Sabidussi [15], Halin [7–9], and Tits [18] on automorphisms of trees. Moreover they proved Tyomkyn's first conjecture holds for locally finite scattered trees. Hamann [10], making use of the monoid of embeddings, deduced the Bonato–Tardif conjecture for trees not satisfying two specific structural properties of that monoid. More recently, Abdi [1] showed that a tree satisfying that first property is stable, and therefore the Bonato–Tardif conjecture also holds in that case.

In a parallel direction, Thomassé's conjecture has been fully verified for countable chains, and its special case also verified for all chains by Laflamme et al. [12], paving the way toward partial orders. A first step was made for direct sums of chains by Abdi [1], and after Hahn et al. [6] proved the special case of the conjecture in the special case of cographs, Abdi [1] extended this result to closely related NE-free posets. Another supporting indication came with the special case of the conjecture for a countable $\aleph_0$-categorical relational structure, proved by Laflamme et al. [14], and extended by Braunfeld et al [4].

Fig. 1 In the category of *connected graphs with loops*, the above structure has exactly 2 siblings

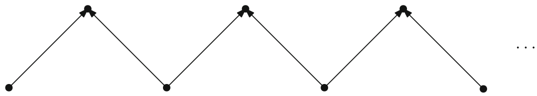


Fig. 2 Similarly in the category of *connected posets*, the one way infinite fence has exactly two siblings

In this paper we revisit Tateno's unpublished ideas [16] to provide a rigorous exposition, constructing locally finite trees having an arbitrary finite number of equimorphy classes. At the same time these examples disprove the above conjectures of Thomassé and Tyomkyn.

Theorem 1 *For each non-zero $\mathfrak{s} \in \mathbb{N}$, there is a locally finite tree $\mathcal{T} = \mathcal{T}_{\mathfrak{s}}$ with exactly $\mathfrak{s}$ siblings (up to isomorphy), considered either as relational structures or trees. Moreover, for $\mathfrak{s} = 1$, the tree is not a ray yet has a non-surjective embedding.*
Thus the conjectures of Bonato–Tardif, Thomassé, and Tyomkyn regarding the sibling number of trees and relational structures are all false.

This result has been a long time coming. Counterexamples had already been produced by Pouzet (see [6, 12]) in the categories of directed graphs and simple graphs with loops (see Figs. 1 and 2), each structure having exactly two siblings. But they are not counterexamples when viewed as binary relations as they now each have infinitely many siblings in that category.

Further, the above trees can be adapted to also provide partial orders with an arbitrary finite number of siblings.

Theorem 2 *For each non-zero $\mathfrak{s} \in \mathbb{N}$, there is a partial order $\mathcal{P}$ with exactly $\mathfrak{s}$ siblings (up to isomorphy).*

The necessary background material can be found in [5]. We warmly thank Maurice Pouzet for bringing these problems to our attention, and for his generosity sharing his insight and expertise over the years on the subject. We also thank Mykhaylo Tyomkyn for making us aware of the claimed counterexample, and Atsuhi Tateno for making his manuscript available to us and agreeing to have us rewrite the ideas. Finally, we further thank the very generous referee for correcting various errors and valuable suggestions greatly improving the presentation.

2 Construction of the locally finite trees

The strategy is to build locally finite trees as a finite set of pairwise non-isomorphic siblings $\langle \mathcal{T}_s : s < \mathfrak{s} \rangle$ for any fixed non-zero $\mathfrak{s} \in \mathbb{N}$, such that any sibling of $\mathcal{T} = \mathcal{T}_0$ (as a binary relational structure) is isomorphic to some $\mathcal{T}_s$. This yields a locally finite tree $\mathcal{T}$ such that $sib(\mathcal{T}) = \mathfrak{s}$. The case $\mathfrak{s} = 2$ will already disprove the conjectures of Bonato–Tardif (and hence Tyomkyn's first conjecture) and of Thomassé since we will show that $T \oplus 1$ does not embed in T. The special case $\mathfrak{s} = 1$ will disprove Tyomkyn's second conjecture. The case $\mathfrak{s} > 2$ is only for additional information, showing that any finite number can be the sibling

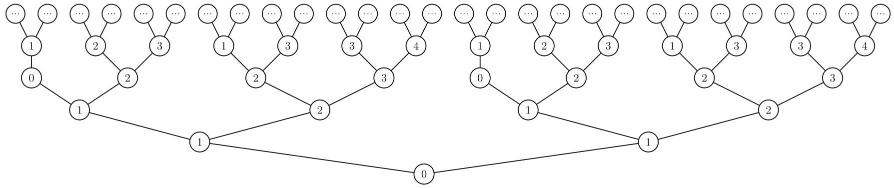


Fig. 3 Rooted Tree $\mathcal{R} = (R, r)$ and the vertex labelling

number of some locally finite tree. These trees will later be adapted to provide similar results for partial orders.

The construction of each $\mathcal{T}_s$ will be done in a similar manner as a countable union of trees, coding the countably many potential siblings within the trees along the way. Moreover $\mathcal{T}_s \backslash \phi(\mathcal{T}_s)$ will be finite for every embedding ϕ, and hence all such differences will be captured after a finite stage of the construction, allowing to eventually show that all siblings have been accounted for. To facilitate the exposition, the construction will initially make use of several non graph properties (labels, type assignments, sign and spin functions), but all will be eventually replaced by genuine graph properties. Thus embeddings will first be assumed to preserve the non graph properties, and it will eventually be shown that these non graph properties are actually preserved by graph embeddings alone with the most delicate case being through the Main Lemma 2.26.

2.1 Rooted tree $\mathcal{R} = (R, r)$

We begin by constructing a rooted tree $\mathcal{R} = (R, r)$ that will be used repeatedly throughout the construction. We will first develop local properties of that tree, and then later extend them to each $\mathcal{T}_s$.

The tree $\mathcal{R}$ is built using a labelling on the vertices $lab : R \to \mathbb{N}$ to guide the construction (see Fig. 3. We first declare $lab(r) = 0$, and then we construct the tree inductively under the following rules:

- If $lab(v) = 0$, then v has exactly two neighbours of label 1.
- If $lab(v) \neq 0$, then v has exactly three neighbours labelled $lab(v)-1$, $lab(v)$, $lab(v)+1$.

We denote by R_0 the 0-labelled vertices of $\mathcal{R} = (R, r)$ and often call them *tree vertices*. Note that these are exactly the vertices of degree 2 in $\mathcal{R}$. Now that the tree has been constructed, one notices that the labelling of a vertex can be recovered from the tree itself as the (graph) distance to the nearest tree vertex (vertex of degree 2).

Observation 2.1 *For any vertex $v \in R$, $lab(v)$ is the distance to the nearest vertex of degree 2.*

Proof Tree vertices are exactly those of degree 2, all other vertices of label $\ell > 0$ have degree 3. Now any vertex of label ℓ has a path of length ℓ with decreasing labels to a tree vertex, hence its distance to the nearest tree vertex is at most ℓ. On the other hand labels decrease by at most 1, therefore no tree vertex can be any closer. □

Thus the labels are a graph property and any (graph) embedding of $\mathcal{R}$ preserves labels. Yet, we will be joining several copies of $\mathcal{R}$ and adding vertices to the tree, so we want to ensure that these labels can be recovered from the eventual graph structure. We do so by encoding

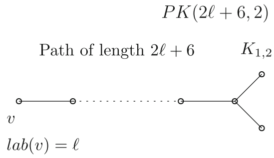


Fig. 4 The gadget $PK(2\ell+6, 2)$: finite rooted tree used to encode label ℓ

these labelled vertices using finite trees (gadgets) rooted at those vertices as follows. First consider the bipartite graph $K_{1,m} = (u, V)$ and call $PK(n, m)$ the finite tree rooted at the initial vertex of a path of length n and with its end vertex identified with the vertex u. Then identify any vertex v with $lab(v) = \ell$ with the root of a copy of $PK(2\ell + 6, 2)$ (see Fig. 4).

It is interesting that the counterexample must have leaves as was shown by Abdi [1], hence the gadgets must be finite. What is also important here is that these gadgets are pairwise non-embeddable as *rooted trees* (mapping roots to roots) for different values of ℓ, and thus play the graph theoretic role of the labels. From this point we want to make the label a graph property by attaching these finite gadgets to vertices. Technically, this results in a tree extension (R^a, r). Note that any graph embedding ϕ of the resulting tree must take vertices in R to vertices in R and it must map the label gadget attached at a vertex v of R to the gadget at $\phi(v)$. Since distinct gadgets do not embed in each other they must be the same. So in fact ϕ must map R^a onto itself, mapping the label gadgets to label gadgets. Having noted this important property we abuse the notation using (R, r) for (R^a, r).

We will henceforth for notational simplicity continue to use the labels $lab(v)$ themselves, with the understanding that the labels are a graph property and preserved by embeddings.

In anticipation of the construction of $\mathcal{T}$ (and each $\mathcal{T}_s$), we note that we will eventually associate a double ray to each tree vertex in a copy of (R, r), and amalgamate the ray and the copy by identifying a single vertex of the ray and the tree vertex. The reason for $2\ell + 6$ above is simply that we will later use a small versions to code type assignments on those ray vertices.

Another important observation is that for any tree vertex v there is a (label preserving) embedding $\phi : \mathcal{R} = (R, r) \to (R, v)$ sending r to v; moreover this is an automorphism since all embeddings of $\mathcal{R}$ are easily seen to be surjective. This remains true even when we make the labels explicitly a graph property by adding the finite tree labels.

Observation 2.2 *All embeddings of $\mathcal{R}$ are surjective, and (R, r) is isomorphic to (R, v) (as rooted trees) for any tree vertex v.*

In fact (R, r) is so symmetric that one main point of the construction of $\mathcal{T}$ is to insert obstructions to control its (graph) embeddings and as a result the number of siblings.

The following notions of colour and height for tree vertices will be used to facilitate the inductive construction of each $\mathcal{T}_s$. We call a pair of adjacent vertices in $\mathcal{R}$ *consecutive* if they have the same label. Note that if a tree vertex $v \in R_0$ is different from r, then the path $P_{r,v}$ from r to v must contain at least one such consecutive pair.

Definition 2.3 For any tree vertex $v \in R_0$, define $col_v : R_0 \to \mathbb{N}$, the colour with respect to v, by

$$col_v(u) = lab(w)$$

where w is a vertex from the last consecutive pair in $P_{v,u}$.
For convenience let $col_v(v) = 0$.

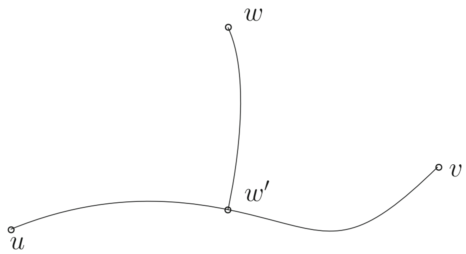


Fig. 5 Lemma 2.4: $col_u(w) = col_v(w)$ for all but finitely many $w \in R_0$

Thus, since labels are preserved by graph embeddings, we have that $col_v(u) = col_{\phi(v)}(\phi(u))$ for any graph embedding $\phi : R \to R$ and tree vertices v, u. However more is true, and this kind of argument will play a central role throughout.

Lemma 2.4 *For any tree vertices $u, v \in R_0$, $col_u(w) = col_v(w)$ for all but finitely many $w \in R_0$.*
The only possible exceptions are tree vertices on paths starting from a vertex in $P_{u,v}$ and with strictly decreasing labels.

Proof We may assume that $w \notin P_{u,v}$ since those are part of the exceptional vertices.

Since R is a tree, $P_{u,w} \cap P_{v,w} = P_{w',w}$ for some unique $w' \in P_{u,v}$ (see Fig. 5). Now if $P_{w',w}$ contains a consecutive pair, then $col_u(w) = col_v(w)$ as desired. Otherwise $P_{w',w}$ must be a path with strictly decreasing labels starting with $lab(w')$. But then there is only one such possible tree vertex w for each (not necessarily tree) vertex w' in the finite path $P_{u,v}$. □

Corollary 2.5 *Let ϕ be an embedding of (R, r) and $v \in R_0$. Then $col_v(w) = col_{\phi(v)}(w) = col_{\phi(v)}(\phi(w))$ for all but finitely many $w \in R_0$.*
The only possible exceptions are vertices originating from a path starting from $P_{v,\phi(v)}$ with strictly decreasing labels.

Proof By Lemma 2.4, $col_v(w) = col_{\phi(v)}(w)$, with only possible exceptions being tree vertices originating from a path starting from $P_{v,\phi(v)}$ with strictly decreasing labels. In addition, as remarked above, $col_v(w) = col_{\phi(v)}(\phi(w))$ for any embedding ϕ. □

The height of any (not necessarily tree) vertex w with respect to another arbitrary vertex v is the maximum label encountered in the path from v to w. Again the construction will be done in stages determined by such maximum height.

Definition 2.6 For $v \in R$, define $ht_v : R \to \mathbb{N}$, the height with respect to v, by

$$ht_v(w) = max\{lab(w') : w' \in P_{v,w}\}.$$

Note that $col_v(w) \leq ht_v(w)$ for any tree vertices v and w. Moreover $col_v(w) = ht_v(w)$ means that the label of the last consecutive pair in $P_{v,w}$ is the maximum label appearing among all vertices of $P_{v,w}$; this is actually the situation that will be used in the inductive construction, and for a given fixed vertex v such a vertex w will be called a *target vertex* (with respect to v).

It may also be worth noting that the corresponding Lemma 2.4 is clearly not true for the height value, meaning that there can be infinitely many (even tree) vertices w such that $ht_v(w) \neq ht_u(w)$. However the difficulty is simply due to labels on $P_{u,v}$ and the following observation will be useful.

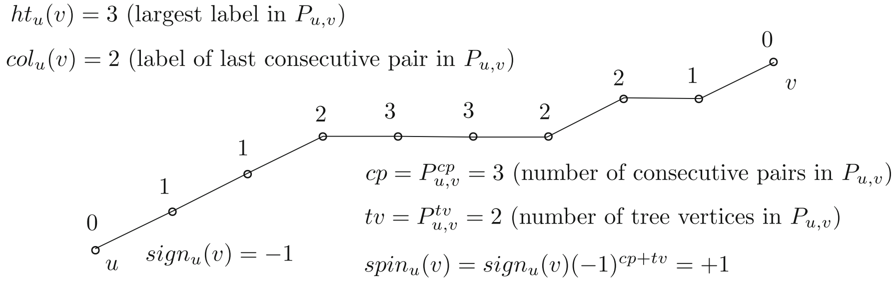


Fig. 6 Definitions 2.3, 2.6 and 2.8: Label, colour, height, sign and spin example

Observation 2.7 *For any vertices $u, v \in R$, $ht_u(w) = ht_v(w)$ for all $w \in R$ such that $min\{ht_u(w), ht_v(w)\} \geq ht_u(v)$.*

Proof Under the given hypothesis we have

$$ht_u(w) \leq max\{ht_v(w), ht_u(v)\} \leq ht_v(w).$$

By symmetry we have $ht_v(w) \leq ht_u(w)$ as well. □

Now for any tree vertex v, there is also an automorphism of $\mathcal{R} = (R, v)$ fixing v and interchanging its two neighbourhoods. In order to keep the number of siblings under control, we will want to prevent not only such automorphisms, but also embeddings mapping one neighbourhood into the other. For this we will define a *sign function* (with respect to v) $sign_v$ which takes values $+1$ on one neighbourhood of v, and -1 on the other neighbourhood, and eventually code these values as graph properties so they are preserved by embeddings. Moreover, within a neighbourhood, we will similarly want to control the various tree vertices and we will define a *spin function* (with respect to v) for that purpose.

The following will define these functions simultaneously, first by defining $sign_r$, $spin_r$, then $sign_v$ and finally $spin_v$ for all other tree vertices v (see Fig. 6 for an example).

Definition 2.8 We write $R_0^v = R_0 \setminus \{v\}$ and $R^v = R \setminus \{v\}$.
Now arbitrarily assign $sign_r(u) = +1$ to every vertex u in one neighbourhood of r, and $sign_r(u) = -1$ to every vertex u in the other neighbourhood of r.

- Let $v \in R_0$ and $sign_v$ a sign function which assigns values ± 1 to each neighbourhood of v.
  Then define $spin_v : R_0^v \to \pm 1$, the spin with respect to v, by:

$$spin_v(u) = sign_v(u)(-1)^{cp+tv}$$

  where

  * $cp = P^{cp}_{v,u}$ is the number of consecutive pairs in $P_{v,u}$
  * $tv = P^{tv}_{v,u}$ is the number of tree vertices in $P_{v,u}$.

- Let $v \neq r$ be a tree vertex, then define $sign_v : R^v \to \pm 1$ by:

  * $sign_v(u) = spin_r(v)$ if u and r belong to the same neighbourhood of v, and
  * $sign_v(u) = -spin_r(v)$ otherwise.

Note that indeed $sign_v$ assigns the value $+1$ on one of its neighbourhood and -1 on the other. Moreover, the spin of r with respect to a tree vertex v can be recovered from the sign of v with respect to the root r, and hence also the spin at any other tree vertex with respect to v as follows.

Lemma 2.9 *Let $v \in R_0^r$. Then:*

1. $spin_v(r) = sign_r(v)$.
2. $spin_v(w) = spin_r(w)$ *for all* $w \in R_0 \backslash P_{r,v}$, *and*
3. $spin_v(w) = -spin_r(w)$ *for all* $w \in (R_0 \cap P_{r,v}) \backslash \{r, v\}$.

Proof By Definition 2.8, $sign_v(r) = spin_r(v)$ since r and r trivially belong to the same neighbourhood of v. Thus, writing $cp = P_{v,r}^{cp} = P_{r,v}^{cp}$ and $tv = P_{v,r}^{tv} = P_{r,v}^{tv}$, we have $spin_v(r) = sign_v(r)(-1)^{cp+tv} = spin_r(v)(-1)^{cp+tv} = \left[sign_r(v)(-1)^{cp+tv}\right](-1)^{cp+tv} = sign_r(v)(-1)^{2cp+2tv} = sign_r(v)$.

For (2), consider $w \in (R_0^r \cap R_0^v) \setminus P_{r,v}$. Then $P_{r,w} \cap P_{v,w} = P_{w',w}$ for some $w' \in P_{r,v}$.

Assume first that $w' = v$, which means that v is on the path from r to w (see Fig. 7). Hence $P_{r,w}^{cp} = P_{r,v}^{cp} + P_{v,w}^{cp}$, and $P_{r,w}^{tv} = P_{r,v}^{tv} + P_{v,w}^{tv} - 1$ because v is counted twice as a tree vertex. Moreover $sign_r(w) = sign_r(v)$ because w and v are in the same neighbourhood of r, and on the other hand $sign_v(w) = -spin_r(v)$ by Definition 2.8 because w and r are in opposite neighbourhoods of v. Thus we get:

$$
\begin{aligned}
spin_r(w) &= sign_r(w)(-1)^{P_{r,w}^{cp}+P_{r,w}^{tv}} \\
&= sign_r(w)(-1)^{P_{r,v}^{cp}+P_{v,w}^{cp}+P_{r,v}^{tv}+P_{v,w}^{tv}-1} \\
&= -\left[sign_r(v)(-1)^{P_{r,v}^{cp}+P_{r,v}^{tv}}\right](-1)^{P_{v,w}^{cp}+P_{v,w}^{tv}} \\
&= -spin_r(v)(-1)^{P_{v,w}^{cp}+P_{v,w}^{tv}} \\
&= sign_v(w)(-1)^{P_{v,w}^{cp}+P_{v,w}^{tv}} = spin_v(w).
\end{aligned}
$$

Similarly, $w' = r$ means that r is on the path from v to w. Hence $P_{v,w}^{cp} = P_{v,r}^{cp} + P_{r,w}^{cp}$, and $P_{v,w}^{tv} = P_{v,r}^{tv} + P_{r,w}^{tv} - 1$ because here r is counted twice as a tree vertex. Moreover $sign_v(w) = spin_r(v)$ because w and r are in the same neighbourhood of v, and on the other hand $sign_r(w) = -sign_r(v)$ because v and w are in opposite neighbourhoods of r. Thus, also observing that $P_{r,v}^{cp} = P_{v,r}^{cp}$ and $P_{r,v}^{tv} = P_{v,r}^{tv}$, we get:

$$
\begin{aligned}
spin_v(w) &= sign_v(w)(-1)^{P_{v,w}^{cp}+P_{v,w}^{tv}} \\
&= sign_v(w)(-1)^{P_{v,r}^{cp}+P_{r,w}^{cp}+P_{v,r}^{tv}+P_{r,w}^{tv}-1} \\
&= spin_r(v)(-1)^{P_{v,r}^{cp}+P_{r,w}^{cp}+P_{v,r}^{tv}+P_{r,w}^{tv}-1} \\
&= \left[sign_r(v)(-1)^{P_{r,v}^{cp}+P_{r,v}^{tv}}\right](-1)^{P_{v,r}^{cp}+P_{r,w}^{cp}+P_{v,r}^{tv}+P_{r,w}^{tv}-1} \\
&= sign_r(v)(-1)^{2P_{r,v}^{cp}+2P_{r,v}^{tv}+P_{r,w}^{cp}+P_{r,w}^{tv}-1} \\
&= -sign_r(v)(-1)^{P_{r,w}^{cp}+P_{r,w}^{tv}} \\
&= sign_r(w)(-1)^{P_{r,w}^{cp}+P_{r,w}^{tv}} = spin_r(w).
\end{aligned}
$$

Finally assume that $w' \in P_{r,v} \setminus \{r, v\}$. Note that in this case $w' \notin R_0$ since tree vertices have degree 2; this yields $P_{r,w}^{cp} = P_{r,w'}^{cp} + P_{w',w}^{cp}$ and $P_{r,w}^{tv} = P_{r,w'}^{tv} + P_{w',w}^{tv}$, and similarly $P_{v,w}^{cp} = P_{v,w'}^{cp} + P_{w',w}^{cp}$ and $P_{v,w}^{tv} = P_{v,w'}^{tv} + P_{w',w}^{tv}$. Moreover $sign_v(w) = spin_r(v)$ since w

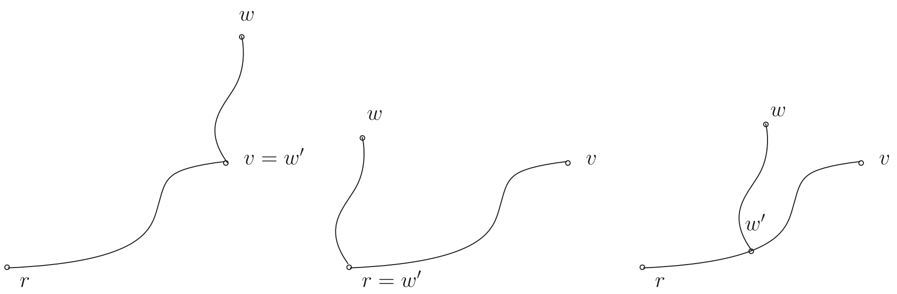


Fig. 7 Lemma 2.9: the three cases: $w' = v$, $w' = r$, and $w' \in P_{r,v} \setminus \{r, v\}$

and r belong to the same neighbourhood of v. So we get:

$$\begin{aligned}
spin_v(w) &= sign_v(w)(-1)^{P^{cp}_{v,w}+P^{tv}_{v,w}}\\
&= spin_r(v)(-1)^{P^{cp}_{v,w}+P^{tv}_{v,w}}\\
&= \left[sign_r(v)(-1)^{P^{cp}_{r,v}+P^{tv}_{r,v}}\right](-1)^{P^{cp}_{v,w}+P^{tv}_{v,w}}\\
&= sign_r(v)(-1)^{P^{cp}_{r,v}+P^{cp}_{v,w}+P^{tv}_{r,v}+P^{tv}_{v,w}}\\
&= sign_r(v)(-1)^{P^{cp}_{r,w'}+P^{cp}_{w',v}+P^{cp}_{v,w'}+P^{cp}_{w',w}+P^{tv}_{r,w'}+P^{tv}_{w',v}+P^{tv}_{v,w'}+P^{tv}_{w',w}}\\
&= sign_r(w)(-1)^{P^{cp}_{r,w}+P^{tv}_{r,w}} = spin_r(w).
\end{aligned}$$

The proof of (3) follows a similar analysis. Note here that $w \in R_0\backslash\{r, v\}$, and since w and r are in the same neighbourhood of v, we have:

$$\begin{aligned}
spin_v(w) &= sign_v(w)(-1)^{P^{cp}_{v,w}+P^{tv}_{v,w}}\\
&= spin_r(v)(-1)^{P^{cp}_{v,w}+P^{tv}_{v,w}}\\
&= \left[sign_r(v)(-1)^{P^{cp}_{r,v}+P^{tv}_{r,v}}\right](-1)^{P^{cp}_{v,w}+P^{tv}_{v,w}}\\
&= \left[sign_r(v)(-1)^{P^{cp}_{r,w}+P^{cp}_{w,v}+P^{tv}_{r,w}+P^{tv}_{w,v}-1}\right](-1)^{P^{cp}_{v,w}+P^{tv}_{v,w}}\\
&= -sign_r(v)(-1)^{2P^{cp}_{v,w}+2P^{tv}_{v,w}+P^{cp}_{r,w}+P^{tv}_{r,w}}\\
&= -sign_r(w)(-1)^{P^{cp}_{r,w}+P^{tv}_{r,w}}\\
&= -spin_r(w)
\end{aligned}$$

This completes the proof of the lemma. □

We can further correlate the spin between any two vertices.

Corollary 2.10 *Let $u, v \in R_0$. Then*

1. $spin_u(w) = spin_v(w)$ *for all* $w \in (R^u_0 \cap R^v_0)\backslash P_{u,v}$, *and*
2. $spin_u(w) = -spin_v(w)$ *for all* $w \in (R^u_0 \cap R^v_0) \cap P_{u,v}$, *with the only exception of* $w \neq r$ *and* $P_{r,u} \cap P_{r,v} = P_{r,w}$, *in which case* $spin_u(w) = spin_v(w)$.

Proof The proof follows by carefully using Lemma 2.9. Consider $w \in (R^u_0 \cap R^v_0)$.

If for a first case $w \notin P_{u,r} \cup P_{v,r}$ (and hence $w \notin P_{u,v}$), then $spin_u(w) = spin_r(w) = spin_v(w)$ by Lemma 2.9 (2).

Next assume that $w \in P_{u,r} \cap P_{v,r}$. If $w \neq r$, then $w \in P_{r,u}\backslash\{r, u\}$ and $w \in P_{r,v}\backslash\{r, v\}$, hence $spin_u(w) = -spin_r(w) = spin_v(w)$ by Lemma 2.9 (3). Note that this includes the case that $P_{r,u} \cap P_{r,v} = P_{r,w}$ with $w \in P_{u,v}$. Now if $w = r$, then either $w \notin P_{u,v}$ and thus u and v are in the same neighbourhood of $w = r$, so by Lemma 2.9 (1), $spin_u(w) =$

$spin_u(r) = sign_r(u) = sign_r(v) = spin_v(r) = spin_v(w)$; or else $w = r \in P_{u,v}$, then u and v are in opposite neighbourhood of $w = r$, so again by Lemma 2.9 (1), $spin_u(w) = spin_u(r) = sign_r(u) = -sign_r(v) = -spin_v(r) = -spin_v(w)$.

So finally assume without loss of generality that $w \in P_{u,r} \setminus P_{v,r}$, and hence $w \in P_{u,v}$. Then $spin_u(w) = -spin_r(w)$ by Lemma 2.9 (3) since $w \in P_{r,u} \setminus \{r, u\}$. Also $spin_v(w) = spin_r(w)$ by Lemma 2.9 (2) since $w \in (R_0^r \cap R_0^v) \setminus P_{r,v}$. Thus $spin_u(w) = -spin_v(w)$. □

Recall that labels have been encoded by graph properties (with the gadgets), and thus are preserved by any (rooted tree) embedding $\phi : (R, r) \to (R, v)$. Hence the height and colour functions are also preserved. If we further ask to preserve the spin (and hence sign) function, then such an embedding is unique (except for possibly interchanging the leaves of gadgets, which is immaterial for our purpose); we will see later how the construction will yield a tree actually preserving the spin through graph properties.

2.2 Ray vertices

The second type of structures that will be used in the construction are double rays (two-way infinite rays) with two kinds of vertex types. Eventually, each tree vertex in all copies of the graph R will be amalgamated with a vertex on a double ray.

Let D be a double ray, which we will normally identify as $D = \{v_i : i \in \mathbb{Z}\}$ with edges (v_i, v_{i+1}) for $i \in \mathbb{Z}$. We will equip D with *type assignments* of the form $tp : V(D) \to \{0, 1\}$, and we write $\mathcal{D}_{tp} = (D, tp)$ for the resulting structure. Here an embedding $\phi : \mathcal{D}_{tp} \to \mathcal{D}_{tp'}$ is a graph embedding of D such that $tp(v_i) \leq tp'(\phi(v_i))$ for each $i \in \mathbb{Z}$. What will be important is that an embedding of $\mathcal{D}_{tp}$ can send type 0 vertices into type 1 vertices, but not the other way around. Again we encode these structures as graph properties so that these embeddings become real graph embeddings. We do so by identifying every vertex $v \in D$ with the root of a copy of $PK(2, 2)$ if $tp(v) = 0$, and with the root of $PK(2, 3)$ if $tp(v) = 1$. Since $PK(2, 2)$ embeds in $PK(2, 3)$ but not vice versa, this yields exactly what is needed.

We remark that eventually every tree vertex v will be identified with a vertex in a double ray equipped with a type assignment, and so will have two finite gadgets attached to it: one that encodes its zero label as a tree vertex, that is a copy of $PK(2 * 0 + 6, 2)$, and the other identified with the root of a copy of either $PK(2, 2)$ or $PK(2, 3)$ matching its type assignment as a ray vertex. Graph embeddings will necessarily send label attachments and type attachments to attachments of the same kind.

We will loosely call $\mathcal{D}_{tp}$ a double ray even though it comes equipped with vertices of type 0 or 1, and we will continue to use the symbol D to denote a regular double ray (without any type assignment). Now consider the special type assignment tp_0 such that:

$$tp_0(v_i) = \begin{cases} 0 \text{ for } i \leq 0 \\ 1 \text{ for } i > 0. \end{cases}$$

and let $\mathcal{D}_0 = \mathcal{D}_{tp_0}$ be the resulting double ray. Note that v_0 is the first (only in this case) vertex of type 0 followed by a type 1 vertex, and we call it the centre $z = z_0$ of $\mathcal{D}_0$. Observe that all siblings of $\mathcal{D}_0$ are of the form $\mathcal{D}_{tp}$ for some type assignment tp consisting of a finite modification of the above type tp_0. Hence there are exactly countably many (up to isomorphy) pairwise non-isomorphic siblings of $\mathcal{D}_0$ not isomorphic to $\mathcal{D}_0$, and all will have a vertex of type 1 followed by a type 0 vertex. We select $\langle \mathcal{D}_s : 0 < s < \mathfrak{s} \rangle$ with pairwise non-isomorphic type assignments $\langle tp_s : 0 < s < \mathfrak{s} \rangle$. To be specific we select tp_s as follows:

$$tp_s(v_j) = \begin{cases} 0 \text{ for } j < 0 \text{ or } 1 \leq j \leq s \\ 1 \text{ for } j = 0 \text{ or } j > s. \end{cases}$$

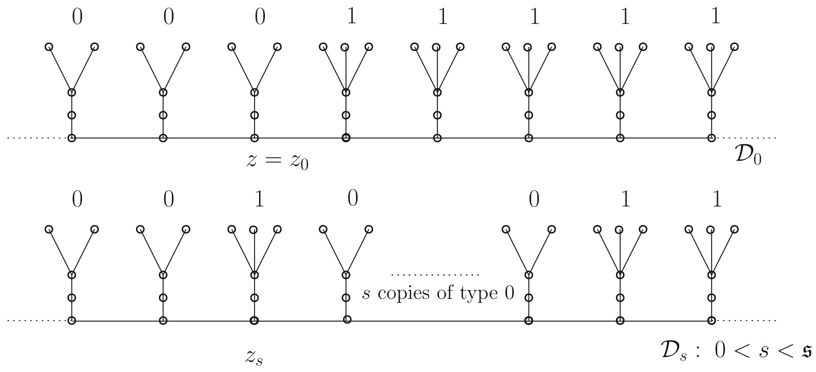


Fig. 8 Double rays $\mathcal{D}_{tp_s} : s < \mathfrak{s}$ with their (graph version) gadget 0–1 type assignments

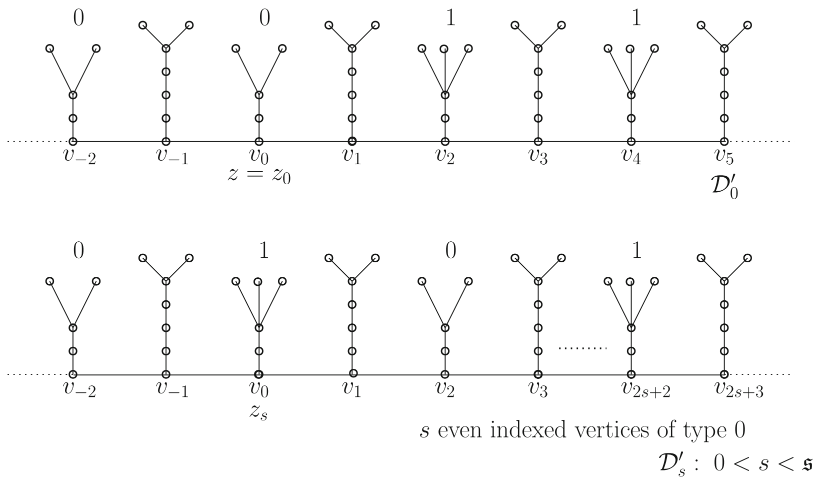


Fig. 9 Double rays $\mathcal{D}_s : s < \mathfrak{s}$ for Posets with gadget 0–1 type assignments on even index vertices

We again let $z_s = v_0$ be the centre of $\mathcal{D}_s$ (see Fig. 8).

These choices are designed so no embedding from $\mathcal{D}_s$ to $\mathcal{D}_{s'}$ can send z_s to $z_{s'}$ for any $s \neq s'$. These will also form the centres of the resulting trees $\mathcal{T}_s$ and will be used to ensure embeddings preserve the sign and spin functions as promised earlier.

We will call *ray vertices* those vertices v_i's on the double rays. Because ray vertices of type 1 cannot embed in ray vertices of type 0, then these double rays are equipped with a natural direction dictated by embeddings, reflecting the positive direction of the indexing along $\mathbb{Z}$.

Finally if D is a double ray in a tree T and $v \in D$, we define for later convenience T_v^D the connected component of T containing v without its two neighbours on the double ray D.

Before we move on to other required properties of these trees, we note that the case of partial orders will require that the double ray type assignments above be done on even indexed vertices only, and that the odd indexed vertices all be equipped with a gadget that does not embed in any of the type assignments gadgets, and vice-versa: using $PK(4, 2)$ on odd indexed vertices for example will do (which is why we left that gadget available), see Fig. 9.

We urge the reader to take note that the construction below can be done using either kind of type assignments on the double rays with similar results; this will be used for the case of partial orders.

2.3 Global colouring, spin and height

Each of the trees $\langle \mathcal{T}_s : s < \mathfrak{s}\rangle$ will be built in a similar manner, and we will do so by first constructing a common "spine" $\mathcal{S}^p$. This will be done by first assembling disjoint copies of (R, r) along a (plain) double ray $\mathcal{D}$, identifying each ray vertex of $\mathcal{D}$ with the root of a disjoint copy of (R, r). Then each new tree vertex will be identified with a vertex of a disjoint copy of a double ray, and again each new ray vertex will be identified with the root of a disjoint copy of (R, r), etc. The only thing missing on the spine $\mathcal{S}^p$ to obtain each tree $\mathcal{T}_s$ for $s < \mathfrak{s}$ are judiciously chosen type assignments on each of those double rays so to control all embeddings.

Every copy (R', r') of (R, r) inherits its own local labelling, colour, height, sign and spin functions, which will be denoted by $lab^{R'}$, $ht^{R'}$, $col^{R'}$, $sign^{R'}$, and $spin^{R'}$ respectively, although we may omit the superscript R' if the context is clear. We wish to extend these notions globally to $\mathcal{S}^p$ (and eventually each $\mathcal{T}_s$) so they apply across copies of (R, r), and we will introduce a centre vertex of $\mathcal{S}^p$ when the need arises.

We will proceed to build $\mathcal{S}^p$ in stages $\mathcal{S}^p(k)$ which will later be reflected in the constructions of each $\mathcal{T}_s$. First we define a global height $\widehat{ht}$ in such trees formed by assembling disjoint copies of (R, r). Recall that each such copy comes equipped with its corresponding label function, and the global height is simply the maximum label encountered in a path. This will be used in particular to determine the stages of the construction.

Definition 2.11 Let $\mathcal{S}$ be a tree formed by assembling disjoint copies of (R, r) along double rays. Then for $v, w \in \mathcal{S}$, define

$$\begin{aligned}\widehat{ht}_v(w) = \quad & max\{lab^{R'}(w') : w' \in P_{v,w}, \text{ and} \\ & w' \text{ is in a copy } (R', r') \text{ of } (R, r)\}.\end{aligned}$$

Note that $\widehat{ht}_v(w) = ht_v^{R'}(w)$ in case v and w belong to the same copy (R', r') of (R, r). We are now ready to define the spine $\mathcal{S}^p$.

Definition 2.12
- We *activate* a ray vertex by identifying that vertex with the root r' of a disjoint copy (R', r') of (R, r).
- We *amalgamate* a tree vertex by identifying that vertex with one of the vertices of a disjoint copy of a double ray.
- Define $\mathcal{S}^p(0)$ by activating every ray vertex of a double ray, and declare $c = v_0$ the center of the spine.
  Given $\mathcal{S}^p(k-1)$, let $\mathcal{S}_0^p(k) = \mathcal{S}^p(k-1)$, and $\mathcal{S}_{\ell+1}^p(k)$ be obtained from $\mathcal{S}_\ell^p(k)$ by amalgamating every non-amalgamated tree vertex of global height at most k with respect to c, followed by activating every new ray vertex.
  Define $\mathcal{S}^p(k) = \bigcup_\ell \mathcal{S}_\ell^p(k)$.
  Finally let $\mathcal{S}^p = \bigcup_k \mathcal{S}^p(k)$.

Observe that every ray vertex of $\mathcal{S}^p$ is *activated* (with the root of a disjoint copy of (R, r)), thus we can think of every vertex of $\mathcal{S}^p$ as being in a copy of (R, r). Moreover every tree vertex is *amalgamated* with a vertex of a double ray, and this is through those double rays

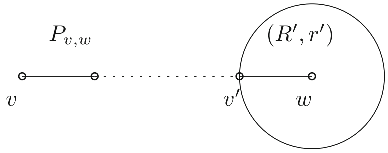


Fig. 10 Definition 2.14: The global colour function: $\widehat{col}_v(w) = col_{v'}^{R'}(w)$

that one navigates from one copy of (R, r) to another. Also observe that $\mathcal{S}^p$ is a locally finite tree.

Now as before an embedding ϕ of $\mathcal{S}^p$ must preserve labels, that is $lab^{R'}(v) = lab^{R''}(\phi(v))$ where v belongs to a copy (R', r') and $\phi(v)$ to a copy (R'', r''); this is simply because the finite gadgets attached to vertices in copies of (R, r) do not embed into each other for different labels. But this means that ϕ actually preserves copies since moving from one copy to another requires a path through at least two consecutive ray vertices, which are activated to tree vertices of label 0. We take note of this in the following important observation.

Observation 2.13 *Any embedding of $\mathcal{S}^p(k)$ or $\mathcal{S}^p$ is surjective, preserves labels, ray and tree vertices, and copies of (R, r).*

We next extend the colour function globally to $\mathcal{S}^p$, and this is a bit more delicate. First note that new copies of (R, r) are created by identifying a ray vertex with the root r' of a disjoint copy (R', r') of (R, r). But once that copy is created, a path originating from outside it could enter that copy through a different tree vertex $v' \neq r'$ that was later activated.

Definition 2.14
1. Let $\widehat{R}_0 = \{v \in \mathcal{S}^p : v$ belongs to a copy (R', r') of (R, r) and $lab^{R'}(v) = 0\}$.
2. For $v \in \widehat{R}_0$, define $\widehat{col}_v : \widehat{R}_0 \to \mathbb{N}$ by $\widehat{col}_v(w) = col_{v'}^{R'}(w)$ where w belongs to a copy (R', r') of (R, r) and v' is the first vertex of $P_{v,w} \cap R'$.

We continue to call tree vertices those in $\widehat{R}_0$.

Thus $v' = v$ and $\widehat{col}_v(w) = col_v^{R'}(w)$ in case v and w belong to the same copy (R', r') of (R, r), see Fig. 10. Also $\widehat{col}_v(w) = 0$ if $v' = w$ since in this case $col_w^{R'}(w) = 0$. We now show that Lemma 2.4 generalizes globally as follows.

Lemma 2.15 *For any two tree vertices $u, v \in \widehat{R}_0$, $\widehat{col}_u(w) = \widehat{col}_v(w)$ for all but finitely many $w \in \widehat{R}_0$.*
The only possible exceptions are vertices originating from a path starting from $P_{u,v}$ with strictly decreasing labels.

Proof Let $u, v, w \in \widehat{R}_0$, w belonging to a copy (R', r') of (R, r), u' the first vertex of $P_{u,w} \cap R'$, and v' the first vertex of $P_{v,w} \cap R'$.

Then by definition $\widehat{col}_u(w) = col_{u'}^{R'}(w)$ and $\widehat{col}_v(w) = col_{v'}^{R'}(w)$. Now by Lemma 2.4, $col_{u'}^{R'}(w) = col_{v'}^{R'}(w)$ with possible exceptions being vertices originating from a path starting from $P_{u',v'}$ with strictly decreasing labels. The result follows. □

It will later be useful to understand the global colouring through embeddings, we do so in two parts (Fig. 11).

Corollary 2.16 *Let $v \in \widehat{R}_0$ be a tree vertex, and ϕ an embedding of $\mathcal{S}^p$. Then $\widehat{col}_v(w) = \widehat{col}_{\phi(v)}(w) = \widehat{col}_{\phi(v)}(\phi(w))$ for all but finitely many $w \in \widehat{R}_0$.*
The only possible exceptions are vertices originating from a path starting from $P_{v,\phi(v)}$ with strictly decreasing labels.

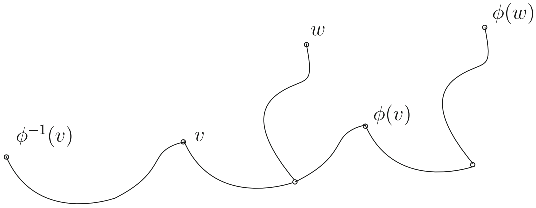


Fig. 11 Corollaries 2.16 and 2.17: $\widehat{col}_v(w)$, $\widehat{col}_{\phi(v)}(w)$, $\widehat{col}_{\phi(v)}(\phi(w))$, and $\widehat{col}_v(\phi(w))$

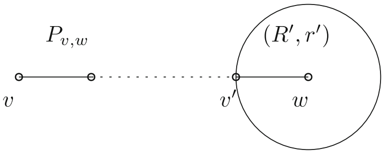


Fig. 12 Definition 2.18: The global sign and spin functions: $\widehat{sign}_v(w) = sign^{R'}_{v'}(w)$ and $\widehat{spin}_v(w) = spin^{R'}_{v'}(w)$

Proof $\widehat{col}_v(w) = \widehat{col}_{\phi(v)}(\phi(w))$ since ϕ preserves labels, and $\widehat{col}_v(w) = \widehat{col}_{\phi(v)}(w)$ by Lemma 2.15 assuming that w does not originate from a path starting from $P_{v,\phi(v)}$ with strictly decreasing labels. □

Corollary 2.17 *Let $v \in \widehat{R}_0$ be a tree vertex, and ϕ an embedding of $\mathcal{S}^p$. Then $\widehat{col}_v(w) = \widehat{col}_v(\phi(w))$ for all but finitely many $w \in \widehat{R}_0$.*
The only possible exceptions are vertices originating from a path starting from $P_{\phi^{-1}(v),v}$ with strictly decreasing labels.

Proof Let $v \in \widehat{R}_0$ be a tree vertex, ϕ an embedding of $\mathcal{S}^p$, and w not originating from a path starting from $P_{\phi^{-1}(v),v}$ with strictly decreasing labels. Recall by Observation 2.13 that ϕ is surjective. Hence, by Corollary 2.16, $\widehat{col}_{\phi^{-1}(v)}(w) = \widehat{col}_{\phi\circ\phi^{-1}(v)}(w) = \widehat{col}_{\phi\circ\phi^{-1}(v)}(\phi(w))$. Thus we conclude that $\widehat{col}_v(w) = \widehat{col}_v(\phi(w))$. □

Similarly we define a global sign and spin functions. First recall that the $sign_v$ and $spin_v$ functions are undefined at the vertex v itself and we also want to avoid defining their global versions on ray vertices other than the center.

Definition 2.18 (a) Let $v \in \widehat{R}_0$, and first define:

1. $\widehat{R}^v = \{w \in \mathcal{S}^p : w$ belongs to a copy (R', r') of (R, r) and w is not the first vertex of $P_{v,w} \cap R'\}$.
2. $\widehat{R}^v_0 = \widehat{R}^v \cap \widehat{R}_0$.

(b) 1. Define $\widehat{sign}_v : \widehat{R}^v \to \pm 1$ by $\widehat{sign}_v(w) = sign^{R'}_{v'}(w)$ where w belongs to a copy (R', r') of (R, r) and $v'(\neq w)$ is the first vertex of $P_{v,w} \cap R'$.
2. Similarly define $\widehat{spin}_v : \widehat{R}^v_0 \to \pm 1$ by $\widehat{spin}_v(w) = spin^{R'}_{v'}(w)$ where w belongs to a copy (R', r') of (R, r) and $v'(\neq w)$ is the first vertex of $P_{v,w} \cap R'$.

Thus in the case that v and w belong to the same copy of (R, r), then $v' = v \neq w$, $\widehat{sign}_v(w) = sign_v(w)$ and $\widehat{spin}_v(w) = spin_v(w)$ (Fig. 12).

Lemma 2.19 *For any tree vertices $u, v \in \widehat{R}_0$, $\widehat{spin}_u(w) = \widehat{spin}_v(w)$ for all but finitely many $w \in \widehat{R}^u_0 \cap \widehat{R}^v_0$, in fact for all $w \notin P_{u,v}$.*

Proof Suppose $w \notin P_{u,v}$, (R', r') a copy of (R, r) containing w, u' is the first vertex of $P_{u,w} \cap R'$ and similarly v' is the first vertex of $P_{v,w} \cap R'$ (Fig. fig:GlobalspinPreserving0).

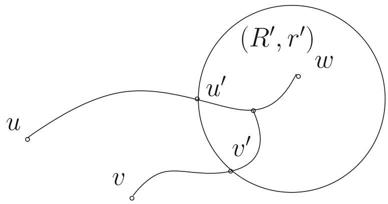


Fig. 13 Lemma 2.19: $\widehat{spin}_u(w) = spin^{R'}_{u'}(w) = spin^{R'}_{v'}(w) = \widehat{spin}_v(w)$

But now $w \notin P_{u',v'}$ and so $w \neq u', v'$. Hence $\widehat{spin}_u(w) = spin^{R'}_{u'}(w) = spin^{R'}_{v'}(w) = \widehat{spin}_v(w)$ by Corollary 2.10 and Definition 2.18. $\square$

Observation 2.20 *Let ϕ be an embedding of $\mathcal{S}^p$, then $\widehat{ht}_v(u) = \widehat{ht}_{\phi(v)}(\phi(u))$ for all $u, v \in \mathcal{D}$.*

As observed before, any graph embedding of $\mathcal{S}^p$ is surjective and hence an automorphism, and thus $sib(\mathcal{S}^p) = 1$. The trees $\mathcal{T}_s$ will be obtained from $\mathcal{S}^p$ by judiciously setting type assignments to ray vertices of $\mathcal{S}^p$, and as a result embeddings of $\mathcal{T}_s$ will preserve the global sign and global spin functions. This will be the main tool in showing that $sib(\mathcal{T}) = \mathfrak{s}$. We are now ready to undertake the construction of the trees $\mathcal{T}_s$.

2.4 The trees $\langle \mathcal{T}_s(k) : s < \mathfrak{s} \rangle$

For a non-zero $\mathfrak{s} \in \mathbb{N}$, the final trees $\langle \mathcal{T}_s : s < \mathfrak{s} \rangle$ we are seeking to produce will be constructed in similar manners to $\mathcal{S}^p$, as a countable union of trees $\mathcal{T}_s(k)$ for $k \in \mathbb{N}$. The trees $\mathcal{T}_s(k)$ will consist of the spine $\mathcal{S}^p(k)$ together with type assignments to double rays amalgamated to tree vertices of global height at most k; interestingly, the only difference among the various $\mathcal{T}_s(k)$'s is the original type assignment to the first double ray $\mathcal{D}_s$. We will often simply write $\mathcal{T}$ for $\mathcal{T}_0$ and similarly $\mathcal{T}(k)$ for $\mathcal{T}_0(k)$.

Recall that every vertex of $\mathcal{S}^p$ belongs to a copy (R', r') of (R, r), each inheriting its own corresponding collection of labels, colours, height, sign and spin functions from $\mathcal{R}$. The notions of colours and height are graph properties, and we have seen in Sect. 2.1 how we have encoded the labels as graph properties (through connecting a vertex of label ℓ with the root of a gadget being a path of length $2\ell + 6$ whose end point is identified with $u \in K_{1,2} = (u, V)$; these finite graphs do not embed into each other as rooted trees unless they are equal). We have also seen in Sect. 2.2 how we encode the type assignments as graph properties (through identifying a ray vertex of type 0 with the root of the gadget $PK(2, 2)$, and a ray vertex of type 1 with the root of $PK(2, 3)$; the type 0 gadgets embed into type 1 gadgets as a rooted tree, but not the other way around). It will be important later to note that a type 0 gadget embedding in a type 1 results in one vertex being omitted from the image of the embedding. It will remain to show how the sign and spin functions can be encoded through graph properties so that they are preserved by embeddings. But first we will build the trees and show how to handle their siblings.

We define $\mathcal{T}_s(0)$ by activating every ray vertex of $\mathcal{D}_s$, and we further define the vertex $z_s \in \mathcal{D}_s$ as the *centre* of the tree $\mathcal{T}_s(0)$, (see Fig. 14). These will remain the centres of all trees $\langle \mathcal{T}_s(k) : s < \mathfrak{s} \rangle$ which we are about to define. Observe that the spine of $\mathcal{T}_s(0)$, that is the tree obtained from $\mathcal{T}_s(0)$ by deleting the type assignments on ray vertices (on $\mathcal{D}_s$), is $\mathcal{S}^p(0)$ as previously defined. Moreover, the spine of any sibling of $\mathcal{T}_s(0)$ is the same $\mathcal{S}^p(0)$ (up to isomorphy), and this is because any self-embeddings of $\mathcal{T}_s(0)$ will map $\mathcal{S}^p(0)$ onto itself. Observe further that $\langle \mathcal{T}_s(0) : s < \mathfrak{s} \rangle$ are pairwise non-isomorphic siblings, and have (up to isomorphism) countably many siblings, each one represented by a type assignment

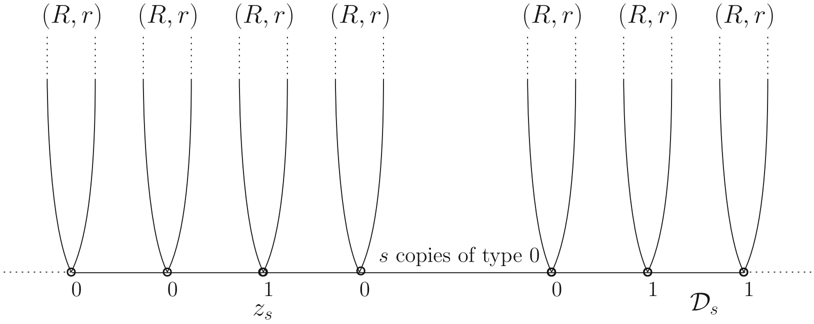
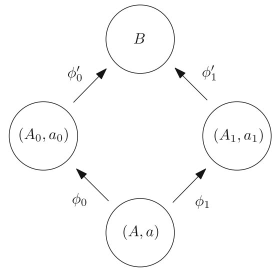


Fig. 14 $\mathcal{T}_s(0)$

Fig. 15 B is an *amalgamation* of (A_0, a_0) and (A_1, a_1) over (A, a)



(consisting of only finitely many modifications of tp_0) on $\mathcal{D}_0$, the ray vertices of the spine $\mathcal{T}_0(0)$.

The inductive construction will use a more delicate operation to *freely amalgamate* two trees over a common subtree. This situation generally occurs for example when two mathematical objects of similar types share a common substructure and the remaining structure does not interfere with each other, hence one may want to combine them together. We define here the specific case we will need (see Fig. 15).

Definition 2.21 Let a_0 be a vertex of a tree A_0 and a_1 a vertex of a tree A_1. Assume further that a is a vertex of a tree A and we have rooted tree embeddings $\phi_i : (A, a) \to (A_i, a_i)$.

Then we say that a tree B is a *(free) amalgamation* of (A_0, a_0) and (A_1, a_1) over (A, a), or (A_0, a_0) and (A_1, a_1) *(freely) amalgamate* (over (A, a)), if there are embeddings $\phi_i' : A_i \to B$ such that:

1. $B = \phi_0'(A_0) \cup \phi_1'(A_1)$,
2. $\phi_0' \circ \phi_0 = \phi_1' \circ \phi_1$, and
3. $\phi_0'(A_0) \cap \phi_1'(A_1) = \phi_0' \circ \phi_0(A) = \phi_1' \circ \phi_1(A)$.

Note that since B is a tree and A is non-empty, then no edges are added to $\phi_0'(A_0) \cup \phi_1'(A_1)$; the two pieces are simply joined together identifying their common copy of A. A simple example is to observe that identifying a ray vertex v from a double ray $\mathcal{D}$ with the root of a copy of (R, r) as we have done can be expressed as amalgamating a double ray $\mathcal{D}$ containing v and (R, r) over v (mapping v to r). But we will more generally amalgamate larger trees in the construction, hence the need for the more general concept above.

The construction will ensure that the $\mathcal{T}_s(k)$ are non-isomophic siblings, and that $sib(\mathcal{T}(k)) = \aleph_0$ for all $k \in \mathbb{N}$ (and hence $sib(\mathcal{T}_s(k)) = \aleph_0$ for all $s < \mathfrak{s}$). Once $\mathcal{T}(k)$ has been constructed, we list (representatives of) the pairwise non-isomorphic siblings of

$\mathcal{T}(k)$ not isomorphic to any $\mathcal{T}_s(k)$) as $\{S_{k,\ell} : \ell \in \mathbb{N}\}$. These are of course graph siblings, and by considering those siblings as substructures of $\mathcal{T}(k)$, they come equipped with labels and type assignments from $\mathcal{T}(k)$ no matter the embedding. This is also the case with the sign and spin functions inherited from $\mathcal{T}(k)$, but those values do depend on the embedding and we will be careful when handling those. We will also select in $S_{k,\ell}$ a double ray non-isomorpic to $\mathcal{D}_0$, and fix the centre $c_{k,\ell}$ of $S_{k,\ell}$ as the first vertex on that double ray having type 1 followed by a vertex of type 0 (as is the case for each z_s for $s > 0$). This is again for the same reason as before to ensure that no embedding of this double ray into $\mathcal{D}_0$ can send centres to centres, and vice versa; this is what will be used to show that the spin is eventually preserved by embeddings. We will later justify the existence of such a double ray in all non-isomorphic siblings of $\mathcal{T}(k)$.

As mentioned, all siblings of the (eventual) tree $\mathcal{T} = \mathcal{T}_0$ will be isomorphic to some $\mathcal{T}_s$, and to do so we will amalgamate approximations of those siblings within each $\mathcal{T}_s(k)$ along the way. It turns out that this will suffice because siblings will differ from $\mathcal{T}$ by only finitely many type assignments and hence will be captured at some stage.

2.4.1 Inductive construction

The following notions will help better describe the construction.

Definition 2.22 • A tree vertex $v \in \mathcal{T}_s(k)$ is called a *target* vertex (of global height ℓ) if $\widehat{ht}_{z_s}(v) = \widehat{col}_{z_s}(v) = \ell$ for some $\ell \geq 0$.

- A crater (or ℓ-crater) centered at a target vertex v of global height ℓ, written $\mathcal{C}(v)$, consists of all vertices of global height less than ℓ from v. That is $\mathcal{C}(v) = \{u \in \mathcal{T}_s(k) : \widehat{ht}_v(u) < \ell\}$.
- We say that a tree vertex $v \in \mathcal{T}_s(k)$ has been *amalgamated* if it was part of an amalgamation.

Thus a target vertex v of global height ℓ is the end vertex in a copy of (R, r) of a path $P_{z_s,v}$ originating at the centre z_s having its last consecutive pair of highest labels ℓ among the path labels and with decreasing labels from that consecutive pair to v. Then all vertices in its crater $\mathcal{C}(v)$ have global height ℓ with respect to z_s.

When a tree vertex is amalgamated, it will be identified with a ray vertex from a double ray and provided with a type assignment. The terminology to *amalgamate* a tree vertex is thus consistent with that of Definition 2.12.

We identify each center z_s with the root r' of a copy of (R, r) and as such it is a target vertex of (global height 0). For each $s < \mathfrak{s}$ it has been amalgamated to the centre of the double ray $\mathcal{D}_s$, with each other ray vertex activated and thus amalgamated to the root of a copy of (R, r), producing $\mathcal{T}_s(0)$. There are no other target vertices of global height 0 in $\mathcal{T}_s(0)$ and thus we can state that all target vertices of $\mathcal{T}_s(0)$ have been amalgamated up to global height 0 with respect to their centres z_s.

We now define how to extend each tree $\mathcal{T}_s(k-1)$ from stage $k-1$ to $\mathcal{T}_s(k)$ at the next level k; this is done the same way for all $s < \mathfrak{s}$ and as a result all trees rooted at ray vertices on all $\mathcal{D}_s$ will be identical. Assume that the trees $\langle \mathcal{T}_s(k-1) : s < \mathfrak{s} \rangle$ have been constructed for some $k \geq 1$, that all tree vertices are amalgamated up to global height $k-1$ with respect to their centres z_s, and that the spine of any sibling of $\mathcal{T}_s(k-1)$ is (up to isomorphy) $\mathcal{S}^p(k-1)$.

Now fix $s < \mathfrak{s}$ and consider a *target* vertex $v \in \mathcal{T}_s(k-1)$ such that $\widehat{ht}_{z_s}(v) = \widehat{col}_{z_s}(v) = k$, and belonging to a copy (R', r') of (R, r). Thus v is not yet amalgamated. Moreover, since $\widehat{col}_{z_s}(v) = k > 0$, the last consecutive pair on $P_{z_s,v}$ has labels k and therefore all tree vertices having global height less than k with respect to v, namely all the tree vertices in its crater

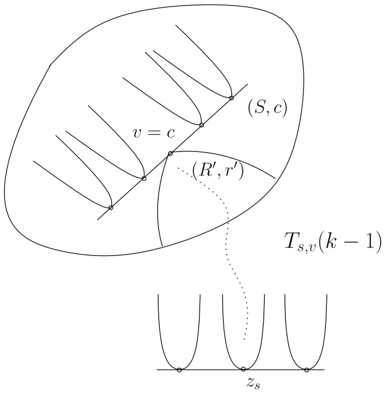


Fig. 16 $\mathcal{T}_{s,v}(k-1)$: amalgamate $(\mathcal{T}_s(k-1), v)$ with (S, c) over (R', r')

$\mathcal{C}(v)$ in $\mathcal{T}_s(k-1)$, are of global height equal to k with respect to z_s, and are thus also not yet amalgamated. Write $k = 2^i(2j+1)$, and consider $S = S_{i,j}$, a sibling of $\mathcal{T}(i)$, with centre $c = c_{i,j}$ lying on a double ray $\mathcal{D}$ (so that no embedding into $\mathcal{D}_0$ can send c to $z = z_0$ and vice versa). Moreover, since S can be viewed as a substructure of $\mathcal{T}(k-1)$, all vertices in S having global height larger than or equal to k with respect to c are also not yet amalgamated. Thus either S or $\mathcal{T}_s(k-1)$ can be amalgamated over (R', r') by identifying v and c (see Fig. 16). This allows to proceed as follows, and either amalgamate:

- $(\mathcal{T}_s(k-1), v)$ with (S, c) over (R', r'), if $\widehat{spin}_{z_s}(v) = +1$.
- $(\mathcal{T}_s(k-1), v)$ with $(\mathcal{T}(k-1), z_0)$ over (R', r'), if $\widehat{spin}_{z_s}(v) = -1$.

If one considers the existing copy (R', r') from $\mathcal{T}_s(k-1)$ as being rooted at v for a moment, it becomes amalgamated with the copy of (R, r) from (S, c) (or $(\mathcal{T}(k-1), z_0)$) rooted at c (or z_0 respectively), in particular identifying v with c, and v becomes amalgamated. Now that (S, c) is embedded in $\mathcal{T}(k-1)$ rooted at $c = v$, it inherits the sign and spin functions from $\mathcal{T}(k-1)$ and we can extend S within $\mathcal{T}(k-1)$ following the inductive construction, and assume that all tree vertices in S are amalgamated up to global height $k-1$ with respect to its centre $c = v$.

The result of the amalgamation will be that all ray vertices in the crater $\mathcal{C}(v)$ will receive a type assignment from S. This creates what we call $\mathcal{T}_{s,v}(k-1)$ (see Fig. 16).

Note to be clear that we indeed amalgamate $(\mathcal{T}(k-1), z_0)$ with $(\mathcal{T}_s(k-1), v)$ for any $s < \mathfrak{s}$, thus the process is the same no matter which s!

Now we repeat the same construction for all target vertices of $\mathcal{T}_s(k-1)$ of global height k with respect to z_s. Observe that any two such distinct target vertices are separated by a consecutive pair of labels k, and therefore the corresponding craters do not intersect and their amalgamations as described above do not interfere with each other. If v and v' are two such target vertices, then $\mathcal{T}_{s,v}(k-1) \cap \mathcal{T}_{s,v'}(k-1) = \mathcal{T}_s(k-1)$. But these amalgamations introduce new target vertices of global height k with respect to z_s in the resulting amalgamated trees, so we repeat until all such vertices have been amalgamated. First define $\mathcal{T}_s^0(k-1) = \mathcal{T}_s(k-1)$, and:

$$\mathcal{T}_s^{\ell+1}(k-1) = \bigcup\{\mathcal{T}_{s,v}^{\ell}(k-1) : v \in \mathcal{T}_s^{\ell}(k-1) \text{ is not yet amalgamated and } \widehat{ht}_{z_s}(v) = \widehat{col}_{z_s}(v) = k\}.$$

Finally define

$$\mathcal{T}_s(k) = \bigcup_{\ell \in \mathbb{N}} \mathcal{T}_s^{\ell}(k-1).$$

This completes the inductive construction.

Obverse that the construction ensures that the spine of $\mathcal{T}_s(k)$ is $\mathcal{S}^p(k)$. At stage k the trees $\mathcal{T}_s(k)$ contain the following types of vertices:

1. Ray vertices of type assignment 0 or 1, and all ray vertices have global height at most k with respect to z_s;
   All ray vertices are amalgamated (activated) with the root of a copy of (R, r);
2. Target tree vertices amalgamated to centres of extended trees of the form $S = S_{i,j}$ where $2^i(2j+1) = \ell \leq k$, or of the form $\mathcal{T}(\ell - 1)$ for some $\ell \leq k$; these are tree vertices v such that $\widehat{ht}_{z_s}(v) = \widehat{col}_{z_s}(v) = \ell$.
3. Amalgamated tree vertices occurring within copies of trees of the same form S or $\mathcal{T}(\ell-1)$, and themselves amalgamated to a target vertex at its centre; these are tree vertices v such that $\widehat{col}_{z_s}(v) < \widehat{ht}_{z_s}(v) = \ell \leq k$.
4. Not yet amalgamated target vertices, these are tree vertices v such that $\widehat{ht}_{z_s}(v) = \widehat{col}_{z_s}(v) > k$.
5. Not yet amalgamated tree vertices, these are tree vertices v such that $\widehat{ht}_{z_s}(v) > k$.
6. Other vertices of positive labels in copies of (R, r).

Thus $\mathcal{T}_s(k)$ consists of its spine $\mathcal{S}^p(k)$ together with type assignments on its ray vertices. These type assignments can be viewed as a disjoint union grouped according to the craters centered at amalgamated target vertices from all the previous constructions, and this will be useful in discussing and creating embeddings. In the next section we will justify the construction, in particular showing that the number of siblings of each $\mathcal{T}_s(k)$ is countable.

2.4.2 Justification of the inductive construction

At this point the trees $\mathcal{T}_s(k)$ are equipped with (finite trees coding) labels on all vertices in copies of (R, r), (finite trees coding) type assignments on ray vertices, and sign and spin functions used for the inductive construction. Due to the finite trees we have seen that the first two notions are graph properties and are thus preserved by (graph) embeddings. We now show that (graph) embeddings also preserve amalgamated and non-amalgamated vertices, and it will remain to show that the (global) sign and spin functions can also be recovered from the graph structure.

Lemma 2.23 *Let ϕ be a (graph) self-embedding of $\mathcal{T}(k)$ (with centre $z = z_0$).*

1. *Then ϕ preserves amalgamated tree vertices; that is maps amalgamated tree vertices to amalgamated tree vertices, and similarly un-amalgamated tree vertices to un-amalgamated tree vertices.*
2. *If $\widehat{ht}_z(\phi(z)) \leq \ell \leq k$, then $\phi \restriction \mathcal{T}(\ell)$ is a self-embedding of $\mathcal{T}(\ell)$.*

Proof Due to the gadgets encoding the labels, tree vertices are sent to tree vertices. Moreover, an amalgamated tree vertex has been identified with a ray vertex and hence has degree 6 (two neighbours as a tree vertex in a copy of (R, r), one neighbour on the finite path corresponding to the gadget encoding label 0, two neighbours as a ray vertex, and one more on the gadget corresponding to its type). Thus it cannot be sent to an un-amalgamated tree vertex which has only degree 3. Now the centre z is itself amalgamated, thus so is $\phi(z)$, and hence $\widehat{ht}_z(\phi(z)) \leq k$. Thus if $v \in \mathcal{T}(k)$ is an un-amalgamated tree vertex, then $\widehat{ht}_z(v) > k$, and hence $\widehat{ht}_{\phi(z)}(\phi(v)) > k$ implying that $\widehat{ht}_z(\phi(v)) > k$ and hence $\phi(v)$ is un-amalgamated.

For (2), assume that $\widehat{ht}_z(\phi(z)) \leq \ell \leq k$. Then $\widehat{ht}_z(v) \leq \ell$ implies that $\widehat{ht}_{\phi(z)}(\phi(v)) \leq \ell$, and hence $\widehat{ht}_z(\phi(v)) \leq \ell$. Thus if $v \in \mathcal{T}(\ell)$, then by construction v is in a copy (R', r') of (R, r) such that $P_{z,v} \cap R' = P_{r',v}$ and $\widehat{ht}_z(r') \leq \ell$. So $\widehat{ht}_z(\phi(r')) \leq \ell$ and hence $\phi(v) \in (\phi(R'), \phi(r')) \subseteq \mathcal{T}(\ell)$. That is $\phi(\mathcal{T}(\ell)) \subseteq \mathcal{T}(\ell)$. $\square$

To further justify the construction, we also need to show that the number of (graph) siblings of $\mathcal{T}(k)$ (and hence of each $\mathcal{T}_s(k)$) is at most countable, and thus we seek to understand embeddings of $\mathcal{T}(k)$. Recall that the spine of $\mathcal{T}(k)$ is $\mathcal{S}^p(k)$, and a graph embedding of $\mathcal{T}(k)$ induces a surjective embedding of $\mathcal{S}^p(k)$, and as noted in Observation 2.13 preserves ray and tree vertices as well as copies of (R, r). Further equipped with sign and spin functions, we now describe the exact nature of self-embeddings of $\mathcal{S}^p(k)$, called *similarities*. As such, embeddings of $\mathcal{T}(k)$ induce a unique similarity on $\mathcal{S}^p(k)$; we will then show this implies that the sign and spin functions are indeed embedded as graph properties, and this will also allow to control the number of (graph) siblings.

We first introduce the *fingerprint* of a path, which is a code for how the path navigates along rays and copies of (R, r), and definite *similarities* as those maps preserving fingerprints at amalgamated vertices.

Definition 2.24 1. The *fingerprint* of a path $P_{u,v} = \langle u = u_0, u_1, \ldots, u_n = v\rangle$ for $u, v \in \mathcal{S}^p(k)$ is the sequence of symbols $\langle f_0, f_1, \ldots, f_n\rangle$ such that for each $i \leq n$:

- $f_i =$ '$sign_{u_i}^{R'}(u_{i+1})$' if $i < n$, both u_i and u_{i+1} are in a copy (R', r') of (R, r) and u_i is the first element of $R' \cap P_{u,v}$;
- $f_i =$ '$lab^{R'}(u_i)$' if u_i is in a copy (R', r') of (R, r), and either $i = n$ or u_i is not the first element of $R' \cap P_{u,v}$;
- $f_i =$ ' $<$ ' (resp. ' $>$ ') if $i < n$, both u_i and u_{i+1} are ray vertices and $u_i < u_{i+1}$ (resp $u_i > u_{i+1}$) (considered as elements of the double ray $\mathbb{Z}$.

2. A map $\Phi(= \Phi_u = \Phi_{u,\Phi(u)})$ of $\mathcal{S}^p(k)$ (resp. $\mathcal{S}^p$) is called a *similarity* at the amalgamated vertex $u \in \widehat{R}_0$ if the fingerprints of $P_{u,v}$ and $P_{\Phi(u),\Phi(v)}$ are equal for all amalgamated $v \in \mathcal{S}^p(k)$ (resp. $\mathcal{S}^p$).

Lemma 2.25 *Let $u, v \in \widehat{R}_0$ be amalgamated vertices of $\mathcal{T}(k)$. Then there is a unique similarity map $\Phi = \Phi_{u,v}$ of $\mathcal{S}^p(k)$ such that $\Phi(u) = v$, and such a similarity map is a self-embedding of $\mathcal{S}^p(k)$.*

Moreover:

1. $\widehat{spin}_u(w) = \widehat{spin}_v(w) = \widehat{spin}_v(\Phi(w))$ *for all* $w \in \widehat{R}_0^u \cap \widehat{R}_0^v$ *except possibly for* $w \in P_{u,v}$*, and further equals* $\widehat{spin}_u(\Phi(w))$ *except possibly for* $w \in P_{\Phi^{-1}(u),u}$.
2. $\widehat{col}_u(w) = \widehat{col}_v(w) = \widehat{col}_v(\Phi(w))$ *for all* $w \in \widehat{R}_0^u \cap \widehat{R}_0^v$ *except possibly those originating from a path starting from* $P_{u,v}$ *with strictly decreasing labels.*

Proof Let $u, v \in \widehat{R}_0$ be amalgamated vertices and set $\Phi(u) = v$. Thus $\widehat{ht}_z(u) \leq k$ and $\widehat{ht}_z(v) \leq k$, and therefore all tree vertices are amalgamated to global height at most k with respect to either z, u, or v. Hence for any vertex w, there exists a unique way to define its image $\Phi(w)$ so that $P_{u,w}$ and $P_{\Phi(u)=v,\Phi(w)}$ have the same fingerprints; this defines the unique similarity $\Phi_{u,v}$. Note that by definition $\Phi_{u,v}$ preserves the sign function and all graph properties.

Now for $w \in \widehat{R}_0^u$, $\widehat{spin}_u(w) = \widehat{spin}_v(\Phi(w))$ simply due to the paths having the same fingerprints. Further, $\widehat{spin}_u(w) = \widehat{spin}_v(w)$ for all $w \notin P_{u,v}$ by Lemma 2.19 and by the same

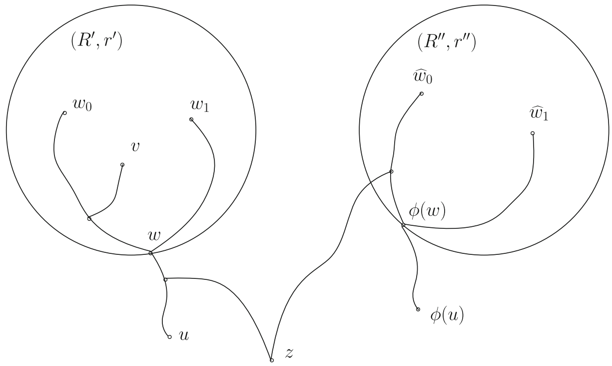


Fig. 17 Lemma 2.26: spin preserving

lemma $\widehat{spin}_u(\Phi(w)) = \widehat{spin}_v(\Phi(w))$ for all $\Phi(w) \notin P_{u,v}$, that is $w \notin P_{\Phi^{-1}(u),\Phi^{-1}(v)} = P_{\Phi^{-1}(u),u}$.

Similarly, by Corollary 2.16, $\widehat{col}_u(w) = \widehat{col}_v(w) = \widehat{col}_v(\Phi(w))$ except possibly those originating from a path starting from $P_{u,v}$ with strictly decreasing labels. □

We now come to the main lemma, showing that the sign function is preserved on amalgamated vertices by graph embeddings of $\mathcal{T}(k)$.

Lemma 2.26 (MAIN lemma) *If $\phi : \mathcal{T}(k) \to \mathcal{T}(k)$ is a (graph) embedding, then $\phi \restriction \mathcal{S}^p(k)$ is a similarity.*
In particular (graph) embeddings of $\mathcal{T}(k)$ preserve the sign function on amalgamated vertices, that is $\widehat{sign}_w(w') = \widehat{sign}_{\Phi(w)}(\Phi(w'))$ for all amalgamated vertices w and all vertices $w' \in \widehat{R}^w$.

Proof We have already observed that (graph) embeddings do preserve labels and the natural direction on double rays. Thus it remains to prove that a graph embedding of $\mathcal{T}(k)$ preserves the sign function on amalgamated vertices.

Let ϕ be a graph embedding of $\mathcal{T}(k)$. We must show, without loss of generality, that if w is the first element in a copy (R', r') of (R, r) on $P_{u,v}$ for some amalgamated vertices u, then $sign$ is preserved at w. Such a w is an amalgamated tree vertex (even if $w = u$), and this implies $\widehat{ht}_z(w) \le k$ and thus $\widehat{ht}_u(w) \le k$. Hence $\phi(w)$ is the first element in a copy (R'', r'') of (R, r) on $P_{\phi(u),\phi(v)}$, $\phi(w)$ is an amalgamated tree vertex (even if $w = u$), and thus $\widehat{ht}_z(\phi(w)) \le k$ as well as $\widehat{ht}_{\phi(u)}(\phi(w)) \le k$. Both w and $\phi(w)$ have two neighbours in R' and R'' respectively. Consider target vertices $w_0, w_1 \in R'$ such that (see Fig. 17):

1. w_0 and w_1 are in different neighbourhoods of w;
2. P_{w,w_0} and P_{w,w_1} have the same label sequence;
3. If $\widehat{w}_0$ and $\widehat{w}_1$ are the vertices in R'' having the same label sequences as w_0 and w_1 from $\phi(w)$, then w_0, w_1, $\widehat{w}_0$ and $\widehat{w}_1$ are not on paths of decreasing labels from $P_{u,z}$, $P_{u,\phi(u)}$, or $P_{z,\phi(u)}$;
4. $\widehat{ht}_u(w_i) = \widehat{col}_u(w_i) = \widehat{ht}_z(w_i) = \widehat{col}_z(w_i) = k$ for each i.

This can be accomplished by choosing w_i along paths formed by ℓ concatenated paths of unimodal labels $\langle 012 \cdots (k-1)kk(k-1) \cdots 10 \rangle$ starting at w in different neighbourhoods of w for some ℓ large enough to satisfy item 2.4.2.

Wlog $spin_w(w_0) = -1$ and hence $spin_w(w_1) = +1$. But $\widehat{spin}_z(w_0) = \widehat{spin}_u(w_0) = spin_w(w_0) = -1$ by Lemma 2.19 since $w_0 \notin P_{u,z}$, and similarly $\widehat{spin}_z(w_1) = \widehat{spin}_u(w_1) = spin_w(w_1) = +1$. This means w_0 is amalgamated to the centre of a copy of the double ray $\mathcal{D}_0$ (from $\mathcal{T}(k-1)$), and w_1 to the centre of a double ray (from some $S = S_{i,j}$) that cannot embed into $\mathcal{D}_0$ preserving their centres, and vice-versa. The corresponding vertices $\widehat{w}_0, \widehat{w}_1 \in R''$ are those starting at $\phi(w)$ with the same label sequence. But $\widehat{spin}_z(\widehat{w}_i) = \widehat{spin}_{\phi(u)}(\widehat{w}_i) = spin_{\phi(w)}(\widehat{w}_i)$ since $\widehat{w}_i \notin P_{z,\phi(u)}$. Thus wlog $\widehat{spin}_z(\widehat{w}_0) = -1$ and hence $\widehat{w}_0$ is amalgamated to the centre of a copy of the double ray $\mathcal{D}_0$, and $\widehat{w}_1$ to the centre of a double ray that cannot embed into $\mathcal{D}_0$ preserving their centres, and vice-versa. Hence there is no alternative but ϕ sending w_0 to $\widehat{w}_0$ and similarly w_1 to $\widehat{w}_1$. But this means that $sign$ is preserved at w as desired. □

We have already observed that any sibling of $\mathcal{T}(k)$ contains (a copy of) $\mathcal{S}^p(k)$, and hence the above result immediately carries to siblings of $\mathcal{T}(k)$.

Corollary 2.27 *If $\mathcal{S}$ and $\mathcal{S}'$ are siblings of $\mathcal{T}(k)$, then any embedding $\phi : \mathcal{S} \to \mathcal{S}'$ induces a similarity on $\mathcal{S}^p(k)$.*

Proof Let $\mathcal{S}$ and $\mathcal{S}'$ be siblings of $\mathcal{T}(k)$, and $\phi : \mathcal{S} \to \mathcal{S}'$ an embedding. Now since $\mathcal{S}$ is a sibling, let $\psi : \mathcal{T}(k) \to \mathcal{S}$ be an embedding, and we may consider $\mathcal{S}'$ as a substructure of $\mathcal{T}(k)$. Hence $\phi \circ \psi : \mathcal{T}(k) \to \mathcal{T}(k)$ is an embedding whose restriction to $\mathcal{S}^p(k)$ is a similarity by Lemma 2.26. But $\psi \restriction \mathcal{S}^p(k)$ is itself a (surjective) similarity, and hence so is ψ^{-1}. Thus $\phi \restriction \mathcal{S}^p(k) = \phi \circ \psi \circ \psi^{-1} \restriction \mathcal{S}^p(k)$ is a similarity. □

Thus any graph embedding of $\mathcal{T}(k)$ (or of any sibling) is a similarity on $\mathcal{S}^p(k)$, and conversely we will see later how to use particular similarities of $\mathcal{S}^p(k)$ to create embeddings of $\mathcal{T}(k)$, meaning how to correctly match the type assignments of ray vertices. As a corollary to Lemma 2.26 we will need the corresponding property for all siblings of $\mathcal{T}(k)$. We first show that the type assignments on double rays can only disagree with the image of an embedding of $\mathcal{T}(k)$ for finitely many ray vertices. That is, only finitely many ray vertices of type 0 are mapped to type 1 ray vertices, or if we recall that type assignments are implemented though finite trees attached to those ray vertices, we show that $\mathcal{T}(k) \setminus \phi(\mathcal{T}(k))$ is finite for any embedding ϕ of $\mathcal{T}(k)$.

There are obvious proper self-embeddings of $\mathcal{T}(k)$ (and each $\mathcal{T}_s(k)$), namely any translation along the double ray $\mathcal{D}_0$, so that all type 1 ray vertices are mapped into type 1 ray vertices. Indeed by construction, all trees attached to ray vertices on $\mathcal{D}_0$ are identical, and thus can be mapped (isomorphically) to the corresponding tree by translation, and hence $\mathcal{T}(k)\setminus\phi(\mathcal{T}(k))$ is finite for such embeddings due to finitely many type 0 ray vertices mapped to type 1 ray vertices. Thus $\mathcal{T}(k)$ is almost equal to its image by a translation embedding. We show that this situation occurs for all embeddings, that is $\mathcal{T}(k)\setminus\phi(\mathcal{T}(k))$ is finite for all embeddings. It is worth observing that this property propagates to all siblings.

Lemma 2.28 *$\mathcal{T}_s(k) \setminus \phi(\mathcal{T}_s(k))$ is finite for any self-embedding ϕ and $s < \mathfrak{s}$. The only possible difference is in a finite number of ray vertices of different type assignments.*

Proof First observe that in general if T is a tree and $T\setminus\phi(T)$ is finite for any self-embedding ϕ of T, then $S\setminus\psi(S')$ is finite for any siblings S and S' of T and embedding $\psi : S' \to S$. In particular since each $\mathcal{T}_s(k)$ is a sibling of $\mathcal{T}(k)$, it suffices to prove the lemma for the latter, and the proof will consist of an analysis of the effect of an embedding on $\mathcal{T}(k)$.

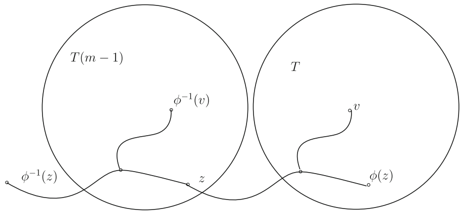


Fig. 18 Lemma 2.28: the first case

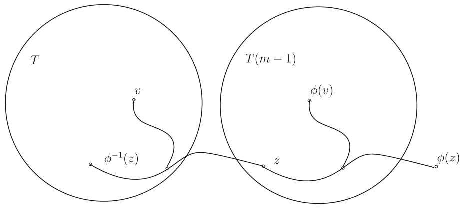


Fig. 19 Lemma 2.28: the second case

We proceed by induction on k. It is easily verified for $k = 0$ since a proper embedding ϕ of $\mathcal{T}(0)$ consists of a (proper) translation of $\mathcal{D}_0$ (in its natural direction). Copies of $(R, r,)$ are moved onto copies of (R, r) and hence $\mathcal{T}(0)\backslash\phi(\mathcal{T}(0))$ consists of finitely many gadget vertices from finitely many ray vertices of type 0 assignment being mapped to type 1 assignments.

We now assume the statement is true for all $\ell < k$, and we consider a self-embedding ϕ of $\mathcal{T}(k)$. By Lemma 2.26, ϕ induces a surjective similarity on $\mathcal{S}^p(k)$, and hence preserves labels, copies of (R, r), amalgamated vertices, ray vertices, the natural direction of double rays, and the sign function on amalgamated vertices. Thus any vertex in $\mathcal{T}(k) \setminus \phi(\mathcal{T}(k))$ must come from ray vertices of type 0 assignment being mapped to type 1 assignments. Recall that these type assignments are set by the amalgamations during the construction, and can be viewed as a disjoint union grouped according to the craters centered at amalgamated target vertices. It thus suffices to show that only finitely many such craters may differ from their image, and that the difference is finite in all those cases where they differ.

We consider two special cases based on $m = \widehat{ht}_z(\phi(z))$, where $z = z_0$ is the center of $\mathcal{T}(k)$, followed by the general case. Note that $m \leq k$ since $\phi(z)$ is amalgamated.

For the first case, $\phi(z)$ is part of a tree T (a copy of $\mathcal{T}(m-1)$ or some extended $S = S_{i,j}$) that was amalgamated with its centre to a target vertex v during the construction of $\mathcal{T}(m)$ (see Fig. 18). Thus $\widehat{ht}_v(\phi(z)) < m$ and $\widehat{ht}_z(v) = \widehat{col}_z(v) = m$. Hence $\widehat{ht}_{\phi(z)}(v) < m$, and the preimage satisfies $\widehat{ht}_z(\phi^{-1}(v)) < m$. But this means that ϕ induces an embedding of $\mathcal{T}(m-1)$ into T, and the induction hypothesis ensures that $T \setminus \phi(\mathcal{T}(m-1))$ is finite.

For the second and similar case, $\phi^{-1}(z)$ is part of a tree T (a copy of $\mathcal{T}(m-1)$ or some extended $S = S_{i,j}$) that was amalgamated with its centre to a target vertex v during the construction of $\mathcal{T}(m)$ (see Fig. 19). Thus $\widehat{ht}_z(v) = \widehat{col}_z(v) = m$, and $\widehat{ht}_v(\phi^{-1}(z)) < m$. Hence $\widehat{ht}_{\phi^{-1}(z)}(v) < m$, and the image satisfies $\widehat{ht}_z(\phi(v)) < m$. But this means that ϕ induces an embedding of T into $\mathcal{T}(m-1)$, and the induction hypothesis ensures that $\mathcal{T}(m-1) \setminus \phi(T)$ is finite.

For the remaining general case, consider an amalgamated target vertex u such that $\widehat{ht}_z(u) = \widehat{col}_z(u) = \ell \geq m$ as well as $\widehat{ht}_{\phi^{-1}(z)}(u) \geq m$ (see Fig. 20). Being amalgamated, we must also have $\ell \leq k$. Then u was activated during the construction of $\mathcal{T}(\ell)$, and was amalgamated with the centre of a tree T (a copy of $\mathcal{T}(\ell-1)$ or some extended $S = S_{i,j}$). Note that as a result

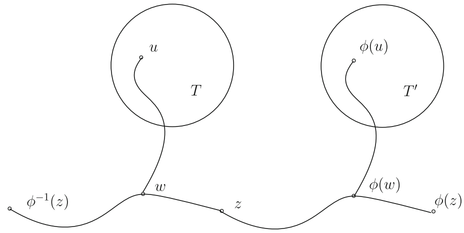


Fig. 20 Lemma 2.28: the general case

all type assignments in the ℓ-crater $C(u)$ are exactly those resulting from this amalgamation of T.

Now $\phi(u)$ is amalgamated and we claim that it is in fact a target vertex such that $\widehat{ht}_z(\phi(u)) = \widehat{col}_z(\phi(u)) = \ell$, this can be shown as follows. Let w be such that $P_{z,u} \cap P_{\phi^{-1}(z),u} = P_{w,u}$. If the last consecutive pair on $P_{z,u}$ (of label ℓ) is on $P_{w,u}$, then it is also on $P_{\phi(w),\phi(u)}$ and we are done. Otherwise, the last consecutive pair must be on $P_{z,w}$ and $P_{w,u}$ consists of decreasing labels strictly less than ℓ. Thus $\ell = m$ since $m = \widehat{ht}_z(\phi^{-1}(z))$. But now $P_{\phi^{-1}(z),w}$ must contain a consecutive pair of labels $\ell = m$ since $\widehat{ht}_{\phi^{-1}(z)}(u) \geq m$, and this shows that $\widehat{ht}_z(\phi(u)) = \widehat{col}_z(\phi(u)) = \ell = m$ in this case as well.

This means that $\phi(u)$ was also activated during the construction of $\mathcal{T}(\ell)$, and was amalgamated with the centre of a tree T' (a copy of $\mathcal{T}(\ell - 1)$ or some extended $S = S_{i,j}$). Now, if $\widehat{spin}_z(u) = \widehat{spin}_z(\phi(u))$, then the same tree $T = T'$ is amalgamated to u and $\phi(u)$ at its centre, and ϕ induces an embedding of that tree, sending the ℓ-crater at u to the ℓ-crater at $\phi(u)$; by uniqueness of the similarity (by Lemma 2.25), ϕ induces an isomorphism of T to $T = T'$ (essentially the identity) and $T' \backslash \phi(T) = \emptyset$. If instead $\widehat{spin}_z(u) \neq \widehat{spin}_z(\phi(u))$, then the tree T' amalgamated at $\phi(u)$ is different than T, but $T' \setminus \phi(T)$ is finite by the induction hypothesis. Note that this latter case can occur only finitely many times by Lemma 2.25.

This completes the proof. □

We can now verify two requirements of the construction. First we show that each non-isomorphic sibling of each $\mathcal{T}(k)$ contains a double ray non-isomorpic to $\mathcal{D}_0$, that is having a type 1 ray vertex followed by a vertex of type 0.

Corollary 2.29 *Every non-isomorphic sibling of each $\mathcal{T}(k)$ contains a double ray non-isomorpic to $\mathcal{D}_0$, that is having a type* 1 *ray vertex followed by a vertex of type* 0.

Proof This is certainly true for $\mathcal{T}(0)$ and the tree $\mathcal{T}(1)$ already contains infinitely many such double rays (from embeddings of $S_{0,0}$). Finally, by Lemma 2.28, any sibling of $\mathcal{T}(k)$ contains all but finitely many of those rays. □

We can also show that each $\mathcal{T}_s(k)$ has countably many siblings.

Corollary 2.30 $sib(\mathcal{T}_s(k)) = \aleph_0$ *for all* k.

Proof By construction each $\mathcal{T}_s(k)$ is countable (this also follows from simply being locally finite trees). Now, by Lemmas 2.25 and 2.26, for any amalgamated vertex v, all self-embedding ϕ such that $\phi(z) = v$ agree on all vertices on its spine $\mathcal{S}^p(k)$ of global height at most k. Now by Lemma 2.28, $\mathcal{T}(k) \backslash \phi(\mathcal{T}(k))$ is finite. So there are only countably many amalgamated vertices, and for each there can be only finitely many siblings $\phi(\mathcal{T}(k)) \subseteq S \subseteq \mathcal{T}(k)$. Hence $sib(\mathcal{T}(k)) = sib(\mathcal{T}_s(k)) \leq \aleph_0$ for all k.

We already noticed that each $\mathcal{T}(k)$ has infinitely many siblings due to translations along $\mathcal{D}_s$, hence $sib(\mathcal{T}_s(k)) = \aleph_0$ exactly. □

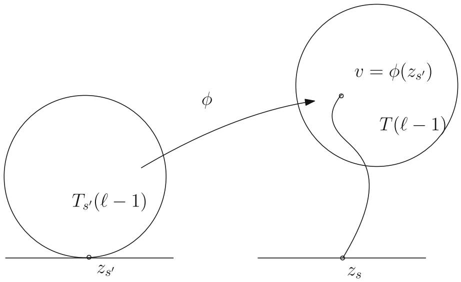


Fig. 21 Lemma 2.31: the case that $\phi : \mathcal{T}_{s'}(k) \to \mathcal{T}_s(k)$ is an isomorphism

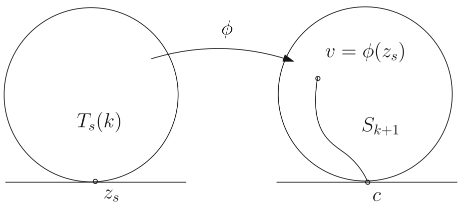


Fig. 22 Lemma 2.31: the case that $\phi : \mathcal{T}_s(k) \to S_{k+1}$ is an isomorphism

We show that all $\mathcal{T}_s(k)$ are pairwise non-isomorphic siblings at every stage, and in fact we prove a bit more so to support the induction argument.

Lemma 2.31 *For each $s \neq s' < \mathfrak{s}$ and $k \in \mathbb{N}$, $\mathcal{T}_s(k) \ncong \mathcal{T}_{s'}(k)$.*
Moreover, if $k+1 = 2^i(2j+1)$, and $S_{i,j}$ as a substructure of $\mathcal{T}_s(i)$ and $\mathcal{T}_s(k)$ was extended to S_{k+1} following the inductive construction so that all tree vertices in S_{k+1} are amalgamated up to global height k with respect to its centre c, then $S_{k+1} \ncong \mathcal{T}_s(k)$.

Proof We prove both statements simultaneously by induction on k. We have already observed that the trees $\mathcal{T}_s(0)$ are pairwise non-isomorphic (since the rays $\mathcal{D}_s$ are not isomorphic). Moreover since $1 = 2^0(2 \cdot 0 + 1)$, $S_1 = S_{0,0}$ and as a substructure of $\mathcal{T}_s(0)$ has all its tree vertices already amalgamated up to global height 0 with respect to its center. By definition $S_{0,0}$ is not isomorphic to $\mathcal{T}_s(0)$ so the base case follows.

Now let $k > 0$ be the smallest counterexample, and suppose first that $\phi : \mathcal{T}_{s'}(k) \to \mathcal{T}_s(k)$ is an isomorphism with $s' \neq s$ (see Fig. 21). Let $v = \phi(z_{s'})$, then v must be amalgamated (since $z_{s'}$ is). Let $\ell = \widehat{ht}_{z_s}(v) \leq k$. If $\ell = 0$, then this means that $\phi(z_{s'}) \in \mathcal{D}_s$, but this is impossible by construction since the double rays $\mathcal{D}_{s'}$ and $\mathcal{D}_s$ are not isomorphic. Thus $\ell > 0$ and write $\ell = 2^i(2j + 1)$. According to the construction, v is either contained in a copy of $\mathcal{T}(\ell - 1)$ (= $\mathcal{T}_0(\ell - 1)$ no matter s by construction), or of an extended sibling S_ℓ of $S_{i,j}$ that was inserted in the tree by amalgamation identifying its centre to a target vertex. But then either $\phi \restriction \mathcal{T}_{s'}(\ell - 1) : \mathcal{T}_{s'}(\ell - 1) \to \mathcal{T}(\ell - 1)$ or $\phi \restriction \mathcal{T}_{s'}(\ell - 1) : \mathcal{T}_{s'}(\ell - 1) \to S_\ell$ is an isomorphism, a contradiction in either case to the induction hypothesis.

Otherwise assume that $\phi : \mathcal{T}_s(k) \to S_{k+1}$ is an isomorphism, where $k + 1 = 2^i(2j + 1)$, and S_{k+1} was extended from $S_{i,j}$ so that all tree vertices in S_{k+1} are amalgamated up to global height k with respect to its centre c (see Fig. 22). Let $v = \phi(z_s)$, and $\ell = \widehat{ht}_c(v)$. If $\ell \leq i$, then v was already amalgamated in $S_{i,j}$ and $\phi \restriction \mathcal{T}_s(i) : \mathcal{T}_s(i) \to S_{i,j}$ is an isomorphism by Lemma 2.23, a contradiction as $S_{i,j}$ was specifically chosen non-isomorphic to $\mathcal{T}_s(i)$ (for any s). If $\ell > i$, then v was amalgamated following the inductive construction by assumption, and hence is part of an amalgamation of some $\mathcal{T}(m)$ (= $\mathcal{T}_0(m)$) or a S_{m+1} for some $i \leq m < \ell$. But then either $\phi \restriction \mathcal{T}_s(m) : \mathcal{T}_s(m) \to \mathcal{T}(m)$ or $\phi \restriction \mathcal{T}_s(m) : \mathcal{T}_s(m) \to S_{m+1}$ is an isomorphism by Lemma 2.23, again a contradiction in either case.

□

2.5 The trees $\langle \mathcal{T}_s : s < \mathfrak{s} \rangle$

We are now ready to define $\langle \mathcal{T}_s : s < \mathfrak{s} \rangle$.

Definition 2.32 Define $\mathcal{T}_s = \bigcup_k \mathcal{T}_s(k)$ for each $s < \mathfrak{s}$.

Recall the spine $\mathcal{S}^p$ defined in Definition 2.12. Since $\mathcal{S}^p(k)$ is the spine of each $\mathcal{T}_s(k)$, then clearly $\mathcal{S}^p$ (up to isomorphism) is the spine of each $\mathcal{T}_s$. Moreover we have the following result arising from known properties of $\mathcal{T}(k)$.

Lemma 2.33 *Let ϕ be a self-embedding of $\mathcal{T}$ into $\mathcal{T}$ (with centre $z = z_0$), then:*

1. *ϕ induces a similarity on $\mathcal{S}^p$.*
2. *If $\widehat{ht}_z(\phi(z)) \leq k$, $\phi \restriction \mathcal{T}(k)$ is a self-embedding of $\mathcal{T}(k)$.*
3. *$\mathcal{T} \setminus \phi(\mathcal{T})$ is finite. The only possible difference is in a finite number of ray vertices of different type assignments.*

Proof Items 1 and 2 follow as in Lemma 2.23 and Corollary 2.27.

To prove item 3, let $k = \widehat{ht}_z(\phi(z))$, and thus $\phi \restriction \mathcal{T}(k)$ is a self-embedding of $\mathcal{T}(k)$ by part 2 and $\mathcal{T}_s(k) \setminus \phi(\mathcal{T}_s(k))$ is finite by Lemma 2.28. Now for a target vertex $v \in \mathcal{T} \setminus \mathcal{T}(k)$, $\widehat{spin}_z(v) = \widehat{spin}_z(\phi(v))$ by Lemma 2.25, and thus the same trees are amalgamated at v and $\phi(v)$ in the construction, and ϕ restricted to that tree must be an isomorphism by uniqueness again by Lemma 2.25. Hence ϕ is an isomorphism on $\mathcal{T} \setminus \mathcal{T}(k)$. The result follows. □

Note that in the last case a type 0 ray vertex being mapped to a type 1 means that a single leaf is not in the image, this will be useful later.

We first confirm that we have at least $\mathfrak{s}$ siblings.

Proposition 2.34 *$\mathcal{T}_s \ncong \mathcal{T}_{s'}$ for any $s \neq s' < \mathfrak{s}$.*

Proof Suppose that $\phi : \mathcal{T}_{s'} \to \mathcal{T}_s$ is an isomorphism, and let $k = \widehat{ht}_{z_s}(\phi(z_{s'}))$. Then $\phi \restriction \mathcal{T}_{s'}(k) : \mathcal{T}_{s'}(k) \to \mathcal{T}_s(k)$ is an isomoprhism by Lemma 2.33(2), contradicting Lemma 2.31. □

And finally we are ready to conclude that $\mathcal{T}$ has exactly $\mathfrak{s}$ siblings.

Proposition 2.35 *If $\mathcal{S} \approx \mathcal{T}$, then $\mathcal{S} \cong \mathcal{T}_s$ for some $s < \mathfrak{s}$.*

Proof Let $\mathcal{S}$ be a sibling of $\mathcal{T}$. Being all siblings, we may assume that $\mathcal{T} = \mathcal{T}_0 \supseteq \mathcal{T}_s \supseteq \mathcal{S} \supseteq \theta(\mathcal{T})$ for all $s < \mathfrak{s}$ and some self-embedding $\theta : \mathcal{T} \to \mathcal{T}$. By Lemma 2.33, we can find n such that:

1. $z_s \in \mathcal{T}(n)$ for each $s < \mathfrak{s}$,
2. $\theta(z_0) \in \mathcal{T}(n)$,
3. $\mathcal{T} \setminus \theta(\mathcal{T}) \subseteq \mathcal{T}(n)$.

Define $\mathcal{S}(n) = \mathcal{S} \cap \mathcal{T}(n)$. So $\theta(\mathcal{T}(n)) \subseteq \mathcal{S}(n) \subseteq \mathcal{T}(n)$, and hence $\mathcal{S}(n)$ is a sibling of $\mathcal{T}(n)$. Thus by construction either $\mathcal{S}(n) \cong \mathcal{T}_s(n)$ for some $s < \mathfrak{s}$, or else $\mathcal{S}(n) \cong S := S_{n,j}$ for some j.

First assume the latter, and we shall show that $\mathcal{S} \cong \mathcal{T}$. We do so by extending an isomorphism from S to $\mathcal{S}(n)$ mapping craters to craters attached to target vertices of the same spin, and hence have the same type assignments on all ray vertices of those craters, producing the required isomorphism from $\mathcal{T}$ to $\mathcal{S}$ (see Fig. 23). Let c be the centre of S and $k = 2^n(2j+1)$.

Now call u the tree vertex of $\mathcal{T}(k)$ at the end of the path $P_{z,u}$ starting at $z = z_0$ with unimodal labels $\langle 01 \cdots kk \cdots 10\rangle$ increasing to k and back to 0, and such that $sign_z(u) = -1$ and thus $spin_z(u) = +1$. Note that u and z are in the same copy (R', r') of (R, r), and u is a target vertex of (global) height k (with respect to z). Thus at stage k of the construction of $\mathcal{T}(k)$, S was extended (following the inductive construction) to S' so that all tree vertices in S' are amalgamated up to global height $k - 1$ (with respect to c) before (S', c) was amalgamated with $(\mathcal{T}(k-1), u)$ over (R', r'). In particular we consider S' (and S) as a substructure of $\mathcal{T}(k)$ with c identified with u.

Fix an isomorphism $\phi : S \to \mathcal{S}(n)$, and let $\phi(u) = v \in \mathcal{S}(n)$. By Corollary 2.27, ϕ induces a similarity map $\hat{\Phi} = \phi \restriction \hat{\mathcal{S}^p}(n)$ on a copy $\hat{\mathcal{S}^p}(n)$ of $\mathcal{S}^p(n)$ as a substructure of $S \subseteq \mathcal{T}$, and by Lemma 2.25 it is the unique such similarity sending u to v. Now from the global point of view of $\mathcal{T}$, there is also a similarity map $\Phi = \Phi_{u,v}$ on $\mathcal{S}^p$ (as a substructure of $\mathcal{T}$), and it turns out that $\hat{\Phi}$ and Φ agree on $\hat{\mathcal{S}^p}(n)$. The reason is that, by definition, both maps preserve the fingerprint of a path $P_{u,u'}$ for $u' \in S$. Those fingerprints may be different from the point of view of S and $\mathcal{T}$: of course the labelling and ray orderings agree, but the sign values are computed on one hand from the point of view of S (and its centre before being embedded into $\mathcal{T}$), and on the other hand from the point of view of $\mathcal{T}$ and its centre z. Yet the fact that $\hat{\Phi}$ and Φ preserve the corresponding fingerprints implies that both will agree on their image of u'. Hence we must find the required embedding from $\mathcal{T}$ to $\mathcal{S}$ by extending $\hat{\Phi}$ to Φ and from there to the type assignments at all ray vertices outside S: we will show that Φ preserves the spin at target vertices outside S, and hence the same trees are amalgamated in the craters of target vertices v and $\Phi(v)$ allowing to extend the embedding outside S as desired. To do so, we will go through z by considering $\Phi_{u,v} = \Phi_{z,v} \circ \Phi_{u,z}$.

First consider $\Phi_{u,z}$, and let w be a target vertex of global height $\ell > n$ with respect to u. Since $\Phi_{u,z}$ preserves labels, its image is also a target vertex of global height $\ell > n$ with respect to z. Now by Lemma 2.25, $\widehat{spin}_u(w) = \widehat{spin}_z(w) = \widehat{spin}_z(\Phi_{u,z}(w))$ except possibly for $w \in P_{u,z}$. This means that for all target vertices w of global height $\ell > n$ with respect to u and not on $P_{u,z}$, $\Phi_{u,z}(w)$ is also a target vertex with respect to z of the same spin, and thus the same tree is amalgamated at w and $\Phi_{u,z}(w)$. Hence $\Phi_{u,z}$ can be extended to an isomorphism $\phi^w_{u,z}$ (matching the type assignments at ray vertices) on the ℓ-crater centered at w to the ℓ-crater centred at $\Phi_{u,z}(w)$. Note this includes all target vertices in $S' \setminus S$ which were amalgamated before S' was itself amalgamated to $\mathcal{T}(k)$. Now for $w = z$, which will come up when $\ell = k$, it is the only remaining target tree vertex on $P_{u,z}$ to handle since $\Phi(u) = v$ has already been taken care of by ϕ. By Definition 2.8, $sign_u(z) = spin_z(u) = +1$ implying through similarity that $sign_z(\Phi_{u,z}(z)) = +1$, and thus $\widehat{spin}_z(\Phi_{u,z}(z)) = -1$. Hence a copy of $\mathcal{T}(k-1)$ is amalgamated at z and $\Phi_{u,z}(z)$, and $\Phi_{u,z}$ can again be extended to an isomorphism $\phi^z_{u,z}$ on the k-crater centered at $w = z$ (contained in $(\mathcal{T}(k-1), z)$) to the k-crater centered at $\Phi_{u,z}(w)$ (contained in a copy of $(\mathcal{T}(k-1), z)$. Note this is a crucial part to ensure that $\mathcal{S} \cong \mathcal{T} = \mathcal{T}_0$, and not any other $\mathcal{T}_s$.

Next consider the similarity map $\Phi_{z,v}$ on $\mathcal{S}^p$, and let w' be a target vertex of global height $\ell > n$ with respect to z. Then, by similarity, its image is also a target vertex of global height $\ell > n$ with respect to v, and and since $\widehat{ht}_z(v) \leq n$ also with respect to z. Again, by Lemma 2.25, $\widehat{spin}_z(w') = \widehat{spin}_z(\Phi_{z,v}(w'))$ except possibly for $w' \in P_{z,v}$, which in this case cannot happen since $\widehat{ht}_z(v) \leq n$. This means that for all target vertices w' of global height $\ell > n$ with respect to z, then $\Phi_{z,v}(w')$ is again a target vertex of the same spin with respect to z, and thus the same tree is amalgamated at w' and $\Phi_{z,v}(w')$. Hence $\Phi_{z,v}$ can be extended to an isomorphism $\phi^{w'}_{z,v}$ (matching the type assignments at ray vertices) on the ℓ-crater centered at w' to the ℓ-crater centered at $\Phi_{z,v}(w')$.

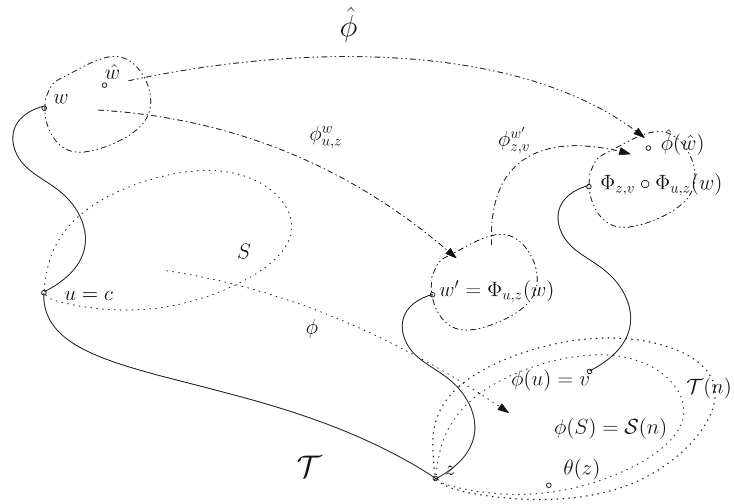


Fig. 23 Proposition 2.35: the case $\mathcal{S}(n) \cong T = S_{n,j}$

Combining the maps, the required isomorphism $\hat{\phi} : \mathcal{T} \to \mathcal{S}$ can be summarized as follows. For $\hat{w} \in \mathcal{T}$:

$$\begin{array}{lll} \hat{\phi}(\hat{w}) & = \phi(\hat{w}) & \text{if } \widehat{ht}_u(\hat{w}) \le n, \\ & = \phi^{w'}_{z,v} \circ \phi^{w}_{u,z}(\hat{w}) & \text{if } \ell = \widehat{ht}_u(\hat{w}) > n, \\ & & \hat{w} \text{ belongs to the } \ell\text{-crater of the target vertex } w, \\ & & \text{and } w' = \Phi_{u,z}(w). \end{array}$$

Note that there are vertices in the spine of $\mathcal{S}$ of height larger than n, and by uniqueness of the similarities their image could have been defined using either case; for simplicity these are now handled in the second case.

The case that $\mathcal{S}(n) \cong \mathcal{T}_s(n)$ for some $s < \mathfrak{s}$ is similar but relatively simpler; here we show that $\mathcal{T}_s \cong \mathcal{S}$ (see Fig. 24). Fix an isomorphism $\phi : \mathcal{T}_s(n) \to \mathcal{S}(n)$, and let $v = \phi(z_s) \in \mathcal{S}(n) \subseteq \mathcal{T}(n)$. By Corollary 2.27, ϕ induces a similarity map $\hat{\Phi} = \phi{\restriction}\mathcal{S}^p(n)$. By Lemma 2.25, it is the unique similarity from $\mathcal{S}^p(n)$ sending z_s to v, and must therefore readily agree with the similarity $\Phi = \Phi_{z_s,v}$ on $\mathcal{S}^p$. We will show that Φ preserves the spin at target vertices of all craters outside $\mathcal{T}_s(n)$. Note here that the spin in $\mathcal{T}_s$ is determined by its centre z_s, and the spin in $\mathcal{S} \subseteq \mathcal{T}$ is determined by the centre z_0.

Thus let $w \in \mathcal{T}_s$ be a target vertex of global height $\ell > n$ with respect to z_s. Since $\Phi_{z_s,v}$ preserves labels, its image is also a target vertex of global height $\ell > n$ with respect to v, and hence a target vertex of global height $\ell > n$ with respect to z_0 since $\widehat{ht}_{z_0}(v) \le n$. Again from Φ being a similarity, $\widehat{spin}_{z_s}(w) = \widehat{spin}_v(\Phi(w))$ for all $w \in \widehat{R}_0^{z_s} \cap \widehat{R}_0^{v}$. Moreover, by Lemma 2.19, $\widehat{spin}_{z_0}(\Phi(w)) = \widehat{spin}_v(\Phi(w))$ as long as $\Phi(w) \notin P_{z_0,v}$, which in this case cannot happen since $\widehat{ht}_v(\Phi(w)) = \ell > n$. This means that the same tree is amalgamated at w and $\Phi(w)$ and hence Φ can be extended to an isomorphism ϕ^w (matching the type assignments at ray vertices) on the ℓ-crater centered at w in $\mathcal{T}_s$ to the ℓ-crater centered at $\Phi(w)$ in $\mathcal{T}$, but since $\mathcal{S} \backslash \mathcal{S}(n) = \mathcal{T} \backslash \mathcal{S}(n)$ this is the same as ℓ-crater centered at $\Phi(w)$ in $\mathcal{S}$.

Combining the maps, the required isomorphism $\hat{\phi} : \mathcal{T}_s \to \mathcal{S}$ can be summarized as follows. For $\hat{w} \in \mathcal{T}_s$:

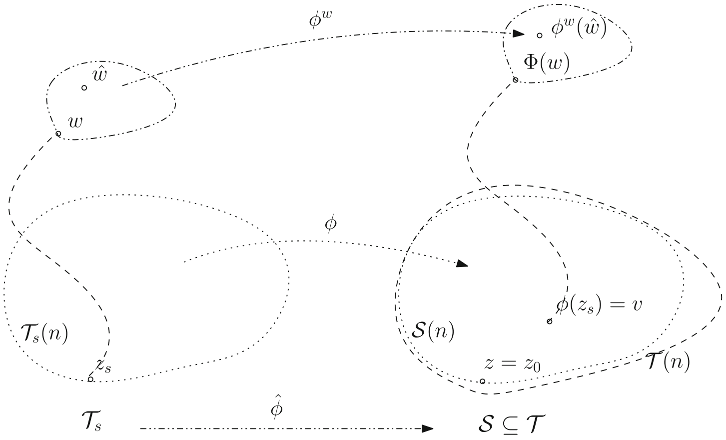


Fig. 24 Proposition 2.35: the case $\mathcal{S}(n) \cong \mathcal{T}_s(n)$

$$\begin{aligned} \hat{\phi}(\hat{w}) &= \phi(\hat{w}) && \text{if } \widehat{ht}_{z_s}(\hat{w}) \le n; \\ &= \phi^w(\hat{w}) && \text{if } \ell = \widehat{ht}_{z_s}(\hat{w}) > n, \text{ and} \\ & && \hat{w} \text{ belongs to the } \ell\text{-crater of the target vertex } w. \end{aligned}$$

This completes the proof. □

We finally have all the ingredients to prove our first main theorem.

Theorem 1. For each non-zero $\mathfrak{s} \in \mathbb{N}$, there is a locally finite tree $\mathcal{T}$ with exactly $\mathfrak{s}$ siblings, considered either as relational structures or trees. Moreover, for $\mathfrak{s} = 1$, the tree is not a ray, yet it has a non-surjective embedding.

Thus the conjectures of Bonato–Tardif, Thomassé, and Tyomkyn regarding the sibling number of trees and relational structures are all false.

Proof By Propositions 2.34 and 2.35, $\mathcal{T}$ is a locally finite tree with $sib(\mathcal{T}) = \mathfrak{s}$, hence disproving the Bonato–Tardif conjecture in the case $\mathfrak{s} \ge 2$, and hence also Tyomkyn's first conjecture.

By Lemma 2.33, any sibling of $\mathcal{T} = \mathcal{T}_0$ viewed as a substructure of $\mathcal{T}$ differs from $\mathcal{T}$ by a finite set of ray vertices of different type assignments, meaning a single leaf has been removed from finitely many type 1 vertices to become type 0 vertices. But this means that $\mathcal{T} \oplus 1$ does not embed in $\mathcal{T}$, and hence any sibling of $\mathcal{T}$, viewed as a binary relational structure, is connected and thus a tree. In this case Thomassé's conjecture is equivalent to Bonato–Tardif's conjecture, and thus also false.

Finally consider the special case $\mathfrak{s} = 1$ so that $sib(\mathcal{T}) = 1$. The embedding ϕ of $\mathcal{D}_0$ given by translation $\phi(v_i) = v_{i+1}$, and its natural extension to $\mathcal{T}$, is a proper embedding (v_1 of type 1 has become type 0), and $\mathcal{T}$ is certainly not a ray. Thus in this case $\mathcal{T}$ disproves Tyomkyn's second conjecture. □

3 Siblings of partial orders

The above construction of locally finite trees with prescribed finite number of siblings can be adapted to provide a similar construction of partial orders with a prescribed finite number of siblings, hence our second main theorem.

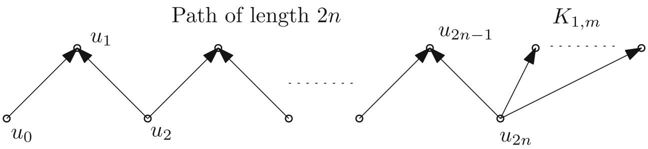


Fig. 25 The gadget $PK(2n, m)$ as a partial order

Theorem 2. For each non-zero $\mathfrak{s} \in \mathbb{N}$, there is a partial order $\mathcal{P}$ with exactly $\mathfrak{s}$ siblings (up to isomorphy).

We briefly outline the main ingredients for the proof. This will be done by following the construction using the modified double rays $\mathcal{D}'$ as described earlier (see Fig. 9), to obtain locally finite trees $\mathcal{T}'$ with a prescribed finite number of siblings. Then a partial order is defined on $\mathcal{T}'$ to create a partial order $\mathcal{P}$ so to have the same monoid of embeddings (either as a tree or as a partial order), and hence the result follows.

3.1 Partial ordering on the gadgets

In the construction of $\mathcal{T}$ (and $\mathcal{T}'$), we have used various gadgets of the form $PK(2n, m)$, and we define a partial order on $PK(2n, m)$ in the form of a fence as follows.

Definition 3.1 On $PK(2n, m)$, the finite gadget formed by connecting $u \in K_{1,m} = (u, V)$ to the end vertex u_{2n} of a path $\langle u_0, u_1, \ldots, u_{2n}\rangle$ of length $2n$, define a partial order in the form of a fence as follows (see Fig. 25):

1. $u_{2i} < u_{2i+1}$ for $0 \leq i < n$,
2. $u_{2i} < u_{2i-1}$ for $0 < i \leq n$,
3. $u_{2n} < v$ for all $v \in V$.

We record the following immediate observation.

Observation 3.2 *$PK(2n, m)$ embeds into $PK(2n', m')$ as rooted trees if and only if $n = n'$ and $m \leq m'$.*
Moreover any graph embedding of such a gadget into another one as a rooted tree is an order embedding, and vice-versa.

3.2 Partial ordering on (R, r)

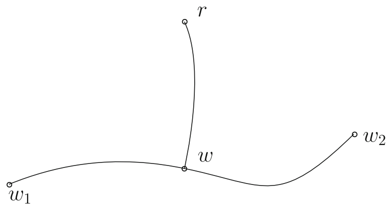
To define a partial ordering on (R, r), we will need the following property.

Lemma 3.3 *Given a path P_{w_1,w_2} in (R, r) between tree vertices w_1, w_2 with unimodal labels $\langle 01 \cdots kk \cdots 10\rangle$ increasing to $k > 0$ and back to 0, then $sign_{w_1}(w_2) = -sign_{w_2}(w_1)$.*

Proof Let w so that $P_{r,w_1} \cap P_{r,w_2} = P_{r,w}$, and we may assume without loss of generality (by interchanging w_1 andw_2 if necessary) that $w \neq w_1$ (see Fig. 26).

First suppose that $w = w_2$. If further $r = w$, then r and w_2 are clearly in the same neighbourhood of w_1, so by Definition 2.8 and definition of $spin$ we have $sign_{w_1}(w_2) = spin_r(w_1) = -sign_{w_2}(w_1)$. Otherwise if $r \neq w$, then w_1 and r belong to opposite neighbourhoods of w_2, and hence $sign_{w_2}(w_1) = -spin_r(w_2)$ according to Definition 2.8. Further in this case, w_2 and r belong to the same neighbourhood of w_1, and hence

Fig. 26 Lemma 3.3: $P_{r,w_1} \cap P_{r,w_2} = P_{r,w}$



$sign_{w_1}(w_2) = spin_r(w_1)$ again according to Definition 2.8. But $spin_r(w_1) = spin_r(w_2)$ since the path P_{w_1,w_2} contains a single consecutive pair and one more tree vertex beyond w_2. Hence $sign_{w_1}(w_2) = -sign_{w_2}(w_1)$ again in this case.

Now assume that $w \neq w_2, w_1$. Then w_1 and r belong to the same neighbourhoods of w_2, and similarly w_2 and r belong to the same neighbourhoods of w_1. Hence $sign_{w_2}(w_1) = spin_r(w_2)$ and $sign_{w_1}(w_2) = spin_r(w_1)$ by Definition 2.8. But the single consecutive pair of P_{w_1,w_2} will be counted exactly once in either $spin_r(w_2)$ or $spin_r(w_1)$, and hence since the number of tree vertices is the same in both cases we get $spin_r(w_2) = -spin_r(w_1)$. Thus again $sign_{w_1}(w_2) = -sign_{w_2}(w_1)$. □

Lemma 3.3 justifies item (2) of the following definition, showing that the choice of the tree vertex w closest to either u or v yields the same order.

Definition 3.4 Consider adjacent vertices $u, v \in R$.

1. If $lab(u) = n$ and $lab(v) = n + 1$, then let $w \in R_0$ be the tree vertex nearest to u (or v), and define:
   $u < v$ if $sign_w(v) = +1$,
   $u > v$ if $sign_w(v) = -1$.
2. If $lab(u) = lab(v)$, then let $w \in R_0$ be the tree vertex nearest to u, and define:
   $u > v$ if $sign_w(u) = +1$,
   $u < v$ if $sign_w(u) = -1$.

This provides the partial order we need on R.

Observation 3.5 *The transitive closure of the above relations makes $(R, <)$ a partial order.*

Now since a graph embedding of (R, r) is a similarity by Lemma 2.26, in particular it preserves the $sign$ function at tree vertices, and as a result is an order preserving embedding as defined above. Conversely we claim that an order embedding ϕ of $(R, <)$ is a graph embedding. First consider an edge uv with $lab(v) = lab(u) + 1$, and therefore u and v are comparable. Then $u' = \phi(u)$ and $v' = \phi(v)$ have the same labels as u and v respectively due to the gadgets. But if v'' is the neighbour of u' with $lab(v'') = lab(v)$, then any non-trivial path from v' to v'' would contain a consecutive pair, and thus $v' \neq v''$ would imply that u' is incomparable to v'. Similarly if uv is a consecutive pair and thus again u and v must be comparable. Let w_1 and w_2 be the tree vertices closest to u and v respectively. From the above argument the paths $P_{w_1,u}$ and $P_{w_2,v}$ are mapped to the paths $P_{\phi(w_1),\phi(u)}$ and $P_{\phi(w_2),\phi(v)}$ respectively. Since $\phi(w_1) \neq \phi(w_2)$, the path from $\phi(u)$ to $\phi(v)$ must contain a consecutive pair, and $\phi(u)$ is incomparable to $\phi(v)$ unless it is the edge $\phi(u)\phi(v)$ we are looking for.

We record this discussion as follows.

Lemma 3.6 *The monoid of embeddings of (R, r) as a tree is the same as that of $(R, <)$ as a partial order.*

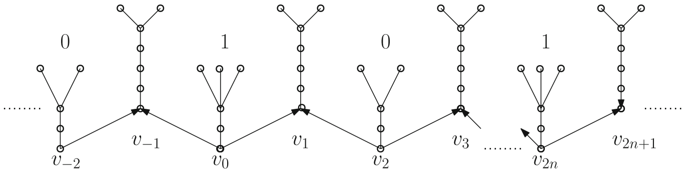


Fig. 27 The partial order on a typical double ray $\mathcal{D}'$ (plus the order on the gadgets as above)

3.3 Partial ordering on double rays

This is where we use the special double rays $\mathcal{D}'$ described in Fig. 27. We order such a double ray as a fence, and together with the ordering of gadgets above yields a partial ordering of double rays.

Again we record the correspondence between graph and order embeddings.

Lemma 3.7 *The monoid of embeddings of a double ray $\mathcal{D}'$ as a tree is the same as that of $(\mathcal{D}', <)$ as a partial order.*

3.4 The posets $\langle \mathcal{P}_s : s < \mathfrak{s} \rangle$

For a non-zero $\mathfrak{s} \in \mathbb{N}$, the posets $\langle \mathcal{P}_s : s < \mathfrak{s} \rangle$ we are seeking to produce consist of the trees $\langle \mathcal{T}_s' : s < \mathfrak{s} \rangle$ constructed as before but now using the special double rays $\mathcal{D}_s'$ (with type assignments on even indexed vertices), equipped with the (transitive closure) of the above orderings on copies of (R, r) and double rays.

It is now clear that order embeddings of $\mathcal{P}_s$ will preserve copies of (R, r) and double rays. First note that two vertices in a copy of (R, r) are connected through a path in the comparability graph of $\mathcal{P}_s$, hence so is their image; but if their image belongs to different copies of (R, r) that path would go through an odd-indexed vertex of a double ray, which is impossible due to the gadgets being mutually non-embedable. Similarly two vertices on a double ray cannot be mapped to different double rays since again the image of their comparability path would be required to contain a vertex of positive label, which is again impossible due to the gadgets being mutually non-embedable.

Hence order embeddings of $\mathcal{P}_s$ coincide with graph embeddings of $\mathcal{T}_s'$. Thus $\mathcal{P} = \mathcal{P}_0$ has indeed exactly $\mathfrak{s}$ siblings up to isomorphism. This completes the proof of the second main theorem.

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