



A model for random chain complexes

Michael J. Catanzaro¹ · Matthew J. Zabka²

Received: 11 March 2021 / Accepted: 21 September 2021 / Published online: 9 November 2021
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Abstract

We introduce a model for random chain complexes over a finite field. The randomness in our complex comes from choosing the entries in the matrices that represent the boundary maps uniformly over \mathbb{F}_q , conditioned on ensuring that the composition of consecutive boundary maps is the zero map. We then investigate the combinatorial and homological properties of this random chain complex.

Keywords Homological algebra · Chain complex

Mathematics Subject Classification 55U15 · 18G35 · 60K99 · 60D99

1 Introduction

There have been a variety of attempts to randomize topological constructions. Most famously, Erdős and Rényi introduced a model for random graphs [6]. This work spawned an entire industry of probabilistic models and tools used for understanding other random topological and algebraic phenomenon. These include various models for random simplicial complexes, random networks, and many more [7, 13]. Further, this has led to beautiful connections with statistical physics, for example through percolation theory [1, 3, 12]. Our ultimate goal is to understand higher dimensional topological constructions arising in algebraic topology from a random perspective. In this manuscript, we begin to address this goal with the much simpler objective of understanding an algebraic construction commonly associated with topological spaces, known as a chain complex.

Chain complexes are used to measure a variety of different algebraic, geometric, and topological properties. Their usefulness lies in providing a pathway for homological algebra computations. They arise in a variety of contexts, including commutative algebra, algebraic

Communicated by Frau Richter.

✉ Matthew J. Zabka
matthew.zabka@smsu.edu

Michael J. Catanzaro
mjcatanz@iastate.edu

¹ Iowa State University, Ames, IA, USA

² Southwest Minnesota State University, Marshall, MN, USA

geometry, group cohomology, Hochschild homology, de Rham cohomology, and of course algebraic topology [2, 4, 9–11]. Specifically, chain complexes measure the relationship between cycles and boundaries of a topological space. This relationship uncovers many topological properties of interest, and is precisely what homology reveals. Furthermore, the singular chain complex of a topological space provides a canonical method of associating a chain complex to a topological space.

Let R be a ring. A *chain complex* $C_* = (C_m, \delta_m)$ with coefficients in R is a sequence of R -modules, denoted C_m , together with a sequence of linear transformations

$$\cdots \xrightarrow{\delta_{m+1}} C_m \xrightarrow{\delta_m} C_{m-1} \xrightarrow{\delta_{m-1}} \cdots$$

such that $\delta_{m-1}\delta_m = 0$ for all $m \in \mathbb{Z}$. The maps δ_m are called the boundary maps of the chain complex, and the equation $\delta_{m-1}\delta_m = 0$ is known as the boundary condition; see [5] for further details.

The boundary condition $\delta_{m-1}\delta_m = 0$ forces $\text{im } \delta_m \subseteq \ker \delta_{m-1}$. The *homology* of a chain complex measures the deviation of this containment from equality:

$$H_m(C_*; R) = \frac{\ker \delta_{m-1}}{\text{im } \delta_m}.$$

When the chain complex arises by taking singular chains on a topological space, homology can be a very powerful tool in algebraic topology [10].

We work over the field with q -elements $R = \mathbb{F}_q$ and consider the chain complex whose R -modules are given by finite dimensional vector spaces, $C_m = \mathbb{F}_q^{n_m}$, where each $n_m \in \mathbb{N}$. After fixing the standard basis for \mathbb{F}_q , the boundary maps can be regarded as $n_{m-1} \times n_m$ matrices, which we denote by A_m . Homology can then be understood in terms of dimension

$$\beta_m := \dim_{\mathbb{F}_q} \frac{\ker A_{m-1}}{\text{im } A_m},$$

where β_m is known as the *m*th Betti number.

1.1 Main Results

Let q be a prime number. We build a random chain complex inductively with coefficients in \mathbb{F}_q as follows (see Definition 2 for a precise statement). Given a sequence of non-negative integers $\{n_m\}$, where $m \in \mathbb{Z}$, we iteratively construct random linear transformations

$$A_m : \mathbb{F}_q^{n_m} \longrightarrow \mathbb{F}_q^{n_{m-1}},$$

for all m . The transformations are subject to the constraint $A_{m-1}A_m = 0$. That is, after fixing the standard basis for $\mathbb{F}_q^{n_m}$ and constructing A_{m-1} , it suffices to construct the random $n_{m-1} \times n_m$ matrix A_m that satisfies $A_{m-1}A_m = 0$. We do so by choosing matrix entries i.i.d. from the uniform distribution on \mathbb{F}_q , subject to the constraint that $A_{m-1}A_m = 0$. We then say that the pair $(\mathbb{F}_q^{n_m}, A_m)$ is a *random chain complex*. We restrict our attention to bounded below chain complexes (see Remark 4).

Our first result is an explicit formula for the distribution of the Betti numbers.

Theorem A *Let β_m be the m -th Betti number of a random chain complex $(\mathbb{F}_q^{n_m}, A_m)$. Then*

$$\mathbb{P}[\beta_m = b] = \sum_{i_m=0}^{n_m} P_{i_m}^m(i_m - b) \sum_{i_{m-1}=0}^{n_{m-1}} P_{i_{m-1}}^{m-1}(n_m - i_m) \cdots \sum_{i_1=0}^{n_1} P_{i_1}^1(n_2 - i_2) P_{n_0}^0(n_1 - i_1),$$

where $P_k^m(r)$ is given in Eq. (4).

As Theorem A gives a formula for computing the distribution of the Betti numbers, it also leads to formulas for other probabilistic properties of β_m , such as its moments and variance.

Our second main result show that, asymptotically, the m -th Betti number of a random chain complex concentrates in a single value. Set

$$(n)_+ = \max(0, n),$$

to be the *positive part* of n . Define

$$B_m = (-n_{m+1} + (n_m - (n_{m-1} - (\cdots - (n_1 - n_0)_+ \cdots)_+)_+)_+). \tag{1}$$

Theorem B For a random chain complex $(\mathbb{F}_q^{n_m}, A_m)$ with B_m defined as in Eq. (1),

$$\mathbb{P}[\beta_m = B_m] \rightarrow 1 \text{ as } q \rightarrow \infty.$$

Remark 1 As a special case of the above theorem, consider when $\{n_m\}$ is constant or increasing. In this case, $B_m = 0$, and the homology is trivial in probability as $q \rightarrow \infty$ (see Corollary 2).

1.2 Related work

Others have considered different methods of applying randomness to chain complexes. In [8], Ginzburg and Pasechnik investigate a different notion of a random chain complex than the one we have described above. Given a finite dimensional vector space V over \mathbb{F}_q , they consider chain complexes of the form

$$\cdots \xrightarrow{D} V \xrightarrow{D} V \xrightarrow{D} \cdots,$$

for a randomly chosen linear operator D such that $D^2 = 0$. They choose the operator D uniformly over all such possible choices. In particular, our construction is distinct from theirs, since they use the same operator D at each stage of the complex. The first of their main results [8, Thm 2.1] states that the rank of homology concentrates in the lowest possible dimension as $q \rightarrow \infty$. In the special case when $n_m \equiv n$ is constant, we also obtain minimal rank homology (see Remark 1).

The second author has introduced and studied the properties of a random Bockstein operation [15]. In homological algebra, the Bockstein is a connecting homomorphism associated with a short exact sequence of abelian groups, which are then used as the coefficients in a chain complex. Given a random boundary operator of a chain complex, the distribution of compatible random Bockstein operations is given in [15, Thm 5.2].

1.3 Outline

The paper is organized as follows. In Sect. 2, we discuss preliminary results useful for the combinatorial aspects of our results. We give a precise definition of a model for a random chain complex in Sect. 3, as well as prove Theorem A. In Sect. 4, we complete the proof of Theorem B.

2 Preliminaries

This section consists of lemmas that are necessary to prove our main results. The first four lemmas count the number of elements in various sets related to finite vector spaces over \mathbb{F}_q . Lemmas 1, 2, and 3 are well known, but the statements and proofs are provided here for the sake of completeness. An interested reader can also see [14] for further details. The last lemma of this section, Lemma 5, gives the asymptotic behavior of a useful conditional probability and will be used several times in the remainder of the paper.

Lemma 1 *The number of ordered, linearly independent k -tuples of vectors in \mathbb{F}_q^n is*

$$\prod_{j=0}^{k-1} (q^n - q^j) = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{k-2})(q^n - q^{k-1}).$$

Proof Since first vector in the k -tuple may be any vector except for the zero vector, there are $q^n - 1$ choices for the first vector. More generally, for $1 \leq m \leq k$, the m -th vector in the k -tuple may be any vector that is not a linear combination of the previously chosen $m - 1$ vectors. So there are $q^n - q^{m-1}$ choices for the m -th vector. \square

Lemma 2 *The number of k -dimensional subspaces of \mathbb{F}_q^n is*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=0}^{k-1} \frac{q^n - q^j}{q^k - q^j}.$$

Proof Let $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denote the number of k -dimensional subspaces of \mathbb{F}_q^n and $N(q, k)$ be the number of ordered, linear independent k -tuples of vectors in \mathbb{F}_q^n . Then Lemma 1 gives

$$N(q, k) = \prod_{j=0}^{k-1} (q^n - q^j). \tag{2}$$

We may also find $N(q, k)$ another way: First choose a k -dimensional subspace and then choose the independent vectors in our k -tuple from the chosen subspace. There are $\begin{bmatrix} n \\ k \end{bmatrix}_q$ k -dimensional subspaces of \mathbb{F}_q^n . There are $q^k - 1$ choices for the first vector in the k -tuple, and more generally, for $1 \leq m \leq k$, there are $q^k - q^{m-1}$ vectors for the m -th vector in the k -tuple. Thus

$$N(q, k) = \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1} q^k - q^j. \tag{3}$$

Equations (2) and (3) give the desired result. □

The number $\begin{bmatrix} n \\ k \end{bmatrix}_q$ defined above is known as the q -binomial coefficient [14]. Lemmas 1 and 2 combine to count the number of matrices of a given rank.

Lemma 3 The number of $m \times n$ matrices of rank r with entries in \mathbb{F}_q is given by

$$\prod_{j=0}^{r-1} \frac{(q^m - q^j)(q^n - q^j)}{q^r - q^j}.$$

Proof Let W be a fixed r -dimensional subspace of \mathbb{F}_q^n . The number of matrices whose column space is W is given by the number of $r \times n$ matrices with rank r . This number is given by Lemma 1. The number of r -dimensional subspaces of \mathbb{F}_q^n is $\begin{bmatrix} n \\ r \end{bmatrix}_q$, as stated in Lemma 2.

The product of these is the number of $m \times n$ rank r matrices. □

Definition 1 Let n_m be a sequence of natural numbers. Let A_m be a sequence of random $(n_m) \times (n_{m-1})$ matrices whose entries are chosen i.i.d. uniformly from \mathbb{F}_q . Let r be a non-negative integer. Define

$$P_k^m(r) := \mathbb{P}[\text{rank}(A_{m+1}) = r \mid A_m A_{m+1} = 0, \text{nul}(A_m) = k].$$

Remark 2 Several comments are in order. First, the conditional probability of Definition 1 requires that A_{m+1} depend on A_m . Thus, one can regard the sequence of random matrices as being constructed iteratively, with A_0 constructed first, followed by A_1 , and so on. Second, for impossible cases like $r > k$ or $k < 0$, we have $P_k^m(r) = 0$. Finally, since at every iteration we have a finite probability space, all of the probabilities involved are strictly positive. Thus we need not concern ourselves with conditioning on events of probability 0.

Lemma 4 With A_m defined as in Definition 1, we have that

$$P_k^m(r) = q^{-kn_{m+1}} \prod_{j=0}^{r-1} \frac{(q^{n_{m+1}} - q^j)(q^k - q^j)}{q^r - q^j}. \tag{4}$$

Proof Let $k = \text{nul}(A_m)$. The linear transformation A_{m+1} maps $\mathbb{F}_q^{n_{m+1}}$ into a k -dimensional subspace of $\mathbb{F}_q^{n_m}$. By changing basis, A_{m+1} can be represented by an $k \times n_{m+1}$ matrix. There are $q^{kn_{m+1}}$ total $k \times n_{m+1}$ matrices over \mathbb{F}_q , and by Lemma 3, there are

$$\prod_{j=0}^{r-1} \frac{(q^{n_{m+1}} - q^j)(q^k - q^j)}{q^r - q^j}$$

such matrices of rank r . □

Remark 3 We adopt the convention that the empty product is 1. With this, Lemma 4 implies that $P_0^m(0) = 1$.

Lemma 5 Fix m and k . Then

$$\lim_{q \rightarrow \infty} P_k^m(r) = \begin{cases} 1 & \text{if } r = \min(k, n_{m+1}), \\ 0 & \text{else.} \end{cases}$$

Proof Suppose $\min(k, n_{m+1}) = k$. Then

$$\begin{aligned} P_k^m(k) &= q^{-kn_{m+1}} \prod_{j=0}^{k-1} \frac{(q^{n_{m+1}} - q^j)(q^k - q^j)}{q^k - q^j} \\ &= q^{-kn_{m+1}} \prod_{j=0}^{k-1} (q^{n_{m+1}} - q^j) \\ &= \prod_{j=0}^{k-1} (1 - q^{j-n_{m+1}}). \end{aligned}$$

Suppose $\min(k, n_{m+1}) = n_{m+1}$. Then

$$\begin{aligned} P_k^m(n_{m+1}) &= q^{-kn_{m+1}} \prod_{j=0}^{n_{m+1}-1} \frac{(q^{n_{m+1}} - q^j)(q^k - q^j)}{q^{n_{m+1}} - q^j} \\ &= \prod_{j=0}^{n_{m+1}-1} (1 - q^{j-k}). \end{aligned}$$

In either of the above cases, $P_k^m(r) \rightarrow 1$ as $q \rightarrow \infty$. On the other hand, if $r \neq \min(k, n_{m+1})$, then $P_k^m(r) \rightarrow 0$ since each $P_k^m(r)$ represents a probability by Definition 1. □

3 The homology of a random chain complex

Definition 2 Let q be a prime number and $\{n_m\}$ be a sequence of non-negative integers indexed by $m \in \mathbb{Z}$. Consider the sequence $\{A_m\}$ of $n_{m-1} \times n_m$ random matrices constructed iteratively as follows: Let the entries of A_0 be chosen i.i.d. according to the uniform distribution on \mathbb{F}_q . For $m > 0$ let the entries of A_m be chosen i.i.d subject to the condition that $A_{m-1}A_m = 0$. The pair $(\mathbb{F}_q^{n_m}, A_m)$ is then said to be a **random chain complex** over the field \mathbb{F}_q .

Remark 4 We are interested in *bounded from below* chain complexes, so we set $A_m = 0$ for all $m < 0$ for the remainder of the manuscript.

We wish to investigate the probabilistic properties of the homology of a random chain complex. We are primarily interested in the distribution of the Betti numbers $\beta_m = \text{nul } A_m - \text{rank } A_{m+1}$.

Remark 5 If $\{A_m\}$ is the sequence of maps from a random chain complex, Definition 1 immediately gives

$$P_k^m(r) = \mathbb{P}[\beta_m = k - r \mid \text{nul}(A_m) = k].$$

Theorem 1 Let $(\mathbb{F}_q^{n_m}, A_m)$ be a random chain complex and $A_0 : \mathbb{F}_q^{n_0} \rightarrow 0$. Then

$$\begin{aligned} & \mathbb{P}[\text{rank}(A_m) = n_m - k] \\ &= \sum_{i_{m-1}=0}^{n_{m-1}} P_{i_{m-1}}^{m-1}(n_m - k) \sum_{i_{m-2}=0}^{n_{m-2}} P_{i_{m-2}}^{m-2}(n_{m-1} - i_{m-1}) \cdots \sum_{i_1=0}^{n_1} P_{i_1}^1(n_2 - i_2) P_{n_0}^0(n_1 - i_1). \end{aligned}$$

Proof The proof is by induction on m . For the base case $m = 1$, we have

$$\begin{aligned} \mathbb{P}[\text{rank}(A_1) = n_1 - k] &= \sum_{i_0=0}^{n_0} \mathbb{P}[\text{rank}(A_1) = n_1 - k \mid \text{nul}(A_0) = i_0] \mathbb{P}[\text{nul}(A_0) = i_0] \\ &= \mathbb{P}[\text{rank}(A_1) = n_1 - k \mid \text{nul}(A_0) = n_0] \\ &= P_{n_0}^0(n_1 - k). \end{aligned}$$

The first equality follows by the Law of Total Probability, and the second equality follows because A_0 is the zero map.

For the inductive step, suppose that

$$\begin{aligned} & \mathbb{P}[\text{rank}(A_{m-1}) = n_{m-1} - i_{m-1}] \\ &= \sum_{i_{m-2}=0}^{n_{m-2}} P_{i_{m-2}}^{m-2}(n_{m-1} - i_{m-1}) \sum_{i_{m-3}=0}^{n_{m-3}} P_{i_{m-3}}^{m-3}(n_{m-2} - i_{m-2}) \cdots \sum_{i_1=0}^{n_1} P_{i_1}^1(n_2 - i_2) P_{n_0}^0(n_1 - i_1). \end{aligned}$$

As in the base case, we have

$$\begin{aligned} & \mathbb{P}[\text{rank}(A_m) = n_m - k] \\ &= \sum_{i_{m-1}=0}^{n_{m-1}} \mathbb{P}[\text{rank}(A_m) = n_m - k \mid \text{nul}(A_{m-1}) = i_{m-1}] \mathbb{P}[\text{nul}(A_{m-1}) = i_{m-1}] \\ &= \sum_{i_{m-1}=0}^{n_{m-1}} P_{i_{m-1}}^{m-1}(n_m - k) \mathbb{P}[\text{rank}(A_{m-1}) = n_{m-1} - i_{m-1}]. \end{aligned}$$

The desired result now follows by the induction hypothesis. □

Theorem A now follows from Theorem 1 and Lemma 5 in a straightforward manner. We give an explicit proof for completeness.

Proof(Proof of Theorem A) By the law of total probability, we have

$$\begin{aligned}
 \mathbb{P}[\beta_m = b] &= \mathbb{P}[\text{rank}(A_{m+1}) = \text{nul}(A_m) - b] \\
 &= \sum_{k=0}^{n_m} \mathbb{P}[\text{rank}(A_{m+1}) = k - b \mid \text{nul}(A_m) = k] \mathbb{P}[\text{nul}(A_m) = k] \\
 &= \sum_{k=0}^{n_m} P_k^m(k - b) \mathbb{P}[n_m - \text{rank}(A_m) = k] \\
 &= \sum_{k=0}^{n_m} P_k^m(k - b) \mathbb{P}[\text{rank}(A_m) = n_m - k].
 \end{aligned}$$

By Theorem 1,

$$\begin{aligned}
 &\sum_{k=0}^{n_m} P_k^m(k - b) \mathbb{P}[\text{rank}(A_m) = n_m - k] \\
 &= \sum_{k=0}^{n_m} P_k^m(k - b) \sum_{i_{m-1}=0}^{n_{m-1}} P_{i_{m-1}}^{m-1}(n_m - k) \cdots \sum_{i_1=0}^{n_1} P_{i_1}^1(n_2 - i_2) P_{n_0}^0(n_1 - i_1),
 \end{aligned}$$

as desired. □

4 Proof of theorem B

In this section, we analyze Theorem A under the limit $q \rightarrow \infty$.

Proposition 1 *Let $I_m := \{0, 1, \dots, n_m\}$ and let $I^{(j)} := I_1 \times \dots \times I_j$. Then for every natural number j , there exists exactly one $\mathbf{i}^* = (i_1^*, \dots, i_j^*)$ in $I^{(j)}$ such that*

$$P_{i_{j-1}^*}^{j-1}(n_j - i_j^*) \cdots P_{i_1^*}^1(n_2 - i_2^*) P_{n_0}^0(n_1 - i_1^*) \rightarrow 1,$$

as $q \rightarrow \infty$. In particular, set $i_0^* = n_0$. Then for ℓ in $\{1, 2, \dots, j\}$, we have $i_\ell^* = (n_\ell - i_{\ell-1}^*)_+$.

Proof The proof is by induction on j .

Base step ($j = 1$). By Lemma 5, we have $P_{n_0}^0(n_1 - i_1^*) \rightarrow 1$ as $q \rightarrow \infty$ if and only if $n_1 - i_1^* = \min(n_0, n_1)$. That is, $i_1^* = (n_1 - n_0)_+ = (n_1 - i_0^*)_+$.

Inductive step. Assume there exists exactly one $(i_1^*, \dots, i_{j-1}^*)$ in $I^{(j-1)}$, with $i_\ell = (n_\ell - i_{\ell-1}^*)_+$ for ℓ in $\{1, 2, \dots, j-1\}$, such that

$$P_{i_{j-2}^*}^{j-2}(n_{j-1} - i_{j-1}^*) \cdots P_{i_1^*}^1(n_2 - i_2^*) P_{n_0}^0(n_1 - i_1^*) \rightarrow 1,$$

as $q \rightarrow \infty$. By Lemma 5, $P_{i_{j-1}^*}^{j-1}(n_j - i_j^*) \rightarrow 1$ as $q \rightarrow \infty$ if and only if $n_j - i_j^* = \min(i_{j-1}^*, n_j)$.

That is, $i_j^* = (n_j - i_{j-1}^*)_+$. For $\mathbf{i} = (i_1^*, \dots, i_{j-1}^*, i_j^*)$ in $I^{(j)}$, we have

$$P_{i_{j-1}^*}^{j-1}(n_j - i_j^*) P_{i_{j-2}^*}^{j-2}(n_{j-1} - i_{j-1}^*) \cdots P_{i_1^*}^1(n_2 - i_2^*) P_{n_0}^0(n_1 - i_1^*) \rightarrow 1$$

as $q \rightarrow \infty$, as desired. □

Proof(Proof of Theorem B) By Theorem A, it is sufficient to show

$$P_{i_m^*}^m(i_m^* - b)P_{i_{m-1}^*}^{m-1}(n_m - i_m^*) \cdots P_{i_1^*}^1(n_2 - i_2^*)P_{n_0}^0(n_1 - i_1^*) \rightarrow 1$$

as $q \rightarrow \infty$ for a single sequence $\mathbf{i}^* = (i_0^*, \dots, i_m^*)$ and a single value of b . After choosing \mathbf{i}^* as in Proposition 1, the value of b is easily determined from Lemma 5 to be

$$\begin{aligned} b &= i_m^* - \min(i_m^*, n_{m+1}) \\ &= (-n_{m+1} + i_m^*)_+ \\ &= (-n_{m+1} + (n_m - i_{m-1}^*)_+)_+ \\ &= (-n_{m+1} + (n_m - (n_{m-1} - (\cdots (n_1 - n_0)_+ \cdots)_+)_+)_+)_+ \\ &= B_m. \end{aligned}$$

□

Proposition 1 and Theorem B have a number of immediate consequences.

Corollary 1 Let $(\mathbb{F}_q^{n_m}, A_m)$ be a random chain complex. Then

$$\mathbb{P}[\text{rank}(A_m) = n_m - (n_m - (n_{m-1} - (\cdots - (n_1 - n_0)_+ \cdots)_+)_+)_+] \rightarrow 1$$

as $q \rightarrow \infty$.

Proof Using Lemma 5, this follows by a similar argument to the Proof of Theorem B.

□

Corollary 2 If $\{n_m\}$ is a monotone increasing sequence, then

$$\lim_{q \rightarrow \infty} \mathbb{P}[\beta_m = 0] = 1.$$

Proof By direct inspection, we have

$$(n_m - (n_{m-1} - (\cdots (n_1 - n_0)_+ \cdots)_+)_+)_+ \leq n_m,$$

and hence $B_m = 0$.

□

Corollary 3 The t -th moments of the random variable β_m satisfy

$$\lim_{q \rightarrow \infty} \mathbb{E}[\beta_m^t] = B_m^t.$$

Acknowledgements The first author would like to thank Peter Bubenik for helpful discussions.

Funding Information Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Bollobás, B., Riordan, O.: *Percolation*. Cambridge University Press, Cambridge (2006)
2. Bott, R., Tu, L.W.: *Differential Forms in Algebraic Topology*, vol. 82. Springer, Berlin (2013)
3. Broadbent, S.R., Hammersley, J.M.: Percolation processes: I. Crystals and mazes. *Math. Proc. Camb. Philos. Soc.* **53**(3), 629–641 (1957)
4. Brown, K.S.: *Cohomology of Groups*, vol. 87. Springer, Berlin (2012)
5. Cartan, H., Eilenberg, S.: *Homological Algebra (PMS-19)*, vol. 19. Princeton University Press, Princeton (2016)
6. Erdős, P., Rényi, A.: On random graphs I. *Publ. Math. Debrecen* **6**, 290–297 (1959)
7. Erdős, P., Rényi, A.: On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci* **5**(1), 17–60 (1960)
8. Ginzburg, V.L., Pasechnik, D.V.: Random chain complexes. *Arnold Math J.* 1–8 (2017)
9. Hartshorne, R.: *Algebraic geometry*, vol. 52. Springer, Berlin (2013)
10. Hatcher, A.: *Algebraic Topology*. Cambridge University Press, Cambridge (2002)
11. Hochschild, G.: On the cohomology groups of an associative algebra. *Ann. Math.* 58–67(1945)
12. Kesten, H.: *Percolation Theory for Mathematicians*, vol. 423. Springer, Berlin (1982)
13. Linia, N., Peled, Y.: Random simplicial complexes: around the phase transition. In: Loebel, M., Nese-tril, J., Thomas, R. (eds.) *A Journey Through Discrete Mathematics: A Tribute to Jiří Matoušek*, pp. 543–570. Springer, Berlin (2017)
14. Stanley, R.P.: *Enumerative Combinatorics*, vol. 1. Cambridge University Press, Cambridge (2011)
15. Zabka, M.: A random Bockstein operator. *Algebra Discrete Math.* **25**(2), 311–321 (2018)

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