Abh. Math. Semin. Univ. Hambg. (2022) 92:1–25
https://doi.org/10.1007/s12188-021-00252-0

Set recognition of decomposable graphs and steps towards their reconstruction

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Received: 2 January 2021 / Accepted: 19 October 2021 / Published online: 3 January 2022

Abstract
It is proved that decomposable graphs are set recognizable and that the index graph of the canonical decomposition as well as the graphs induced on the maximal autonomous sets of vertices are set reconstructible. From these results, we obtain set reconstructibility for many decomposable graphs as well as a concise description of the decomposable graphs for which set reconstruction remains an open problem.

Keywords Graph · Decomposable · Card · Set recognition · Set reconstruction

Mathematics Subject Classification 05C60

1 Introduction

The (deck) reconstruction problem is a classical open problem in graph theory. Hundreds of publications and multiple surveys are dedicated to its investigation. We refer the reader to [6] as the most recent expository work on this subject, which provides a comprehensive overview and a thorough list of references, and to [1] and [13] for recent works on variations of this problem. A natural generalization of the (deck) reconstruction problem is the set reconstruction problem. In set reconstruction, we ask whether every graph with at least 4 vertices is uniquely reconstructible from its *set* of isomorphism types of one-vertex-deleted subgraphs. More precisely, we have the following. (Throughout this paper, all graphs are assumed to be finite.)

Definition 1.1 Let $G=(V,E)$ be a graph and let $x \in V$. We call the induced subgraph $G-x := G[V \setminus \{x\}]$ a **card** of G.

Communicated by Nathan Bowler.

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Definition 1.2 For any finite graph $H = (W, F)$, let $[H]$ denote the set of all graphs H' on the vertex set $\{1, \dots, |W|\}$ such that H is isomorphic to H'. For any graph G, we call $\{[G - x] : x \in V\}$ the **set of unlabelled cards** of G.

Open Question 1.3 **The Set Reconstruction Problem.** Let G and H be two finite graphs with at least 4 vertices and equal sets of unlabelled cards. Must G be isomorphic to H?

Similar to the premise in [6], which states that symmetry properties hold a key to reconstruction, it was suggested in Sect. 8.5 of [10] that set reconstruction should be tackled by first proving structural results about graphs and to subsequently obtain set reconstruction results as a consequence. We take this approach here, as we use a structural result by Schmerl and Trotter, see [11] or Theorem 2.2 here, to prove set recognizability of decomposable graphs.

Definition 1.4 A class $\mathcal{C}$ of graphs is called **set recognizable** iff, for all $G = (V, E)$ in $\mathcal{C}$ and all graphs $H = (W, F)$, we have that $\{[G - x] : x \in V\} = \{[H - y] : y \in W\}$ implies that H is in $\mathcal{C}$, too.

Recognition of a class of graphs is a natural step towards the eventual reconstruction of its members. Set reconstruction of decomposable graphs, the focus of this paper, is a natural target, because many of the graphs that distinguish set reconstruction from (deck) reconstruction are decomposable (also see Sect. 2 of [12], which discusses this question for ordered sets).

Definition 1.5 Let $G = (V, E)$ be a graph, let $v \in V$ and let $A \subseteq V$. We write $v \sim A$ iff, for all $a \in A$, we have that $v \sim a$.

Definition 1.6 Let $G = (V, E)$ be a graph. The set of vertices $A \subseteq V$ is called **autonomous** (see, for example, [5]) iff, for all $v \in V \setminus A$, we have that, if there is an $a \in A$ such that $v \sim a$, then $v \sim A$. An autonomous set of vertices A is called **trivial** iff $|A| \in \{0, 1, |V|\}$.

Definition 1.7 A graph is called **decomposable** iff it contains a nontrivial autonomous set of vertices. Otherwise, it is called **indecomposable**.

We first prove the following theorem, which generalizes the main result of [4] to the set recognition of decomposable undirected simple graphs without any restrictions on the number of vertices. We note that our results hold for all classes of relations that have a Schmerl-Trotter type theorem (see Theorem 2.2) and a Gallai-type canonical decomposition (see Theorem 4.1 here). In particular, all results given here have natural analogues for the set recognition and set reconstruction of ordered sets.

Theorem 1.8 *Decomposable graphs with at least* 4 *vertices are set recognizable.*

By Theorem 4.12 in [7] or Theorem 7 in [9], disconnected graphs are set reconstructible. Therefore, for the remainder of this paper, we can concentrate on connected graphs G with connected complements G^c.

Let G be connected, decomposable and so that G^c is connected. If at least 3 vertices of G are contained in nontrivial autonomous sets of vertices, then all cards of G are decomposable. Hence, the only indecomposable graphs that could have the same set of unlabelled cards are the critically indecomposable graphs defined below.

Definition 1.9 An indecomposable graph $G = (V, E)$ is called **critically indecomposable** (see [11]) iff, for every $x \in V$, the card $G - x$ is decomposable.

Let G be connected, decomposable, let G^c be connected and assume that fewer than 3 vertices of G are contained in nontrivial autonomous sets of vertices. Then G has exactly one autonomous set of vertices, and this set contains exactly 2 vertices. It is easy to see that, in this case, the only indecomposable graphs that could have the same set of unlabelled cards are the pseudo-autonomous-doubleton indecomposable graphs defined below.

Definition 1.10 An indecomposable graph $G = (V, E)$ is called **pseudo-autonomous-doubleton indecomposable** iff G has a decomposable card, G has an indecomposable card, the indecomposable cards of G are pairwise isomorphic, and one of the following holds.

1. Every decomposable card of G contains an autonomous set of vertices A that consists of 2 independent vertices. In this case, G is called an **independent pseudo-autonomous-doubleton indecomposable** graph. Each such set A is called a **preferred nonedge**.
2. Every decomposable card of G contains an edge A that is an autonomous set of vertices. In this case, G is called an **edge pseudo-autonomous-doubleton indecomposable** graph. Each such set A is called a **preferred edge**.

Clearly, (not necessarily indecomposable) graphs with decks as stated in Definitions 1.9 and 1.10 are set recognizable. Hence the proof of Theorem 1.8 (see Sect. 4) consists of the set recognition of critically indecomposable graphs among graphs whose decks are as in Definition 1.9 (the main structural results are given in Sect. 2) and of the set recognition of pseudo-autonomous-doubleton indecomposable graphs among graphs whose decks are as in Definition 1.10 (the main structural results are given in Sect. 3).

With set recognizability established, we proceed in Sects. 5-7 with the reconstruction of the isomorphism types of the index graph in the canonical decomposition and with the set reconstruction of the graphs induced on the maximal autonomous sets of vertices of a decomposable graph. The set reconstruction of these parameters relies on the structure of critically indecomposable graphs. Availability of these parameters allows the set reconstruction of many decomposable graphs, described in Sect. 8, which, in the conclusion, allows us to give a description of the decomposable graphs whose (set) reconstruction remains as yet to be established.

Because the property of being an autonomous set of vertices is invariant under complementation, so are the properties of being decomposable, indecomposable, critically indecomposable, or pseudo-autonomous-doubleton indecomposable. Hence, throughout this paper, we will be free to work with complements of graphs when needed or desired.

2 Critically indecomposable graphs

In [11], Schmerl and Trotter characterized the critically indecomposable graphs.

Definition 2.1 For $k \in \mathbb{N}$, let $\mathcal{G}_k = (V_k, E_k)$ be the bipartite graph with the vertex set V_k that is the disjoint union of the independent vertex sets $\{\ell_1, \dots, \ell_k\}$ and $\{u_1, \dots, u_k\}$ such that, for $i, j \in \{1, \dots, k\}$, we have that $\ell_i \sim u_j$ iff $i \geq j$. See Fig. 1 for a visualization.

Theorem 2.2 (See [11], Corollary 5.8 (1).) *The graphs $\mathcal{G}_k$ and their complements $\mathcal{G}_k^c$, where $k \geq 2$, are the only critically indecomposable finite graphs.*

As is often the case with families of graphs with very specific structure, we can prove the set reconstructibility of graphs $\mathcal{G}_k$ and $\mathcal{G}_k^c$, see Lemma 2.4 below. Recall that a **cutvertex** in a graph $G = (V, E)$ is a vertex $c \in V$ such that $G - c$ is disconnected.

Lemma 2.3 *Let $G = (V, E)$ be a connected graph, let $c \in V$ be a cutvertex of G, and let $A' \subset V$ be a nontrivial autonomous set of vertices in G. Then the following hold.*

1. $c \notin A'$.
2. *A' is contained in a component of $G - c$ or A' is a union of components of $G - c$.*

Proof Note that, if A and B are distinct components of $G - c$, then no vertex of A is adjacent to any vertex of B. Because A' is nontrivial, at least one component of $G - c$ intersects A'.

Claim *If A' intersects more than one component of $G - c$, then A' contains every component of $G - c$ that intersects A'.*

To prove the *Claim*, let A' intersect more than one component of $G - c$ and let A be a component of $G - c$ such that $A' \cap A \neq \emptyset$. By assumption, there is another component $B \neq A$ of $G - c$ such that $A' \cap B \neq \emptyset$. If there was an $a_1 \in A \setminus A'$, then there would be an $a_2 \in A \setminus A'$ that is adjacent to an $a' \in A' \cap A$. Because A' is an autonomous set of vertices, a_2 would be adjacent to all elements of $A' \cap B$, which is not possible. Thus $A \setminus A' = \emptyset$, that is, $A \subset A'$. Hence, if A' intersects more than one component of $G - c$, then A' contains every component of $G - c$ that intersects A'. The *Claim* is therefore established.

To prove part 1, suppose, for a contradiction, that $c \in A'$. Let A be a component of $G - c$ that intersects A'. Let $B \neq A$ be another component of $G - c$ and let $b \in B$ be adjacent to $c \in A'$. Because b is not adjacent to any vertex in $A \cap A' \neq \emptyset$, because $c \in A'$ and because A' is an autonomous set of vertices in G, we must have that $b \in A'$. We conclude that B intersects A'. Hence, if $c \in A'$, then every component of $G - c$ intersects A'. By the *Claim* above, this means that all components of $G - c$ are contained in A'. By

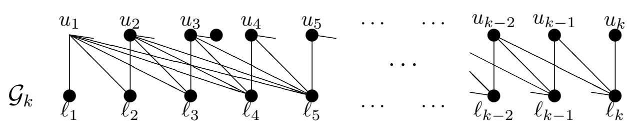

Fig. 1 The bipartite graphs $\mathcal{G}_k$ from Definition 2.1

assumption, we have $c \in A'$, which means $A' = V$, contradicting that A' is a nontrivial autonomous set of vertices in G. Hence $c \notin A'$.

To prove part 2, let A' be not contained in any single component of $G - c$. By the *Claim* above, A' contains every component of $G - c$ that intersects A' and, by part 1, $c \notin A'$. Hence A' is a union of components of $G - c$. □

Lemma 2.4 *For $k \geq 2$, the graphs $\mathcal{G}_k$ and $\mathcal{G}_k^c$ are set-reconstructible.*

Proof Because we are free to work with the complement, we only need to prove set-reconstructibility of $\mathcal{G}_k$. Set reconstructibility of $\mathcal{G}_2$ can be proved by direct inspection, so we can assume $k \geq 3$.

Suppose, for a contradiction, that there is a decomposable graph $G = (V, E)$ with the same set of unlabelled cards as $\mathcal{G}_k$. By Theorem 4.12 in [7] or Theorem 7 in [9], disconnected graphs are set reconstructible. Thus, G and G^c are connected. Because G has the same set of unlabelled cards as $\mathcal{G}_k$, there is a $c \in V$ such that $G - c$ is disconnected with two connected components, a singleton $\{s\}$ and a component C such that $G[C]$ is isomorphic to a graph $\mathcal{G}_{k-1}$. In particular, both components are indecomposable.

Let $A' \subseteq V$ be a nontrivial autonomous set of vertices in G. By part 1 of Lemma 2.3, we have $c \notin A'$. Because $|A'| \geq 2$, we have $A' \cap C \neq \emptyset$. Because $G[C]$ is indecomposable and because $A' \cap C$ is an autonomous set of vertices in $G[C]$, we have $A' = C$ or $|A' \cap C| = 1$. Because $k \geq 3$, there is a $v \in C$ such that $c \nsim v$. If $v \notin A'$, then trivially $C \neq A'$, whereas, if $v \in A'$, then no vertex in C that is adjacent to c is in A', and in this case, too, we have $C \neq A'$. Thus $|A' \cap C| = 1$. However, by part 2 of Lemma 2.3, we obtain $A' \subseteq C$, a contradiction.

Hence, there is no decomposable graph $G = (V, E)$ with the same set of unlabelled cards as $\mathcal{G}_k$. By Theorem 2.2, any indecomposable graph $G = (V, E)$ with the same set of unlabelled cards as $\mathcal{G}_k$ must be isomorphic to $\mathcal{G}_k$. Thus $\mathcal{G}_k$ is set reconstructible. □

3 Pseudo-autonomous-doubleton indecomposable graphs

It would be quite satisfying to either show that there are no pseudo-autonomous-doubleton indecomposable graphs or to classify them in a similar way to that in Theorem 2.2. However, Example 3.1 and Remark 3.2 below indicate that the characterization or reconstruction of this rather technical class of graphs is probably quite technical itself.

Example 3.1 Pseudo-autonomous-doubleton indecomposable graphs exist.

Consider the graphs G and H in Fig. 2. It is easy to check that both G and H are indecomposable. Because both graphs are rotationally symmetric, for the decks, we only need to consider the vertices w_1, z_1, w_2, z_2 in G and the vertices $w_1, z_1, w_2, z_2, \widetilde{z_1}, \widetilde{z_2}$ in H. We first consider G, then H.

For the card $G - z_1$, note that $\{z_2, w_2\}$ is the unique nontrivial autonomous set of vertices in $G - z_1$. Then note that $G - z_1$ and $G - z_2$ are isomorphic and decomposable and that $G - w_1$ and $G - w_2$ are isomorphic and indecomposable. This proves that G is a pseudo-autonomous-doubleton indecomposable graph.

For the card $H - z_1$, note that $\{z_2, w_2\}$ is the unique nontrivial autonomous set of vertices in $H - z_1$. Similarly, $\{\widetilde{z_3}, w_3\}$ is the unique nontrivial autonomous set of vertices in

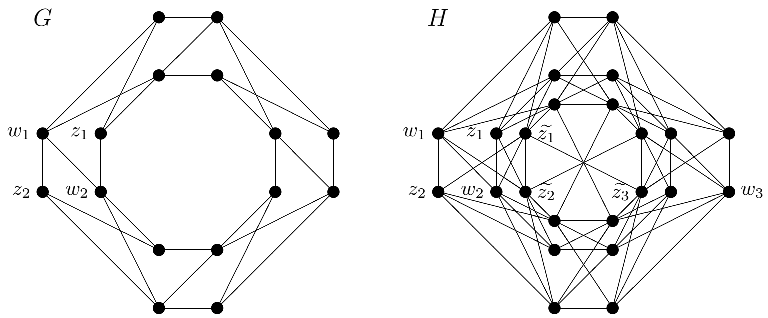

Fig. 2 Some pseudo-autonomous-doubleton indecomposable graphs

$H - \widetilde{z_1}$. Then note that $H - z_1$ and $H - z_2$ are isomorphic and decomposable, that $H - \widetilde{z_1}$ and $H - \widetilde{z_2}$ are isomorphic and decomposable, and that $H - w_1$ and $H - w_2$ are isomorphic and indecomposable. This proves that H is a pseudo-autonomous-doubleton indecomposable graph, too. □

Remark 3.2 The construction of the graph G in Example 3.1 can be executed with any directed even cycle with more than 4 vertices as follows: Replace every vertex v with a 4-path $\{v_1, v_2, v_3, v_4\}$ and, for every arc (v, w), introduce the adjacencies $\{v_2, v_4\} \sim \{w_1, w_3\}$.

The author is quite certain that the construction of G can be executed for a significant subclass of the class of vertex-transitive directed graphs. However, the author is also quite certain that the description of this class will be rather technical: The hypotheses would need to ascertain that symmetries can be used as in Example 3.1, and, for $G - z_1$ being isomorphic to $G - z_2$, this becomes unwieldy. Similarly, the construction of H should be expandable, but even for directed even cycles spelling out the details looks to be quite technical. □

Hence, the author believes that the class of pseudo-autonomous-doubleton indecomposable graphs is rather large and that the (set) reconstruction of all its members would be cumbersome.

In this section, we will derive enough properties of these graphs to allow the set recognition of decomposable graphs. It is easily checked directly that every pseudo-autonomous-doubleton indecomposable graph must have more than 4 vertices, so we will feel free to assume that there are at least 5 vertices throughout. We start by analyzing in detail the mechanism by which a set of vertices can be not autonomous in a pseudo-autonomous-doubleton indecomposable graph G, but autonomous on a card of G.

Definition 3.3 Let $G = (V, E)$ be a graph, let $z \in V$ be so that $G - z$ is decomposable and let $A \subset V$ be a nontrivial autonomous set of vertices of $G - z$ that is not autonomous in G. Then we say that z **binds** A.

Lemma 3.4 *Let $G = (V, E)$ be a graph such that $z \in V$ binds $A \subseteq V$. Then z is adjacent to some, but not all, vertices of A.*

Proof Because A is an autonomous set of vertices in $G - z$, for all $v \in V \setminus (A \cup \{z\})$, we have that, if there is an $a \in A$ such that $v \sim a$, then $v \sim A$. Suppose, for a contradiction, that $z \sim A$ or that z is not adjacent to any vertex in A. Then, in either case, A would be an autonomous set of vertices in G, which is a contradiction to A not being autonomous in G. Thus, z is adjacent to some, but not all, vertices of A. □

Clearly, by Lemma 3.4, no two distinct vertices z_1 and z_2 can bind the same set A.

Note that any pseudo-autonomous-doubleton indecomposable graph can be edge pseudo-autonomous-doubleton indecomposable, independent pseudo-autonomous-doubleton indecomposable, or both. Because we are free to work with the complement, we will focus on independent pseudo-autonomous-doubleton indecomposable graphs.

Lemma 3.5 *Let $G = (V, E)$ be an independent pseudo-autonomous-doubleton indecomposable graph and let A be a preferred nonedge that is bound by the vertex z. Then there is a $z' \in A$ which binds a preferred nonedge A'.*

Proof Let $A = \{z', a\}$ and suppose, for a contradiction, that neither vertex in A binds any subset of V. Then both $G - z'$ and $G - a$ are indecomposable. However, because, by Lemma 3.4, the degrees of z' and a are not equal, we obtain that $G - z'$ and $G - a$ are not isomorphic, a contradiction to the fact that the indecomposable cards of pseudo-autonomous-doubleton indecomposable graphs are pairwise isomorphic.

Thus, we can assume, without loss of generality, that $G - z'$ is decomposable, and, because G is independent pseudo-autonomous-doubleton indecomposable, we can further assume that z' binds a preferred nonedge A'. □

Lemma 3.6 *Let $G = (V, E)$ be an indecomposable graph. Let $z \in V$ bind $A \subseteq V$, let $z' \in V$ bind $A' \subseteq V$, and assume that $z' \in A$ and $z \notin A'$. Then $A \cap A' \neq \emptyset$.*

Proof Suppose, for a contradiction, that $A' \cap A = \emptyset$. By Lemma 3.4, $z' \in A$ is adjacent to some, but not all, vertices in A'. Because A is a nontrivial autonomous set of vertices in $G - z$, we infer that $A \setminus \{z'\} \neq \emptyset$. Let $a \in A \setminus \{z'\}$. Because $A' \cap A = \emptyset$, we have that $a \in V \setminus (A' \cup \{z'\})$. Because A is an autonomous set of vertices in $G - z$, because $z' \in A$ and because $z \notin A'$, we conclude that a is adjacent to some, but not all, vertices in A'. Therefore, A' is not an autonomous set of vertices in $G - z'$, a contradiction. Thus we must have $A' \cap A \neq \emptyset$. □

Lemma 3.7 *Let $G = (V, E)$ be an independent pseudo-autonomous-doubleton indecomposable graph. Let $z \in V$ bind the preferred nonedge $A \subseteq V$, and let $z' \in A$ bind the preferred nonedge $A' \subseteq V$. Then $z \in A'$ and $A \cap A' = \emptyset$.*

Proof Let $A = \{z', a\}$.

Suppose, for a contradiction that $z \notin A'$. By Lemma 3.6, we have that $A \cap A' \neq \emptyset$. Because $A = \{z', a\}$ and $z' \notin A'$, we obtain $A \cap A' = \{a\}$. Because $|A'| = 2$, we infer $|A' \setminus A| = 1$. Let a' be the unique vertex in $A' \setminus A$. Because $A = \{z', a\}$ is an independent set of vertices, we have $z' \nsim a$. Because z' binds $A' = \{a, a'\}$, by Lemma 3.4, we infer that $z' \sim a'$. Because $a' \in V \setminus (A \cup \{z\})$ and because A is an autonomous set of vertices in $G - z$, the adjacency $a' \sim z'$ implies $a' \sim a$. This is a contradiction to A' being a preferred nonedge. Hence we must have $z \in A'$.

Now suppose, for a contradiction, that $A \cap A' \neq \emptyset$. We claim that $A' \cup A$ is an autonomous set of vertices in G. Let $v \in V \setminus (A' \cup A)$ be so that there is an $x \in A' \cup A$ with $v \sim x$. Without loss of generality, let $x \in A$. Then, because $v \in V \setminus (A \cup \{z\})$, we have $v \sim A$ and, in particular, because $A \cap A' \neq \emptyset$, v is adjacent to a vertex in A'. Now, because $v \in V \setminus (A' \cup \{z'\})$, we have $v \sim A'$. Thus $v \sim A' \cup A$ and $A' \cup A$ is an autonomous set of vertices in G, a contradiction. Hence $A \cap A' = \emptyset$. □

Lemma 3.8 *Let $G = (V, E)$ be an independent pseudo-autonomous-doubleton indecomposable graph. Let $z \in V$ bind the distinct preferred nonedges $A_1, A_2 \subseteq V$. Then $A_1 \cap A_2 = \emptyset$. Moreover, z is contained in at least 2 preferred nonedges.*

Proof Suppose, for a contradiction, that $A_1 \cap A_2 \neq \emptyset$, say, $A_1 = \{a_1, m\}$ and $A_2 = \{a_2, m\}$.

Because A_1 and A_2 are nonedges, we have that $m \nsim a_1$ and $m \nsim a_2$. Let $v \in V \setminus \{a_1, a_2, m, z\}$ be such that v is adjacent to one of a_1 and a_2. Without loss of generality, say $v \sim a_1$. Because $\{a_1, m\}$ is an autonomous set of vertices in $G - z$, we obtain $v \sim m$ and then, because $\{a_2, m\}$ is an autonomous set of vertices in $G - z$, we obtain $v \sim a_2$.

In case $z \sim a_1$, by Lemma 3.4, we have $z \nsim m$ and then $z \sim a_2$. In case $z \sim m$, by Lemma 3.4, we have $z \nsim a_1$ and $z \nsim a_2$.

We conclude that $\{a_1, a_2\}$ is an autonomous set of vertices in G, a contradiction. Hence $A_1 \cap A_2 = \emptyset$.

Now, by Lemma 3.5, each A_i contains a z_i' that binds a preferred nonedge A_i'. Because $z_1' \neq z_2'$, we have $A_1' \neq A_2'$. By Lemma 3.7, z must be in $A_1' \cap A_2'$, which completes the proof. □

Lemma 3.9 *Let $G = (V, E)$ be an independent pseudo-autonomous-doubleton indecomposable graph. Let $z_1 \in V$ bind the preferred nonedge $A_1 = \{a_1, m\} \subseteq V$, let $z_2 \in V \setminus \{z_1\}$ bind the preferred nonedge $A_2 = \{a_2, m\} \subseteq V$, and assume that m binds a preferred nonedge. Then $\{z_1, z_2\}$ is the only preferred nonedge bound by m.*

Proof By Lemma 3.7 applied to $z' := m$ and $z := z_i$, $i = 1, 2$, any preferred nonedge A' bound by m must contain z_1 and z_2. Hence, every preferred nonedge bound by m is equal to $\{z_1, z_2\}$. □

Lemma 3.10 *Let $G = (V, E)$ be an independent pseudo-autonomous-doubleton indecomposable graph. Then every vertex of G that binds a preferred nonedge binds exactly one preferred nonedge.*

Proof Suppose, for a contradiction, that $z \in V$ binds 2 preferred nonedges. Then, by Lemma 3.8, z is contained in 2 preferred nonedges. However, by Lemma 3.9, this means that z binds exactly one preferred nonedge, a contradiction. □

Although we will not delve deeper into the structure of pseudo-autonomous-doubleton indecomposable graphs, we have determined some key facts about their structure. By Lemmas 3.5 and 3.7, for every preferred nonedge A, there is a preferred nonedge $A' \neq A$ such that (A, A') is a matched pair of preferred nonedges as defined below.

Definition 3.11 Let $G = (V, E)$ be an independent pseudo-autonomous-doubleton indecomposable graph and let A, A' be preferred nonedges. Then (A, A') are called a **matched**

pair of preferred nonedges iff the vertex z that binds A is in A', the vertex z' that binds A' is in A, and $A \cap A' = \emptyset$.

Lemma 3.12 *Let $G = (V, E)$ be an independent pseudo-autonomous-doubleton indecomposable graph. Then there is a $w \in V$ such that $G - w$ is indecomposable and such that there are two distinct vertices z and z' that bind preferred nonedges that do not contain w.*

Proof Let $w \in V$ be such that $G - w$ is indecomposable. If w is contained in 2 distinct preferred nonedges $A_1 =: \{z_1', w\}$ and $A_2 =: \{z_2', w\}$, then, by Lemma 3.7, w is not contained in the preferred nonedges A_1' and A_2' that are bound by z_1' and z_2', respectively.

If w is contained in no preferred nonedge, note that there must be at least one matched pair of preferred nonedges.

If w is contained in exactly one preferred nonedge $\{w, z\}$, we are done if there is a matched pair of preferred nonedges, such that neither of the matched nonedges contains w.

This leaves the case that w is in exactly one preferred nonedge $\{w, z\}$ and, for any matched pair of preferred nonedges, w is in one of the nonedges. Because, by Lemma 3.10, z can only bind one preferred nonedge, G contains exactly one matched pair of preferred nonedges. Because G has at least 5 vertices, there is a vertex w' outside these nonedges, and the card $G - w'$ must be indecomposable. □

We are now ready to prove Theorem 1.8.

4 Proof of Theorem 1.8

For the remaining sections, we need the concept of the canonical decomposition of a connected graph with connected complement. This result is originally due to Gallai (see [3], Satz 1.2 or [5], Theorem 1.2). Recall that an autonomous set of vertices A is called **maximal** iff $A \neq V$ and there is no other autonomous set of vertices B such that $A \subsetneq B \subsetneq V$.

Theorem 4.1 (Canonical Decomposition.) *Let $G = (V, E)$ be a connected graph with connected complement. Then every $v \in V$ is contained in a unique maximal autonomous set of vertices. That is, V can be partitioned into a union of maximal autonomous sets of vertices.* □

Theorem 4.1 means that connected graphs with connected complement can be represented in the following form.

Definition 4.2 Let $T = (W, F)$ be a graph and, for every $w \in W$, let $P_w = (A_w, E_w)$ be a graph. Assume that the vertex sets A_w are pairwise disjoint. The **lexicographic sum** $G = (V, E) = L\{P_w \mid w \in W = V(T)\}$ of the **pieces** P_w over the **index graph** T is the union of the graphs P_w with the following additional edges: If $u, w \in W$ are two distinct indices such that $u \sim_T w$, then, for all $v_u \in A_u$ and $v_w \in A_w$, we have that $v_u \sim v_w$.

In case A_w has exactly one vertex, we let $v(w)$ be the unique vertex of A_w. Moreover, for every $v \in V = \bigcup_{w \in W} A_w$, we let $I(v)$ be the unique index $w \in W$ such that $v \in A_w$.

The easiest way to connect Theorem 4.1 with Definition 4.2 is to let W be the set of all maximal autonomous sets of vertices in G and, for distinct maximal autonomous sets of vertices A, B, to let $\{A, B\} \in F$ iff there are $a \in A$ and $b \in B$ such that $a \sim b$. It is easy to see that the index graph of the canonical decomposition of G is indecomposable.

To prove Theorem 1.8, we need to prove that there is no indecomposable graph that has the same set of unlabelled cards as a decomposable graph. Let $G = (V, E)$ be an indecomposable graph.

If G is neither critically indecomposable, nor pseudo-autonomous-doubleton indecomposable, then, because the deck of every decomposable graph either consists entirely of decomposable cards (indecomposable graphs with this property are by Definition 1.9 critically indecomposable) or of pairwise isomorphic indecomposable cards plus decomposable cards with an autonomous edge or an autonomous independent doubleton (indecomposable graphs with this property are by Definition 1.10 pseudo-autonomous-doubleton indecomposable), there is no decomposable graph with the same set of unlabelled cards as G.

If G is critically indecomposable, then, by Theorem 2.2 and Lemma 2.4, G is set-reconstructible and hence there is no graph, and, in particular, no decomposable graph, with the same set of unlabelled cards as G.

This leaves the case that G is a pseudo-autonomous-doubleton indecomposable graph. Without loss of generality, assume that G is an independent pseudo-autonomous-doubleton indecomposable graph. By Lemma 3.10, every vertex of G binds at most one preferred nonedge. Thus, for every decomposable card $G - x$ of G, among the autonomous sets of vertices of $G - x$, there is exactly one autonomous set of vertices that is an independent set of 2 vertices.

By Lemma 3.12, there is a $w \in V$ that does not bind any subset of V and such that there are distinct $z, z' \in V \setminus \{w\}$ such that each of z, z' binds a preferred nonedge in G that does not contain w. In particular, each of $z, z' \in V \setminus \{w\}$ binds a preferred nonedge in $G - w$.

Suppose for a contradiction that there is a decomposable graph H with the same set of unlabelled cards as G. Let $T = (W, F)$ be the index graph of the canonical decomposition of H. Then T is isomorphic to $G - w$. By the preceding paragraph, there are $z_T, z'_T \in W$ such that the isomorphic image of each binds a doubleton of independent vertices in G and neither doubleton contains w. Hence, each of $z_T, z'_T \in W$ binds a doubleton of independent vertices in T. Without loss of generality, let z_T not be the index of the autonomous nonedge of H and let $v(z_T)$ be the vertex of H that is indexed by z_T. Then $H - v(z_T)$ contains at least two autonomous doubletons of independent vertices. Because every decomposable card of G contains, by Lemma 3.10, exactly one autonomous doubleton of independent vertices, this means that G and H cannot have the same set of unlabelled cards. □

5 NTMA-cards

A typical progression in (set) reconstruction is that the (set) recognition of a class precedes its (set) reconstruction. Although (set) reconstruction of decomposable graphs remains an open problem, it is relatively simple to prove that many decomposable graphs are set reconstructible. We first set reconstruct the isomorphism type of the index graph of the canonical decomposition (see Theorem 5.4 below) and then the isomorphism types of the graphs induced on the nontrivial maximal autonomous subsets and their multiplicities (see Theorem 7.1 below).

Definition 5.1 Let $G=(V,E)$ be a connected decomposable graph with connected complement. We call $x \in V$ an **NTMA-vertex** iff x is contained in a nontrivial maximal autonomous set of vertices. A card $G-x$ such that x is an NTMA-vertex will be called an **NTMA-card**.

Note that, clearly, the index graph of the canonical decomposition of an NTMA-card of a graph G is isomorphic to the index graph of the canonical decomposition of G.

Definition 5.2 Let $G=(V,E)$ be a connected decomposable graph with connected complement. For every $x \in V$ such that $G-x$ is a connected (not necessarily decomposable) graph with connected complement, we define τ_x as the number of vertices in the index graph of the canonical decomposition of $G-x$. For all other $x \in V$, we set $\tau_x := 0$.

Lemma 5.3 *Let $G=(V,E)$ be a connected decomposable graph with connected complement and let $T=(W,F)$ be the index graph of the canonical decomposition of G. Then the NTMA-cards of G are exactly the connected cards $G-x$ with connected complement such that τ_x is maximal among the values $\{\tau_z : z \in V\}$. Therefore, the isomorphism types of the NTMA-cards are identifiable in the set of unlabelled cards.*

Proof Let $G-x$ be an NTMA-card of G. Then $G-x$ is connected with connected complement and the index graph of its canonical decomposition is isomorphic to T. Hence, in this case, $\tau_x = |W| > 0$.

Let $G-y$ be a card of G that is not an NTMA-card and such that $\tau_y \neq 0$. Then all maximal autonomous sets of vertices in G that do not contain y are still autonomous sets of vertices in $G-y$. Hence, the index graph of the canonical decomposition of $G-y$ has at most $|W|-1$ vertices. □

Theorem 5.4 *Let $G=(V,E)$ be a connected decomposable graph with connected complement and let $T=(W,F)$ be the index graph of the canonical decomposition of G. Then the isomorphism type of T is set reconstructible.*

Proof By Lemma 5.3, the isomorphism types of the NTMA-cards of G are identifiable in the set of unlabelled cards. The result follows, because the index graph of the canonical decomposition of an NTMA-card is isomorphic to T. □

6 Lemmas

This section provides lemmas that will be needed for the reconstruction of the isomorphism types of the graphs induced on the maximal autonomous sets of vertices. Primarily, we set reconstruct the number of vertices in nontrivial autonomous sets of vertices and the number of nontrivial autonomous sets of vertices (see Lemma 6.7). At the end of this section, Lemma 6.10 considers the case in which there is exactly one nontrivial autonomous set of vertices.

We start by proving the existence of certain cards for indecomposable graphs.

Lemma 6.1 *Let $T = (W, F)$ be an indecomposable graph, let $d \in W$ and let $m \geq 2$. Then there is an $x \in W \setminus \{d\}$ such that every nontrivial autonomous set of vertices A of the card $T - x$ either contains d or it satisfies $|A| \neq m$.*

Proof Suppose, for a contradiction that every card $T - x$ with $x \neq d$ has a nontrivial autonomous set of vertices A of size m that does not contain the vertex d.

First consider the case that T has $m + 2$ vertices. Let $t \in W \setminus \{d\}$. Then $T - t$ has an autonomous set U of m vertices that does not contain d. This means that $d \sim U$ or that d is not adjacent to any vertex in U. Because we are free to work with the complement, assume without loss of generality that $d \sim U$. Then, because $W = U \cup \{d, t\}$ and T is indecomposable, we have $d \nsim t$. Let $u \in U$. Then $T - u$ has an autonomous set S of m vertices that does not contain d. Because T has $m + 2$ vertices, we infer that $S = W \setminus \{d, u\}$. Now d is adjacent to a vertex in $U \cap S \setminus \{u\} \neq \emptyset$ and $d \nsim t \in S$, a contradiction. Thus T has more than $m + 2 \geq 4$ vertices.

Next note that, if $T - d$ was decomposable, then T would be critically indecomposable and, by assumption on d, all but possibly 1 card has an autonomous set of vertices with m elements. However, by Theorem 2.2, for all critically indecomposable graphs $G = (V, E)$ with $|V| \geq 5$, all but 2 cards have exactly one nontrivial autonomous set of vertices, which happens to have 2 elements, and the remaining cards have exactly one nontrivial autonomous set of vertices, which happens to have $|V| - 2 > 2$ elements. Thus $T - d$ must be indecomposable.

Every card of $T - d$ has an autonomous set of vertices with m elements. Because T has more than $m + 2$ vertices, $T - d$ has more than $m + 1$ vertices, and every card of $T - d$ has a nontrivial autonomous set of vertices. Hence, $T - d$ is critically indecomposable. By Theorem 2.2, examination of the connected cards of $T - d$ with connected complement gives $m = 2$. Now the remaining cards of $T - d$ are disconnected (or have a disconnected complement) and the (complement's) components are a singleton and, because $m = 2$, a 2-vertex set. Hence $T - d$ has 4 vertices, which means we can assume that $T - d$ is a path with 4 vertices $p_1 \sim p_2 \sim p_3 \sim p_4$.

Now $\{p_1, p_2\}$ is the only 2-element autonomous set of vertices in $(T - d) - p_3$, which means it must be the 2-element autonomous set of vertices in $T - p_3$ that does not contain d. Similarly, $\{p_2, p_4\}$ must be the 2-element autonomous set of vertices in $T - p_1$ that does not contain d, and $\{p_4, p_3\}$ must be the 2-element autonomous set of vertices in $T - p_2$ that does not contain d. Because T is connected, d is adjacent to one of the vertices p_i. Because of the sets $\{p_1, p_2\}$, $\{p_2, p_4\}$, $\{p_4, p_3\}$ are autonomous sets of vertices in subgraphs that contain d, this implies that $d \sim \{p_1, p_2, p_3, p_4\}$, which means that T is decomposable, a contradiction. □

Lemma 6.2 *Let $T = (W, F)$ be an indecomposable graph and let $d \in W$. Then there is an $x \in W \setminus \{d\}$ such that every nontrivial autonomous set of vertices A of the card $T - x$ either contains d or it satisfies $|A| \neq 2$. Moreover, for any such x, if the card $T - x$ contains an autonomous set of vertices with 2 elements, then it contains exactly one autonomous set of vertices with 2 elements.*

Proof By Lemma 6.1, there is an $x \in W \setminus \{d\}$ such that every nontrivial autonomous set of vertices A of $T - x$ either contains d or it satisfies $|A| \neq 2$. We are left to prove that, if there is a 2-element autonomous set of vertices in $T-x$, then there is exactly one such set. Note that any 2-element autonomous set of vertices in $T-x$ contains d.

Let $\{a_1, d\}$ and $\{a_2, d\}$ be two autonomous doubletons in $T-x$ and suppose, for a contradiction, that $a_1 \neq a_2$. We claim that $\{a_1, a_2, d\}$ is autonomous in $T-x$: Let $v \in W \setminus \{x, a_1, a_2, d\}$ be adjacent to one of a_1, a_2, d. If $v \sim a_1$, then, because $\{a_1, d\}$ is autonomous in $T-x$, we have $v \sim d$. Similarly, $v \sim a_2$ implies $v \sim d$. Finally, $v \sim d$ implies $v \sim a_1, a_2$. Hence $\{a_1, a_2, d\}$ is autonomous in $T-x$.

Because $\{a_1, a_2\}$ cannot be autonomous in $T-x$, and because $\{a_1, a_2, d\}$ is autonomous in $T-x$, we have that d is adjacent to exactly one of a_1 and a_2, say, $d \sim a_1$ and $d \nsim a_2$. However, then, in case $a_1 \sim a_2$, we have that $\{a_1, d\}$ is not autonomous in $T-x$, whereas, in case $a_1 \nsim a_2$, we have that $\{a_2, d\}$ is not autonomous in $T-x$, a contradiction. Thus $a_1 = a_2$, that is, if $T-x$ has an autonomous doubleton, then it has exactly one. □

The next proposition is key to showing that orbits (see Definition 6.3 below) of the index graph T of a decomposable graph can be matched with each other on NTMA-cards of the graph.

Definition 6.3 Let $G=(V,E)$ be a graph, let M be a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G and let $v \in V$. Then the set $M \cdot v := \{\Phi(v) : \Phi \in M\}$ is called the **orbit of** v **under the action of** M or the M**-orbit** of v. Explicit mention of v can be dropped when specific knowledge of v is not needed. The set of M-orbits is denoted G/M. When no group M is explicitly mentioned, we assume by default that $M = \mathrm{Aut}(G)$.

Proposition 6.4 *Let $G=(V,E)$ and $G'=(V',E')$ be isomorphic graphs and let $O \subseteq V$ be an* $\mathrm{Aut}(G)$*-orbit of G. Then, for any two isomorphisms $\Phi, \Psi : V \to V'$, we have that $\Phi^{-1}\Psi[O] = O$ and, in particular, $\Psi[O] = \Phi[O]$.*

Proof Because $\Phi^{-1}\Psi$ is an automorphism of G and O is an $\mathrm{Aut}(G)$-orbit of G, we conclude $\Phi^{-1}\Psi[O] = O$. □

Definition 6.5 For any graph H, we define m_H to be the number of vertices of H that are contained in nontrivial autonomous sets of vertices in H, and we define n_H to be the number of nontrivial maximal autonomous sets of vertices in H.

Notation 6.6 The following notation will be used throughout the rest of this paper. For the remaining lemmas in this section, the items below are the hypotheses, which will not be explicitly restated.

1. $G=(V,E)$ will be a connected decomposable graph with connected complement.
2. Consistent with Definition 4.2, we let $T=(W,F)$ be the index graph of the canonical decomposition of G and we let $P_w = (A_w, E_w)$ be the pieces.
3. To simplify notation, we will assume that $\{1, \ldots, n_G\} \subseteq W$ and that the indices of the nontrivial maximal autonomous sets of vertices A_w are $1, \ldots, n_G$.
4. For an NTMA-card $C = G - v$, we let $T^C = \left(W^C, F^C\right)$ be the index graph of the canonical decomposition of C and we let $P_w^C = \left(A_w^C, E_w^C\right)$ be the pieces (where w ranges over W^C).
5. We let Φ_C be an isomorphism from T^C to T.

Note that, by Proposition 6.4, for any $\mathrm{Aut}\left(T^{C}\right)$-orbit O, we have that $\Phi_C[O]$ is independent of the choice of Φ_C.

Lemma 6.7 *m_G and n_G are set reconstructible.*

Proof First consider the case that there is an NTMA-card with a nontrivial maximal autonomous set of vertices with more than 2 vertices. (Clearly, this case is set recognizable.) Let C be an NTMA-card such that m_C is maximal among NTMA-cards. Then C was obtained by removal of a vertex from a maximal autonomous set of vertices with more than 2 vertices and $m_G = m_C + 1$ and $n_G = n_C$. For the remainder, we can assume that all maximal autonomous sets of vertices on NTMA-cards have at most 2 vertices.

If all NTMA-cards are indecomposable, then $n_G = 1$ and $m_G = 2$. For the remainder, we can assume that all NTMA-cards have a nontrivial maximal autonomous set of vertices.

Under these circumstances, if there is an NTMA-card C that has $n_C \geq 2$ nontrivial maximal autonomous sets of vertices, then all nontrivial maximal autonomous sets of vertices of G must have exactly 2 vertices. Hence $n_G = n_C + 1$, $m_G = 2n_C + 2$.

The last case to consider is that all NTMA-cards have exactly one nontrivial maximal autonomous set of vertices, and, for each NTMA-card, said nontrivial maximal set of vertices has 2 elements. In this case, either $n_G = 1$ and $|A_1| = 3$ or $n_G = 2$ and $|A_1| = |A_2| = 2$.

If, in case $n_G = 2$ and $|A_1| = |A_2| = 2$, the indices of A_1 and A_2 are in different $\mathrm{Aut}(T)$-orbits O_1 and O_2 of T, then G has two NTMA-cards C_1 and C_2 whose nontrivial autonomous subset is indexed by an index in $\Phi_{C_1}^{-1}[O_1]$ and $\Phi_{C_2}^{-1}[O_2]$, respectively. In case $n_G = 1$ and $|A_1| = 3$ there is exactly one $\mathrm{Aut}(T)$-orbit O such that, for every NTMA-card K, the index of the nontrivial autonomous set of vertices is in $\Phi_K^{-1}[O]$. Thus the two last cases are distinguishable unless the indices of A_1 and A_2 are in the same $\mathrm{Aut}(T)$-orbit of T, which remains the last case to be considered.

When $n_G = 2$ and $|A_1| = |A_2| = 2$ and the indices of A_1 and A_2 are in the same $\mathrm{Aut}(T)$-orbit of T, then, on every non-NTMA-card of G, the sets A_1 and A_2 are autonomous sets of vertices with 2 elements each. Moreover, the T-neighborhoods of the indices of A_1 and A_2 are not equal and, because the indices are in the same $\mathrm{Aut}(T)$-orbit, the T-neighborhoods are of the same size. Hence, there are $v_1, v_2 \in V$ such that $v_1 \sim A_1$, $v_1 \nsim A_2$, $v_2 \sim A_2$, and $v_2 \nsim A_1$. Therefore, in any card $G - v$, the set $(A_1 \cup A_2) \setminus \{v\}$ does not form an autonomous set of vertices.

Now consider the situation when $n_G = 1$ and $|A_1| = 3$. By Lemma 6.1 with $d = 1$, there is a card $T - x$ with $x \neq 1$ of the index graph T such that every nontrivial autonomous set of vertices A of $T - x$ either contains 1 or it satisfies $|A| \neq 2$.

Let $B = \{b_1, b_2\}$ be a 2-element autonomous set of vertices of the non-NTMA-card $K := G - v(x)$. Then $\{I(b_1), I(b_2)\}$ is a singleton, which necessarily is $\{1\}$, or, $\{I(b_1), I(b_2)\}$ is the, by Lemma 6.2 with $d = 1$, unique 2-element autonomous set of vertices in $T - x$, in which case one of $I(b_1)$ and $I(b_2)$ is 1. In particular, this means that, when $n_G = 1$ and $|A_1| = 3$, all autonomous sets of vertices with 2 vertices of the non-NTMA-card $K := G - v(x)$ are contained in the same autonomous set of vertices M, which satisfies $|M| \in \{3, 4\}$.

However, when $n_G = 2$ and $|A_1| = |A_2| = 2$ and the indices of A_1 and A_2 are in the same T-orbit, we have already shown that, in any non-NTMA-card $G - v$, the set $A_1 \cup A_2$ does not form an autonomous set of vertices. Hence we can distinguish the last two cases, which means we have set reconstructed m_G and n_G. □

Lemma 6.8 *The set of unlabelled cards of the pairwise disjoint union of the induced subgraphs $G[A_1], \dots, G[A_{n_G}]$ on the nontrivial maximal autonomous sets of vertices is set reconstructible.*

Proof Let $[C_1], \dots, [C_k]$ be the isomorphism types of the NTMA-cards of G, which are recognizable in the set of unlabelled cards of G. Let $j \in \{1, \dots, k\}$. If $\sum_{\left|A_w^{C_j}\right|>1} \left|A_w^{C_j}\right| = m_G - 1$, we let D_j be the graph obtained as the pairwise disjoint union of the graphs $C_j\left[A_w^{C_j}\right]$ with $\left|A_w^{C_j}\right| > 1$. If $\sum_{\left|A_w^{C_j}\right|>1} \left|A_w^{C_j}\right| < m_G - 1$, then $\sum_{\left|A_w^{C_j}\right|>1} \left|A_w^{C_j}\right| = m_G - 2$, and we let D_j be the graph obtained as the pairwise disjoint union of the graphs $C_j\left[A_w^{C_j}\right]$ with $\left|A_w^{C_j}\right| > 1$ plus an added isolated vertex. Now $\{[D_1], \dots, [D_k]\}$ is the set of unlabelled cards of the the pairwise disjoint union of the induced subgraphs $G[A_1], \dots, G[A_{n_G}]$ on the nontrivial maximal autonomous sets of vertices. □

Lemma 6.9 *Let $G = (V, E)$ be a graph such that $V = A_1 \cup A_2$, $A_1 \setminus A_2 \neq \emptyset$, $A_2 \setminus A_1 \neq \emptyset$ and each A_j is an autonomous set of vertices. Then G is disconnected or its complement G^c is disconnected.*

Proof We will prove that, if G is connected, then the complement of G is disconnected. So let G be connected.

In case $A_1 \cap A_2 = \emptyset$, the claim is clear: There are $a_1 \in A_1$ and $a_2 \in A_2$ such that $a_1 \sim a_2$. Hence $a_1 \sim A_2$ and then $A_1 \sim A_2$.

Now consider the case $S := A_1 \cap A_2 \neq \emptyset$. Without loss of generality, assume that there are $a_2 \in A_2 \setminus S$ and $s \in S \subseteq A_1$ such that $s \sim a_2$. Then $a_2 \sim A_1$. Hence $A_1 \setminus A_2 \sim a_2$ and then $A_1 \setminus A_2 \sim A_2$, which completes the proof. □

Lemma 6.10 *Let $G = (V, E)$ be a connected decomposable graph with connected complement that has exactly one nontrivial maximal autonomous set of vertices $A \subset V$.*

1. *In case $|A| \geq 3$, the graph G is set reconstructible.*
2. *In case $|A| = 2$, the isomorphism type of the induced subgraph $G[A]$ is set reconstructible.*

Proof By Lemma 6.7, graphs G as indicated are set recognizable.

To prove part 1, we first set reconstruct the isomorphism type of $G[A]$. If $G[A]$ or its complement is disconnected, then, because, by Lemma 6.8, the set of unlabelled cards of $G[A]$ is set reconstructible, we have that the isomorphism type of $G[A]$ is set reconstructible by Theorem 7 in [9]. This leaves the case that $G[A]$ is connected with connected complement.

Let d be the index of A in the canonical decomposition $\{P_w \mid w \in W = V(T)\}$ of G. By Lemma 6.1, there is an index $x \in W \setminus \{d\}$, such that every nontrivial autonomous subset B of $T - x$ either contains d or satisfies $|B| \neq m_G \geq 3$. Suppose, for a contradiction, that $C := G - v(x)$, which is a non-NTMA-card, contains two distinct autonomous sets of vertices A_1 and A_2 with m_G elements each such that each $C[A_i]$ is connected with connected complement. Then $d \in I[A_1] \cap I[A_2]$. By Lemma 6.9 applied to $C[A_1 \cup A_2]$, we obtain that

$C[A_1 \cup A_2]$ or its complement is disconnected. However, then one of the $C[A_i]$ or its complement is disconnected, a contradiction.

Thus $C := G - v(x)$, which is a non-NTMA-card, contains exactly one autonomous set of vertices B with m_G elements such that $C[B]$ is connected with connected complement. Now that we have established that such cards exist, let K be any non-NTMA-card that contains exactly one autonomous set of vertices B with m_G elements such that $K[B]$ is connected with connected complement. Then B must be A and $G[A]$ is isomorphic to $K[B]$.

Now that we have reconstructed the isomorphism type of $G[A]$, we reconstruct G as follows. Let $H = G - v = (V_H, E_H)$ be an NTMA-card. Let B be the unique nontrivial maximal autonomous set of vertices of H with $m_G - 1 > 1$ vertices. Then $B = A - v$. We obtain a graph isomorphic to G by replacing the induced subgraph $H[B]$ with an isomorphic copy $J = (V_J, E_J)$ of $G[A]$ such that $x \in V_H \setminus B$ satisfies $x \sim V_J$ iff $x \sim_H B$.

To prove part 2, again, we let d be the index of A in the canonical decomposition $\{P_w \mid w \in W = V(T)\}$ of G. By Lemma 6.1, there is an index $x \in W \setminus \{d\}$, such that every nontrivial autonomous subset B of $T - x$ either contains d or satisfies $|B| \neq m_G = 2$. Thus, for any two-element autonomous set of vertices of the non-NTMA-card $G - v(x)$, at least one element's index is d.

We first consider the case that there is a non-NTMA-card $C = G - y$ of G that has exactly one autonomous set B of vertices with 2 elements. In this case, $B = A$ and $G[A]$ is isomorphic to $C[B]$.

This leaves the case that every non-NTMA-card of G has at least two autonomous subsets with 2 elements. In this case, the card $T - x$ of T must have a nontrivial autonomous set of two vertices that contains d. Now, by Lemma 6.2, the card $T - x$ of T has exactly one nontrivial autonomous set with two vertices and this set must contain d. Let $\{d, z\}$ be this set and let $\{a, b\} := A$. Then the set $\{a, b, v(z)\}$ is autonomous in the non-NTMA-card $G - v(x)$, it contains all 2-element autonomous sets of vertices of $G - v(x)$ and it contains at least two 2-element autonomous sets of vertices of $G - v(x)$. A graph on 3 vertices that contains more than one autonomous set of vertices with 2 elements is either discrete or complete.

Thus, in this case, to set reconstruct the isomorphism type of $G[A]$, we identify a non-NTMA-card $C = G - y$ that has a 3-element autonomous set of vertices B that contains all 2-element autonomous sets of vertices of $G - y$. If this set B is a clique, then $G[A]$ is a complete graph, otherwise $G[A]$ is a discrete graph.

This completes the proof of part 2. □

7 Set reconstructible parameters

We can now set reconstruct the isomorphism types of the graphs induced on the nontrivial autonomous sets of vertices. We then proceed to refine this idea and start proving reconstruction results for certain classes of graphs.

Theorem 7.1 *Let $G = (V, E)$ be a connected decomposable graph with connected complement. Then the isomorphism types of the induced subgraphs $G[A_1], \ldots, G[A_{n_G}]$ on the nontrivial maximal autonomous sets of vertices and their multiplicities are set reconstructible.*

Proof If $n_G = 1$, this is Lemma 6.10. Hence, throughout this proof, we can assume that $n_G \geq 2$.

Let H be the pairwise disjoint sum of the induced subgraphs $G[A_1], \ldots, G[A_{n_G}]$. Because $n_G \geq 2$, the graph H is disconnected, which means that, by Lemma 6.8 and Theorem 7 in [9], the graph H is set reconstructible from information obtained from the set of unlabelled cards of G.

Let $[C_1], \ldots, [C_k]$ be the isomorphism types of the NTMA-cards of G, which by Lemma 5.3 are recognizable in the set of unlabelled cards of G. If there is a $j \in \{1, \ldots, k\}$ such that $\sum_{\left|A_w^{C_j}\right|>1} \left|A_w^{C_j}\right| = m_G - 2$, then $n_{C_j} = n_G - 1$ and we let $B_1, \ldots, B_{n_G-1}$ be equal to the maximal autonomous sets of vertices of C_j. All of these sets are maximal autonomous sets of vertices of G. Otherwise, for all $j \in \{1, \ldots, k\}$, we have $\sum_{\left|A_w^{C_j}\right|>1} \left|A_w^{C_j}\right| = m_G - 1$. In this case, is it possible to choose a j such that there is an $A_{w_0}^{C_j}$ with at least 2 elements such that no other NTMA-card C_ℓ has a nontrivial maximal autonomous set of vertices with fewer than $\left|A_{w_0}^{C_j}\right|$ vertices. We then let $B_1, \ldots, B_{n_G-1}$ be equal to the nontrivial maximal autonomous sets of vertices $A_w^{C_j}$ of C_j with $w \neq w_0$. Again, all of these sets are maximal autonomous sets of vertices of G.

Now eliminate from H a subgraph isomorphic to the pairwise disjoint union of $C_j[B_1], \ldots, C_j[B_{n_G-1}]$ and call the resulting graph F. Then the isomorphism types (with multiplicity) of the induced subgraphs $G[A_1], \ldots, G[A_{n_G}]$ are given by $C_j[B_1], \ldots, C_j[B_{n_G-1}]$, and F. □

By Theorems 5.4 and 7.1, one possible avenue towards set reconstruction of decomposable graphs $G = L\{P_w \mid w \in W = V(T)\}$ is to identify, for each index $w \in W = V(T)$ in the index graph T, the graph $G[P_w]$. In fact, with an NTMA-card $C = G - v$, we could reconstruct G if we could identify the index $I(v)$ of the removed vertex v. We have already seen this idea in the proof of part 1 of Lemma 6.10. By analyzing the automorphism group of the index graph T, it is possible to make significant progress regarding determining the location of $I(v)$.

Definition 7.2 Let $G = (V, E)$ be a connected decomposable graph with connected complement. We will call $w \in W$ **populated** iff $|A_w| > 1$. An $\mathrm{Aut}(T)$-orbit is called **populated** iff one of its vertices is populated.

Theorem 7.3 *Let $G = (V, E)$ be a connected decomposable graph with connected complement and $m_G \geq 3$. Then, for any NTMA-card $C = G - v$, the $\mathrm{Aut}\left(T^C\right)$-orbit O that contains $\Phi_C^{-1}(I(v))$, the isomorphism types of the $P_{\Phi_C(o)}$ with $o \in O$, as well as the isomorphism type of $P_{I(v)}$ can be set reconstructed.*

Proof We start by set recognizing whether there is one or more than one populated $\mathrm{Aut}(T)$ -orbit.

Consider the case that there are two distinct populated $\mathrm{Aut}(T)$-orbits O' and O''. Then, with $w' \in O'$ and $w'' \in O''$ being populated indices, $v' \in A_{w'}$ and $v'' \in A_{w''}$, the NTMA-cards $C' := G - v'$ and $C'' := G - v''$ satisfy $\sum_{o' \in \Phi_{C'}^{-1}[O']} |A_{o'}^{C'}| < \sum_{o \in \Phi_{C''}^{-1}[O']} \left|A_o^{C''}\right|$. Suppose, for a contradiction, that there is an isomorphism Ψ from C' to C''. Because Ψ maps maximal autonomous sets of vertices to maximal autonomous sets of vertices, Ψ induces an isomorphism $\overline{\Psi}$ from $T^{C'}$ to $T^{C''}$ via $\overline{\Psi}(t) := I\left(\Psi\left[A_t^{C'}\right]\right)$. By Proposition 6.4, $\overline{\Psi}$ maps

$\Phi_{C'}^{-1}[O']$ to $\Phi_{C''}^{-1}[O']$. Thus Ψ maps $\bigcup_{o'\in\Phi_{C'}^{-1}[O']} A_{o'}^{C'}$ to $\bigcup_{o\in\Phi_{C''}^{-1}[O']} A_{o}^{C''}$, which is a contradiction to $\sum_{o'\in\Phi_{C'}^{-1}[O']} |A_{o'}^{C'}| < \sum_{o\in\Phi_{C''}^{-1}[O']} \left|A_{o}^{C''}\right|$. We conclude that, when there are two populated $\mathrm{Aut}(T)$-orbits, G has two nonisomorphic NTMA-cards C' and C'' such that there is an $\mathrm{Aut}(T)$-orbit O' such that $\sum_{o'\in\Phi_{C'}^{-1}[O']} |A_{o'}^{C'}| < \sum_{o\in\Phi_{C''}^{-1}[O']} \left|A_{o}^{C''}\right|$. We note that these facts are set recognizable from the set of unlabelled cards.

On the other hand, in case there is exactly one populated $\mathrm{Aut}(T)$-orbit, either the NTMA cards are pairwise isomorphic, or, for any two nonisomorphic NTMA-cards C' and C'' and any $\mathrm{Aut}(T)$-orbit O, we have $\sum_{o\in\Phi_{C'}^{-1}[O]} |A_{o}^{C'}| = \sum_{o\in\Phi_{C''}^{-1}[O]} \left|A_{o}^{C''}\right|$. Again, the preceding is set recognizable from the set of unlabelled cards. Hence, it is set recognizable whether there are (at least) two populated $\mathrm{Aut}(T)$-orbits.

Now let C be an NTMA-card.

In case there is exactly one populated $\mathrm{Aut}(T)$-orbit, there is exactly one populated $\mathrm{Aut}(T^C)$-orbit O and it must contain $\Phi_C^{-1}(I(v))$. The non-singleton $P_{\Phi_C(o)}$ (where $o \in O$) and their multiplicities are set reconstructible via Theorem 7.1. The number of singleton $P_{\Phi_C(o)}$ (where $o \in O$) is the difference between the size $|O|$ of the orbit and the number of non-singleton maximal autonomous sets of vertices A_w in G. The isomorphism type of $P_{I(v)}$ is the unique isomorphism type for which there are fewer copies among the $C[A_o^C]$ than among the $P_{\Phi_C(o)}$ (where $o \in O$).

This leaves the case that there are (at least) two populated $\mathrm{Aut}(T)$-orbits O' and O''. The $\mathrm{Aut}(T^C)$-orbit that contains $\Phi_C^{-1}(I(v))$ is the unique $\mathrm{Aut}(T^C)$-orbit O of T^C such that there is an NTMA-card D such that $\sum_{o\in O} |A_o^C| < \sum_{o\in O} \left|A^D_{\Phi_D^{-1}(\Phi_C(o))}\right|$. The $D\left[A^D_{\Phi_D^{-1}(\Phi_C(o))}\right]$ are a permutation of the $P_{\Phi_C(o)}$ (where $o \in O$). Again, the isomorphism type of $P_{I(v)}$ is the unique isomorphism type for which there are fewer copies among the $C[A_o^C]$ than among the $P_{\Phi_C(o)}$ (where $o \in O$). □

Corollary 7.4 *Let $G = (V, E)$ be a connected decomposable graph with connected complement, let $T = (W, F)$ be the index graph of the canonical decomposition of G, let $O = \{o_1, \dots, o_n\}$ be an* $\mathrm{Aut}(T)$*-orbit of T, and, for $j = 1, \dots, n$, let $A_{o_j} \subset V$ be the maximal autonomous set of vertices in G such that $I[A_{o_j}] = o_j$. If there are a $j \in \{1, \dots, n\}$ and a $z \in A_{o_j}$ such that $|A_{o_j}| > 1$ and $G[A_{o_j}] - z$ is not isomorphic to any of the $G[A_{o_i}]$, $i = 1, \dots, n$, then G is set reconstructible.*

Proof Via Theorem 7.3, because of our hypothesis, we can identify an NTMA-card $C = G - v$ that has an $\mathrm{Aut}\left(T^C\right)$-orbit $O = \{o_1, \dots, o_n\}$ of the index graph T^C of the canonical decomposition of C, such that there is an $i \in \{1, \dots, n\}$ such that $C[A_{o_i}]$ is not among the $G[A_{\Phi_C(o_j)}]$, $j = 1, \dots, n$. This means that $I(v) = \Phi_C(o_i)$. We set reconstruct G by replacing $C[A_{o_i}]$ with $P_{I(v)}$. □

Corollary 7.4 has a multitude of immediate applications. For example, consider the following.

Corollary 7.5 *A decomposable graph G with $m_G \geq 3$ such that the index graph of the canonical decomposition has a populated singleton orbit is set reconstructible. In particular, this means that decomposable graphs with rigid index graph and $m_G \geq 3$ are set reconstructible.* □

Recall that the **lexicographic product** $G \bullet H$ of graphs G and H, which is also called the **wreath product** $G \wr H$, is the lexicographic sum of graphs H_t that are isomorphic to H over the index graph $G = (T, F)$.

Corollary 7.6 *Let G and H be two finite graphs with more than* 1 *vertex each. Then the lexicographic* (*wreath*) *product* $G \bullet H$ ($G \wr H$) *is set reconstructible.*

Proof Clearly, $G \wr H$ is set reconstructible if it or its complement is disconnected. When $G \wr H$ and its complement are connected, because both factors have more than one vertex, every vertex of $G \wr H$ is contained in a nontrivial maximal autonomous set of vertices. We can therefore apply Corollary 7.4 with a vertex v that is contained in a nontrivial maximal autonomous set of vertices of smallest size. □

8 Interdependent orbit unions

Naively speaking, with Theorem 7.3 in hand, if we could just identify the right $o \in O$ such that $\Phi_C(o) = I(v)$, then, replacing $C[A_o^C]$ with the "lost" $P_{I(v)}$ reconstructs the graph. The problem with this idea is that, in general, the right $o \in O$ may not be readily identifiable: If, for every populated orbit O' of T, for every $o' \in O'$ with non-singleton $P_{o'}$, and for every $v \in P_{o'}$, there is an $o'' \in O'$ such that $P_{o''}$ is isomorphic to $P_{o'} - v$, then Corollary 7.4 cannot be applied and there is no simple method to distinguish o' from the o''. We proceed by using a representation of $\mathrm{Aut}(T)$ to make further progress as indicated in Theorem 8.16 below.

If an induced subgraph $H = (V_H, E_H)$ of $G = (V, E)$ was obtained by removing a union of $\mathrm{Aut}(G)$-orbits, then $\mathrm{Aut}(G)$-orbits that are contained in V_H can be strictly contained in $\mathrm{Aut}(H)$-orbits: The $\mathrm{Aut}(G)$-orbits of the graph G in Fig. 3 are marked by ovals. We can see that, for $X \in \{A, B, C, D\}$, the $\mathrm{Aut}(G)$-orbits X and $\widetilde{X}$ are strictly contained in the $\mathrm{Aut}(G - R)$-orbit $X \cup \widetilde{X}$. For this reason (further elaborated later in Remark 8.11 below), dictated orbit structures, see Definition 8.2 below, will be useful for the representation of $\mathrm{Aut}(G)$ in Proposition 8.10.

Definition 8.1 Let $G = (V, E)$ be a graph, let $\Phi : V \to V$ be an automorphism and let $v \in V$. Then $\Phi \cdot v := \{\Phi^n(v) : n \in \mathbb{Z}\}$ is called the **orbit** of v under Φ or an **orbit** of Φ. To distinguish these orbits from the ones in Definition 6.3, the automorphism will always be mentioned.

Definition 8.2 Let $G = (V, E)$ be a graph and let $\mathcal{D}$ be a partition of V. Then $\mathrm{Aut}_{\mathcal{D}}(G)$ is the set of automorphisms $\Phi : V \to V$ such that, for every orbit O of Φ, there is a $D \in \mathcal{D}$ such

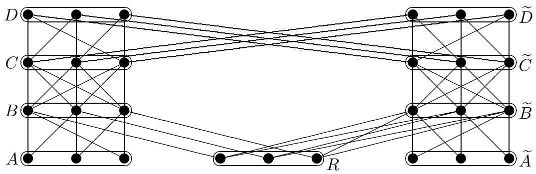

Fig. 3 A graph $G = (V, E)$ with $\mathrm{Aut}(G)$-orbits marked with ovals

that $O \subseteq D$. In this context, $\mathcal{D}$ is called a **dictated orbit structure** for G. $\mathrm{Aut}_{\mathcal{D}}(P)$-orbits will, more briefly, be called $\mathcal{D}$-**orbits**.

The partition of V into its $\mathrm{Aut}(G)$-orbits is called the **natural orbit structure** of G, which will typically be denoted $\mathcal{N}$. When working with the natural orbit structure, explicit indications of the orbit structure, usually via subscripts or prefixes $\mathcal{D}$, will often be omitted.

Clearly, $\mathrm{Aut}_{\mathcal{D}}(G) \neq \mathrm{Aut}(G)$ iff there are an orbit O of G and a $D \in \mathcal{D}$ such that $O \cap D \neq \emptyset$ and $O \nsubseteq D$.

Definition 8.3 Let $G = (V, E)$ be a graph, let $\mathcal{D}$ be a dictated orbit structure for G, and let C, D be two $\mathcal{D}$-orbits of G such that there are a $c_1 \in C$ and a $d_1 \in D$ such that $c_1 \sim d_1$. We will write $C \upharpoonleft\downharpoonright_{\mathcal{D}} D$ and say that C and D are **directly interdependent** iff there are $c_2 \in C$ and $d_2 \in D$ such that $c_2 \nsim d_2$.

Figure 3 shows how (connection through) direct interdependence can allow one orbit to restrict the values of automorphisms on many other orbits: In the graph in Fig. 3, we have $A \upharpoonleft\downharpoonright_{\mathcal{N}} B \upharpoonleft\downharpoonright_{\mathcal{N}} R \upharpoonleft\downharpoonright_{\mathcal{N}} \widetilde{B} \upharpoonleft\downharpoonright_{\mathcal{N}} \widetilde{A}$ and the values of any automorphism on $A \cup B \cup R \cup \widetilde{B} \cup \widetilde{A}$ are determined by the automorphism's values on A.

It should not be surprising that the transitive closure of $\upharpoonleft\downharpoonright_{\mathcal{D}}$ is an equivalence relation.

Proposition 8.4 *Let $G = (V, E)$ be a graph and let $\mathcal{D}$ be a dictated orbit structure for G. The relation $\leftrightsquigarrow_{\mathcal{D}}$, defined to be the transitive closure of the union of $\upharpoonleft\downharpoonright_{\mathcal{D}}$ and the identity relation, is an equivalence relation.*

Proof Reflexivity and transitivity are trivial, because $\leftrightsquigarrow_{\mathcal{D}}$ is the transitive closure of a relation that contains the identity relation.

Symmetry follows from the symmetry of $\upharpoonleft\downharpoonright_{\mathcal{D}}$. □

Definition 8.5 Let $G = (V, E)$ be a graph and let $\mathcal{D}$ be a dictated orbit structure for G. Two $\mathcal{D}$-orbits C and D with $C \leftrightsquigarrow_{\mathcal{D}} D$ will be called **interdependent**. If $\mathcal{E}$ is a $\leftrightsquigarrow_{\mathcal{D}}$ equivalence class, then we will call $\bigcup \mathcal{E}$ an **interdependent** $\mathcal{D}$-**orbit union**.

Definition 8.6 Let $G = (V, E)$ be a graph, let $\mathcal{D}$ be a dictated orbit structure for G, and let $H = (V_H, E_H)$ be an induced subgraph such that, for all $\mathcal{D}$-orbits D, we have $D \subseteq V_H$ or $D \cap V_H = \emptyset$. The dictated orbit structure for H **induced by** $\mathcal{D}$, denoted $\mathcal{D}|H$, is defined to be the set of all $\mathcal{D}$-orbits that are contained in V_H.

For the natural orbit structure $\mathcal{N}$ for G, the dictated orbit structure for H induced by $\mathcal{N}$ will be called the **naturally required** orbit structure $\mathcal{N}|H$.

Proposition 8.7 below now lays the groundwork for representing automorphisms through certain automorphisms on the non-singleton interdependent orbit unions in Proposition 8.10.

Proposition 8.7 *Let $G = (V, E)$ be a graph, let $\mathcal{D}$ be a dictated orbit structure for G, and let U be an interdependent $\mathcal{D}$-orbit union. Then the following hold.*

1. *For all $x \in V \setminus U$ and all $C \in \mathcal{D}|G[U]$, if there is a $c \in C$ such that $c \sim x$, then $C \sim x$.*

2. *For every* $\Phi \in \mathrm{Aut}_{\mathcal{D}}(G)$, *we have that* $\Phi|_U \in \mathrm{Aut}_{\mathcal{D}|G[U]}(G[U])$.
3. *Every* $\Psi \in \mathrm{Aut}_{\mathcal{D}|G[U]}(G[U])$ *can be extended to an automorphism* Ψ^G *of* G *by setting* Ψ^G *equal to* Ψ *on* U *and equal to the identity on* $V \setminus U$.
4. *The* $\mathrm{Aut}_{\mathcal{D}|G[U]}(G[U])$*-orbits are just the sets in* $\mathcal{D}|G[U]$.

Proof To prove part 1, let $x \in V \setminus U$ and $C \in \mathcal{D}|G[U]$ be so that there is a $c \in C$ such that $c \sim x$. Let X be the $\mathcal{D}$-orbit of x. Because $x \notin U$, we must have $X \not\parallel_{\mathcal{D}} C$. Because $C \ni c \sim x \in X$, we must have that $C \sim X$ and hence $C \sim x$.

Part 2 follows directly from the definitions.

To prove part 3, let $\Psi^G(x) := \begin{cases} \Psi(x); & \text{if } x \in U \\ x; & \text{if } x \in V \setminus U \end{cases}$. Clearly, Ψ^G is bijective. To prove that Ψ^G preserves adjacency, let $x \sim y$. If x, y are both in U or both in $V \setminus U$, we obtain $\Psi^G(x) \sim \Psi^G(y)$. In case $x \in V \setminus U$ and $y \in U$, let $Y \in \mathcal{D}|G[U]$ be so that $y \in Y$. Then $\Psi(y) \in Y$. By part 1, we have that $x \sim Y$ and hence $\Psi^G(x) = x \sim Y \ni \Psi^G(y)$. The case $y \in V \setminus U$ and $x \in U$ is handled similarly.

Part 4 follows from parts 2 and 3. □

Definition 8.8 Let $G = (V, E)$ be a graph, let $\mathcal{D}$ be a dictated orbit structure for G and let U be an interdependent $\mathcal{D}$-orbit union. We define $\mathrm{Aut}^G_{\mathcal{D}|G[U]}(G[U])$ to be the set of automorphisms $\Psi^G \in \mathrm{Aut}(G)$ as in part 3 of Proposition 8.7.

Let $G = (V, E)$ be a graph, let $\mathcal{D}$ be a dictated orbit structure for G and let U, U' be disjoint interdependent $\mathcal{D}$-orbit unions. Then clearly, for $\Psi^G \in \mathrm{Aut}^G_{\mathcal{D}|G[U]}(G[U])$ and $\Phi^G \in \mathrm{Aut}^G_{\mathcal{D}'|G[U']}(G[U'])$, we have $\Psi^G \circ \Phi^G = \Phi^G \circ \Psi^G$. Hence the following definition is sensible.

Definition 8.9 Let $G = (V, E)$ be a graph and let $\mathcal{A}_1, \dots, \mathcal{A}_z \subseteq \mathrm{Aut}(G)$ be sets of automorphisms such that, for all pairs of distinct $i, j \in \{1, \dots, z\}$, all $\Phi_i \in \mathcal{A}_i$ and all $\Phi_j \in \mathcal{A}_j$, we have $\Phi_i \circ \Phi_j = \Phi_j \circ \Phi_i$. We define $\bigcirc_{j=1}^{z} \mathcal{A}_j$ to be the set of compositions $\Psi_1 \circ \cdots \circ \Psi_z$ such that, for $j = 1, \dots, z$, we have $\Psi_j \in \mathcal{A}_j$.

Proposition 8.10 *Let* $G = (V, E)$ *be a graph with natural orbit structure* $\mathcal{N}$ *and let* $U_1, \dots, U_z$ *be the non-singleton interdependent orbit unions of* G. *Then* $\mathrm{Aut}(G) = \bigcirc_{j=1}^{z} \mathrm{Aut}^G_{\mathcal{N}|G[U_j]}(G[U_j])$.

Proof The containment $\mathrm{Aut}(G) \supseteq \bigcirc_{j=1}^{z} \mathrm{Aut}^G_{\mathcal{N}|G[U_j]}(G[U_j])$ is clear.

By part 2 of Lemma 8.7, for every $\Phi \in \mathrm{Aut}(G)$ and every $j \in \{1, \dots, z\}$, we have $\Phi|_{U_j} \in \mathrm{Aut}_{\mathcal{N}|G[U_j]}(G[U_j])$. Because Φ fixes all points in $P \setminus \bigcup_{j=1}^{z} U_j$, we have $\Phi = \Phi|^G_{U_1} \circ \cdots \circ \Phi|^G_{U_z}$. Hence $\mathrm{Aut}(G) \subseteq \bigcirc_{j=1}^{z} \mathrm{Aut}^G_{\mathcal{N}|G[U_j]}(G[U_j])$. □

Remark 8.11 Although the naturally required orbit structure $\mathcal{N}|G[U]$ may look more technical than natural, it is indispensable for the representation in Proposition 8.10. Consider the graph in Fig. 3. The natural orbit unions in this graph are $U_1 := A \cup B \cup R \cup \widetilde{B} \cup \widetilde{A}$ and $U_2 := C \cup D \cup \widetilde{D} \cup \widetilde{C}$. However, $G[U_2]$ considered as a graph by itself is transitive, which means it has only one orbit. Hence, we cannot use the automorphism groups $\mathrm{Aut}(G[U_j])$ in place of their subgroups $\mathrm{Aut}_{\mathcal{N}|G[U_j]}(G[U_j])$ in Proposition 8.10. □

Returning to the set reconstruction of decomposable graphs, we can now record the following.

Definition 8.12 Let $G = (V, E)$ be a connected decomposable graph with connected complement, let $T = (W, F)$ be the index graph of the canonical decomposition of G and let $K \subseteq W$. We define the graph G **with** K **collapsed** $G \searrow K$ to have the vertex set $K \cup \bigcup_{w \in W \setminus K} A_w$ and by letting the edges of $G \searrow K$ be the edges induced by G on $\bigcup_{w \in W \setminus K} A_w$, the edges induced by T on K, and by letting there be an edge between $k \in K$ and $v \in \bigcup_{w \in W \setminus K} A_w$ iff there is an edge in T between k and $I(v)$.

Definition 8.13 Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be connected decomposable graphs with connected complement and let $T_G = (W_G, F_G)$ and $T_H = (W_H, F_H)$ be the index graphs of the canonical decompositions of G and H, respectively. Let X_G and X_H be unions of interdependent orbit unions in T_G and T_H, respectively, and let $S_G := \bigcup_{w \in X_G} A_w$ and $S_H := \bigcup_{w \in X_H} A_w$. (Note that $S_G = V_G$ and $S_H = V_H$ are possible.) For any isomorphism $\Phi : S_H \to S_G$ from $H[S_H]$ to $G[S_G]$ that maps maximal autonomous sets of vertices in H to maximal autonomous sets of vertices in G, we define the **induced isomorphism** $\Phi^T : X_H \to X_G$ from the subgraph $T_H[X_H]$ of the index graph T_H to the subgraph $T_G[X_G]$ of the index graph T_G by mapping each $x_H \in X_H$ to the unique $x_G \in X_G$ such that $\Phi[A_{x_H}] = A_{x_G}$.

Lemma 8.14 *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be connected decomposable graphs with connected complement and let $T_G = (W_G, F_G)$ and $T_H = (W_H, F_H)$ be the index graphs of the canonical decompositions of G and H, respectively. Let U_G be an interdependent orbit union in T_G and let U_H be an interdependent orbit union in T_H such that there is an isomorphism $\Phi_\searrow$ from $H \searrow U_H$ to $G \searrow U_G$ that maps U_H to U_G. Let $S_G := \bigcup_{w \in U_G} A_w$ and $S_H := \bigcup_{w \in U_H} A_w$. Let Φ_S be an isomorphism from $H[S_H]$ to $G[S_G]$ that maps maximal autonomous sets of vertices in H to maximal autonomous sets of vertices in G such that the induced isomorphism Φ_S^T from $T_H[U_H]$ to $T_G[U_G]$ maps each* Aut(T_H)*-orbit O in $T_H[U_H]$ to $\Phi_\searrow[O]$, Then the graph H is isomorphic to the graph G.*

Proof For $x \in V_H$, define

$$\Phi(x) := \begin{cases} \Phi_\searrow(x); & \text{for } x \in V_H \setminus S_H, \\ \Phi_S(x); & \text{for } x \in S_H. \end{cases}$$

Clearly, Φ is bijective.

Let $x, y \in V_H$. If both x and y are in $V_H \setminus S_H$ or if both x and y are in S_H, then, because $\Phi_\searrow$ and Φ_S are isomorphisms, we have $x \sim y$ iff $\Phi(x) \sim \Phi(y)$. By symmetry, this leaves the case that $x \in V_H \setminus S_H$ and $y \in S_H$. Let $O_{I(x)}$ and $O_{I(y)}$ be the Aut(T_H)-orbits of $I(x)$ and $I(y)$, respectively. Then $x \sim y$ iff $I(x) \sim I(y)$ iff $O_{I(x)} \sim O_{I(y)}$ iff $\Phi_\searrow^T[O_{I(x)}] \sim \Phi_\searrow^T[O_{I(y)}]$ iff $\Phi_\searrow^T[O_{I(x)}] \sim \Phi_\searrow[O_{I(y)}]$ iff $\Phi_\searrow^T[O_{I(x)}] \sim \Phi_S^T[O_{I(y)}]$ iff $O_{I(\Phi_\searrow(x))} \sim O_{I(\Phi_S(y))}$ iff $I(\Phi_\searrow(x)) \sim I(\Phi_S(y))$ iff $I(\Phi(x)) \sim I(\Phi(y))$ iff $\Phi(x) \sim \Phi(y)$. Hence Φ is an isomorphism.

□

Definition 8.15 Let $G = (V, E)$ be a connected decomposable graph with connected complement and let $T = (W, F)$ be the index graph of the canonical decomposition of G. An interdependent Aut(T)-orbit union is called **populated** iff it contains a populated Aut(T) -orbit.

Theorem 8.16 *Let $G=(V,E)$ be a connected decomposable graph with connected complement and with two distinct populated interdependent* Aut(T)*-orbit unions. Then G is set reconstructible.*

Proof By Theorem 7.3, we can identify two NTMA-cards $C_j = G - x_j$, $j = 1,2$ such that $I(x_1)$ and $I(x_2)$ are not in the same interdependent Aut(T)-orbit union in T. For $j = 1,2$, let U_j be the interdependent Aut(T)-orbit union that contains $I(x_j)$. By Theorem 7.3, for $j = 1,2$, we can identify $\Phi_{C_j}^{-1}[U_j]$ in T^{C_j}.

Let Φ be an isomorphism from T^{C_1} to T^{C_2}. Then, for $j = 1,2$, we have $\Phi\left[\Phi_{C_1}^{-1}[U_j]\right] = \Phi_{C_2}^{-1}[U_j]$. Thus, we can identify $\Phi_{C_2}^{-1}[U_1]$ on C_2 as the Φ-image of the identified set $\Phi_{C_1}^{-1}[U_1]$ in T^{C_1}. Obtain H from C_1 by replacing each maximal autonomous set of vertices A such that $I[A] \in \Phi_{C_1}^{-1}[U_1]$ with the subgraph induced by C_2 on the maximal autonomous set of vertices from C_2 whose index is $\Phi[I[A]]$.

Then, by Lemma 8.14 with $U_G = U_1, U_H = \Phi_{C_1}^{-1}[U_1]$, $\Phi_{\searrow}$ being the natural isomorphism from $C_1 \searrow \Phi_{C_1}^{-1}[U_1]$ to $G \searrow U_1$ and Φ_S being the natural isomorphism from $H\left[\bigcup_{u\in U_1} A_{\Phi(u)}\right]$ to $G\left[\bigcup_{u\in U_1} A_u\right]$, the graph H is isomorphic to G. □

9 Conclusion

By Lemma 6.10, Corollary 7.4 and Theorem 8.16, set reconstructibility of decomposable graphs will be established if the class of graphs with exactly two vertices in nontrivial autonomous sets of vertices can be set reconstructed, and if we can set reconstruct the connected decomposable graphs $G=(V,E)$ with connected complement and the following properties.

1. There are at least two nontrivial autonomous sets of vertices,
2. All indices of nontrivial autonomous sets of vertices are contained in the same interdependent orbit union of the index graph T of the canonical decomposition of G,
3. For every populated T-orbit O, every $w \in O$, every nontrivial induced subgraph $G[A_w]$, and every $v \in A_w$, there is a $w_v \in O$ such that $G[A_{w_v}]$ is isomorphic to $G[A_w] - v$.

Naturally, if these classes could be deck reconstructed, we would achieve deck reconstruction of decomposable graphs. Therefore, although set and deck reconstruction of decomposable graphs remain open problems, we have a very well-defined set of targets for further study.

We should also note that there are simple extensions of the methods presented here. For an example, consider the following.

Theorem 9.1 *Let $G=(V,E)$ be a connected decomposable graph with connected complement, let $T=(W,F)$ be the index graph of the canonical decomposition of G, let P be an orbit of T, let $O=\{o_1,\dots,o_n\} \subseteq P$ be an orbit of $G \searrow P$, and let $A_{o_1},\dots,A_{o_n} \subset V$ be the maximal autonomous sets of vertices of G indexed by vertices in O. If there are a*

$j \in \{1, \ldots, n\}$ and a $z \in A_{o_j}$ such that $G[A_{o_j}] - z$ is not empty and is not isomorphic to any of the $G[A_{o_i}]$ with $i \neq j$, then G is set reconstructible.

Proof (sketch). The proof is a simple extension of the proof of Corollary 7.4. First identify the NTMA-cards $C = G - v$ such that $I(v) \in \Phi_C[P]$. For these NTMA-cards consider the orbits of the graphs $C \searrow \Phi_C^{-1}[P]$, which are isomorphic to $G \searrow P$. Just as in the proof of Theorem 7.3, for each of these C, we can identify the $\mathrm{Aut}\left(C \searrow \Phi_C^{-1}[P]\right)$-orbit O_v such that $v \in \Phi_C[O_v]$. The remainder of the argument is exactly the same as in the proof of Corollary 7.4. □

Theorem 9.1 and its proof provide information that Corollary 7.4 does not, as we can obtain, for example, in the case of discrete orbits, for every NTMA-card $C = G - v$, the isomorphism type of the neighborhood of $A_{I(v)}$. This further restricts the class of decomposable graphs that remain to be reconstructed that was given above. In fact, the idea of Theorem 9.1 can be iterated by populating all $\mathrm{Aut}\left(C \searrow \Phi_C^{-1}[P]\right)$-orbits except the orbit O_v such that $v \in \Phi_C[O_v]$. Although, unfortunately, it is not guaranteed that orbits would split until set reconstruction is achieved, this process provides a significant amount of additional information on the structure of the decomposable graphs with $m_G \geq 3$ that remain to be reconstructed.

Acknowledgements The author thanks the referee for the very thorough review of this paper.

Author Contributions Not applicable.

Funding None.

Declarations

Conflict of interest Not applicable.

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