Abh. Math. Semin. Univ. Hambg. (2021) 91:345–352
https://doi.org/10.1007/s12188-021-00249-9

On a non-local area-preserving curvature flow in the plane

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Received: 26 May 2021 / Accepted: 21 September 2021 / Published online: 30 September 2021

Abstract
In this paper, we consider a kind of area-preserving flow for closed convex planar curves which will decrease the length of the evolving curve and make the evolving curve more and more circular during the evolution process. And the final shape of the evolving curve will be a circle as time $t \to +\infty$.

Keywords Closed convex plane curves · Area-preserving · Existence · Convergence

Mathematics Subject Classification 35K55 · 53A04 · 53E99

1 Introduction

The classical curve shortening flow equation in a plane is

$$\frac{\partial X(u,t)}{\partial t} = \kappa N, \tag{1.1}$$

where $X(u,t) = (x(u,t), y(u,t)) : S^1 \times [0,T) \to \mathbb{R}^2$ is a family of closed planar curves, κ is the curvature and N is the unit normal vector. Gage and Hamilton [1–3] have studied this curve shortening flow and have proved it shrinks to a round point in finite time. Then another natural question arises for expanding evolution flow for curves. Chow–Tsai [4] have studied the expanding flow such as

$$\frac{\partial X(u,t)}{\partial t} = -G\Big(\frac{1}{\kappa}\Big)N, \tag{1.2}$$

where G is a positive smooth function with $G' > 0$ everywhere. Andrews [5] and Tsai [6] have studied more general expanding flows, especially flows with anisotropic speeds. They have obtained deep results too. Later, people began to study curve flow problems preserving some geometric quantities. Gage [7] has considered an area-preserving flow

Communicated by Herr Cortés.

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$$\frac{\partial X(u,t)}{\partial t} = \left(\kappa - \frac{2\pi}{L}\right)N, \tag{1.3}$$

where L is the length of the evolving curve, and he proved that the length of the curve is non-increasing and finally converge to a circle. In [8], Pan-Yang considered a length-preserving curve flow

$$\frac{\partial X(u,t)}{\partial t} = \left(\frac{L}{2\pi} - \frac{1}{\kappa}\right)N. \tag{1.4}$$

They have proved that the enclosed area of curve is non-decreasing and finally converge to a circle.

Motivated by their works, we now investigate a non-local area-preserving curvature flow for closed convex planar curves. Let $X(u,t) = (x(u,t), y(u,t)) : S^1 \times [0,T) \to \mathbb{R}^2$ be a family of closed planar curves with $X(u,0) = X_0(u)$ being a closed convex initial curve, then the evolution problem considered is defined as:

$$\frac{\partial X(u,t)}{\partial t} = \left(P - \frac{2A}{L}\right)N, \tag{1.5}$$

where $P = - < X, N >$ is the Minkowski support function, N is the unit inward pointing normal vector, $A(t)$ and $L(t)$ are the enclosed area and the length of the curve, respectively. The main result of this paper is the following theorem.

Theorem 1.1 *A closed convex plane curve which evolves according to* (1.5) *remains convex*, *decreases its length and preserves the enclosed area*, *becomes more and more circular during the evolution process*, *and finally converges to a finite circle in Hausdorff sense with radius* $\sqrt{\frac{A}{\pi}}$ *as* $t \to \infty$.

2 Limits of evolving curves

In this section we first derive a set of evolution equations along (1.5). Since monotonicity formulas are immediate consequences, we prove them in this section while the existence and convexity will be proved in the next section. Let $s(u, t)$ be the arc-length parameter of $X(u, t)$, then $\frac{\partial s}{\partial u} = |X_u| \frac{\partial}{\partial s}$. Let $T = X_s$ be the unit tangent vector.

Lemma 2.1 *Suppose* $\frac{\partial X}{\partial t} = \phi N$, *then the evolution equations for* ds, T*and* N *are*

$$(ds)_t = -\phi\kappa ds, \quad T_t = \phi_s N, \quad N_t = -\phi_s T.$$

Proof Since $ds = |X_u| du$ one has

$$(ds)_t = \frac{\partial |X_u|}{\partial t} du = \frac{< X_u, X_{ut} >}{|X_u|} du =< T, (\phi N)_u > du =< T, \phi N_u > du = -\phi\kappa ds.$$

By direct computations one has

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{X_{ut}}{|X_u|} - \frac{< X_u, X_{ut} > X_u}{|X_u|^3} \\ &= \frac{1}{|X_u|}\left(X_{tu} - < X_{tu}, T > T\right) \\ &= \frac{\partial}{\partial s}(\phi N) - < \frac{\partial}{\partial s}(\phi N), T > T \\ &= \phi_s N.\end{aligned}$$

Taking time derivative of $< T, N >= 0$ we have $< T_t, N > + < T, N_t >= 0$. Note that $N_t \perp N$ and we get $N_t = -\phi_s T$. □

Lemma 2.2 *Suppose* $\frac{\partial X}{\partial t} = \phi N$, *then the evolution equations for* $L(t)$, $A(t)$ *are*

$$\frac{dL(t)}{dt} = -\int_0^L \phi\kappa ds, \quad \frac{dA(t)}{dt} = -\int_0^L \phi ds. \tag{2.1}$$

Proof The derivative of $L(t)$ is a direct consequence of $(ds)_t = -\phi\kappa ds$. By the Green's formula we have $A = -\frac{1}{2}\int_0^L < X, N > ds$, thus

$$\begin{aligned}\frac{dA(t)}{dt} &= -\frac{1}{2}\int_0^L < X_t, N > + < X, N_t > - < X, N > \phi\kappa ds \\ &= -\frac{1}{2}\int_0^L \phi + < X, -\phi_s T > - < X, \phi T_s > ds \\ &= -\frac{1}{2}\int_0^L \phi + \frac{\partial}{\partial s} < X, -\phi T > + < X_s, \phi T > ds.\end{aligned}$$

Since X is a closed curve we have $\int_0^L \frac{\partial}{\partial s} < X, -\phi T > ds = 0$. Plugging in $X_s = T$ we get the formula for the derivative of $A(t)$. □

Lemma 2.3 *If* $X(u, t)$ *evolves under the equations defined by* (1.5), *then, during the evolution process, the length of the evolving curve is decreasing and the enclosed area keeps constant.*

Proof In (2.1) plug in $\phi = P - \frac{2A}{L}$, one has

$$L'(t) = -\int_0^L \left(P - \frac{2A}{L}\right)\kappa ds = -L + \frac{4\pi A}{L} = \frac{4\pi A - L^2}{L} \le 0, \tag{2.2}$$

and

$$A'(t) = -\int_0^L \left(P - \frac{2A}{L}\right)ds = -2A + 2A = 0, \tag{2.3}$$

where we have used the facts $\int_0^L P\kappa ds = L$, $\int_0^L Pds = 2A$ and the classical isoperimetric inequality $L^2 - 4\pi A \ge 0$. □

Corollary 2.4 *The length of the evolving curve* $X(u, t)$ *is given by*

$$L = \sqrt{4\pi A + \left(L_0^2 - 4\pi A\right)e^{-2t}}, \tag{2.4}$$

where L_0 is the length of initial curve $X_0(u)$, and therefore $L(t)$ goes to a constant $\sqrt{4\pi A}$ as the time t goes to infinity.

Proof From (2.2), we find that $L'(t) = -L + \frac{4\pi A}{L}$ is actually an ordinary differential equation. Since the area enclosed by the evolving curve is constant $A(t) = A_0$ $\left(A_0\right.$ is enclosed area of initial curve $\left.X_0(u)\right)$, then solving this equation yields (2.4) and therefore $L(t)$ goes to a constant $\sqrt{4\pi A}$ as the time t goes to infinity. □

Lemma 2.5 *The isoperimetric deficit $L^2 - 4\pi A$ of the evolving curve is decreasing during the evolution process and converges to zero as the time t goes to infinity.*

Proof

$$\frac{d\left(L^2 - 4\pi A\right)}{dt} = 2LL_t - 4\pi A_t = 2L\left(-L + \frac{4\pi A}{L}\right) - 0 = -2\left(L^2 - 4\pi A\right) \le 0. \tag{2.5}$$

Integrating this yields

$$L^2 - 4\pi A = \left(L_0^2 - 4\pi A\right)e^{-2t}. \tag{2.6}$$

Therefore, when $t \to \infty$, there holds $L^2 - 4\pi A \to 0$. □

By the Bonnesen inequality (see [9])

$$L^2 - 4\pi A \ge \pi^2\left(r_{out} - r_{in}\right)^2,$$

one gets that the difference between the inner and outer radii decreases to zero. Thus the evolving curve converges to a circle in the Hausdorff metric.

3 Existence and convexity

In this section we use Gage-Hamilton's trick in [3]. Let θ be the angle between T and the positive direction of the x axis and $\tau = t$. With parameters θ and τ the evolution equation of curvature κ is

$$\frac{\partial \kappa}{\partial \tau} = \kappa^2\left(\frac{\partial^2}{\partial \theta^2}\left(P - \frac{2A}{L}\right) + P - \frac{2A}{L}\right). \tag{3.1}$$

Lemma 3.1 *A solution $\kappa(\theta,\tau)$ to (3.1) with initial value $\kappa_0(\theta)$ exists for all time. Moreover, if $\kappa_0(\theta) > 0$ then $\kappa(\theta,\tau) > 0$.*

Proof By (3.1) and $P_{\theta\theta} + P = \frac{1}{\kappa}$ one has

$$\begin{aligned} \frac{\partial \kappa}{\partial \tau} &= \kappa^2\left(\frac{\partial^2}{\partial \theta^2}\left(P - \frac{2A}{L}\right) + P - \frac{2A}{L}\right) \\ &= \kappa^2\left(\frac{1}{\kappa} - \frac{2A}{L}\right). \end{aligned}$$

Set $W = (\frac{1}{\kappa} - \frac{L}{2\pi})e^{\tau}$, then we have

$$\begin{aligned}\frac{\partial W}{\partial \tau} &= \left(-\frac{1}{\kappa^2}\frac{\partial \kappa}{\partial \tau} - \frac{L_\tau}{2\pi}\right)e^{\tau} + \left(\frac{1}{\kappa} - \frac{L}{2\pi}\right)e^{\tau} \\ &= \left(\frac{2A}{L} - \frac{1}{\kappa} + \frac{L}{2\pi} - \frac{2A}{L}\right) + \left(\frac{1}{\kappa} - \frac{L}{2\pi}\right)e^{\tau} = 0.\end{aligned}$$

Since $W(\theta, 0) = \frac{1}{\kappa_0} - \frac{L_0}{2\pi}$, then we get

$$W(\theta, \tau) = \frac{1}{\kappa_0} - \frac{L_0}{2\pi}. \tag{3.2}$$

Furthermore, we can write explicitly that

$$\kappa(\theta, \tau) = \left[\left(\frac{1}{\kappa_0} - \frac{L_0}{2\pi}\right)e^{-\tau} + \frac{L}{2\pi}\right]^{-1}, \tag{3.3}$$

combined with (2.4), It can also be expressed as

$$\kappa(\theta, \tau) = \big[(\frac{1}{\kappa_0} - \frac{L_0}{2\pi})e^{-\tau} + \frac{\sqrt{4\pi A + (L_0^2 - 4\pi A)e^{-2\tau}}}{2\pi}\big]^{-1}. \tag{3.4}$$

Now we begin to prove the convexity of evolving curve.

When $\frac{1}{\kappa_0} \geq \frac{L_0}{2\pi}$. Obviously, in this case, $\kappa > 0$ for all time.

When $\frac{1}{\kappa_0} < \frac{L_0}{2\pi}$. Let $\alpha = (\frac{1}{\kappa_0} - \frac{L_0}{2\pi})e^{-\tau}, \beta = \frac{\sqrt{4\pi A + (L_0^2 - 4\pi A)e^{-2\tau}}}{2\pi}$, respectively. (Clearly, $\beta > 0$) Then we calculate:

$$\begin{aligned}\beta^2 - \alpha^2 &= \frac{4\pi A + \left(L_0^2 - 4\pi A\right)e^{-2\tau}}{4\pi^2} - \left(\frac{1}{\kappa_0} - \frac{L_0}{2\pi}\right)^2 e^{-2\tau} \\ &= \frac{A}{\pi} + \frac{L_0^2 e^{-2\tau}}{4\pi^2} - \frac{Ae^{-2\tau}}{\pi} - \frac{e^{-2\tau}}{\kappa_0^2} - \frac{L_0^2 e^{-2\tau}}{4\pi^2} + \frac{L_0 e^{-2\tau}}{\pi\kappa_0} \\ &= \frac{1}{\kappa_0}\left(\frac{L_0}{\pi} - \frac{1}{\kappa_0}\right)e^{-2\tau} + \frac{A}{\pi}\left(1 - e^{-2\tau}\right).\end{aligned}$$

Since $\kappa_0 > 0$, then $\beta^2 - \alpha^2 > 0$. Combined with $\beta > 0$, we can easily get $\beta > \alpha$. By (3.4), then we have $\kappa > 0$. This completes the proof. □

Furthermore, when $\tau \to \infty$, from (3.4), we obtain

$$\lim_{\tau \to \infty} \kappa(\theta, \tau) = \sqrt{\frac{\pi}{A}}.$$

Since the radius of curvature $\rho = \frac{1}{\kappa}$, then we have $\lim_{\tau\to\infty} \rho(\theta, \tau) = \sqrt{\frac{A}{\pi}}$.

Next, we study the following curve flow problem that is equivalent to (1.5) and prove its existence. To this end, we will deal equivalently with the evolution equation for support function of the evolving curve.(see [10, 11])

$$\frac{\partial X(\theta,\tau)}{\partial \tau} = \left(P - \frac{2A}{L}\right)N(\theta) - \frac{\partial\left(P - \frac{2A}{L}\right)}{\partial \theta}T(\theta). \tag{3.5}$$

Lemma 3.2 *For a solution $X(\theta,\tau)$ to (3.5), the evolution equation of the support function satisfies*

$$\frac{\partial P}{\partial \tau} = \frac{2A}{L} - P. \tag{3.6}$$

Proof By the definition of the support function $P(\theta,\tau) = -< X, -N >$ and $\frac{\partial N}{\partial \tau} = 0$, we get

$$\begin{aligned}\frac{\partial P}{\partial \tau} &= -< \frac{\partial X}{\partial \tau}, N > + < F, \frac{\partial N}{\partial \tau} > \\ &= -< \left(P - \frac{2A}{L}\right)N - \frac{\partial\left(P - \frac{2A}{L}\right)}{\partial \theta}T, N > \\ &= \frac{2A}{L} - P.\end{aligned}$$

□

Lemma 3.3 *The support function P satisfies*

$$P(\theta,\tau) = \left(P(\theta,0) + \frac{L_0}{2\pi}\right)e^{-\tau} + \frac{\sqrt{4\pi A + \left(L_0^2 - 4\pi A\right)e^{-2\tau}}}{2\pi}. \tag{3.7}$$

Proof Set $U = (P - \frac{L}{2\pi})e^{\tau}$, then we have

$$\begin{aligned}\frac{\partial U}{\partial \tau} &= \left(P_\tau - \frac{L_\tau}{2\pi}\right)e^{\tau} + \left(P - \frac{L}{2\pi}\right)e^{\tau} \\ &= \left(\frac{2A}{L} - P + \frac{L}{2\pi} - \frac{2A}{L}\right) + \left(P - \frac{L}{2\pi}\right)e^{\tau} = 0.\end{aligned}$$

Since $U(\theta,0) = P(\theta,0) - \frac{L_0}{2\pi}$, then we get

$$U(\theta,\tau) = P(\theta,0) - \frac{L_0}{2\pi},$$

combined with (2.4), we can complete the proof. □

What's more, when $\tau \to \infty$, from (3.7), we obtain

$$\lim_{\tau\to\infty} P(\theta,\tau) = \sqrt{\frac{A}{\pi}}.$$

Lemma 3.4 *Suppose $P(\theta,\tau) : [0,2\pi] \times [0,\infty) \to R$ is the smooth solution of the Eq. (3.7), the radius of curvature $P_{\theta\theta} + P > 0$ and the initial curve $X_0 = -P(\theta,0)N(\theta) + \frac{\partial}{\partial\theta}P(\theta,0)T(\theta)$. Then there exist a unique solution $X(u, t)$ satisfying the Eq. (1.5) and the support function of curve is $P(\theta,\tau)$.*

Proof We know that any convex curve on the plane can be uniquely represented by the support function, so $\widetilde{X} : [0, 2\pi] \times [0, \infty)$ can be expressed as:

$$\widetilde{X} = -PN + \frac{\partial P}{\partial \theta} T.$$

Because the unit tangent T and the unit normal vector N are independent of time τ, then we can get:

$$\begin{aligned}
\frac{\partial \widetilde{X}}{\partial \tau} &= -\frac{\partial P}{\partial \tau} N - \frac{\partial N}{\partial \tau} P + \frac{\partial^2 P}{\partial \tau \partial \theta} T + \frac{\partial P}{\partial \theta} \frac{\partial T}{\partial \tau} \\
&= -\frac{\partial P}{\partial \tau} N + \frac{\partial^2 P}{\partial \tau \partial \theta} T \\
&= \left(P - \frac{2A}{L}\right) N + \frac{\partial}{\partial \theta}\left(\frac{2A}{L} - P\right) T.
\end{aligned}$$

We make a parameter transformation, let $\theta = \theta(u, t)$, $\tau = t$, then θ satisfies the following equation:

$$\begin{cases} \frac{\partial \theta}{\partial t} = -\kappa \frac{\partial}{\partial \theta}\left(\frac{2A}{L} - P\right) \\ \theta(u, 0) = u \end{cases} \tag{3.8}$$

We see that $\theta(u, t)$ is the only solution of the above equation, then we have

$$\begin{aligned}
\frac{\partial X(u,t)}{\partial t} &= \frac{\partial \widetilde{X}(\theta, \tau)}{\partial t} \\
&= \frac{\partial \widetilde{X}}{\partial \theta} \frac{\partial \theta}{\partial t} + \frac{\partial \widetilde{X}}{\partial \tau} \frac{\partial \tau}{\partial t} \\
&= \left(-P_\theta N - P N_\theta + P_{\theta\theta} T + P_\theta T_\theta\right) \frac{\partial \theta}{\partial t} + \frac{\partial \widetilde{X}}{\partial \tau} \\
&= -\left(-P_\theta N + PT + P_{\theta\theta} T + P_\theta N\right) \kappa \frac{\partial}{\partial \theta}\left(\frac{2A}{L} - P\right) \\
&\quad + \left(P - \frac{2A}{L}\right) N + \frac{\partial}{\partial \theta}\left(\frac{2A}{L} - P\right) T \\
&= \left(P - \frac{2A}{L}\right) N.
\end{aligned}$$

This completes the proof. □

Acknowledgements I'm grateful to the anonymous referee for his or her careful reading of the original manuscript of this short paper and giving me many helpful suggestions and invaluable comments.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author has no conflicts of interest to declare that are relevant to the content of this article.

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