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Local positivity and effective Diophantine approximation

Matthias Nickel[1]

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Abstract
In this paper we present a new approach to prove effective results in Diophantine approximation. This approach involves measures of local positivity of divisors combined with Faltings's version of Siegel's lemma instead of a zero estimate such as Dyson's lemma. We then use it to prove an effective theorem on the simultaneous approximation of two algebraic numbers satisfying an algebraic equation with complex coefficients.

Keywords Diophantine approxmiation · Positivity of divisors · Seshadri constants

Mathematics Subject Classification 11J68 · 14C20

1 Introduction

Positivity concepts for divisors play a crucial role in algebraic geometry. Among these concepts is *ampleness*, which can also be interpreted intersection theoretically via the Nakai–Moishezon–Kleiman criterion. A weaker form of positivity is *bigness*: a divisor D is big iff the growth of the dimension of global sections of its multiples is maximal. The rate of this growth is then measured by the volume of the divisor [35, Sect. 2.1] and for ample divisors this is simply the top self-intersection by the asymptotic Riemann–Roch theorem [35, Theorem 1.1.24]. In [20] Demailly introduces a measure of local positivity of a divisor at a point, the Seshadri constant, in order to study the Fujita conjecture.

The connection between Diophantine approximation and positivity concepts is central to many results on Diophantine geometry. It is a key element in Vojta's proof of Mordell's conjecture [51] and in Faltings's proof of the Mordell–Lang conjecture [25]. In [29] it has been shown that the constants showing up in Diophantine approximations can be obtained as the expectation of certain random variables coming from filtrations on the graded ring of sections of a divisor. Later [18, 22, 23, 27] showed that these constants can be obtained via different geometric invariants. In [39] Diophantine approximation constants are shown to be related to volumes of divisors. This is shown to be true also in the function field

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✉ Matthias Nickel
manickel@uni-mainz.de

1 Institut für Mathematik, Johannes Gutenberg-Universität, Staudingerweg 9, 55128 Mainz, Germany

case in [30] using an effective Schmidt subspace theorem over function fields [52]. Finally [31, 44, 45] treat the more general case where not only points but closed subschemes are approximated.

Most results on Diophantine approximation rely on the construction of an auxiliary polynomial having a certain order of vanishing at given points. In this paper we present a new approach that follows Faltings's proof of the Mordell–Lang conjecture [25] using information on local positivity at these points to study the vector spaces of suitable auxiliary polynomials.

One of the most important results in Diophantine approximation is Roth's theorem on the approximation of algebraic numbers by rationals [43]. It states that for a given algebraic number α and a given $\varepsilon > 0$ there are only finitely many rational numbers $p/q \in \mathbb{Q}$ such that

$$\left|\alpha - \frac{p}{q}\right| \leq q^{-(2+\varepsilon)}\,. \tag{1}$$

The proof of this theorem consists of two steps:

1. First an auxiliary polynomial $P \in \mathbb{Z}[X_1, \dots, X_n]$ having a certain order of vanishing at $(\alpha, \dots, \alpha)$ is constructed, which is then shown to vanish to a suitable order at $(p_1/q_1, \dots, p_n/q_n)$ where p_i/q_i are solutions to (1). Here one usually uses a version of Siegel's lemma [48].
2. Next, one shows that there exists an upper bound for the order of P at the point $(p_1/q_1, \dots, p_n/q_n)$ obtaining a contradiction. This upper bound may be either of geometric (Dyson's lemma [21] or rather its generalization by Esnault and Viehweg [24]) or of arithmetic nature (Roth's lemma [43] and Faltings's product theorem [25]).

Note that there are closely related methods in transcendence theory employing a different strategy that does not require Siegel's lemma, in particular Laurent's interpolating determinants [34] and Bost's slope method [11], see also [16].

There are also many results on the simultaneous approximation of algebraic numbers by rationals. The generalization of Roth's theorem in this context is due to Schmidt [46, Corollary to Theorem 1]. Suppose that $\alpha_1, \dots, \alpha_r$ are algebraic numbers such that $1, \alpha_1, \dots, \alpha_r$ are linearly independent over $\mathbb{Q}$. Then for every $\varepsilon > 0$ there exist only finitely many r-tuples of rational numbers $(p_1/q, \dots, p_r/q)$ such that

$$\left|\alpha_i - \frac{p_i}{q}\right| \leq q^{-(1+1/r+\varepsilon)} \tag{2}$$

holds for all $1 \leq i \leq r$.

The theorems of Roth and Schmidt are not effective in that there is no bound for q for the rational numbers p/q and p_i/q satisfying (1) and (2) respectively. The earliest effective result in the approximation of a single algebraic number is the theorem of Liouville [36], which is similar to Roth's theorem with exponent the degree d of the algebraic number in question instead of 2. Fel'dman [26] obtained an improvement of Liouville's theorem, in which the exponent is strictly smaller than d, however, the difference is extremely small. A different approach to this problem is Bombieri's Thue–Siegel principle [2, 6–10, 15]. For improvements see for example [5, 13]. In the case of simultaneous approximation there are effective results where the tuple of algebraic numbers is given by rational powers of rational numbers [1, 4, 41, 42].

Here we discuss a different strategy linking methods from positivity and Diophantine approximation that follows Faltings's proof of the Mordell–Lang conjecture [25]. For a detailed discussion of the strategy of Faltings's proof see [40]. We consider homogeneous polynomials in two variables having large index at the point (α_1, α_2), see Definition 8, and *a priori* small index at $(p_1/q, p_2/q)$ where p_i/q is a suitably good rational approximation of α_i for $i = 1, 2$.

Using Faltings's Siegel lemma we can then ensure that we can find such a polynomial with suitably bounded coefficients in $\mathbb{Z}$. Finally we give a bound for q involving the index of P at (α_1, α_2) and $(p_1/q, p_2/q)$.

The novelty of this approach is that it avoids providing a zero estimate: we only need to suitably bound the dimension of the space of polynomials with given degree and given index at (α_1, α_2), all of its conjugates and $(p_1/q, p_2/q)$. Therefore we only need a partial understanding of the volume function on blowups of $\mathbb{P}^2$. The fact that we only consider one solution $(p_1/q, p_2/q)$ will finally make our theorem effective.

We obtain the following theorem.

Theorem 1 *Let α_1, α_2 be algebraic numbers and let $d := [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$. Suppose that (α_1, α_2) and all of its conjugates are nonsingular points of an irreducible curve of degree m defined over $\mathbb{C}$. Then there exists for all $\delta \in \mathbb{Q}$ with $\delta > \max\{m, d/m\}$ an effectively computable constant $C_0(\alpha_1, \alpha_2, \delta, m)$ depending only on (α_1, α_2), m and δ such that for all pairs of rational numbers $(p_1/q, p_2/q)$ satisfying*

$$\left|\alpha_i - \frac{p_i}{q}\right| \leq q^{-\delta} \text{ for i=1,2} \tag{3}$$

we have $q \leq C_0(\alpha_1, \alpha_2, \delta, m)$.

The proof of Theorem 1 yields the following corollary on a possible choice for $C_0(\alpha_1, \alpha_2, \delta, m)$.

Corollary 2 *Using the notation of the previous theorem let α_0 be defined as $\alpha_1 + M_0\alpha_2$ where M_0 is the smallest natural number that $\alpha_1 + M_0\alpha_2$ is a primitive element of $\mathbb{Q}(\alpha_1, \alpha_2)$ (such a M_0 always exists by the proof of the primitive element theorem [33, Theorem V.4.6]). Let α be defined as $M_1\alpha_0$ where M_1 is the smallest natural number such that $M_1\alpha_0$ is an algebraic integer. Now let N be the smallest natural number such that $N\alpha_1$ and $N\alpha_2$ can be expressed as*

$$N\alpha_i = c_1^i\alpha^{d-1} + \cdots + c_{d-1}^i\alpha + c_d^i \text{ for } i = 1, 2$$

where $c_h^i \in \mathbb{Z}$ and let M be defined as $\max\{|c_h^i| \mid h = 1, \ldots, d \text{ and } i = 1, 2\}$. Let Q be defined as the denominator of

$$\theta := \frac{1/\delta + \min\{1/m, m/d\}}{2}$$

and let

$$\theta_0 := \min\left\{\frac{\min\{1/m, m/d\} - 1/\delta}{4}, \frac{1}{Ql(\theta)}\right\}$$

where

$$l(\theta) := \left\lceil \frac{1}{2} \left\lfloor \frac{((-3+d\theta)Q + 4Q^2(1-d\theta^2)+1)^2}{Q^2(1-d\theta^2)} + 3 \right\rfloor \right\rceil.$$

Then Theorem 1 holds with

$$C_0(\alpha_1, \alpha_2, \delta, m) := \left(64(2^{10}(dM(|m_\alpha| + 1)^d)^3)^{1/\theta_0^2} \max\{1, |\alpha_1|, |\alpha_2|\} N^3\right)^{\frac{1}{\delta(\theta-\theta_0)-1}}.$$

1.1 Notation

In the remainder of this article we will denote by α_1 and α_2 algebraic numbers and let $d := [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$.

2 Seshadri constants on blow-ups of $\mathbb{P}^2$

In this section we will be only concerned with varieties over $\mathbb{C}$.

We begin by discussing Seshadri constants. These constants measuring local positivity of divisors were first defined by Demailly in [20] and their name is due to the Seshadri criterion for ampleness [47, Remark 7.1].

Definition 3 Let X be a smooth projective surface, let M be a nef $\mathbb{R}$-divisor on X, let x be a point in X and let $\pi_x : X' \to X$ be the blowup of X at x and E its exceptional divisor. Then the Seshadri constant of M at x is defined as

$$\varepsilon(X, M; x) := \sup\{t \geq 0 \mid \pi_x^* M - tE \text{ is a nef } \mathbb{R} - \text{divisor on } X'\}.$$

Let us recall some properties of Seshadri constants.

Lemma 4 ([35, Example 5.1.4, Example 5.1.6]) *Let X, X' and x be as above and let M be nef and integral. Then*:

1. *The Seshadri constant is homogenous*:

$$\varepsilon(X, lM; x) = l\,\varepsilon(X, M; x)$$

   for all $l \in \mathbb{N}$.
2. *If M is very ample then*

$$\varepsilon(X, M; x) \geq 1\,.$$

For more about Seshadri constants the reader may consult [3] and [35, Chapter 5].

We will need the following statement about ample divisors on the blowup of $\mathbb{P}^2$ at points that lie on an irreducible curve of degree m.

Proposition 5 *Let $x_1, \ldots, x_d$ be distinct points lying on an irreducible curve D of degree m in $\mathbb{P}^2$ such that* $\text{mult}_{x_i} D = 1$ *for all i, let L be a line in $\mathbb{P}^2$ and consider the* *blow-up* $\pi : S \to \mathbb{P}^2$ *of* $\mathbb{P}^2$ *at* $x_1, \dots, x_d$ *with exceptional divisors* $E_1, \dots, E_d$. *Then for every* $0 < t < \min\{1/m, m/d\}$ *the* $\mathbb{R}$*-divisor* $\pi^*L - t(E_1 + \cdots + E_d)$ *is ample.*

Proof The strict transform D' of D is linearly equivalent to the divisor

$$C := m\,\pi^*L - (E_1 + \cdots + E_d)$$

on S. This implies that for $0 \le t \le 1/m$ we have

$$L_t := \pi^*L - t(E_1 + \cdots + E_d) = (1 - mt)\pi^*L + tC = (1 - mt)\pi^*L + t(C - D') + tD'.$$

The intersection of π^*L and the strict transform of any irreducible curve on $\mathbb{P}^2$ is positive and $C - D'$ is numerically trivial. Further, D' intersects all irreducible curves on S except possibly itself nonnegatively. Using

$$\begin{aligned} L_tE_i &= t \\ L_tD' &= m - dt \\ L_t^2 &= 1 - dt^2 \end{aligned}$$

we conclude that for $0 < t < \min\{1/m, m/d, 1/\sqrt{d}\}$ it holds that $L_t^2 > 0$ and that L_t intersects all irreducible curves on S positively. Furthermore, as $1/m \le m/d$ is equivalent to $1/m \le 1/\sqrt{d}$, we conclude that the equality $\min\{1/m, m/d, 1/\sqrt{d}\} = \min\{1/m, m/d\}$ holds. By the real version of the Nakai–Moishezon criterion [17] the statement of the proposition holds. □

In what follows we will need to have a lower bound for the Seshadri constant of $\pi^*L - t(E_1 + \cdots + E_d)$ for $0 \le t < \min\{1/m, m/d\}$ at another point. In order to do this we will employ an effective version of Matsusaka's big theorem [37, 38] for surfaces by Fernández del Busto [19]. Note that Siu has given an effective version of Matsusaka's big theorem valid in higher dimensions [49, 50].

Theorem 6 [19] *Let A be an ample divisor on a smooth projective algebraic surface X. Then lA is very ample for every*

$$l > \frac{1}{2}\left\lfloor \frac{(A(K_X + 4A) + 1)^2}{A^2} + 3 \right\rfloor.$$

Using this we are now ready to prove the following geometric theorem, which will be essential in the proof of the main theorem.

Theorem 7 *Let* $x_1, \dots, x_d$ *be distinct points lying on an irreducible curve D of degree m in* $\mathbb{P}^2$ *such that* $\operatorname{mult}_{x_i} D = 1$ *for all i, let* $x_{d+1} \in \mathbb{P}^2$, *let L be a line in* $\mathbb{P}^2$ *and consider the blow-up* $\pi : X \to \mathbb{P}^2$ *of* $\mathbb{P}^2$ *at* $x_1, \dots, x_{d+1}$ *with exceptional divisors* $E_1, \dots, E_{d+1}$. *Let us define for* $Q > 0$

$$l(\theta) := \left\lceil \frac{1}{2}\left\lfloor \frac{((-3 + d\theta)Q + 4Q^2(1 - d\theta^2) + 1)^2}{Q^2(1 - d\theta^2)} + 3 \right\rfloor \right\rceil.$$

Then for all $\theta \in \mathbb{Q}$ *with denominator* $Q \in \mathbb{N}$ *satisfying* $\theta < \min\{1/m, m/d\}$ *and for every* $0 \le \mu \le \frac{1}{Ql(\theta)}$ *we have that*

$$\operatorname{vol}_X(\pi^*L - \theta(E_1 + \cdots + E_d)) - \operatorname{vol}_X(\pi^*L - \theta(E_1 + \cdots + E_d) - \mu E_{d+1}) = \mu^2 .$$

Proof By Proposition 5 above we know that $M := (\pi^*L - \theta(E_1 + \cdots + E_d))$ is ample. Note that $l(\theta) = \left\lceil \frac{1}{2} \left\lfloor \frac{(QM(K_X+4QM)+1)^2}{(QM)^2} + 3 \right\rfloor \right\rceil$. By Theorem 6 and because QM is an integral ample divisor, we now know that the divisor $l(\theta)QM$ is very ample and therefore $\varepsilon(\mathrm{Bl}_{x_1,\ldots,x_d}(\mathbb{P}^2), M; x_{d+1}) \geq \frac{1}{Ql(\theta)}$ by the properties of Seshadri constants in Lemma 4 (note that the homogenity of Seshadri constants immediately extends to $\mathbb{Q}$-divisors). Therefore $\pi^*L - \theta(E_1 + \cdots + E_d) - \mu E_{d+1}$ is nef for $0 \leq \mu \leq \frac{1}{Ql(\theta)}$ and the statement of the theorem follows by the asymptotic Riemann–Roch theorem [35, Corollary 1.4.41]. □

3 Bounding the denominator of a good approximation

In this section we will bound the denominator q of a good approximation $(p_1/q, p_2/q)$ of (α_1, α_2) using a polynomial $P \in \mathbb{Z}[X_1, X_2]$ with suitably bounded coefficients and suitable index at (α_1, α_2) and $(p_1/q, p_2/q)$. This chapter closely follows [32, §D.5]. For the convenience of the reader we give proofs as we need slightly different statements than those in [32].

Definition 8 Let

$$P = \sum_{j \in \mathbb{N}_0^2} a_{j_1, j_2} X_1^{j_1} X_2^{j_2}$$

be a polynomial with coefficients in $\mathbb{C}$.

1. For a multi-index $j \in \mathbb{N}_0^2$ we define a differential operator ∂_j via

$$\partial_j P := \frac{1}{j_1! j_2!} \frac{\partial^{j_1+j_2}}{\partial X_1^{j_1} \partial X_2^{j_2}} P.$$

2. If $P \neq 0$, we define the index of P at $x = (x_1, x_2)$ with respect to the weights $(r_1, r_2) \in \mathbb{N}^2$ to be the nonnegative real number

$$\operatorname{ind}_{(x_1, x_2; r_1, r_2)}(P) := \min\{j_1/r_1 + j_2/r_2 \mid j \in \mathbb{N}_0^2, \partial_j P(x) \neq 0\}.$$

3. If $P \in \mathbb{Z}[X_1, X_2]$, the naive height of P is defined as

$$|P| := max\{|a_j| \mid j \in \mathbb{N}_0^2\} .$$

Let us summarize some properties of the index and the differential operators ∂_j for later use.

Lemma 9 ([32, Lemmas D.3.1 and D.3.2]) *Let $P \in \mathbb{C}[X_1, X_2]$ and let $j \in \mathbb{N}_0^2$.*

1. *If $\partial_j P \neq 0$, then* $\operatorname{ind}_{(x_1, x_2; r_1, r_2)}(\partial_j P) \geq \operatorname{ind}_{(x_1, x_2; r_1, r_2)}(P) - j_1/r_1 - j_2/r_2$,

2. *if* $P \in \mathbb{Z}[X_1, X_2]$, *we have* $\partial_j P \in \mathbb{Z}[X_1, X_2]$,
3. *if* $\deg P \leq k$, *then* $|\partial_j P| \leq 4^k |P|$.

From now on we will make the assumption $r_1 = r_2 = k$. In particular we have that $\operatorname{ind}_{(\alpha_1,\alpha_2;k,k)}(P) = \operatorname{ord}_{(\alpha_1,\alpha_2)}(P)/k$.

In the following two lemmas we will provide a bound on the denominator and absolute value of derivatives of a polynomial $P \in \mathbb{Z}[X_1, X_2]$ at $(p_1/q, p_2/q)$.

Lemma 10 *Let* $P \in \mathbb{Z}[X_1, X_2]$ *be a polynomial of degree less or equal k and let* $j \in \mathbb{N}_0^2$ *be a multi-index. Then*

$$q^k \partial_j P(p_1/q, p_2/q) \in \mathbb{Z}\,.$$

Proof Using Lemma 9 we obtain that the the coefficients of $\partial_j P$ are in $\mathbb{Z}$. Therefore $\partial_j P(p_1/q, p_2/q)$ is a sum of terms whose denominators are divisors of q^k giving us the desired bound. □

Lemma 11 *Let* $P \in \mathbb{Z}[X_1, X_2]$ *of degree less or equal k with* $k \geq 4$ *and let* $j \in \mathbb{N}_0^2$ *be a multi-index. Let* θ *be the index of P at* (α_1, α_2) *with respect to* (k, k), *let* $0 < \theta_0 < \theta$, *let* $\delta > 0$ *and let* $N \in \mathbb{N}$.

Then it holds that for $(p_1/q, p_2/q) \in \mathbb{Q}$ *satisfying*

$$\left|\alpha_i - \frac{p_i}{q}\right| \leq Nq^{-\delta} \text{ for } i = 1, 2 \tag{4}$$

and for every $j = (j_1, j_2) \in \mathbb{N}_0^2$ *such that* $\frac{j_1+j_2}{k} \leq \theta_0$ *we have*

$$\left|\partial_j P(p_1/q, p_2/q)\right| \leq 64^k |P| (\max\{1, |\alpha_1|, |\alpha_2|\})^k N^{2k} q^{-k\delta(\theta-\theta_0)}.$$

Proof The claim of the lemma is evident for $\partial_j P = 0$. We may therefore assume that $\partial_j P \neq 0$. First note that for all $i \in \mathbb{N}_0^2$ we have that $\partial_i \partial_j P(\alpha_1, \alpha_2)$ is a sum of at most $1/2\,(k+1)(k+2) \leq 2^k$ terms because $k \geq 4$. These terms are of the form $c_{i_1,i_2} \alpha_1^{i_1} \alpha_2^{i_2}$ with $c_{i_1,i_2} \in \mathbb{Z}$ by Lemma 9 and $i_1 + i_2 \leq k$ and are themselves bounded by

$$\left|c_{i_1,i_2} \alpha_1^{i_1} \alpha_2^{i_2}\right| \leq \left|\partial_i \partial_j P\right| (\max\{1, |\alpha_1|, |\alpha_2|\})^k \leq 16^k |P| (\max\{1, |\alpha_1|, |\alpha_2|\})^k$$

where we have used Lemma 9 two times. We may now expand $\partial_j P$ around (α_1, α_2) and use that Lemma 9 implies $\operatorname{ind}_{(\alpha_1,\alpha_2;k,k)}(\partial_j P) \geq \theta - \theta_0$ to obtain

$$\partial_j P(p_1/q, p_2/q) = \sum_{\substack{0 \leq i_1, i_2 \leq k \\ \theta - \theta_0 \leq (i_1+i_2)/k \leq 1}} (\partial_i \partial_j P)(\alpha_1, \alpha_2)(p_1/q - \alpha_1)^{i_1} (p_2/q - \alpha_2)^{i_2}$$

and by assumption (4), the fact that the number of terms above is bounded by 2^k and the bounds above we have

$$\left|\partial_j P(p_1/q, p_2/q)\right| \leq 64^k |P| (\max\{1, |\alpha_1|, |\alpha_2|\})^k N^{2k} q^{-k\delta(\theta-\theta_0)}\,.$$

□

Using the results above we obtain a bound for the denominator of a good approximation as follows.

Lemma 12 *Let $k \geq 4$ be a positive integer. Let $0 < \theta_0 < \theta$ be given and suppose that $(p_1/q, p_2/q) \in \mathbb{Q}^2$ is a solution of inequality (4) for given $\delta > 1/(\theta - \theta_0)$, $N \in \mathbb{N}$. Now assume that $P \in \mathbb{Z}[X_1, X_2]$ satisfies the following properties:*

1. *the degree of P is at most k,*
2. *the index of P at (α_1, α_2) with respect to the weights (k, k) satisfies*

$$\text{ind}_{(\alpha_1,\alpha_2;k,k)}(P) \geq \theta,$$

3. *$|P| \leq B^k$, where B depends only on (α_1, α_2), k and δ.*

Let

$$C(\alpha_1, \alpha_2, \delta, N) := \left(64B \max\{1, |\alpha_1|, |\alpha_2|\} N^2\right)^{\frac{1}{\delta(\theta-\theta_0)-1}}.$$

Then it holds that if

$$\text{ind}_{(p_1/q, p_2/q;k,k)}(P) < \theta_0,$$

we have $q \leq C(\alpha_1, \alpha_2, \delta, N)$.

Proof Assume $\text{ind}_{(p_1/q,p_2/q;k,k)}(P) < \theta_0$ and let $j \in \mathbb{N}_0^2$ with $\frac{j_1+j_2}{k} < \theta_0$ be such that $\partial_j P(p_1/q, p_2/q) \neq 0$, say $\partial_j P(p_1/q, p_2/q) = s/m$ with $s \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{N}$ and s and m coprime. Now Lemma 11 and the bound on $|P|$ give us that

$$|\partial_j P(p_1/q, p_2/q)| \leq \left(64B \max\{1, |\alpha_1|, |\alpha_2|\} N^2 q^{-\delta(\theta-\theta_0)}\right)^k.$$

We use the principle that there is no integer strictly between 0 and 1 to obtain

$$1/m \leq \left(64B \max\{1, |\alpha_1|, |\alpha_2|\} N^2 q^{-\delta(\theta-\theta_0)}\right)^k.$$

Finally Lemma 10 gives

$$q^{-k} \leq \left(64B \max\{1, |\alpha_1|, |\alpha_2|\} N^2 q^{-\delta(\theta-\theta_0)}\right)^k$$

and after taking k-th roots and simplifying we obtain

$$q^{\delta(\theta-\theta_0)-1} \leq 64B \max\{1, |\alpha_1|, |\alpha_2|\} N^2$$

and the claimed inequality follows. □

4 Finding a suitable global section

For this section we fix an embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ and consider the line $L = \mathbb{P}^2 \setminus \mathbb{A}^2$ such that the global sections of $\mathcal{O}_{\mathbb{P}^2}(kL)$ restricted to $\mathbb{A}^2$ are the polynomials of degree less or equal k, and view (α_1, α_2), all of its conjugates and $(p_1/q, p_2/q)$ as elements of $\mathbb{P}^2$ via this embedding. In this chapter we will always indicate which base field we are working over.

We now state Faltings's version of Siegel's lemma.

Lemma 13 ([25, Proposition 2.18]) *Let V, W be two finite dimensional normed $\mathbb{R}$-vector spaces and let $M \subset V$ and $N \subset W$ be $\mathbb{Z}$-lattices of maximal rank. Let further $\phi : V \to W$ be a linear map such that $\phi(M) \subset N$. Let $b := \dim(V)$ and $a := \dim(\operatorname{Ker}(\phi))$ and assume that there exists a constant $C \geq 2$ such that*

1. *M is generated by elements of norm at most C,*
2. *the norm of ϕ is bounded by C,*
3. *all non-trivial elements of M and N have norm at least $1/C$.*

For $1 \leq i \leq b$ set

$$\lambda_i := \inf\{\lambda > 0 \mid \exists i \text{linearly independent vectors of norm} \leq \lambda \text{ in } \operatorname{Ker}(\phi) \cap M\}.$$

Then it holds that

$$\lambda_{i+1} \leq (C^{3b} b!)^{1/(a-i)}.$$

We will need the following number theoretical lemma.

Lemma 14 ([32, Lemma D.3.4]) *Let $\alpha \in \overline{\mathbb{Q}}$ be an algebraic integer of degree $d_\alpha := [\mathbb{Q}(\alpha) : \mathbb{Q}]$ over $\mathbb{Q}$ and let $m_\alpha \in \mathbb{Q}[X]$ be the minimal polynomial of α over $\mathbb{Q}$. Then we have $\alpha^l = a_1^{(l)} \alpha^{d_\alpha - 1} + \cdots + a_{d_\alpha}^{(l)}$ with $a_i^{(l)} \in \mathbb{Z}$ satisfying $\left| a_i^{(l)} \right| \leq (|m_\alpha| + 1)^l$.*

The following Lemma now clarifies how we intend to use Faltings's version of Siegel's lemma. In it we will make an assumption that implies that α_1, α_2 are algebraic integers. Note that we can always satisfy this assumption by considering $N\alpha_i$ instead of α_i for a suitable $N \in \mathbb{N}$.

Lemma 15 *Let $k \geq 4$ be a positive integer, let $B_k := H^0(\mathcal{O}_{\mathbb{P}^2_{\mathbb{Q}}}(kL))$, which we will identify with the polynomials of degree less or equal k in $\mathbb{Q}[X_1, X_2]$, and let A_k be the subspace of sections whose index at (α_1, α_2) with respect to the weights (k, k) is at least θ. Choose an algebraic integer α which is a primitive element for $\mathbb{Q}(\alpha_1, \alpha_2)$ and assume that α_1 and α_2 can be expressed as*

$$\alpha_i = c_1^i \alpha^{d-1} + \cdots + c_{d-1}^i \alpha + c_d^i \text{ for } i = 1, 2$$

where $c_h^i \in \mathbb{Z}$ *and let* M *be defined as* $\max\{|c_h^i| \mid h = 1, \dots, d \text{ and } i = 1, 2\}$. *Then there exists a linear map* $\phi_k : B_k \otimes \mathbb{R} \to \mathbb{Q}(\alpha_1, \alpha_2)^{l_k} \otimes \mathbb{R}$ *where*

$$l_k = \#\{j \in \mathbb{N}_0^2 \mid \frac{j_1 + j_2}{k} < \theta\}$$

such that

1. $\mathrm{Ker}\,(\phi_k) = A_k \otimes \mathbb{R}$,
2. ϕ_k, *the lattice inside* $B_k \otimes \mathbb{R}$ *generated by monomials and the lattice in* $\mathbb{Q}(\alpha_1, \alpha_2)^{l_k} \otimes \mathbb{R}$ *generated by* α^i *for* $i = 0, \dots, d-1$ *in every component satisfy the conditions in Lemma 13 with* $C = B^k$ *where* $B > 0$ *is the following constant*

$$B := 8dM(|m_\alpha| + 1)^d .$$

Proof Define the linear map

$$\begin{aligned} \phi_k : B_k \otimes \mathbb{R} &\to \mathbb{Q}(\alpha_1, \alpha_2)^{l_k} \otimes \mathbb{R} \\ P \otimes 1 &\mapsto (\partial_j P)(\alpha_1, \alpha_2) \otimes 1 \end{aligned}$$

where j ranges over all pairs of non-negative integers satisfying $(j_1 + j_2)/k < \theta$. Consider the basis of $V := B_k \otimes \mathbb{R}$ which consists of monomials, the basis of $W := \mathbb{Q}(\alpha_1, \alpha_2)^{l_k} \otimes \mathbb{R}$ consisting of α^i for $i = 0, \dots, d-1$ in every component and the lattices L_V and L_W generated by these bases.

By Lemma 9 and the assumptions on α we have that $\phi_k(L_V) \subset L_W$. We now identify $V \cong \mathbb{R}^{\dim_\mathbb{Q} B_k}$ and $W \cong \mathbb{R}^{dl_k}$ using the above bases and equip these $\mathbb{R}$-vector spaces with the maximum norm $|\cdot|_\infty$. It is then clear that L_V is generated by elements of norm 1 and all non-trivial elements of L_V and L_W have norm greater or equal 1. Therefore we only need to give a bound on the norm of ϕ_k.

To achieve this, we consider a polynomial $P \in V$, note that P is a sum of at most $1/2\,(k+1)(k+2) \le 2^k$ terms and use Lemma 9 to obtain that the coefficients of P are bounded by $|\partial_j P| \le 4^k |P|$. Then by using the assumptions of Lemma 15 to expand $\alpha_1^u \alpha_2^v$ where $u + v \le k$ into a $\mathbb{Z}$-linear combination of powers of α we obtain a sum of $d^{u+v} \le d^k$ terms $R\alpha^l$ with $l \le (u+v)d \le kd$ and $R \le M^{u+v} \le M^k$. By Lemma 14 we have that α^l is then a $\mathbb{Z}$-linear combination of $1, \alpha, \dots, \alpha^{d-1}$ with coefficients bounded by $(|m_\alpha| + 1)^{kd}$. Therefore it holds that

$$|\phi_k(P)|_\infty \le |P|(8dM(|m_\alpha| + 1)^d)^k$$

and this implies the statement of the lemma. □

Lemma 16 *Let us keep the notation and assumptions of the previous lemma and let* $\pi : X_\mathbb{C} \to \mathbb{P}^2$ *be the blowup of* $\mathbb{P}^2_\mathbb{C}$ *in* (α_1, α_2) *and all of its conjugates with corresponding exceptional divisors* $E_1, \dots, E_d$ *and in* $(p_1/q, p_2/q)$ *with corresponding exceptional divisor* E_{d+1}. *Letting* $b_k := \dim_\mathbb{Q} B_k$, $a_k := \dim_\mathbb{Q} A_k$, $i_k := \dim_\mathbb{Q} U_k$ *where* U_k *is the linear subspace of* A_k *of sections* $s \in A_k$ *with* $\mathrm{ind}_{(p_1/q, p_2/q; k, k)} s \ge \theta_0$ *we have that*

$$\lim_{k\to\infty} \frac{b_k}{k^2/2} = \mathrm{vol}_{\mathbb{P}^2_{\mathbb{C}}}(\pi^* L) = 1$$
$$\lim_{k\to\infty} \frac{a_k}{k^2/2} = \mathrm{vol}_{X_{\mathbb{C}}}(\pi^* L - \theta(E_1 + \cdots + E_d))$$
$$\lim_{k\to\infty} \frac{i_k}{k^2/2} = \mathrm{vol}_{X_{\mathbb{C}}}(\pi^* L - \theta(E_1 + \cdots + E_d) - \theta_0 E_{d+1}).$$

Proof For the first statement note that $b_k = 1/2\,(k+1)(k+2)$.

Regarding the second and third statement note that θ and θ_0 are real numbers and that the volume function for real divisors is defined by extending the volume function on $\mathbb{Q}$-divisors [35, Corollary 2.2.45]. However by [28, Theorem 3.5] it holds that for a $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on a projective variety V we have

$$\mathrm{vol}_V(D) = \lim_{k\to\infty} \frac{h^0(\lfloor kD \rfloor)}{k^{\dim(V)}/\dim(V)!}\,.$$

The vector space $H^0(\mathcal{O}_{X_{\mathbb{C}}}(\lfloor k\pi^* L - k\theta(E_1 + \cdots + E_d)\rfloor))$ is the space of complex polynomials of degree k vanishing at (α_1, α_2) and all of its conjugates with multiplicity at least $\lfloor k\theta \rfloor$. We may view this space as the linear subspace of $\mathbb{C}^{b_k} \cong H^0(\mathcal{O}_{X_{\mathbb{C}}}(k\pi^* L))$ given as the solution set of the equations $\partial_j P(\alpha_1, \alpha_2) = 0$ where $P \in H^0(\mathcal{O}_{X_{\mathbb{C}}}(k\pi^* L))$ and j ranges over all pairs of non-negative integers satisfying $(j_1 + j_2)/k < \theta$. The coefficients of these equations are algebraic and therefore there exists a basis consisting of algebraic elements of $\mathbb{C}^{b_k}$ and the dimension is equal to the dimension of the solution set of the same equations in $\overline{\mathbb{Q}}^{b_k}$. The absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ acts on the coefficient vectors by permutating them and therefore the solution space in $\overline{\mathbb{Q}}^{b_k}$ is stable under $G_{\mathbb{Q}}$. By [12, Corollary on page V.63] the dimension of this space equals the dimension of the solution set intersected with $\mathbb{Q}^{b_k}$ and this number is equal to a_k. The same argument yields $\dim_{\mathbb{C}} H^0(\mathcal{O}_{X_{\mathbb{C}}}(\lfloor k\pi^* L - k\theta(E_1 + \cdots + E_d) - k\theta_0 E_{d+1}\rfloor)) = i_k$ and the statement of the lemma follows. □

5 Proof of the main theorem

We are now ready to conclude the proof of the main theorem. In this section we will use the notation of Theorem 1 and Corollary 2.

Proof of the main theorem Let us consider the asymptotics obtained in the Lemma 16 above and use Faltings's version of Siegel's lemma. In order to use Lemma 15 we replace α_i by $N\alpha_i$. After this replacement we have that

$$\left| N\alpha_i - \frac{Np_i}{q} \right| \leq Nq^{-\delta} \text{ for i=1,2}\,.$$

We have by Lemma 15 that B^k satisfies the assumptions on C in Lemma 13 and therefore

$$\lambda_{i_k+1} \leq ((8dM(|m_\alpha| + 1)^d)^{3kb_k} b_k!)^{1/(a_k - i_k)} \leq ((8dM(|m_\alpha| + 1)^d)^{3k} b_k)^{b_k/(a_k - i_k)}\,. \quad (5)$$

By the choice of θ and θ_0, Lemma 16 and Theorem 7 the exponent on the right hand side of (5) satisfies

$$\begin{aligned}\lim_{k\to\infty} \frac{b_k}{(a_k - i_k)} &= \frac{\operatorname{vol}_{\mathbb{P}^2}(L)}{\operatorname{vol}_X(L - \theta(E_1 + \cdots + E_d)) - \operatorname{vol}_X(L - \theta(E_1 + \cdots + E_d) - \theta_0 E_{d+1})} \\ &= 1/\theta_0^2 .\end{aligned}$$

Now Lemma 13 shows the existence of $i_k + 1$ linearly independent elements of A_k such that their norm is bounded by

$$((8dM(|m_\alpha| + 1)^d)^{3k} b_k)^{b_k/(a_k - i_k)} \leq (2^{10}(dM(|m_\alpha| + 1)^d)^3)^{kb_k/(a_k - i_k)}$$

where we have used that $b_k = 1/2\,(k+1)(k+2) \leq 2^k$. In particular, for $k \gg 0$ at least one of those elements P is not an element of U_k. Noting that

$$\delta(\theta - \theta_0) \geq 1/4\,\delta \min\{1/m, m/d\} + 3/4 > 1 ,$$

we conclude that P satisfies all of the conditions for Lemma 12 and we obtain that

$$q \leq \left(64(2^{10}(dM(|m_\alpha| + 1)^d)^3)^{b_k/(a_k - i_k)} \max\{1, |\alpha_1|, |\alpha_2|\} N^3\right)^{\frac{1}{\delta(\theta - \theta_0) - 1}} .$$

Finally we take the limit for $k \to \infty$ and obtain

$$q \leq \left(64(2^{10}(dM(|m_\alpha| + 1)^d)^3)^{1/\theta_0^2} \max\{1, |\alpha_1|, |\alpha_2|\} N^3\right)^{\frac{1}{\delta(\theta - \theta_0) - 1}} ,$$

which finishes the proof. ☐

Remark 17 The effective constant of Corollary 2 is likely not close to being optimal. On the one hand there exist versions of Siegel's lemma like [14] that are not directly applicable in the situation of Lemma 15, but have a better dependence on the involved quantities. This suggests that an improvement of Lemma 15 might be possible. On the other hand one might hope that there exist better lower bounds on Seshadri constants than the ones used in the proof of Theorem 7. Both improvements would yield a better estimate for the effective constant of Corollary 2.

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