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An equivalent to the Riemann hypothesis in the Selberg class

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Abstract
In 2020 S. M. Gonek, S. W. Graham and Y. Lee formulated the Lindelöf hypothesis for prime numbers and proved that it is equivalent to the Riemann Hypothesis. In this note we show that their result holds in the Selberg class of L-functions.

Keywords Selberg class · Riemann hypothesis · Lindelöf hypothesis

Mathematics Subject Classification 11M26 · 11M41

1 Introduction

As usual we denote $s = \sigma + it$, a non-trivial zero of $F(s)$ by $\rho = \beta + i\gamma$ and by p a prime number.

A function $F(s)$ belongs to the Selberg class S if it satisfies the following properties:

(1) For $\sigma > 1$, $F(s)$ is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

(2) For some integer $m \geq 0$, $(s-1)^m F(s)$ is an entire function of finite order.

(3) $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})}$$

where

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$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s),$$

with $Q > 0$, $\lambda_j > 0$, $\Re \mu_j \geq 0$ and $|\omega| = 1$.

(4) (Ramanujan hypothesis) For every $\varepsilon > 0$, $a(n) \ll n^{\varepsilon}$.

(5) (Euler product) For σ sufficiently large,

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

where $b_n = 0$ unless $n = p^k$ for $k \in \mathbb{N}$ and $b_n \ll n^{\theta}$ for some $\theta < 1/2$.

The data Q, λ_j, μ_j and ω does not determine $F(s)$ uniquely, however $d_F = 2\sum_{j=1}^{r} \lambda_j$ is an invariant called the degree of $F(s)$. Let m_F be the order of the pole of $F(s)$ at $s = 1$.

The zeros of $F(s)$ that come from the poles of the Gamma function in the functional equation are called trivial. We say that $F(s)$ satisfies the Riemann Hypothesis (RH) if all of its non-trivial zeros $\rho = \beta + i\gamma$ have $\beta = 1/2$. More about the Selberg class see Kaczorowski and Perelli [3].

Gonek et al. [4] proved that the Riemann Hypothesis is equivalent to the following relation

$$\sum_{p \leq x} p^{-it} = \int_2^x \frac{u^{-it}}{\log u} du + O(x^{1/2}|t|^{\varepsilon}),$$

for all $\varepsilon, B > 0$ and $2 \leq x \leq |t|^B$. For further development see Banks [1]. In [2, Corollary 6] a similar equivalent was considered in the case of the Lindelöf hypothesis for the Lerch zeta-function.

In this short note we show that Gonek's, Graham's and Lee's result holds for all functions from S.

Theorem 1 *Let $F(s) \in S$ and $d_F \geq 1$. Then $F(s)$ satisfies RH if and only if*

$$\sum_{n \leq x} b_n n^{-it} = m_F \int_2^x \frac{u^{-it}}{\log u} du + O(x^{1/2}|t|^{\varepsilon}), \tag{1}$$

for all $\varepsilon, B > 0$ and $2 \leq x \leq |t|^B$.

2 Lemmas and proof of Theorem 1

Let $F(s) \in S$ and denote $\Lambda_F(n) = b_n \log n$, then

$$-\frac{F'}{F}(s) = \sum_{n=1} \frac{\Lambda_F(n)}{n^s}.$$

Lemma 2 *Let $F(s) \in S$, $\varepsilon > 0$ and let $\mu_F : \mathbb{R} \to \mathbb{R}$ be such that*

$$F(\sigma + it) \ll |t|^{\mu_F(\sigma)+\varepsilon}.$$

Then

$$\mu_F(\sigma) = \begin{cases} 0, \text{ if } \sigma > 1, \\ (1/2)d_F(1-\sigma), \text{ if } 0 \leq \sigma \leq 1, \\ (1/2-\sigma)d_F, \text{ if } 0 < \sigma. \end{cases}$$

Proof See Steuding [6, Theorem 6.8]. □

Lemma 3 *Let* $T > 0$ *and suppose that* $x > 0$ *is half an odd integer. Then,*

$$\sum_{n \leq x} \Lambda_F(n) n^{-it} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} -\frac{F'}{F}(w+it)\frac{x^w}{w} dw + O\left(\frac{x^2}{T} + 1\right).$$

Proof Lemma can be proved by using the argumentation presented in the proof of Lemma 3.12 of Titchmarsh [7] by fixing $c = 2$ and noticing that

$$\sum_{n=1}^{\infty} \frac{|\Lambda_F(n)|}{n^2} \leq \sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}}$$

converges. □

Lemma 4 *Let* $0 \leq \delta < 1/4$ *be such that* $F(s) \in S$ *has no trivial zeros with* $\sigma = -1 + \delta$. *Then*

$$\int_{-1+\delta-iT}^{-1+\delta+iT} -\frac{F'}{F}(w)\frac{x^w}{w} dw = O\left(\frac{\log^2 T}{x^{1-\delta}}\right)$$

for any $T > 0$

Proof By Hadamard theory we have (see Smajlović [5, proof of Lemma 5.1])

$$\frac{F'}{F}(s) = \sum_{|\gamma - T| < 1} \frac{1}{s-\rho} + O(\log T). \tag{2}$$

For non-trivial zeros we have $0 \leq \beta \leq 1$ and there are $O(\log T)$ zeros of $F(s)$ with $|\gamma - T| < 1$ (see [3]). Thus,

$$\frac{F'}{F}(s) = O(\log t),$$

when $\sigma = -1 + \delta$. Then,

$$\int_{-1+\delta-iT}^{-1+\delta+iT} -\frac{F'}{F}(w)\frac{x^w}{w} dw \ll x^{-1+\delta} \int_{-1+\delta-iT}^{-1+\delta+iT} \frac{\log w}{w} dw \ll \frac{\log^2(T)}{x^{1-\delta}}$$

□

Lemma 5 *Let* $0 \leq \delta < 1/4$ *be such that* $F(s) \in S$ *has no trivial zeros with* $\sigma = -1 + \delta$. *Then*

$$\int_{-1+\delta+iT}^{2+iT} -\frac{F'}{F}(w)\frac{x^w}{w} dw = O\left(\frac{x^2 \log^2 T}{T}\right)$$

for any $T>0$ *such that it is not an ordinate of a non-trivial zero of* $F(s)$.

Proof Moving by a finite distance we can pick T such that $|T-\gamma| \gg 1/\log T$. Then with such a choice of T using (2) we obtain

$$\int_{-1+\delta+iT}^{2+iT} -\frac{F'}{F}(w)\frac{x^w}{w}dw = O\left(\frac{x^2\log^2 T}{T}\right).$$

Moving the line of integration by a bounded amount we may cross at most $O(\log T)$ zeros $F(s)$ (counting with multiplicities) and they will contribute residues of total size at most $O\big(x^2\log^2 T/T\big)$. Hence, noting Lemma 4, we obtain

$$\int_{-1+\delta+iT}^{2+iT} -\frac{F'}{F}(w)\frac{x^w}{w}dw = O\left(\frac{x^2\log^2 T}{T}\right)$$

for any $T>0$ such that it is not an ordinate of a non-trivial zero of $F(s)$. □

Proof of Theorem 1 The proof is pretty much the same as the one in Gonek's, Graham's and Lee's paper with additional consideration given for greater generality.

By Abel's summation formula we see that (1) is equivalent to

$$\sum_{n\le x}\Lambda_F(n)n^{-it} = m_F\frac{x^{1-it}}{1-it} + O(x^{1/2}|t|^{\varepsilon}) \tag{3}$$

for all $\varepsilon, B>0$ and $2\le x\le |t|^B$.

Thus, it is enough to prove that RH for $F(s)$ is equivalent to (3)

Suppose $F(s)$ satisfies RH. Let $x\ge 5/2$ be half an odd integer and $T=|t|^C$, where $C>1$ will be chosen later. By Lemma 3 we have

$$\sum_{n\le x}\Lambda_F(n)n^{-it} = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} -\frac{F'}{F}(w+it)\frac{x^w}{w}dw + O\left(\frac{x^2}{T}+1\right). \tag{4}$$

Choose $0\le\delta<1/4$ such that $F(s)$ would have no trivial zeros with $\sigma=-1+\delta$. Replacing the line of integration in (4) by one consisting of the three leftmost sides of the rectangle with vertices $2-iT$, $-1+\delta-iT$, $-1+\delta+iT$ and $2+iT$ and using Lemmas 4 and 5 we see that

$$\sum_{n\le x}\Lambda_F(n)n^{-it} = m_F\frac{x^{1-it}}{1-it} - \sum_{|\gamma-t|<T}\frac{x^{\rho-it}}{\rho-it} + O\left(\frac{x^2}{T} + \frac{x^2\log^2(|t|+T)}{T} + \frac{\log^2(|t|+T)}{x^{1-\delta}} + 1\right).$$

Note that the sum is over the non-trivial zeros of $F(s)$. It might happen that we pass over trivial zeros of $F(s)$, however there are only finitely many of them in $\sigma>-1+\delta$ and for each of them we have $\beta\le 0$, thus they contribute a term of size $O(1)$.

By RH, using Abel's summation formula, we obtain

$$\sum_{|\gamma-t|<T}\frac{x^{\rho-it}}{\rho-it} \ll x^{1/2}\sum_{\substack{|\gamma-t|<T\\ \beta=1/2}}\frac{1}{1+|t-\gamma|} \ll x^{1/2}\log^2(|t|+T).$$

Now, suppose that $5/2 \le x \le |t|^B$ and choose $C > \max(1, 3B/2)$. Then $T = |t|^C \ge \max(|t|, x^{3/2})$ and we obtain

$$\sum_{n \le x} \Lambda_F(n) n^{-it} = m_F \frac{x^{1-it}}{1-it} + O(x^{1/2}|t|^{\varepsilon}). \tag{5}$$

We have assumed until now that $x \ge 5/2$ is half an odd integer. If we relax this condition and just assume that $x \ge 2$, then such x is always within $O(1)$ of half an odd integer. Changing x by this amount in (5) changes the left-hand side by no more than $O(x^{1/2}\log x)$ and the right-hand side by at most $O(|t|^{\varepsilon})$. Since $x^{1/2}\log x \ll x^{1/2}|t|^{\varepsilon}$, (5) holds for $2 \le x \le |t|^B$.

Next we prove that (3) implies RH for $F(s)$. Write

$$\psi(x,t) = \sum_{n \le x} \Lambda_F(n) n^{-it}$$

and

$$R(x,t) = \psi(x,t) - m_F \frac{x^{1-it}}{1-it}.$$

Then by our assumption

$$R(x,t) \ll x^{1/2}|t|^{\varepsilon} \tag{6}$$

for $2 \le x \le |t|^B$, where $\varepsilon > 0$ and B is arbitrarily large but fixed.

First we show that for all $s \ne 1$

$$\int_1^{\infty} \frac{R(x,t)}{x^s} dx = -\left(\frac{1}{s-1} \frac{F'}{F}(s+it-1) + \frac{m_F}{(1-it)(s+it-2)} \right). \tag{7}$$

Suppose that $\sigma > 2$. Then we see that

$$\int_1^{\infty} \frac{\psi(x,t)}{x^s} dx = \sum_{n=2}^{\infty} \frac{\Lambda_F(n)}{n^{it}} \int_n^{\infty} x^{-s} dx = -\frac{1}{s-1} \frac{F'}{F}(s+it-1).$$

Integrating the other term and combining we get (7) for $\sigma > 2$, the right hand side of which defines a meromorphic continuation of the left hand side which has a simple pole at $s = 1$.

Define

$$H(s) = \int_1^{\infty} \frac{R(x,t)}{x^s} dx = - \left(\frac{(1-it)(s+it-2)F'(s+it-1) + m_F(s-1)F(s+it-1)}{(s-1)(1-it)(s+it-2)F(s+it-1)} \right).$$

Assume, by way of contradiction, that $\rho_0 = \beta_0 + i\gamma_0$ is a zero of $F(s)$ with $\beta_0 > 1/2$. Let m be the multiplicity of ρ_0, and define

$$h(s) = \frac{(s+it-2)F(s+it-1)}{(s+it-\rho_0-1)^m (s+it+1)^{4d_F}}.$$

For real u, define

$$w(u)=\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}h(s)e^{us}ds$$

and consider the integral

$$\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}h(s)H(s)e^{s\log x}ds=\int_{1}^{\infty}R(y,t)w(\log x-\log y)dy. \tag{8}$$

We move the line of integration in the integral of the left hand side to left to $\sigma=5/4$ and pass two poles at $s=2-it$ and $s=\rho_0+1-it$. The residue at $s=2-it$ is equal to 0 and the other residue is

$$-x^{\rho_0+1-it}\frac{(\rho_0-1)F^{(m)}(\rho_0)}{(m-1)!(\rho_0-it)(\rho_0+2)^{4d_F}}.$$

Using the bounds $F(1/4+iv)\ll(1+|v|^{1/2d_F})$ and $F'(1/4+iv)\ll(1+|v|^{1/2d_F})\log(2+|v|)$, the left hand side is

$$\begin{aligned}&=x^{\rho_0+1-it}\frac{(\rho_0-1)F^{(m)}(\rho_0)}{(m-1)!(\rho_0-it)(\rho_0+2)^{4d_F}}\\&\quad+O\left(x^{5/4}\int_{-\infty}^{\infty}\frac{(1+|t+v|^{1/2d_F})\log(2+|v|)}{(1+|v+t-\gamma_0|)^m(1+|v+t|)^{4d_F}}dv\right)\\&=x^{\rho_0+1-it}\frac{(\rho_0-1)F^{(m)}(\rho_0)}{(m-1)!(\rho_0-it)(\rho_0+2)^{4d_F}}+O\big(x^{5/4}\big).\end{aligned}$$

Next we estimate $w(u)$. If $u\le 0$ we pull the contour right to ∞. Since

$$h(s)e^{us}\ll\frac{e^{u\sigma}}{|s+it-\rho_0|^m|s+it|^{4d_F-1}}$$

for $\sigma\ge 3$, we see that $w(u)=0$. If $u>0$, we pull the contour left to $-5/4$. We pass a pole of $h(s)$ at $s=-1-it$ of order $4d_F$ which contributes a residue of size $O(1)$. The integral on the new line is

$$\begin{aligned}\int_{-5/4-i\infty}^{-5/4+i\infty}h(s)e^{us}ds&\ll\int_{-\infty}^{\infty}e^{-5/4u}\frac{(1+|v+t|)|F(-9/4+i(v+t))|}{|1+(v+t-\gamma_0)|^m|1+(v+t)|^{4d_F}}dv\\&\ll\int_{-\infty}^{\infty}e^{-5/4u}\frac{(1+|v+t|)(1+|v+t|^{23/8d_F})}{|1+(v+t-\gamma_0)|^m|1+(v+t)|^{4d_F}}dv\ll 1.\end{aligned}$$

Thus,

$$w(u)=\begin{cases}0 \text{ if } u\le 0,\\ O(1) \text{ if } u>0.\end{cases}$$

Collecting the estimates in the previous discussion and applying them to (8) we see that for ρ_0 fixed

$$x^{\beta_0+1}\ll_{\rho_0}\int_1^x|R(y,t)|dy+x^{5/4}.$$

Then, by assumption, setting $B = 2/(\beta_0 - 1/2)$ we get

$$x^{\beta_0+1} \ll_{\rho_0} x^{3/2}|t|^{\varepsilon},$$

for $2 \le x \le |t|^{2/(\beta_0-1/2)}$. In other words,

$$x \ll_{\rho_0} |t|^{(1+\varepsilon)/(\beta_0-1/2)}.$$

This contradiction implies that $\beta_0 = 1/2$. This completes the proof of the theorem. □

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