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On curves on Hirzebruch surfaces

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Abstract
We call a smooth irreducible projective curve a Castelnuovo curve if it admits a birational map into the projective r-space such that the image curve has degree at least 2r+1 and the maximum possible geometric genus (which one can calculate by a classical formula due to Castelnuovo). It is well known that a Castelnuovo curve must lie on a Hirzebruch surface (rational ruled surface). Conversely, making use of a result of W. Castryck and F. Cools concerning the scrollar invariants of curves on Hirzebruch surfaces we show that curves on Hirzebruch surfaces are Castelnuovo curves unless their genus becomes too small w.r.t. their gonality. We analyze the situation more closely, and we calculate the number of moduli of curves of fixed genus g and fixed gonality k lying on Hirzebruch surfaces, in terms of g and k.

Keywords Castelnuovo curves · Hirzebruch surfaces · Scrollar invariants · Moduli of curves

Mathematics Subject Classification 14H45; Secondary 14J26

1 Introduction

C always denotes a smooth irreducible projective curve of genus $g > 1$ defined over $\mathbb{C}$, and we let $k \geq 2$ be its gonality. Among the curves C with fixed g and k those lying on a Hirzebruch surface X (i.e. a rational ruled surface, [1], V, Sect. 2) form a distinguished class for which these invariants are determined by the geometry of X. For g this is clear by the adjunction formula; as for k, if f denotes a fibre of the natural projection $X \to \mathbb{P}^1$ we can arrange [2] that the linear series $|f|_C|$ on C (cut out by the ruling $|f|$ of X) is a "gonality pencil" g^1_k of C. Associated with a gonality pencil g^1_k on (in fact, any curve) C are its *scrollar invariants* $e_1 \leq e_2 \leq \ldots \leq e_{k-1}$ which determine the function $\dim|ng^1_k|$ $(0 \leq n \in \mathbb{Z})$; more precisely, we have $\dim|(n+1)g^1_k| = \dim|ng^1_k| + i$ for $e_{i-1} < n \leq e_i$ $(i \in \mathbb{Z}, 1 \leq i < k;$ $e_0 := -1)$, and $|(n+1)g^1_k|$ is non-special for $n > e_{k-1}$. In particular, $i = 1$ yields $e_1 + 1 = \mathrm{Max}\{0 \leq n \in \mathbb{Z} : \dim|ng^1_k| = n\}$.

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The invention of the e_i is apparently due to Christoffel [3]. (Concerning their name we recall: For the canonical model of C in $\mathbb{P}^{g-1}$ the divisors in g^1_k span secant spaces of dimension $k-2$ whose union constitutes a $(k-1)$-dimensional rational normal scroll S of degree $g-(k-1)$ in $\mathbb{P}^{g-1}$, and then $S \cong \mathbb{P}(£)$ where £ is the vector bundle $\mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_{k-1})$ of rank $k-1$ over $\mathbb{P}^1$.)

For C on a Hirzebruch surface $X = X_e$ of invariant e Castryck and Cools [4, 10.2] observed that the distribution of the scrollar invariants e_i $(1 \leq i < k)$ of the chosen pencil g^1_k on C is determined by the ambient space X: they are equidistantly distributed with distance e, i.e. we have $e_i = e_1 + (i-1)e$. Since $e_1 + \cdots + e_{k-1} = g - (k-1)$ holds it follows that

$$e_1 + 1 = \frac{g}{k-1} - \frac{k-2}{2}e.$$

Hence e_1 does not depend on the choice of the gonality pencil g^1_k on C. Consequently, $e_1, \ldots, e_{k-1}$ and for $k \geq 3$ also e are even invariants of the curve $C \subset X$.

For short, we call the latter formula for e_1 the *CC-formula*.

In this paper we study "how much" the geometry of X may affect the geometry of $C \subset X$. Since for $k = 3$ this is already performed by Maroni's theory of trigonal curves (e.g., [5], Sect. 1) we may assume that $k > 3$. In the next section we apply the CC-formula for a simple proof of the fact that $C \subset X$ cannot be a general k-gonal curve of genus g. In Sect. 3 we see that X induces, by sections, on C a certain finite set of very ample linear series (which in Sect. 5 is shown to depend on C only). We use this set in Sect. 4 (cf. Theorem 4.1) to find in it (and only in it) a series making C an *extremal curve*, in the sense of Arbarello et al. [6], III, provided that $g >> k$, and we discuss related questions, then (results 4.2–4.6). In the final Sect. 5 we ascertain the number of moduli of curves on Hirzebruch surfaces for fixed g and k thereby making the result of Sect. 2 much more precise.

2 On general k-gonal curves

Proposition 2.1 *For $k > 3$ a general k-gonal curve cannot lie on a Hirzebruch surface.*

Proof According to Ballico [7], for a general k-gonal curve of genus g (so $g \geq 2k-3$, by Meis' bound for the gonality k) the scrollar invariant e_1 of a pencil of degree k satisfies $e_1 = \left[\frac{g}{k-1}\right] - 1$, i.e. e_1 attains its maximum possible value.

Claim: Let C be a curve of genus g and gonality $k > 3$ admitting a g^1_k with $e_1 = \left[\frac{g}{k-1}\right] - 1$. Assume that C lies on a Hirzebruch surface X_e of invariant e. Then $e = 0$.

To prove the Claim we use the terminology of Hartshorne [1], V, Sect. 2. So let $\mathrm{Pic}(X_e) = \mathbb{Z}C_0 \oplus \mathbb{Z}f$ where $C_0 \subset X_e$ is a section of self-intersection $C_0^2 = -e$ and f is a fibre of the natural projection $X_e \to \mathbb{P}^1$ whence $f^2 = 0$ and $C_0 \cdot f = 1$. By [2] we may assume that $k = C \cdot f$; then $C \sim kC_0 + xf$ for some integer $x > 0$. Write $g = m(k-1) + \epsilon$ with $m := \left[\frac{g}{k-1}\right]$ (so $\epsilon \in \mathbb{Z}$ such that $0 \leq \epsilon < k-1$); note that $m = e_1 + 1$, by our hypothesis. By the CC-formula we know that $e_1 + 1 = \frac{g}{k-1} - \frac{k-2}{2}e$, and it follows that $(k-1)(k-2)e = 2g - 2(k-1)(e_1+1) = 2g - 2(k-1)m = 2g - 2(g-\epsilon) = 2\epsilon \leq 2(k-2)$, i.e. $e \leq \frac{2}{k-1} < 1$ for $k > 3$. Thus we obtain $e = 0$ (and since e is even, $k-1$ divides g). The Claim is proved.

For C as in the Claim we easily can calculate x: since $X := X_e$ has the canonical divisor $K_X \sim -2C_0 - (2+e)f$ we know, by adjunction, that C has the canonical divisor $K_C \sim (K_X + C)|_C = ((k-2)C_0 + (x-2-e)f)|_C$. Hence $2g - 2 = (K_X + C) \cdot C = -(k-2)ke + k(x-2-e) + x(k-2) = 2(k-1)x - (k-1)ke - 2k$. By the Claim, $e = 0$;

so it follows that $x = \frac{g}{k-1} + 1$ and that $x = C_0 \cdot C$. Hence the second ruling $|C_0|$ of X cuts out on C a complete and base point free pencil g^1_x. Clearly, then, $x \geq k$, and since $C_0 \cdot f = 1$ we have $g^1_x \neq g^1_k$.

We observe that $k > 3$ implies that $k \leq x = \frac{g}{k-1} + 1 < \frac{g}{2} + 1$. Since on the general curve of genus g and gonality $k < \frac{g}{2} + 1$ its g^1_k is the only complete and base point free pencil of degree strictly smaller than $\frac{g}{2} + 1$ [8, 2.6] we see that the existence of our $g^1_x \neq g^1_k$ on C forces C to be a special curve w.r.t. moduli of k-gonal curves of genus g. □

Note that the proof makes the meaning of "general k-gonal curve" more transparent here: For $k > 3$ a k-gonal curve having a g^1_k with maximal e_1 but lacking a complete and base point free pencil of degree $e_1 + 2$ cannot exist on a Hirzebruch surface.

In Sect. 5 we clarify "how special" curves on Hirzebruch surfaces are w. r. t. moduli.

3 A criterion

We call a complete linear series $|D|$ on a curve C *very special* if its index of speciality $h^1(D)$ is at least 2, i.e. if the dual series $|K_C - D|$ of $|D|$ contains a pencil. Inspired by [9], we note

Lemma 3.1 *Let C be a curve of genus g and gonality $k > 3$. Choosing a pencil g^1_k on C the following statements are equivalent:*

(i) *C is contained in a Hirzebruch surface X such that $g^1_k = |f|_C|$ for a fibre f of the natural map $X \to \mathbb{P}^1$.*
(ii) *C has a complete, very special and very ample linear series G on which the g^1_k imposes only two conditions (i.e. $\dim|G - g^1_k| = \dim(G) - 2$).*

Proof (ii) implies (i) according to (the proof of) [10], 3.1 if $\dim(G) > 2$. If $\dim(G) = 2$ then C is a smooth plane curve, $k = \deg(G) - 1$; so $|G - g^1_k|$ is a point P_0 of C and $\mathbb{P}^2$ functions as the cone with vertex P_0 over a line whence by blowing up P_0 we obtain C (more precisely the strict transform of C) as a smooth curve on the Hirzebruch surface X_1 of invariant 1 [1, V, 2.11.5].

To see that (i) implies (ii) let $C \subset X$, $X = X_e$ a Hirzebruch surface of invariant e. As in the proof of Proposition 2.1 we have $C \sim kC_0 + xf$ for some $x \in \mathbb{Z}$, and since C is (smooth and) irreducible we have $x \geq ke$ and $x > 0$ [1, V, 2.18]. The linear series $|C_0 + ef|$ on X is base point free of dimension $e + 1$, and for $e > 0$ the morphism it defines maps X birationally onto a cone over a rational normal curve in $\mathbb{P}^e$ (by blowing down C_0 to the vertex of the cone, [1], V, 2.11.4). Let $\Gamma := |(C_0 + ef)|_C|$; we claim that $\dim(\Gamma) = e + 1$, i.e. that $|C_0 + ef|$ cuts out on C the complete and base point free linear series Γ being then a g^{e+1}_x which is (by the above) for $e > 0$ also a simple series on C. To compute $\dim(\Gamma)$ we note that $x - e \geq (k-1)e$ implies that the linear series $|C - (C_0 + ef)| = |(k-1)C_0 + (x-e)f|$ contains a (smooth and) irreducible curve D [1, V, 2.18]; in particular, then, $h^0(X, -D) = 0$. So the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

implies that $h^1(X, -D) = h^0(D, \mathcal{O}_D) - h^0(X, \mathcal{O}_X) = 0$ since $h^1(X, \mathcal{O}_X) = 0$ and $h^0(D, \mathcal{O}_D) = 1 = h^0(X, \mathcal{O}_X)$. Hence from the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X(C_0 + ef) \to \mathcal{O}_C(C_0 + ef) \to 0$$

we conclude that $\dim(\Gamma) + 1 = h^0(C, (C_0 + ef)|_C) = h^0(X, C_0 + ef) = e + 2$, as wanted.

In the proof of Proposition 2.1 we observed already that $2g-2=2(k-1)x-(k-1)ke-2k$. Hence we have $g=(k-1)(x-1-\frac{1}{2}ke)$ and $x=\frac{g}{k-1}+1+\frac{1}{2}ke=\left(\frac{g}{k-1}-1-\frac{k-2}{2}e\right)+2+(k-1)e=e_1+2+(k-1)e$, the latter by the CC-formula. (This formula for x will be useful also later on.) Hence we see that $K_X+C\sim(k-2)C_0+(x-2-e)f=(k-2)(C_0+ef)+(x-2-(k-1)e)f=(k-2)(C_0+ef)+e_1f$ whence $|K_C|=|(K_X+C)|_C|=|(k-2)\Gamma+e_1g^1_k|$ since $|f|_C|=g^1_k$. In particular, this implies that $|\Gamma+e_1g^1_k|=|K_C-(k-3)\Gamma|$ is very special since we assume $k>3$.

Claim: $\dim|(k-3)\Gamma|=\dim|(k-3)(C_0+ef)|=\frac{1}{2}(k-3)((k-2)e+2)$.

In fact, as before for Γ we obtain the first of these two equalities. Concerning the second one the theorem of Riemann–Roch for X shows that for an integer $\lambda\geq 0$ we have

$h^0(\lambda(C_0+ef))=\frac{1}{2}\lambda(C_0+ef)(\lambda(C_0+ef)-K_X)+1+h^1(\lambda(C_0+ef))=\frac{1}{2}\lambda((\lambda+1)e+2)+1+h^1(\lambda(C_0+ef))$

which for $\lambda=k-3$ gives us the desired result provided that $h^1(\lambda(C_0+ef))=0$. Now, by Serre-duality and Ramanujam's vanishing theorem (e.g., [11], 3.5) we have $h^1(\lambda(C_0+ef))=h^1(K_X-\lambda(C_0+ef))=h^1(-((\lambda+2)C_0+(\lambda e+e+2)f))=0$ since $|(\lambda+2)C_0+(\lambda e+e+2)f|$ contains a numerically 2-connected divisor: in fact, $|(\lambda+1)C_0+(\lambda e+e+2)f|$ contains an irreducible curve E since $\lambda e+e+2>(\lambda+1)e$ [1, V, 2.18]; so $(\lambda+2)C_0+(\lambda e+e+2)f\sim C_0+E$ with $C_0\cdot E=2$. This proves the Claim. (And, by the way, more generally we can prove this way that any irreducible curve $D\neq C_0$ on X is non-special, i.e. satisfies $h^1(D)=0$.)

As a consequence of the Claim we obtain that $\dim|\Gamma+e_1g^1_k|=\dim|K_C-(k-3)\Gamma|=g-1-(k-3)x+\frac{k-3}{2}((k-2)e+2)=g-1-(k-3)\left(x-\frac{k-2}{2}e-1\right)$, and since we observed that $x=\frac{g}{k-1}+1+\frac{1}{2}ke$ we see that $\dim|\Gamma+e_1g^1_k|=g-1-(k-3)\left(\frac{g}{k-1}+e\right)=\frac{2g}{k-1}-1-(k-3)e$ where, by the CC-formula, $\frac{2g}{k-1}=2e_1+2+(k-2)e$. Hence we find that $\dim|\Gamma+e_1g^1_k|=e+1+2e_1$. Since $|C_0+(e+i)f|$ is very ample for $i>0$ [1, V, 2.17] so is $|\Gamma+ig^1_k|$, and this implies that $\dim|\Gamma+ig^1_k|\geq\dim|\Gamma+(i-1)g^1_k|+2$ $(i>0)$. From $\dim|\Gamma+e_1g^1_k|=e+1+2e_1$ it follows that $\dim|\Gamma+ig^1_k|=\dim(\Gamma)+2i$ for $i=0,1,\ldots,e_1$. In particular, for $e_1>0$ the very special series $G:=|\Gamma+g^1_k|$ on C is very ample and satisfies $\dim|G-g^1_k|=\dim(G)-2$.

It remains the case $e_1=0$. In that case, recalling that $ke\leq x=e_1+2+(k-1)e$ we obtain $0<e\leq e_1+2=2$, and since $0=e_1=\frac{g}{k-1}-1-\frac{k-2}{2}e$ we have $x=k+1$, $\Gamma=g^2_{k+1}$, $g=\frac{1}{2}(k-1)k$ for $e=1$, resp. $x=2k$, $\Gamma=g^3_{2k}$, $g=(k-1)^2$ for $e=2$. Since $e>0$ the series Γ is simple and therefore here, according to Castelnuovo's genus bound ([12], or [6], III, Sect. 2), even very ample; so C is a smooth plane curve of degree $k+1$ and $\dim|\Gamma-g^1_k|=0$, resp. $\Gamma=|2g^1_k|$ embeds C in a quadric cone in $\mathbb{P}^3$ and $|\Gamma-g^1_k|=g^1_k$. □

We add some comments.

Remark (i) The genus of C in the Lemma is rather big for its gonality; in fact, we have $g\geq\frac{(k-1)k}{2}$, and equality holds if and only if C is isomorphic to a smooth plane curve. To see this we use the series $\Gamma=g^{e+1}_x$ from the proof of the Lemma. We know that $x\geq ke$ and $x\geq k$, and $2g=(k-1)(2x-2-ke)$. Hence we obtain $g\geq(k-1)^2$ for $e\neq 1$, and then $g>\frac{(k-1)k}{2}$ (since $k>2$). For $e=1$ we have $\Gamma=g^2_x$ whence $x>k$ which implies then that $g\geq\frac{(k-1)k}{2}$. Let $g=\frac{(k-1)k}{2}$. Then $e=1$, $(k-1)k=2g=(k-1)(2x-2-k)$, i.e. $x=k+1$, and so $\Gamma=g^2_{k+1}$ must be very ample (i.e. C a smooth plane curve) since otherwise C would have a g^1_{k-1}. (Conversely, a smooth plane curve of degree $d\geq 4$ satisfies $k=d-1$, $g=\frac{(d-2)(d-1)}{2}=\frac{(k-1)k}{2}$, and we observed already that it lies on X_1; so the CC-formula implies that $e_1=0$ and that any Hirzebruch surface containing it must have invariant $e = 1$.) (ii) Any complete and very ample linear series g^r_d on a k-gonal curve C satisfying $\dim|g^r_d - g^1_k| = r - 2$ maps C birationally onto a (linearly normal) smooth curve on a rational normal scroll (maybe, a cone) of degree $r - 1$ in $\mathbb{P}^r$. On the related Hirzebruch surface X_e the g^r_d is cut out on $C \subset X_e$ by $|C_0 + nf|$ for some $n \geq e$ such that $r - 1 = 2n - e$ ([1], V, 2.17, 2.19). Hence we have $g^r_d = |\Gamma + ig^1_k|$ with $i = n - e \geq 0$, and so $\dim|\Gamma + ig^1_k| = r = 2n - e + 1 = e + 1 + 2i = \dim(\Gamma) + 2i$. In the proof of the Lemma we obtained the basic relation $\dim|\Gamma + ig^1_k| = \dim(\Gamma) + 2i$ ($i \geq 0$) for $0 \leq i \leq e_1$; in fact, it is still true for $i = e_1 + 1$ since it can be shown that $h^0(K_C - (\Gamma + (e_1 + 1)g^1_k)) = h^0(((k-3)(C_0 + ef) - f)|_C) = h^0((k-3)(C_0 + ef) - f)) = h^0((k-3)C_0 + ((k-3)e - 1)f)) = h^0((k-4)C_0 + ((k-3)e - 1)f)) = \frac{(k-3)(k-2)}{2}e$. (Observe, however, that for $e = 0$ the series $|\Gamma + (e_1 + 1)g^1_k|$ is non-special.)

Already for $i = e_1 + 2$ this dimension relation fails. For $e = 0$ this simply follows from the Riemann–Roch theorem for C: more generally, since $\Gamma = g^{e+1}_x$ with $x = e_1 + 2 + (k-1)e$ we compute $\dim|\Gamma + (e_1 + j)g^1_k| = \dim|K_C - ((k-3)\Gamma - jg^1_k)| = g - 1 - ((k-3)x - jk) + \dim|(k-3)\Gamma - jg^1_k| \geq g - 2 - ((k-3)x - jk) = g - 2 - (k-3)(e_1 + 2 + (k-1)e) + jk = g - 2 - (k-1)(e_1 + 2) + 2(e_1 + 2) - (k-3)(k-1)e + jk = g - 2 - (k-1)\left(\frac{g}{k-1} + 1 - \frac{k-2}{2}e\right) + 2(e_1 + 2) - (k-3)(k-1)e + jk = -k - 1 + \frac{(k-2)(k-1)}{2}e + 2(e_1 + 2) - (k-3)(k-1)e + jk = (j-1)(k-2) + 2(e_1 + j) + 1 - \frac{k-1}{2}(k-4)e = \dim(\Gamma) + 2(e_1 + j) - e + (j-1)(k-2) - \frac{k-1}{2}(k-4)e = \dim(\Gamma) + 2(e_1 + j) + (j-1)(k-2) - \frac{1}{2}(k-3)(k-2)e$; hence we obtain $\dim|\Gamma + (e_1 + j)g^1_k| > \dim(\Gamma) + 2(e_1 + j)$ if $e < \frac{2j-2}{k-3}$.

Furthermore, since $|\Gamma + ig^1_k| = |(C_0 + (e + i)f)|_C|$ and $\dim|C_0 + (e + i)f| = 2(e + i) - e + 1 = e + 1 + 2i$ for $i \geq 0$ we note that the failure of our dimension relation just means that the series cut out on C by $|C_0 + (e + i)f|$ is incomplete.

4 Reduction to Castelnuovo curves

Now we return to the description of k-gonal curves on a Hirzebruch surface. To begin with, we recall that, according to Arbarello et al. [6], an integral and non-degenerate curve of degree $d > 2r$ in $\mathbb{P}^r$ ($r \geq 2$) is called an *extremal curve* in $\mathbb{P}^r$ if its geometric genus attains its maximum possible value, expressed by a well-known genus bound $\pi(d, r)$ due to Castelnuovo ([6, 12], III, Sect. 2; [13], 3.7). An extremal curve in $\mathbb{P}^r$ is smooth and linearly normal (even projectively normal), and for $r > 2$ it lies on a scroll of degree $r - 1$ in $\mathbb{P}^r$ (hence on a Hirzebruch surface X_e of invariant $e < r$) unless $r = 5$ in which case it is also possible that it lies on a Veronese surface in $\mathbb{P}^5$ and is then isomorphic to a smooth plane curve of degree $\frac{d}{2}$ which we can embed into X_1. And an extremal curve in $\mathbb{P}^2$ ($r = 2$) is a smooth plane curve whence it lies on X_1.

We call a curve a *Castelnuovo curve* if it is an extremal curve in $\mathbb{P}^r$ for some integer $r \geq 2$. So we may rephrase the above by simply saying that a Castelnuovo curve lies on a Hirzebruch surface.

A trigonal curve C lies always on a Hirzebruch surface, and if $g \geq 6$ it is easy to see that it is a Castelnuovo curve via the series $|K_C - g^1_3|$ (in fact, via $|K_C - ng^1_3|$ for $n = 1, ..., \left[\frac{g-3}{3}\right]$).

Theorem 4.1 *Let C be a curve of gonality $k \geq 4$ and genus $g > \frac{(k-2)(k-1)^2}{2}$ which lies on a Hirzebruch surface. Then C is, in a definite way, a Castelnuovo curve.*

Before the proof we introduce resp. recall some terminology. Let $C = C_e \subset X_e$, $C \sim kC_0 + xf$, $g^1_k = |f|_C|$, $\Gamma = |(C_0 + ef)|_C| = |eg^1_k + C_0|_C|$ (with $h^0(C, C_0|_C) = h^0(X, C_0) =$

1 for $e > 0$ since $C_0^2 = -e < 0$), and

$$W_e = W_e(g_k^1) := \{|\Gamma + ig_k^1| : i = 0, 1, ..., e_1\}$$

be the data considered in the proof of Lemma 3.1. We observed that the members of W_e are very special and very ample series g_{x+ik}^{e+1+2i} for $0 < i \leq e_1$ (and for $e_1 = 0$), and $\Gamma = g_x^{e+1}$ is base point free and for $e > 0$ also simple. Moreover, by construction the series in W_e are primitive, i.e. also their dual series are base point free. To prove the Theorem we constructively specify a g_d^r in W_e such that C becomes, via this series, an extremal curve of degree d in $\mathbb{P}^r$. For short, we call a series making C an extremal curve an *extremal series* on C.

Proof For fixed k and g but increasing e we consider W_e subject to the conditions $0 \leq e_1(e) = \frac{g}{k-1} - 1 - \frac{k-2}{2}e \in \mathbb{Z}$ (the CC-formula) and $x(e) = e_1(e) + 2 + (k-1)e$ (as noticed in the proof of the Lemma). We have $e_1(e) - e_1(e+2) = k-2$ and $x(e+2) - x(e) = k$. Hence $x(e+2i) = x(e) + ik$ and so $|g_{x(e)}^{e+1} + ig_k^1| = g_{x(e)+ik}^{e+1+2i} = g_{x(e+2i)}^{(e+2i)+1}$ for $0 \leq i \leq e_1(e)$, i.e. the $(i+1)$-th member of W_e has the same degree and dimension as the first member of W_{e+2i} (as long as $e_1(e+2i) \geq 0$). It follows that W_{e+2i} is "numerically contained" in W_e, i.e. to every $g_d^r \in W_{e+2i}$ (on C_{e+2i}) we find a $g_d^r \in W_e$ (on C_e). Formally we can proceed like this until the last (i.e. the smallest non-negative) value $e_1^{(0)}$ of e_1 is reached. Since $e_1(e) - e_1(e+1) = \frac{k-2}{2}$ we have $e_1^{(0)} \leq \frac{k-4}{2}$ for even k resp. $e_1^{(0)} \leq k-3$ for odd k; for even k this implies that $e_1^{(1)} \leq k-3$ if $e_1^{(1)}$ is the last but one value of e_1 in this proceeding, and for odd k we see that all e's must have the same parity.

Now, let $g > \frac{(k-2)(k-1)^2}{2}$ and e be defined by $e_1(e) = e_1^{(0)}$. (So we choose e to be maximal for leaving $e_1(e)$ non-negative.) Since $e_1^{(0)} \leq k-3$ and $e_1(e) + 1 = \frac{g}{k-1} - \frac{k-2}{2}e$ it follows that $e = \frac{2}{k-2}\left(\frac{g}{k-1} - (e_1^{(0)} + 1)\right) > k-3 \geq e_1^{(0)}$; in particular we have $e \geq 2$ (since $k \geq 4$). Furthermore, we see that $x(e) = e_1(e) + 2 + (k-1)e \geq 2 + (k-1)e > 2(e+1)$. Following [12] we write $x(e) = m \cdot e + q$ for suitable $q = 2, 3, ..., e+1$. Then $m = k-1$ and $q = e_1^{(0)} + 2 \leq e+1$. Applying [12], 3.3, Castelnuovo's genus bound w.r.t. the simple series $g_{x(e)}^{e+1}$ (the first member of W_e, for our special choice for e) is the number $\pi(x(e), e+1) := m(m-1)\frac{e}{2} + m(q-1) = (k-1)((k-2)\frac{e}{2} + e_1^{(0)} + 1) = (k-1)((k-2)\frac{e}{2} + \frac{g}{k-1} - \frac{k-2}{2}e) = g$. Hence the $g_{x(e)}^{e+1}$ is extremal, and we observed before that such a series occurs in all $W_{e'}$ with $e' \leq e$, $e' \equiv e \bmod 2$. If k is odd then $e' \equiv e \bmod 2$ holds and so all $W_{e'}$ contain an extremal series. For even k we also consider e to be defined by $e_1(e) = e_1^{(1)}$; since $e_1^{(1)} \leq k-3$ for even k we can repeat the above argument (with $e_1^{(1)}$ instead of $e_1^{(0)}$), and we see that the simple series $g_{x(e)}^{e+1}$, for e with $e_1(e) = e_1^{(1)}$, is also extremal, and such a series occurs in those $W_{e'}$ left out before. Consequently, all $W_{e'}$ contain an extremal series, regardless if k is odd or even.

However, there is still a word to be said. Making e run until $e_1(e)$ becomes negative (as we did) does not completely fit into the situation since we have (by [1], V, 2.18) that $ke \leq x(e) = e_1 + 2 + (k-1)e$, i.e. $e \leq e_1(e) + 2$; so the W_e become of only computational appearance if e grows beyond $e_1(e)+2$ (i.e. if $\frac{k}{2}e > \frac{g}{k-1} + 1$). But since the related series $g_{x(e)}^{e+1}$ just considered above nevertheless are numerically contained in "honest" $W_{e'}$ for smaller e' such series exist and are simple whence our conclusion concerning their extremality persists. (That is, our procedure just makes the calculation w.r.t. Castelnuovo's genus bound more amenable.) ◻

The genus bound in Theorem 4.1 is not the best possible; there may exist extremal series on C which we did not consider in the proof of the Theorem. For instance, for $k = 4$ we need no bound at all (as is also seen from [9]), for $k = 5$ we must exclude only $g = 14$ and for $k = 6$ only $g = 20$, $g = 25$ and $g = 35$. In fact, a refinement of the discussion in the proof of Theorem 4.1 shows that we may replace the genus bound $g > \frac{(k-2)(k-1)^2}{2}$ by $g \geq \frac{(k-2)^2(k-1)}{2}$. (Only in the cases $g = \frac{(k-2)(k-1)^2}{2}$ and $g = \frac{(k-2)^2(k-1)}{2}$ the choice of the series made in the proof of Theorem 4.1 has partly to be modified, by adding the g^1_k once, then.) We omit the related details since the new bound is still of degree 3 in k. Instead, we explain these facts by two examples.

Example 1 ($k = 6$, $g = \frac{(k-2)^2(k-1)}{2} = 40$): let C be a curve of gonality $k = 6$ and genus $g = 40$ on a Hirzebruch surface X_e. Again using the formulae $e_1 = \frac{g}{k-1} - 1 - \frac{k-2}{2}e$ and $x = e_1 + 2 + (k-1)e$ we obtain $e_1 = 7 - 2e$ and so $e \leq 3$ and $x = (7-2e) + 2 + 5e = 9 + 3e$. Hence using the terminology of the proof of Theorem 4.1 we have $W_0 = \{g^1_9, g^3_{15}, g^5_{21}, ...\}$, $W_1 = \{g^2_{12}, g^4_{18}, ...\}$, $W_2 = \{g^3_{15}, g^5_{21}, ...\}$ and $W_3 = \{g^4_{18}, g^6_{24}\}$; so $e^{(0)}_1 = 1$ ($e = 3$), $e^{(1)}_1 = 3$ ($e = 2$), and the $g^4_{x(3)} = g^4_{18}$ is extremal but the $g^3_{x(2)} = g^3_{15}$ is not. (The $g^3_{15} \in W_0$ is very ample and moves $C = C_0$ into a smooth quadric in $\mathbb{P}^3$ thus being of type $(6, 9)$ thereon whereas the $g^3_{15} \in W_2$ is not very ample moving $C = C_2$ in a quadric cone whose vertex becomes a triple point of the image curve.) However, in W_2 the second series $g^5_{21} = |g^3_{15} + g^1_6|$ turns out to be extremal. Thus C is (for any e) a Castelnuovo curve.

Example 2 ($k = 5$, $g = 14$): let C be a 5-gonal curve of genus $g = 14$ on a Hirzebruch surface X_e. Then $e_1 = \frac{g}{4} - 1 - \frac{3}{2}e$ whence $e = 1 = e_1$ and $x = e_1 + 2 + 4e = 7$; so $W_1 = \{g^2_7, g^4_{12}\}$. In particular, C is birational to a plane septic with a single double point (an ordinary node or cusp).

Conversely, let C be the normalization of an integral plane septic with a single double point. Then C has genus 14 and a base point free g^1_5 (obtained by this double point), and if C would have gonality $k < 5$ then C would have genus $g \leq (5-1)(k-1) \leq 12$, a contradiction. The series $|g^2_7 + g^1_5|$ is a g^r_{12} with $r \geq 4$, and if $r > 4$ then C has Clifford index $c \leq 12 - 2r \leq 2$, a contradiction. Since the g^2_7 is simple so is the g^4_{12} obtained. The g^4_{12} is not extremal but, according to its construction (or by Harris [13], 3.15 i), it moves C into a cubic scroll in $\mathbb{P}^4$. Since C is 5-gonal this scroll cannot be a cone; hence it is isomorphic to X_1, and the image curve $C' \sim 5C_0 + 7f$ of C upon it has arithmetic genus 14 whence it is isomorphic to C. (Writing $C' \sim 7(C_0 + f) - 2C_0$ we observe that C' likewise is the blowing-up of our plane septic at its singularity, [1], V, 4.8.1.)

Some elementary calculation concerning Castelnuovo's genus bound shows that a Castelnuovo curve of genus 14 must be trigonal. Consequently, if C is birational to a plane septic with a single double point then it is not a Castelnuovo curve though it lies on a Hirzebruch surface.

In the exceptional cases $g = 20, 25, 35$ for $k = 6$ mentioned above there exist 6-gonal curves on Hirzebruch surfaces which are not Castelnuovo curves: for $g = 20$ always; for $g = 25$ iff $e = 1$, and for $g = 35$ iff $e \neq 1$. This follows easily from the following Proposition implying that extremal series on C_e cannot be found outside W_e unless $g = \frac{(k-1)k}{2}$ (a fact already hinted at by remark (ii) in Sect. 3).

Proposition 4.2 *Let C be a curve of gonality $k \geq 4$ which is not a smooth plane curve. Then an extremal series on C (thus moving C in a Hirzebruch surface X_e such that $\frac{k-2}{2}e = \frac{g}{k-1} - (e_1 + 1)$) is one of the $e_1 + 1$ series in W_e.*

Proof Let g^r_d on C be extremal, and assume that C is not a smooth plane curve of degree $k+1 \geq 5$. Then $r \geq 3$, and C lies via this series on a scroll S of degree $r-1$ in $\mathbb{P}^r$, hence on a Hirzebruch surface X_e such that for some section $H \sim C_0 + nf$ $(n \geq e)$ the series $|H|$ on X_e maps X_e birationally on S (even $S \cong X_e$ for $n > e$); so H corresponds to a hyperplane section of S and $|H|_C| = g^r_d$. In particular, $r-1 = \deg(S) = H^2 = 2n-e$ and $d = C \cdot H = (kC_0 + xf) \cdot (C_0 + nf) = x + k(n-e)$.

Assume that $g^r_d \notin W_e$.

Since $g^r_d = |(C_0+nf)|_C| = |((C_0+ef)+(n-e)f)|_C| = |\Gamma + (n-e)g^1_k|$ we see that (according to the definition of W_e) $n-e > e_1$; then the scroll S is smooth, and we have $n \leq n + e_1 < n + (n-e) = 2n - e = r - 1$. There is also a lower bound for n: The (complete) g^r_d has Clifford index $\mathrm{cliff}(g^r_d) = d - 2r = (x + k(n-e)) - 2(2n-e+1) = x + (k-4)n - (k-2)e - 2 = (e_1 + 2 + (k-1)e) + (k-4)n - (k-2)e - 2 = e_1 + e + (k-4)n < n + (k-4)n = (k-3)n$, i.e. we have $n > \frac{1}{k-3}\mathrm{cliff}(g^r_d)$.

Since $H \cdot f = 1$ we may write $C \sim kH + \beta f$ (as, in fact, is done in [6], p. 121, or [13], p. 91; note that $\beta = x - kn$ may be negative); then $d = C \cdot H = k(r-1) + \beta$, and (as is noted loc. cit.) the g^r_d is extremal iff $-(r-2) \leq \beta \leq 1$. Since our g^r_d is extremal we have $\beta \geq 2-r$, and so we obtain $n > \frac{d-2r}{k-3} = \frac{k(r-1)+\beta-2r}{k-3} = r - 1 + \frac{r-3+\beta}{k-3} \geq r - 1 - \frac{1}{k-3} \geq r-2$, a contradiction.

If, however, C is a smooth plane curve of degree $d = k+1 \geq 5$ then $e = 1$, $e_1 = 0$, $W_1 = \{g^2_d\}$, and $|2g^2_d| = g^5_{2d} \notin W_1$ is extremal (moving C in a Veronese surface) unless $k = 4$ in which case $|2g^2_d| = g^5_{10} = |K_C|$, and according to our definition the canonical series is not considered as being extremal. ◻

The following Corollary generalizes Example 2.

Corollary 4.3 *A Hirzebruch surface of invariant 1 contains for every integer $k \geq 5$ a k-gonal curve which is not a Castelnuovo curve.*

Proof The linear series $|kC_0 + (k+2)f| = |(k+2)(C_0+f) - 2C_0|$ on X_1 contains for $k \geq 2$ a smooth irreducible curve C ([1], V, 2.18; C is the blowing-up of an integral plane curve of degree $d = k+2$ at its single double point, an ordinary node or cusp ([1], V, 4.8.1)). Hence C is k-gonal [2, Corollary 1] of genus $g = (k-1)\big((k+2) - 1 - \frac{1}{2}ke\big) = \frac{(d-3)d}{2} = \frac{(d-2)(d-1)}{2} - 1 > 1$, and $e_1 = \frac{g}{k-1} - 1 - \frac{k-2}{2}e = \frac{d}{2} - 1 - \frac{d-4}{2} = 1$. For $k \geq 4$ we thus have $W_1 = \{g^2_d, g^4_{2d-2}\}$. One easily computes that $\pi(2d-2, 4) \geq \frac{(d-3)(2d-3)}{3} > \frac{(d-3)d}{2} = g$ for $d > 6$, and by Proposition 4.2 this suffices to conclude that C is for $k \geq 5$ never a Castelnuovo curve. ◻

Note that $g = \frac{(k-1)k}{2} + (k-1)$ in Corollary 4.3; this is near to the lower bound for g (cf. the Remark (i) in Sect. 3). One observes (by combining [6], III, Theorem 2.5 and [2], Corollary 1) that an extremal curve of degree d in $\mathbb{P}^r$ has gonality k satisfying $k = m_0 + 1$ resp. $m_0 \leq k \leq m_0 + 1$ with $m_0 := \left[\frac{d-1}{r-1}\right]$ if $(r-1) \nmid (d-1)$ resp. $(r-1) \mid (d-1)$; hence computing m_0 for g^r_d in W_e it is easy to check that for $k \geq 5$ there always is another generalization of Example 2, of genus g near to the bound $\frac{(k-2)^2(k-1)}{2}$ stated directly after Theorem 4.1: Namely we have that a k-gonal curve of genus $g = \frac{(k-2)^2(k-1)}{2} - (k-1)$ on a Hirzebruch surface of invariant $e = k-4$ (then $e_1 = e \geq 1$) cannot be a Castelnuovo curve. In particular, it follows that for even k the lower genus-bound $\frac{(k-2)^2(k-1)}{2}$ obtained for the validity of Theorem 4.1 is the best possible. To be more precise: $g = \frac{(k-1)k}{2} + (k-1)$ resp. $g = \frac{(k-2)^2(k-1)}{2} - (k-1)$ *is the smallest resp. greatest genus of a k-gonal non-Castelnuovo* *curve on a Hirzebruch surface, for* $k \geq 5$. (For even k we just have seen this. For odd k one still has to exclude the cases $g = \frac{(k-1)k}{2} + \frac{k-1}{2}$ and $g = \frac{(k-2)^2(k-1)}{2} - \frac{k-1}{2}$; in fact, in the first case we have $e = 0$, $\Gamma = g^1_{\frac{k+3}{2}}$ which is impossible since $\frac{k+3}{2} < k$, and in the second case $\Gamma = g^{e+1}_x$ for e satisfying $e_1(e) = e_1^{(0)}$ is an extremal $g^{k-2}_{\frac{(k-3)(2k-1)}{2}+1}$.)

To apply these results we note the

Example 3 We inspect the curves C of genus $g = 3p$ for a prime number $p > 2$ which lie on a Hirzebruch surface X_e and have gonality $k > 3$. Since $g = (k-1)\big(x - 1 - \frac{1}{2}ke\big)$ we clearly have $(k-1) \mid 2g = 6p$, and by the Remark (i) in Sect. 3 we know that $(k-1)k \leq 2g = 6p$. Thus it follows that $k = 4$ or $k = 7$ unless $p = 5$ in which case C is quadrigonal or a smooth plane septic. By the Theorem, C is a Castelnuovo curve for $k = 4$, and if $k = 7$ it is a Castelnuovo curve for all primes $p \geq 25$. Let $k = 7$; then $g \geq 21$, i.e. $p \geq 7$. Checking the few prime numbers between 7 and 23 it turns out (using Proposition 4.2) that C is a Castelnuovo curve if and only if $p \in \{7, 17, 19\}$. Consequently, C is always a Castelnuovo curve unless we have $k = 7$ and $g = 33, 39, 69$.

Having dealt with the question if $C \subset X_e$ actually has extremal series we conclude this section by clarifying how to find them all (in particular, how many there are, if any). By Proposition 4.2 they lie in W_e (unless $g = \frac{(k-1)k}{2}$), and we will show that they "stick together" therein, i.e. are consecutive members in W_e. More precisely, we supplement Proposition 4.2 by the

Lemma 4.4 *A series in* W_e *of dimension* $e' + 1$ *(then* $e' \equiv e$ *mod 2) is extremal if and only if* $\frac{2g}{(k-1)k} \leq e' \leq \frac{2g}{(k-2)(k-1)}$.

Proof We recall (from the proof of Theorem 4.1) that the series in W_e of dimension $e' + 1$ has degree $x(e') = e_1(e') + 2 + (k-1)e'$. By the CC-formula, $e_1(e') + 1 = \frac{g}{k-1} - \frac{k-2}{2}e'$; hence we see that $\frac{2g}{(k-1)k} \leq e' \leq \frac{2g}{(k-2)(k-1)}$ just means $e' \geq e_1(e') + 1 \geq 0$.

Assume that $e' \geq e_1(e') + 1 \geq 0$. If $e' = 0$ it follows that $e_1(e') + 1 = 0$ and, then, $x(e') = 1$, a contradiction. Hence $e' > 0$, and so our $g^{e'+1}_{x(e')} \in W_e$ is simple. As in the proof of Theorem 4.1 we follow [12] and write $x(e') = me' + q$ for some integer $q = 2, ..., e' + 1$. If $e_1(e') \geq 0$ we thus have $m = k-1$, $q = e_1(e') + 2$, and as in the proof of Theorem 4.1 this implies that $g^{e'+1}_{x(e')}$ is extremal. Let $e_1(e') = -1$. Then $x(e') = 1 + (k-1)e' = (k-2)e' + e' + 1$ whence $m = k-2$, $q = e' + 1$, and by [12], 3.3 we obtain $\pi(x(e'), e' + 1) = m(m-1)\frac{e'}{2} + m(q-1) = (k-2)((k-3)\frac{e'}{2} + e') = \frac{(k-2)(k-1)}{2}e'$. But $-1 = e_1(e') = \frac{g}{k-1} - 1 - \frac{k-2}{2}e'$ shows that $\pi(x(e'), e' + 1) = g$ again, and so our $g^{e'+1}_{x(e')}$ is extremal in this case, too.

Conversely, assume that our $g^{e'+1}_{x(e')}$ is extremal. Then $e' > 0$, and we observed earlier that $m_0 := \big[\frac{x(e')-1}{e'}\big]$ is $k-1$ if $e' \nmid (x(e') - 1)$ resp. is $k-1$ or k if $e' \mid (x(e') - 1)$. Since $x(e') - 1 = (k-1)e' + e_1(e') + 1$ it follows that we must have $0 \leq \frac{e_1(e')+1}{e'} < 1$ in the case $m_0 = k-1$, i.e. $0 \leq e_1(e') + 1 < e'$. In the case $m_0 = k$ we know that $e' \mid (x(e') - 1)$ whence $k = \big[\frac{x(e')-1}{e'}\big] = \frac{x(e')-1}{e'} = k - 1 + \frac{e_1(e')+1}{e'}$ and so $e_1(e') + 1 = e' > 0$. □

Example 4 Let $g = 60$, $k = 5$, and assume that C lies on X_e. Then $e \in \{0, 2, 4, 6\}$ (by the CC-formula, and since $e \leq e_1(e) + 2$), and C has always precisely three extremal series, namely g^7_{31} ($e' = 6$ in Lemma 4.4), g^9_{36} ($e' = 8$; this is the series coming from Theorem 4.1) and g^{11}_{41} ($e' = 10$).

Conversely, an extremal curve of degree 31 in $\mathbb{P}^7$ is a 5- or 6-gonal curve of genus 60. If it is 5-gonal it has (besides the g^7_{31}) exactly two further extremal series, of degree 36 resp. 41 (as we have seen just before). If it is 6-gonal (then $e \in \{0, 2, 4\}$) it has always precisely one further extremal series, namely a g^5_{25} (the series alluded to in Theorem 4.1).

Corollary 4.5 *Let $C \subset X_e$, $k \geq 4$ and $g \neq \frac{(k-1)k}{2}$. Then there is an extremal series of dimension $e' + 1$ on C if and only if e' in an integer between $\frac{2g}{(k-1)k}$ and $\frac{2g}{(k-2)(k-1)}$ satisfying $e' \equiv e$ mod 2.*

Proof By Proposition 4.2 and Lemma 4.4 it suffices to show that for $C \subset X_e$ any integer e' (if any) in the closed intervall $\left[\frac{2g}{(k-1)k}, \frac{2g}{(k-2)(k-1)}\right]$ satisfies $e \leq e' \leq e + 2e_1$, i.e. $e' + 1$ is in the range of the dimensions of the series in W_e. In fact, since we always have $e \leq e_1 + 2 = \frac{g}{k-1} + 1 - \frac{k-2}{2}e$ it follows that $e \leq \frac{2g}{(k-1)k} + \frac{2}{k} \leq e' + \frac{2}{k} < e' + 1$, i.e. $e' \geq e$. And since $\frac{g}{k-1} = e_1 + 1 + \frac{k-2}{2}e$ we see that $e' \leq \frac{2g}{(k-2)(k-1)} = \frac{2}{k-2} \cdot \frac{g}{k-1} = \frac{2e_1+2}{k-2} + e$; so if $\frac{2e_1+2}{k-2} \leq 2e_1$ we obtain $e' \leq e + 2e_1$, as wanted. But $\frac{2e_1+2}{k-2} \leq 2e_1$ means $e_1 \geq \frac{1}{k-3}$, and for $e_1 > 0$ we are done. So let $e_1 = 0$. Then we are in the situation discussed at the end of the proof of Lemma 3.1, and so we have $e = 2$ and $g = (k-1)^2$. It follows that $e' = 2 = e$ unless $k = 4$ (and $e' = 3$). But for $e_1 = 0$, $e = 2$, $k = 4$, $g = 9$ we have $W_2 = \{g^3_8\}$, and by Proposition 4.2 the claim of our Corollary is obvious in this case. □

Example 5 Let $C \subset X_e$ be of *odd* gonality $k \geq 5$ and genus $g = \frac{(k-2)^2(k-1)}{2} - (k-1)$. For that genus, the only integer e' between $\frac{2g}{(k-1)k} = k - 4 + \frac{2}{k}$ and $\frac{2g}{(k-2)(k-1)} = k - 2 - \frac{2}{k-2}$ is $k - 3$. Since the CC-formula shows that for odd k also e is odd it follows that $e' \not\equiv e$ mod 2, and so C cannot be a Castelnuovo curve.

More generally, by the CC-formula we noticed (in the proof of Theorem 4.1) that for curves of fixed *odd* gonality and fixed genus which lie on Hirzebruch surfaces the invariants of these surfaces all have the same parity. Hence it follows that the answer to the question if such a curve is a Castelnuovo curve does not depend on the specific value of the invariant of the Hirzebruch surface containing it; so *either all or none of these curves are Castelnuovo curves*. Conversely we note

Corollary 4.6 *If no curve C of fixed gonality $k \geq 4$ and fixed genus $g \geq \frac{(k-3)(k-1)k}{4}$ lying on some Hirzebruch surface is a Castelnuovo curve then k is odd.*

Proof We may assume that $k \geq 5$. Recall that for $C \subset X_e$ the number $n := \frac{2g}{k-1}$ is an integer. For the integral part $\left[\frac{n}{k-2}\right]$ of $\frac{n}{k-2}$ we have $\left[\frac{n}{k-2}\right] = \frac{n-\delta}{k-2} \geq \frac{n-(k-3)}{k-2}$ for some integer $0 \leq \delta \leq k-3$; thus if $\frac{n-k+3}{k-2} \geq \frac{n}{k}$, i.e. if $n \geq \frac{(k-3)k}{2}$, then $\left[\frac{n}{k-2}\right] \geq \frac{n}{k}$, and so there is an integer e' between $\frac{2g}{(k-1)k} = \frac{n}{k}$ and $\frac{2g}{(k-2)(k-1)} = \frac{n}{k-2}$. Clearly, $n \geq \frac{(k-3)k}{2}$ is just our genus bound $g \geq \frac{(k-3)(k-1)k}{4}$.

Assume that k is even. Then we can find such a curve C on a Hirzebruch surface X_{e_0} whose invariant e_0 satisfies $e_0 \equiv e'$ mod 2. (In fact, $g \geq \frac{(k-3)(k-1)k}{4}$ implies that $g \geq (k-1)^2$ for $k \geq 6$, and since we noticed in the proof of Corollary 4.5 that $e \leq \frac{2g}{(k-1)k} + \frac{2}{k}$ and since k is even it follows that any integer between 0 and $\frac{2}{k}\left(\frac{g}{k-1} + 1\right) \geq 2$ is possible for e. So we can choose $e_0 = 0$ resp. $e_0 = 1$ if e' is even resp. odd.) Hence according to Corollary 4.5 C_{e_0} (on X_{e_0}) is a Castelnuovo curve. □

5 Moduli

Theorem 4.1 implies the

Corollary 5.1 *If a curve C of gonality $k \geq 3$ and genus $g > \frac{(k-2)(k-1)^2}{2}$ lies on a (not assigned) Hirzebruch surface then it is represented by a point in a locus of codimension $\frac{k-3}{k-1}g$ in the moduli space $M_g(k)$ of k-gonal curves of genus g.*

Proof We may assume that $k \geq 4$. The extremal series on C used in the proof of Theorem 4.1 are $g^{e+1}_{x(e)}$ with $e > e_1(e)$, for $e \geq 2$ such that $e_1(e) = e_1^{(0)}$ (resp. also $e_1(e) = e_1^{(1)}$ in the case of even k). Again writing (as we did there) $x(e) = me + q$ with $m = k - 1$ and $2 \leq q = e_1(e) + 2 \leq e + 1$ we apply the last formula in [12], section 6, counting the moduli of extremal curves. Here this number is

$D = g + m(e+2) + 2q - (e+1) - 4 = g + (k-1)(e+2) + 2(e_1(e)+2) - e - 5 = g + (k-1)(e+2) + 2\big(\frac{g}{k-1} + 1 - \frac{k-2}{2}e\big) - e - 5 = g + \frac{2g}{k-1} + 2k - 5 = (2g + 2k - 5) - \frac{k-3}{k-1}g$, and since $\dim(M_g(k)) = 2g + 2k - 5$ we are done. (In the case $q = e + 1 \geq 4$ there are also extremal curves of degree $x(e)$ in $\mathbb{P}^{e+1}$ having gonality $m + 2 = k + 1$. But in the Hilbert scheme of extremal curves of degree $x(e)$ in $\mathbb{P}^{e+1}$ they only constitute a lower dimensional irreducible component, cf. [13], 3.12 ii).) □

Our moduli count for Castelnuovo curves can be generalized, which is useful for computing the number of moduli in the low genus cases left out in Corollary 5.1. To begin with, we recall the (omitted) count for an extremal curve C in $\mathbb{P}^2$, i.e for a smooth plane curve of degree $d > 4$: it has gonality $k = d - 1$, genus $g = \frac{(k-1)k}{2}$, and since it has merely one g^2_d it depends on only $(h^0(\mathbb{P}^2, C) - 1) - \dim(PGL_3(\mathbb{C})) = \frac{1}{2}d(d+3) - 8 = \frac{1}{2}(k+1)(k+4) - 8 = \frac{1}{2}k(k+1) + 2k - 6 = (k+1)\frac{g}{k-1} + 2k - 6 = (2g - \frac{k-3}{k-1}g) + 2k - 6 = \dim(M_g(k)) - \big(\frac{k-3}{k-1}g + 1\big)$ moduli.

Having dealt with the case of smallest genus $g = \frac{(k-1)k}{2}$ for $C \subset X$ we turn to curves of bigger genus. First we prove an auxiliary general result, namely the

Lemma 5.2 *For an integer $k \geq 3$ let g^1_k be a complete and base point free pencil and $L_1, ..., L_{k-2}$ be complete, base point free and simple linear series of dimension at least 2 on a curve C. Then $\dim|L_1 + \cdots + L_{k-2} + g^1_k| \geq \dim|L_1 + \cdots + L_{k-2}| + k - 1$.*

Proof Let $E = P_1 + \cdots + P_k$ be a general divisor in g^1_k; in particular, these points P_j of C are pairwise different. Since L_i ($i = 1, ..., k-2$) is base point free and simple of dimension at least 2 the series $|L_i - P_i|$ is base point free; hence we can take $F_i \in |L_i - P_i|$ such that P_j is not contained in the support of F_i, for all $i = 1, ..., k-2$ and $j = 1, ..., k$. Since $|L_1 + \cdots + L_{k-2}|$ is base point free the complete series $|F_1 + \cdots + F_{k-2} + E| = |L_1 + \cdots + L_{k-2} + P_{k-1} + P_k|$ can only have P_{k-1} or P_k as base points; but since the g^1_k is base point free the subpencil $F_1 + \cdots + F_{k-2} + g^1_k$ in $|F_1 + \cdots + F_{k-2} + E|$ has base points only in the support of the divisor $F_1 + \cdots + F_{k-2}$ which does not contain P_{k-1} or P_k. Hence $|L_1 + \cdots + L_{k-2} + P_{k-1} + P_k|$ is base point free which implies that $\dim|L_1 + \cdots + L_{k-2} + P_{k-1} + P_k| \geq \dim|L_1 + \cdots + L_{k-2}| + 1$.

Assume that we have already shown that $\dim|L_1 + \cdots + L_{k-2} + P_{k-i} + P_{k-i+1} + \cdots + P_{k-1} + P_k| \geq \dim|L_1 + \cdots + L_{k-2}| + i$ for some integer i with $1 \leq i \leq k-2$.

The series $|L_1 + \cdots + L_{k-2} + P_{k-i-1} + P_{k-i} + \cdots + P_{k-1} + P_k| = |(F_1 + P_1) + \cdots + (F_{k-i-2} + P_{k-i-2}) + L_{k-i-1} + \cdots + L_{k-2} + P_{k-i-1} + P_{k-i} + \cdots + P_{k-1} + P_k| = |F_1 + \cdots + F_{k-i-2} + L_{k-i-1} + \cdots + L_{k-2} + E|$ contains the subseries $F_1 + \cdots + F_{k-i-2} + L_{k-i-1} + \cdots + L_{k-2} + g^1_k$ for which P_{k-i-1} is not a base point since the series $L_{k-i-1} + \cdots + L_{k-2} + g^1_k$ is base point free and P_{k-i-1} is not in the support of the divisor $F_1 + \cdots F_{k-i-2}$. Consequently,

P_{k-i-1} is not a base point of $|L_1 + \cdots + L_{k-2} + P_{k-i-1} + P_{k-i} + \cdots + P_{k-1} + P_k|$, and so we have $\dim|L_1 + \cdots + L_{k-2} + P_{k-i-1} + P_{k-i} + \cdots + P_{k-1} + P_k| \geq \dim|L_1 + \cdots + L_{k-2} + P_{k-i} + \cdots + P_{k-1} + P_k| + 1 \geq (\dim|L_1 + \cdots + L_{k-2}| + i) + 1 = \dim|L_1 + \cdots + L_{k-2}| + i + 1$. □

In our situation this Lemma has a remarkable consequence.

Theorem 5.3 *For $C \subset X_e$ of gonality $k \geq 4$ the set $W_e = W_e(g_k^1)$ is an intrinsic notion (i.e. does not depend on the specific Hirzebruch surface containing C).*

Proof Let $C \sim kC_0 + xf$ on $X = X_e$ and (as before) $\Gamma = |(C_0 + ef)|_C| = |eg_k^1 + C_0|_C|$. Let X' be another Hirzebruch surface containing C; then X' has the same invariant e as X (by the CC-formula), and the corresponding set $W'_e(g_k^1) = \{|\Gamma' + ig_k^1| : i = 0, ..., e_1\}$ formed w.r.t. X' is designed by the same gonality pencil g_k^1 of C as for $W_e(g_k^1)$ (so $\Gamma' = |eg_k^1 + C'_0|_C|$). Assume that Γ and Γ' are different g_x^{e+1} on C; recall that $x \geq ke$.

Let $e > 0$. Then Γ and Γ' are base point free and simple of dimension at least 2. Applying Lemma 5.2 we obtain $\dim|(k-3)\Gamma + \Gamma' + g_k^1| \geq \dim|(k-3)\Gamma + \Gamma'| + k - 1$. By the Riemann-Roch theorem for C we thus have $\dim|K_C - ((k-3)\Gamma + \Gamma' + g_k^1)| \geq \dim|K_C - ((k-3)\Gamma + \Gamma')| - 1$.

Let $\dim|K_C - ((k-3)\Gamma + \Gamma' + g_k^1)| = \dim|K_C - ((k-3)\Gamma + \Gamma')|$. Then $|(k-3)\Gamma + \Gamma'|$ is non-special, and so $\dim|(k-3)\Gamma + \Gamma'| = (k-2)x - g = (k-2)x - (k-1)(x - 1 - \frac{k}{2}e) = (k-1)\frac{k}{2}e + (k-1) - x \leq (k-1)\frac{k}{2}e + (k-1) - ke = (k-3)\frac{k}{2}e + k - 1$. But since we have $x \geq (k-2)e + 2 = (k-3)\dim(\Gamma) + \dim(\Gamma') + 1 - (k-3)$ and $x \geq e + 1 = \dim(\Gamma')$ the hypotheses for applying [12], 4.2 are satisfied, and so we conclude that $\dim|(k-3)\Gamma + \Gamma'| \geq \left(\frac{(k-3)(k-2)}{2}(e+1) - \frac{(k-4)(k-3)}{2}\right) + (e+1) + (k-3)(e+1) = \frac{(k-2)(k-1)}{2}(e+1) - \frac{(k-4)(k-3)}{2}$. Thus we obtain the contradiction $k - 4 + e \leq 0$.

Let $\dim|K_C - ((k-3)\Gamma + \Gamma' + g_k^1)| = \dim|K_C - ((k-3)\Gamma + \Gamma')| - 1$. Then, according to [14], Lemma 1.8 the base point free part of $|K_C - ((k-3)\Gamma + \Gamma')|$ is $m \cdot g_k^1$ with $m = \dim|K_C - ((k-3)\Gamma + \Gamma')| \geq 0$, and we obtain $2g - 2 - (k-2)x = \deg(K_C - ((k-3)\Gamma + \Gamma')) \geq m \cdot k = k(g - 1 - (k-2)x + \dim|(k-3)\Gamma + \Gamma'|)$ whence $\dim|(k-3)\Gamma + \Gamma'| \leq \frac{k-2}{k}((k-1)x - g + 1) = (k-2)(1 + \frac{k-1}{2}e)$. But by the inequality obtained above by [12], 4.2 it follows that $k \leq 3$, a contradiction.

Let $e = 0$. Then we have $g = (k-1)(x-1)$, and Γ, Γ' are two (by assumption different) base point free pencils of degree $x \geq k$. Let $L := |\Gamma + g_k^1|$ and $L' := |\Gamma' + g_k^1|$; then these series (in $W_0(g_k^1)$ resp. $W'_0(g_k^1)$) are two different very ample webs of degree $x + k$. Using these webs we proceed as in the case $e > 0$. By Lemma 5.2 we have $\dim|(k-3)L + L' + g_k^1| \geq \dim|(k-3)L + L'| + k - 1$. Since $x \geq k$ we can apply [12], 4.2 and obtain $\dim|(k-3)L + L'| \geq (3\frac{(k-3)(k-2)}{2} - \frac{(k-4)(k-3)}{2}) + 3 + 3(k-3) = k^2 - k - 3$. If $|(k-3)L + L'|$ is non-special we have $\dim|(k-3)L + L'| = (k-2)(x+k) - g = (k-2)(x+k) - (k-1)(x-1) = k^2 - k - 1 - x$ which leads to the contradiction $x \leq 2$. If $|(k-3)L + L'|$ is special then (again by [14], Lemma 1.8) $|K_C - ((k-3)L + L')| = m \cdot g_k^1 + F$ with $m = \dim|K_C - ((k-3)L + L')| \geq 0$ and some effective divisor F of C whence $2g - 2 - (k-2)(x+k) = \deg(K_C - ((k-3)L + L')) \geq k(g - 1 - (k-2)(x+k) + \dim|(k-3)L + L'|)$ which implies that $\dim|(k-3)L + L'| \leq (k-2)k$; then $(k-2)k \geq \dim|(k-3)L + L'| \geq k^2 - k - 3$, and we obtain the contradiction $k \leq 3$.

After all, we see that $\Gamma' = \Gamma$ and, then, $W'_e(g_k^1) = W_e(g_k^1)$. □

Remark (i) If C is a Castelnuovo curve the claim of Theorem 5.3 is just a consequence of Proposition 4.2 since a curve cannot have more than one extremal series of a fixed degree.

(ii) Our last two results Lemma 5.2 and Theorem 5.3 generalize the last part of the proof of the main result (Theorem 3.1) of [9] (cf. Claim 2 etc. there). Moreover, [9] contains a natural interpretation of W_e for $k = 4$, based on the description of complete and very ample series on quadrigonal curves.

(iii) The proof of Theorem 5.3 shows not only the uniqueness of Γ but, e.g. for $e > 0$, more generally that a complete, base point free and simple g_x^{e+1} on C is unique, and $|g_x^{e+1} - eg_k^1| \neq \emptyset$.

(iv) At least if the gonality of $C \subset X_e$ is a prime number the series Γ is of maximal index of speciality among all complete and base point free linear series on C which are not compounded of a fixed gonality pencil g_k^1 of C. In fact, let g_d^r be complete and base point free on C. Then, according to the Remark in Sect. 3 of [15], $d \leq e_{k-1}+r$ implies for prime k that $g_d^r = rg_k^1$, and $d \leq e_{k-1}+r$ just means $h^1(g_d^r) > h^1(\Gamma)$ since $e_{k-1} = e_1+(k-2)e$ (cf. Section 1), i.e. $e_{k-1} = x-2-e = x-h^0(\Gamma) = g-1-h^1(\Gamma)$.

Corollary 5.4 *A curve C of gonality $k \geq 4$ which is not a smooth plane curve can have only finitely many complete and very ample linear series moving C into a rational normal surface scroll.*

Proof We observed in Remark (ii) in Sect. 3 that such a series g_d^r (thus satisfying $\dim|g_d^r - g_k^1| = r-2$) is of form $|\Gamma + ig_k^1|$ for some integer i with $0 \leq i \leq e_1+1$, and $C \subset X$ has only one gonality pencil g_k^1 unless C is a smooth plane curve (which has infinitely many gonality pencils, but this case is excluded here) or a curve of type (k,k) on a smooth quadric in $\mathbb{P}^3$ (in which case it has exactly two gonality pencils); cf. [2]. Hence our claim follows from Theorem 5.3. □

In generalization of Corollary 5.1 we now have

Corollary 5.5 *The curves C of fixed gonality $k \geq 3$ and fixed genus $g \neq \frac{(k-1)k}{2}$ lying on Hirzebruch surfaces constitute a locus of codimension $\frac{k-3}{k-1}g$ in $M_g(k)$.*

Proof We may assume that $k \geq 4$. There is a complete and very ample linear series g_d^r on $C \subset X$ embedding C in a rational normal scroll S in $\mathbb{P}^r$. Since, according to Remark (i) in Sect. 3, C is not a smooth plane curve we have $r \geq 3$, and by Corollary 5.4 there are only finitely many (in fact, at most two) such g_d^r on C. We write $C \sim kH + \beta f$ (as in the proof of Proposition 4.2) where the curve H on X corresponds to a hyperplane section of S. Then, within the Hilbert scheme $I_{d,g,r}$ of integral and non-degenerate curves of arithmetic genus g and degree d in $\mathbb{P}^r$, C belongs to the locus $I_{k,\beta}$ made up by such curves in $|kH+\beta f|$ on some scroll of degree $r-1$ in $\mathbb{P}^r$, and according to [13], p. 91/92 we have $\dim(I_{k,\beta}) = (h^0(X,C)-1) + \dim(\text{space of rational normal surface scrolls in } \mathbb{P}^r) = \left(\frac{k(k+1)}{2}(r-1) + (k+1)(\beta+1) - 1\right) + ((r-1)(r+3)-3) = \left(\frac{k(k+1)}{2} + r + 3\right)(r-1) + (k+1)(\beta+1) - 4$. (If g is sufficiently near to Castelnuovo's genus bound $\pi(d,r)$ then $I_{k,\beta}$ is an irreducible component of $I_{d,g,r}$; cf. [13], 3.13–3.16.) Observing that $g = \frac{(k-1)k}{2}(r-1) + (k-1)(\beta-1)$ [13, p. 91] it follows that C depends on $\dim(I_{k,\beta})-\dim(PGL_{r+1}(\mathbb{C})) = \dim(I_{k,\beta}) - ((r+1)^2-1) = \frac{k(k+1)}{2}(r-1) + (k+1)(\beta+1) - 7 = (k+1)\left(\frac{k}{2}(r-1) + \beta - 1\right) + 2k - 5 = (k+1)\frac{g}{k-1} + 2k - 5 = (2g+2k-5) - \frac{k-3}{k-1}g = \dim(M_g(k)) - \frac{k-3}{k-1}g$ moduli. □

To clarify the deviation in the number of moduli observed for a smooth plane curve C (though C lies on a Hirzebruch surface) we add the

Remark By blowing up $\mathbb{P}^2$ at a point $P_0 \in C$ a smooth plane curve C of degree $d \geq 5$ lies on X_1 with class $(k+1)(C_0+f) - C_0 = kC_0 + (k+1)f$ ($k = \text{gon}(C) = d-1$), and if we embed X_1 via $H \sim C_0 + 2f$ as a smooth cubic scroll in $\mathbb{P}^4$ then $|H|_C|$ is a very ample g^4_{2k+1}. (Note that $|H|_C| = |(C_0 + 2f)|_C| = |(C_0 + f)|_C + f|_C| = |g^2_d + g^1_k| = |\Gamma + g^1_k| \notin W_1$ with $g^1_k := |f|_C| = |(C_0 + f)|_C - C_0|_C| = |g^2_d - P_0|$, and that already $|(C_0 + 3f)|_C| = |g^2_d + 2g^1_k| = |3g^2_d - 2P_0|$ has dimension $7 > 6 = \dim|C_0 + 3f|$.)

Conversely, any k-gonal smooth, irreducibe and non-degenerate curve of degree $2k+1$ in $\mathbb{P}^4$ which lies on a cubic scroll S is isomorphic to a smooth plane curve of degree $d = k+1$ since it has genus $g = (k-1)((2k+1) - 1 - \frac{1}{2}k(4-1)) = \frac{(k-1)k}{2}$ ([10], Lemma 3.1, and Remark (i) in Sect. 3), and S cannot be a cone (otherwise the projection off its vertex gives a contradiction); thus S is smooth, i.e. $S \cong X_1$.

As in the proof of Corollary 5.5, we can compute the number of moduli of C, by using here our series $g^4_{2k+1} = |2g^2_d - P_0|$. But then we have to take into account that for a smooth plane curve C of degree d the one-dimensional variety $W^4_{2k+1} = \{|2g^2_d - P| : P \in C\} = g^2_d + W^1_k$ entirely consists of very ample series each moving C into a smooth cubic scroll in $\mathbb{P}^4$. Thus our total freedom of choice for the point $P_0 \in C$ blown up to obtain $C \subset X_1$ gives rise to a violation of the claim of Corollary 5.4, and this explains the different outcome in the number of moduli in this special case.

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