An Unexpected Cyclic Symmetry of $I\mathfrak{u}_n$

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Abstract

We find and discuss an unexpected (to us) order n cyclic group of automorphisms of the Lie algebra $I\mathfrak{u}_n := \mathfrak{u}_n \ltimes \mathfrak{u}_n^*$, where $\mathfrak{u}_n$ is the Lie algebra of upper triangular $n \times n$ matrices. Our results also extend to $\mathfrak{gl}_{n+}^\epsilon$, a "solvable approximation" of $\mathfrak{gl}_n$, as defined within.

Keywords Lie algebras · Lie bialgebras · Lie algebra automorphism · Solvable approximation · Triangular matrices

Mathematics Subject Classification 17B40 · 57M25

Given any Lie algebra $\mathfrak{a}$ one may form its "inhomogeneous version" $I\mathfrak{a} := \mathfrak{a} \ltimes \mathfrak{a}^*$, its semidirect product with its dual $\mathfrak{a}^*$ where $\mathfrak{a}^*$ is considered as an Abelian Lie[1] algebra and $\mathfrak{a}$ acts on $\mathfrak{a}^*$ via the coadjoint action. (Over $\mathbb{R}$ if $\mathfrak{a} = \mathfrak{so}_3$ then $\mathfrak{a}^* = \mathbb{R}^3$ and so $I\mathfrak{a} = \mathfrak{so}_3 \ltimes \mathbb{R}^3$ is the Lie algebra of the Euclidean group of rotations and translations, explaining the name).

In general, we care about $I\mathfrak{a}$. It is a special case of the Drinfel'd double / Manin triple construction [12, 13] when the cobracket is 0. These Lie algebras occur in the study of the Kashiwara-Vergne problem [1, 7] and they provide the simplest quantum algebra context for the Alexander polynomial [2, 6]. We care especially for the case where $\mathfrak{a}$ is a Borel subalgebra of a semi-simple Lie algebra (e.g., upper triangular matrices) as then the algebras $I\mathfrak{a}$ are the $\epsilon = 0$ "base case" for "solvable approximation" [3–5, 8, 9], and their automorphisms are expected to become symmetries of the resulting knot invariants.

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[1] Two Norwegians!

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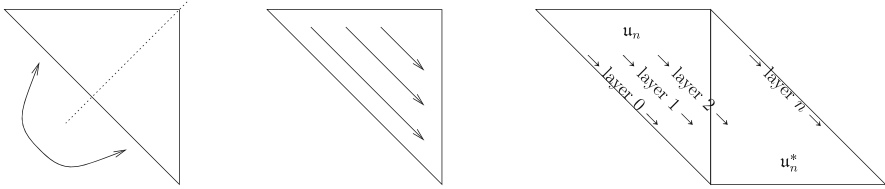
Fig. 1 An expected anti-automorphism (left), an unexpected automorphism (middle), and an alternative presentation of the "layers" table (right)

Let $\mathfrak{u}_n$ be the Lie algebra of upper triangular $n \times n$ matrices over a field in which 2 is invertible. Beyond inner automorphisms, $\mathfrak{u}_n$ and hence $I\mathfrak{u}_n$ has one obvious and expected anti-automorphism Φ corresponding to flipping matrices along their anti-main-diagonal, as shown in the first image of Fig. 1. With x_{ij} denoting the $n \times n$ matrix with 1 in position (ij) and zero everywhere else ($i \leq j$ in $\mathfrak{u}_n$), Φ is given by $x_{ij} \mapsto x_{n+1-j,n+1-i}$.

There clearly isn't an automorphism of $\mathfrak{u}_n$ that acts by "sliding down and right parallel to the main diagonal", as in the second image in Fig. 1. Where would the last column go? Yet the sliding map, when restricted to where it is clearly defined ($\mathfrak{u}_n$ with the last column excluded), does extend to an automorphism of $I\mathfrak{u}_n$ as in the theorem below.

Theorem 1 *With the basis $\{x_{ij}\}_{1 \leq i < j \leq n} \cup \{a_i = x_{ii}\}_{1 \leq i \leq n}$ for $\mathfrak{u}_n$ and dual basis $\{x_{ji}\}_{1 \leq i < j \leq n} \cup \{b_i\}_{1 \leq i \leq n}$ for $\mathfrak{u}_n^*$ (and duality $\langle x_{kl}, x_{ij} \rangle = \delta_{li}\delta_{jk}$, $\langle b_i, a_j \rangle = 2\delta_{ij}$,[2] and $\langle x_{ji}, a_k \rangle = \langle b_k, x_{ij} \rangle = 0$), the map $\Psi : I\mathfrak{u}_n \to I\mathfrak{u}_n$ defined by "incrementing all indices by 1 mod n" (precisely, if ψ is the single-cycle permutation $\psi = (123 \ldots n)$ then Ψ is defined by $\Psi(x_{ij}) = x_{\psi(i)\psi(j)}$, $\Psi(a_i) = a_{\psi(i)}$, and $\Psi(b_i) = b_{\psi(i)}$) is a Lie algebra automorphism of $I\mathfrak{u}_n$.*

Note that our choice of bases, using similar symbols x_{ij}/x_{ji} for the non-diagonal matrices and their duals, hides the intricacy of Ψ; e.g., $\Psi : x_{n-1,n} \mapsto x_{n1}$ maps an element of $\mathfrak{u}_n$ to an element of $\mathfrak{u}_n^*$ (also see Fig. 1, right).

It may be tempting to think that Ψ has a simple explanation in $\mathfrak{gl}_n$ language: $\mathfrak{u}_n$ is a subset of $\mathfrak{gl}_n$, $\mathfrak{gl}_n$ has a metric (the Killing form) such that the dual of x_{ij} is x_{ji} as is the case for us, and every permutation of the indices induces an automorphism of $\mathfrak{gl}_n$. But this explains nothing and too much: nothing because the bracket of $I\mathfrak{u}_n$ simply isn't the bracket of $\mathfrak{gl}_n$ (even away from the diagonal matrices), and too much because *every* permutation of indices induces an automorphism of $\mathfrak{gl}_n$, whereas only ψ and its powers induce automorphisms of $I\mathfrak{u}_n$.

Proof of Theorem 1 Recall that as a vector space $I\mathfrak{u}_n = \mathfrak{u}_n \oplus \mathfrak{u}_n^*$, yet with bracket $[(x, f), (y, g)] = ([x, y], x \cdot g - y \cdot f)$ where $\cdot$ denotes the coadjoint action, $(x \cdot f)(v) = f([v, x])$. With that and some case checking and explicit computations, the commutation relations of $I\mathfrak{u}_n$ are given by

$$[x_{ij}, x_{kl}] = \chi_{\lambda(x_{ij})+\lambda(x_{kl})<n}(\delta_{jk}x_{il} - \delta_{li}x_{kj}) \quad \text{unless both } j = k \text{ and } l = i,$$

$$[x_{ij}, x_{ji}] = \frac{1}{2}(b_i - b_j), \quad [a_i, x_{jk}] = (\delta_{ij} - \delta_{ik})x_{jk}, \quad [b_i, x_{jk}] = 0, \tag{1}$$

$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 0.$$

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[2] The awkward factor of 2 in $\langle b_i, a_j \rangle$ is irrelevant for Theorem 1 yet crucial for Theorem 2. Removing this factor removes the factor $\frac{1}{2}$ in (1), see also Chapter 4 of [13]

Here χ is the indicator function of truth (so $\chi_{5<7} = 1$ while $\chi_{7<5} = 0$), and $\lambda(x_{ij})$ is the "length" of x_{ij}, defined by $\lambda(x_{ij}) := \begin{cases} j - i & i < j \\ n - (i - j) & i > j \end{cases}$.

It is easy to verify that the length $\lambda(x_{ij})$ is Ψ-invariant, and hence everything in (1) is Ψ-equivariant. □

$I\mathfrak{u}_n$ is a solvable Lie algebra (as a semi-direct product of solvable with Abelian, and as will be obvious from the table below). It is therefore interesting to look at the structure of its commutator subalgebras. This structure is summarized in the following table (an alternative view is in Fig. 1):

layer 0	$\mathfrak{g} = I\mathfrak{u}_n$	$a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow$		$a_{n-2} \rightarrow$	$a_{n-1} \rightarrow$	$a_n \rightarrow$
layer 1	$\mathfrak{g}'_1 = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$	$x_{12} \rightarrow x_{23} \rightarrow \cdots \rightarrow$	$x_{n-2,n-1} \rightarrow$	$x_{n-1,n} \rightarrow$	$x_{n1} \rightarrow$
layer 2	$\mathfrak{g}'_2 = [\mathfrak{g}', \mathfrak{g}'_1]$	$x_{13} \rightarrow x_{24} \rightarrow \cdots \rightarrow$	$x_{n-2,n} \rightarrow$	$x_{n-1,1} \rightarrow$	$x_{n2} \rightarrow$
layer 3	$\mathfrak{g}'_3 = [\mathfrak{g}', \mathfrak{g}'_2]$	$x_{14} \rightarrow x_{25} \rightarrow \cdots \rightarrow$	$x_{n-2,1} \rightarrow$	$x_{n-1,2} \rightarrow$	$x_{n3} \rightarrow$
$\vdots$	$\vdots$	$\vdots$ $\vdots$	$\vdots$	$\vdots$	$\vdots$
layer $(n-1)$	$\mathfrak{g}'_{n-1} = [\mathfrak{g}', \mathfrak{g}'_{n-2}]$	$x_{1n} \rightarrow x_{21} \rightarrow \cdots \rightarrow$	$x_{n-2,n-1} \rightarrow$	$x_{n-1,n-2} \rightarrow$	$x_{n,n-1} \rightarrow$
layer n	$\mathfrak{g}'_n = [\mathfrak{g}', \mathfrak{g}'_{n-1}]$	$b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow$	$b_{n-2} \rightarrow$	$b_{n-1} \rightarrow$	$b_n \rightarrow$

In this table (all assertions are easy to verify):

- Apart for the treatment of the a_i's and the b_i's, layer=length= $\lambda(x_{ij})$.
- The layers indicate a filtration; each layer should be considered to contain all the ones below it. The generators marked at each layer generate it modulo the layers below.
- The bracket of an element at layer p with an element of layer q is in layer $p + q$ (and it must vanish if $p + q > n$).
- If $p \geq 2$, every generator in layer p is the bracket of a generator in layer 1 with a generator in layer $p - 1$.
- In layer p, the first $n - p$ generators indicated belong to $\mathfrak{u}_n$ and the last p belong to $\mathfrak{u}_n^*$. So as we go down, $\mathfrak{u}_n^*$ slowly "overtakes" the table.
- The automorphism Ψ acts by following the arrows and shifting every generator one step to the right (and pushing the rightmost generator in each layer back to the left).
- The anti-automorphism Φ acts by mirroring the $\mathfrak{u}_n$ part of each layer left to right and by doing the same to the $\mathfrak{u}_n^*$ part, without mixing the two parts.
- Note that $I\mathfrak{u}_n$ can be metrized by pairing the $\mathfrak{u}_n$ summand with the $\mathfrak{u}_n^*$ one. The metric only pairs generators indicated in layer p with generators indicated in layer $(n - p)$.

Note also that the brackets of the generators indicated in layer 1 yield the generators indicated in layer 2 as follows:
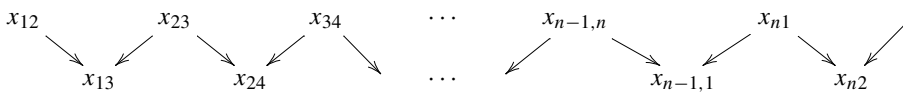
(with the diagram continued cyclically). The symmetry group of the above cycle is the dihedral group D_n and this strongly suggests that the group of outer automorphisms and anti-automorphisms of $I\mathfrak{u}_n$ (all automorphisms and anti-automorphisms modulo inner automorphisms) is D_n, generated by Φ and Ψ. We did not endeavor to prove this formally.

Extension. The Drinfel'd double / Manin triple construction [12, 13], when applied to $\mathfrak{u}_n$, is a way to reconstruct $\mathfrak{gl}_n$ from its subalgebras of upper triangular matrices $\mathfrak{u}_n$ and lower

triangular matrices $\mathfrak{l}_n$. Precisely, one endows the vector space $\mathfrak{g} = \mathfrak{u}_n \oplus \mathfrak{l}_n$ with a non-degenerate symmetric bilinear form by declaring that the subspaces $\mathfrak{u}_n$ and $\mathfrak{l}_n$ are isotropic ($\langle \mathfrak{u}_n, \mathfrak{u}_n \rangle = \langle \mathfrak{l}_n, \mathfrak{l}_n \rangle = 0$) and by setting $\langle x_{kl}, x_{ij} \rangle = \delta_{li}\delta_{jk}$, $\langle b_i, a_j \rangle = 2\delta_{ij}$, and $\langle x_{ji}, a_k \rangle = \langle b_k, x_{ij} \rangle = 0$ as in Theorem 1 and where a_i stands for the diagonal matrix x_{ii} considered as an element of $\mathfrak{u}_n$ and b_i stands for the same matrix as an element of $\mathfrak{l}_n$. There is then a unique bracket on $\mathfrak{g}$ that extends the brackets on the summands $\mathfrak{u}_n$ and $\mathfrak{l}_n$ and relative to which the inner product of $\mathfrak{g}$ is invariant[3] With our judicious choice of bilinear form, this bracket on $\mathfrak{g}$ satisfies the Jacobi identity and turns $\mathfrak{g}$ into a Lie algebra isomorphic to $\mathfrak{gl}_{n+} = \mathfrak{gl}_n \oplus \mathfrak{h}'_n$, where $\mathfrak{h}'_n$ denotes a second copy of the diagonal matrices in $\mathfrak{gl}_n$.

We let $\mathfrak{gl}_{n+}^\epsilon$ be the Inonu-Wigner [14] contraction of $\mathfrak{g}$ along its $\mathfrak{l}_n$ summand, with parameter ϵ.[4] All that this means is that the bracket of $\mathfrak{l}_n$ gets multiplied by ϵ to give $\mathfrak{l}_n^\epsilon$, and then the Drinfel'd double / Manin triple construction is repeated starting with $\mathfrak{u}_n \oplus \mathfrak{l}_n^\epsilon$, without changing the bilinear form. The result is a Lie algebra $\mathfrak{gl}_{n+}^\epsilon$ over the ring of polynomials in ϵ which specializes to $I\mathfrak{u}_n$ at $\epsilon = 0$ and which is isomorphic to $\mathfrak{gl}_n \oplus \mathfrak{h}'_n$ when ϵ is invertible.[5] We care about $\mathfrak{gl}_{n+}^\epsilon$ a lot [3–5, 8–10, 18]; when reduced modulo $\epsilon^{k+1} = 0$ for some natural number k it becomes solvable, and hence a "solvable approximation" of $\mathfrak{gl}_n$ with applications to computability of knot invariants.

Theorem 2 *With the same conventions as in Theorem 1 the map Ψ is also a Lie algebra automorphism of $\mathfrak{gl}_{n+}^\epsilon$.*

Proof By some case checking and explicit computations, the commutation relations of $\mathfrak{gl}_{n+}^\epsilon$ are given by

$$[x_{ij}, x_{kl}] = \chi_{\lambda(x_{ij})+\lambda(x_{kl})<n}^\epsilon (\delta_{jk}x_{il} - \delta_{li}x_{kj}) \quad \text{unless both } j=k \text{ and } l=i,$$

$$[x_{ij}, x_{ji}] = \frac{1}{2}(b_i - b_j) + \frac{\epsilon}{2}(a_i - a_j), \quad [a_i, x_{jk}] = (\delta_{ij} - \delta_{ik})x_{jk},$$

$$[b_i, x_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})x_{jk},$$

$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 0,$$

where $\chi_{\text{True}}^\epsilon = 1$ and $\chi_{\text{False}}^\epsilon = \epsilon$. These relations are clearly Ψ-equivariant. □

Note 3 There is of course an "$\mathfrak{sl}$" version of everything, in which linear combinations $\sum \alpha_i a_i$ and $\sum \beta_i b_i$ are allowed only if $\sum \alpha_i = \sum \beta_i = 0$, with obvious modifications throughout.

Note 4 At $n = 2$ and $\epsilon = 0$, the algebra $\mathfrak{sl}_{2+}^0$ is the "diamond Lie algebra" of [15, Chapter 4.3], which is sometimes called "the Nappi-Witten algebra" [17]. With $a = (a_1 - a_2)/2$, $x = x_{12}$, $y = x_{21}$, and $b = (b_1 - b_2)/2$, it is

$$\langle a, x, y, b \rangle / ([a, x] = x, \ [a, y] = -y, \ [x, y] = b, \ [b, -] = 0).$$

Here $\Phi: (a, x, y, b) \mapsto (-a, x, y, -b)$ and $\Psi: (a, x, y, b) \mapsto (-a, y, x, -b)$.

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[3] Indeed we only need to determine $[u, l]$ for $u \in \mathfrak{u}_n$ and $l \in \mathfrak{l}_n$. Writing $[u, l] = u' + l'$ with $u' \in \mathfrak{u}_n$ and $l' \in \mathfrak{l}_n$, we determine u' using the non-degeneracy of the inner product from the relation $\langle u', l'' \rangle = \langle [u, l], l'' \rangle = \langle u, [l, l''] \rangle$ which holds for every $l'' \in \mathfrak{l}_n$ due to the invariance of $\langle, \rangle$. Similarly l' is determined from $\langle l', u'' \rangle = \langle [u, l], u'' \rangle = \langle l, [u'', u] \rangle$.

[4] Alternatively, make $\mathfrak{u}_n$ into a Lie bialgebra with cobracket δ using its given duality with $\mathfrak{l}_n$, and double it as in [12, 13] but using the cobracket $\epsilon\delta$.

[5] Hence $\mathfrak{gl}_{n+}^\epsilon \rightarrow I\mathfrak{u}_n$ is a counter-example to the feel-true statement "a contraction of a direct sum is a direct sum". Indeed with notation as in Theorem 2, as $\epsilon \rightarrow 0$ the decomposition $\mathfrak{gl}_{n+}^\epsilon = \mathfrak{gl}_n \oplus \mathfrak{h}'_n = \langle x_{ij}, b_i + \epsilon a_i \rangle \oplus \langle b_i - \epsilon a_i \rangle$ collapses.

Note 5 Upon circulating this paper as an eprint we received a note from A. Knutson informing us of [16, esp. sec. 2.3], where the algebra $I\mathfrak{u}_n$ (except reduced modulo $\langle b_i \rangle$ and considered globally rather than infinitesimally) is considered from a different perspective. It is shown to be a subquotient of the affine algebra $\widehat{\mathfrak{gl}}_n$ in a manner preserved by its automorphisms corresponding to its Dynkin diagram, which is a cycle. Similar comments apply to the other algebras considered here.

Note 6 A day later we received a note [11] from M. Bulois and N. Ressayre reporting on an explanation of Theorem 1 in terms of affine Kac-Moody Lie algebras, similarly to Note 5.

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