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A short note on sign changes and non-vanishing of Fourier coefficients of half-integral weight cusp forms

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Abstract
We study sign changes and non-vanishing of a certain double sequence of Fourier coefficients of cusp forms of half-integral weight.

Keywords Modular form · Half-integral weight · Sign changes

Mathematics Subject Classification 11F30 · 11F37

1 Introduction

Starting with the paper [6] many authors have investigated sign change properties of Fourier coefficients of cusp forms, in various directions. In particular, the case of half-integral weight has been the focus of much research. If g is a cusp form of half-integral weight $k+\frac{1}{2}$ with real Fourier coefficients $c(m)\,(m\geq 1)$ and in addition g is a Hecke eigenform, then there are at least two important themes in this area: on the one hand the study of sign changes of $(c(tn^2))_{n\geq 1}$ where t is a fixed positive integer, and on the other hand the corresponding question for the sequence $(c(t))_{t\geq 1 squarefree}$ where t runs over positive squarefree integers only. Of course, similar questions can be studied for forms of weight $k+\frac{1}{2}$ in the plus subspace in which case t has to be replaced by $|D|$ where D is a fundamental discriminant with $(-1)^kD>0$. For a good (at least partial) survey the reader may look up the literature given in [4].

Note that sign change results trivially imply corresponding non-vanishing results and in general non-vanishing properties of Fourier coefficients a priori are easier to handle. We recall that non-vanishing of products of Fourier coefficients was studied in [3].

In this short note we will investigate sign change and non-vanishing properties of the double sequence $(c(4n+r^2))_{n\geq 1, r\in \mathbf{Z}}$ where g is a cusp form of weight $k+\frac{1}{2}$ with k even and level 4 in the plus subspace $S^+_{k+1/2}$ (so $c(m)=0$ unless $m\equiv 0,1 \pmod 4$, see [7]). These

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coefficients turn up naturally when one considers the adjoint linear map with respect to the Petersson scalar products of (essentially) the linear map "multiplication with θ", where

$$\theta(z) = \sum_{r\in\mathbf{Z}} q^{r^2}$$

is the standard theta function of weight $\frac{1}{2}$ and level 4. Here as throughout $q = e^{2\pi iz}$ for $z \in \mathcal{H}$, the complex upper half-plane.

Our results will be stated in the next section; the proofs will be given in section 3. They rely on a detailed study of the above mentioned adjoint map, on growth properties of Fourier coefficients of cusp forms of integral weight due to Ram Murty and on a strong bound for the Fourier coefficients of cusp forms of half-integral weight due to Blomer-Harcos. Detailed references will be given below.

2 Statement of results

If $M \subset \mathbf{Z}$ we denote by $\#M$ the cardinality of M (thus $\#M$ is either a non-negative integer or ∞).

By k we always understand a positive even integer. We let S_k be the space of cusp forms of weight k on $\Gamma_1 := SL_2(\mathbf{Z})$. There is a linear map

$$L : S_k \to S^+_{k+1/2}, \quad f(z) \mapsto f(4z)\theta(z).$$

Note that in general L is not Hecke equivariant.

We denote by $L^* : S^+_{k+1/2} \to S_k$ the linear map adjoint to L with respect to the Petersson scalar products. Note that since L is injective, L^* is surjective.

Let $g \in S^+_{k+1/2}$ be fixed, with Fourier coefficients $c(m)$ $(m \geq 1)$. For each $n \in \mathbf{N}$ we then put

$$\alpha_n := \#\{r \in \mathbf{Z} \mid c(4n + r^2) \neq 0\}$$

and if in addition the $c(m)$ are real

$$\alpha^+_n := \#\{r \in \mathbf{Z} \mid c(4n + r^2) > 0\}, \quad \alpha^-_n := \#\{r \in \mathbf{Z} \mid c(4n + r^2) < 0\}.$$

Theorem 1 *Let $g \in S^+_{k+1/2}$ with real Fourier coefficients $c(m)$ $(m \geq 1)$ and suppose that L^*g is a normalized Hecke eigenform. Then there are sequences $(n_\nu)_{\nu\geq 1}$ and $(m_\mu)_{\mu\geq 1}$ in $\mathbf{N}$ such that for any $\sigma < \frac{1}{16}$ one has $\lim_{\nu\to\infty} \frac{\alpha^+_{n_\nu}}{n^\sigma_\nu} = \infty$ and $\lim_{\mu\to\infty} \frac{\alpha^-_{m_\mu}}{m^\sigma_\mu} = \infty$. In particular one has $\lim_{\nu\to\infty} \alpha^+_{n_\nu} = \infty$ and $\lim_{\mu\to\infty} \alpha^-_{m_\mu} = \infty$.*

Remark It is easy to see that for any normalized Hecke eigenform $F \in S_k$ there exists $g \in S^+_{k+1/2}$ with real Fourier coefficients such that $F = L^*g$.

If we drop the assumption that L^*g is an eigenform, we still can get non-vanishing results for the Fourier coefficients. Let us put $V := imL$ and denote by $V^\perp$ the orthogonal complement of V in $S^+_{k+1/2}$.

Theorem 2 *Let* $g \in S^+_{k+1/2}$ *with real Fourier coefficients* $c(m)\,(m \geq 1)$ *and suppose that* g *is not contained in* $V^\perp$. *Then there exists a sequence* $(n_\nu)_{\nu\geq 1}$ *in* $\mathbf{N}$ *such that for any* $\sigma < \frac{1}{16}$ *one has* $\lim_{\nu\to\infty} \frac{\alpha_{n_\nu}}{n_\nu^\sigma} = \infty$. *In particular one has* $\lim_{\nu\to\infty} \alpha_{n_\nu} = \infty$.

Remark Applying the above result with g replaced by $g - g_0$ where $g_0 \in V^\perp$ has Fourier coefficients $c_0(m)$, we obtain a corresponding statement with "$c(4n + r^2) \neq 0$" replaced by "$c(4n + r^2) \neq c_0(4n + r^2)$" in the definition of α_n. A corresponding assertion *mutatis mutandis* (and in the case where the $c_0(m)$ are real) of course is valid also in the context of Theorem 1.

3 Proof of results

We start with briefly indicating the explicit construction of the map L^* adjoint to L following [9, sect. 5], and [8], *mutatis mutandis*.

Let $g \in S^+_{k+1/2}$. The n-th Fourier coefficient of L^*g is given by

$$a(L^*g, n) = \frac{(4\pi n)^{k-1}}{(k-2)!}\langle L^*g, P_{k,n}\rangle$$

by the usual Petersson formula, where $P_{k,n}$ denotes the n-th Poincaré series in S_k.

By definition

$$\begin{aligned}\langle L^*g, P_{k,n}\rangle &= \langle g(z), P_{k,n}(4z)\theta(z)\rangle \\ &= \int_{\mathcal{F}} G(z)\overline{P_{k,n}(4z)}y^k dV\end{aligned}$$

where $z = x + iy, dV = \frac{dxdy}{y^2}$ is the invariant measure, $\mathcal{F}$ is a fundamental domain for $\Gamma_0(4) \subset \Gamma_1$ and $G(z) := \sqrt{y}\, g(z)\overline{\theta(z)}$ behaves like a modular form of weight k under $\Gamma_0(4)$. Recall that $\Gamma_0(4)$ consists of those matrices in Γ_1 whose left lower component is divisible by 4. The integral in the last line above can be computed by the usual unfolding argument.

Altogether one finds that

$$a(L^*g, n) = C_k \cdot n^{k-1} \cdot \ell(g, n) \tag{1}$$

where C_k is a real positive constant depending only on k and

$$\ell(g, n) := \sum_{r\in\mathbf{Z}} \frac{c(4n + r^2)}{(4n + r^2)^{k-1/2}}. \tag{2}$$

The convergence of the sum is clear by the usual Hecke estimate for the coefficients $c(m)$ (observe that we may assume that $k \geq 4$, otherwise $S^+_{k+1/2} = \{0\}$). This gives an explicit description of the map L^*.

Since the $P_{k,n}\,(n \geq 1)$ generate S_k, we also see that $V^\perp = kerL^*$ consists of those g with the property that $\ell(g, n) = 0$ for all $n \geq 1$.

For the proof of our results we also need Ω-results for the Fourier coefficients $a(n)\,(n \geq 1)$ of cusp forms $f \in S_k$. Recall that for arithmetic functions v, w with $w(n)$ ultimately strictly positive, one defines

$$v(n) = \Omega(w(n))$$

if

$$\limsup_{n\to\infty} \frac{|v(n)|}{w(n)} > 0,$$

and if in addition v is real-valued

$$v(n) = \Omega_+(w(n))$$

if

$$\limsup_{n\to\infty} \frac{v(n)}{w(n)} > 0,$$

and

$$v(n) = \Omega_-(w(n))$$

if

$$\liminf_{n\to\infty} \frac{v(n)}{w(n)} < 0.$$

Now recall that for $f \neq 0$ it was proved in [11] that

$$a(n) = \Omega\Big(n^{(k-1)/2} \exp(c\frac{\log n}{\log\log n})\Big), \tag{3}$$

and if in addition f is a normalized Hecke eigenform

$$a(n) = \Omega_\pm\Big(n^{(k-1)/2} \exp(c_\pm\frac{\log n}{\log\log n})\Big), \tag{4}$$

where $c_, c_\pm$ are positive constants depending only on f.

We shall now prove the first assertion of Theorem 1. We put $F := L^*g$ and denote by $A(n)\,(n \geq 1)$ the Fourier coefficients of F. According to (4) (applied with Ω_+) we can choose a sequence $(n_\nu)_{\nu\geq 1}$ in $\mathbf{N}$ such that

$$A(n_\nu) > 0 \tag{5}$$

for all ν and

$$\lim_{\nu\to\infty} \frac{A(n_\nu)}{n_\nu^{(k-1)/2}} \exp(-c_+\frac{\log n_\nu}{\log\log n_\nu}) > 0. \tag{6}$$

We claim that

$$\lim_{\nu\to\infty} \frac{\alpha^+_{n_\nu}}{n_\nu^\sigma} = \infty,$$

for any $\sigma < \frac{1}{16}$.

Suppose that this is not true, for a given σ. Then we can find a sequence $n_{\nu_1} < n_{\nu_2} < \ldots$ and $K > 0$ such that

$$\frac{\alpha^+_{n_{\nu_\mu}}}{n^\sigma_{\nu_\mu}} \leq K, \tag{7}$$

for all $\mu \geq 1$.

It follows from (1) and (2) that

$$\begin{aligned} A(n_{\nu_\mu}) &= C_k \cdot n^{k-1}_{\nu_\mu} \cdot \left(\sum\nolimits_r^+ \frac{c(4n_{\nu_\mu} + r^2)}{(4n_{\nu_\mu} + r^2)^{k-1/2}} + \sum\nolimits_r^- \frac{c(4n_{\nu_\mu} + r^2)}{(4n_{\nu_\mu} + r^2)^{k-1/2}} \right) \\ &\leq C_k \cdot n^{k-1}_{\nu_\mu} \cdot \sum\nolimits_r^+ \frac{c(4n_{\nu_\mu} + r^2)}{(4n_{\nu_\mu} + r^2)^{k-1/2}}, \end{aligned} \tag{8}$$

where r in $\sum_r^+$ runs over those $r \in \mathbf{Z}$ with $c(4n_{\nu_\mu} + r^2) > 0$ and r in $\sum_r^-$ runs over those r with $c(4n_{\nu_\mu} + r^2) \leq 0$. Note that the sum $\sum_r^+$ is non-empty by (1) and (5) and for each fixed μ is finite by (7).

By [1] the Fourier coefficients $c(m)$ of g can be estimated by

$$c(m) \ll_{g,\epsilon} m^{k/2-\delta+\epsilon} \quad (\epsilon > 0) \tag{9}$$

where one can take $\delta = \frac{1}{16}$. This estimate is slightly better than the Weil bound with $\delta = 0$. It is important to us that the bound (9) holds for all $m \geq 1$. Bounds better than the Weil bound for m squarefree were obtained in [2, 5, 10].

Inserting (9) into (8) we obtain

$$\begin{aligned} A(n_{\nu_\mu}) &\ll_{g,\epsilon} n^{k-1}_{\nu_\mu} \cdot \sum\nolimits_r^+ \frac{1}{(4n_{\nu_\mu} + r^2)^{k/2-1/2+\delta-\epsilon}} \\ &\ll_{g,\epsilon} n^{k-1}_{\nu_\mu} \cdot \frac{\alpha^+_{n_{\nu_\mu}}}{(4n_{\nu_\mu})^{k/2-1/2+\delta-\epsilon}} \\ &\ll_{g,\epsilon,K} n^{k/2-1/2-\delta+\epsilon+\sigma}_{\nu_\mu} \end{aligned}$$

where in the last line we have used (7). Choosing $\epsilon = \delta - \sigma = \frac{1}{16} - \sigma$ we therefore find that

$$A(n_{\nu_\mu}) \ll_{g,\epsilon,K} n^{(k-1)/2}_{\nu_\mu}.$$

Letting μ going to ∞ we obtain a contradiction to (6).

This proves the assertion of Theorem 1 regarding α_n^+. To obtain the assertion with α_n^- one proceeds in the same way, *mutatis mutandis*, using (4) with Ω_-. Finally to prove Theorem 2, one again proceeds in the same way, using (3). Note that the assumption that $g \notin V^\perp$ is used to guarantee that $L^* g \neq 0$.

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