A-hypergeometric series and a *p*-adic refinement of the Hasse-Witt matrix

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Abstract

We identify the *p*-adic unit roots of the zeta function of a projective hypersurface over a finite field of characteristic *p* as the eigenvalues of a product of special values of a certain matrix of *p*-adic series. That matrix is a product $F(\Lambda^p)^{-1}F(\Lambda)$, where the entries in the matrix $F(\Lambda)$ are *A*-hypergeometric series with integral coefficients and $F(\Lambda)$ is independent of *p*.

Keywords *A*-hypergeometric series · Hasse-Witt matrix · *p*-adic cohomology

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1 Introduction

Dwork [7] expressed the unit root of the zeta function of an ordinary elliptic curve in the Legendre family in characteristic *p* in terms of the Gaussian hypergeometric function $_2F_1(1/2, 1/2, 1, \lambda)$. There have since been a number of generalizations and extensions of that result: see [3, Sect. 1]. The point of the current paper is to prove such a result for projective hypersurfaces where the zeta function has multiple unit roots by using the matrix of *A*-hypergeometric series that appeared in [1]. We note that related results have recently been obtained for hypersurfaces in the torus by Beukers and Vlasenko [4, 5] using different methods.

Let $\{x^{\mathbf{b}_k}\}_{k=1}^N$ be the set of *all* monomials of degree *d* in variables $x_0, \dots, x_n$ $\left(so\ N = \begin{pmatrix} d+n \\ n \end{pmatrix} \right)$ and let

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$$f_\lambda(x_0, \ldots, x_n) = \sum_{k=1}^{N} \lambda_k x^{\mathbf{b}_k} \in \mathbb{F}_q[x_0, \ldots, x_n] \tag{1.1}$$

be a homogeneous polynomial of degree d over the finite field $\mathbb{F}_q$, $q = p^a$, p a prime. Let $X_\lambda \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ be the projective hypersurface defined by the vanishing of f_λ and let $Z(X_\lambda/\mathbb{F}_q, t)$ be its zeta function. Define a rational function $P_\lambda(t)$ by the equation

$$P_\lambda(t) = \left(Z(X_\lambda/\mathbb{F}_q, t)(1-t)(1-qt)\cdots(1-q^{n-1}t)\right)^{(-1)^n} \in 1 + t\mathbb{Z}[[t]].$$

When X_λ is smooth, $P_\lambda(t)$ is the characteristic polynomial of Frobenius acting on middle-dimensional primitive cohomology. In this case, $P_\lambda(t)$ has degree

$$d^{-1}\left((d-1)^{n+1} + (-1)^{n+1}(d-1)\right).$$

In the general case, we know only that $P_\lambda(t)$ is a rational function.

We regard f_λ as a polynomial with fixed exponents and variable coefficients, giving rise to a family of rational functions $P_\lambda(t)$. The p-adic unit (reciprocal) roots of $P_\lambda(t)$ all occur in the numerator (Proposition 1.1) and for generic λ it has the maximal possible number of p-adic unit (reciprocal) roots (by the generic invertibility of the Hasse-Witt matrix, see below). Our goal in this paper is to give a p-adic analytic formula for these unit roots in terms of A-hypergeometric series.

Write $\mathbf{b}_k = (b_{0k}, \ldots, b_{nk})$ with $\sum_{i=0}^{n} b_{ik} = d$. It will be convenient to define an augmentation of these vectors. For $k = 1, \ldots, N$, put

$$\mathbf{a}_k = (\mathbf{b}_k, 1) = (b_{0k}, b_{1k}, \ldots, b_{nk}, 1) \in \mathbb{N}^{n+2}$$

(where $\mathbb{N}$ denotes the nonnegative integers). Let $A = \{\mathbf{a}_k\}_{k=1}^{N}$. Note that the vectors $\mathbf{a}_k$ all lie on the hyperplane $\sum_{i=0}^{n} u_i = du_{n+1}$ in $\mathbb{R}^{n+2}$. Let

$$U = \left\{ u = (u_0, \ldots, u_n, 1) \in \mathbb{N}^{n+2} \mid \sum_{i=0}^{n} u_i = d \text{ and } u_i > 0 \text{ for } i = 0, \ldots, n \right\}.$$

Note that $\{x_0^{u_0} \cdots x_n^{u_n} \mid u \in U\}$ is the set of all monomials of degree d that are divisible by the product $x_0 \cdots x_n$, so $|U| = \binom{d-1}{n}$. We assume throughout that $d \geq n+1$, which implies $U \neq \emptyset$. (If $d < n+1$, then U is empty and none of the reciprocal roots of $P_\lambda(t)$ is a p-adic unit. If one defines the determinant of an empty matrix to be 1, then Theorem 1.1 below is trivially true in this case.)

We recall the definition of the Hasse-Witt matrix of f_λ, a $(|U| \times |U|)$-matrix with rows and columns indexed by U. Let $\Lambda_1, \ldots, \Lambda_N$ be indeterminates and let $H(\Lambda) = \left[H_{uv}(\Lambda)\right]_{u,v \in U}$ be the matrix of polynomials:

$$H_{uv}(\Lambda) = \sum_{\substack{v \in \mathbb{N}^N \\ \sum_{k=1}^{N} v_k \mathbf{a}_k = pu - v}} \frac{\Lambda_1^{v_1} \cdots \Lambda_N^{v_N}}{v_1! \cdots v_N!} \in \mathbb{Q}[\Lambda_1, \ldots, \Lambda_N]. \tag{1.2}$$

Since the last coordinate of each $\mathbf{a}_k$, u, and v equals 1, the condition on the summation implies $\sum_{k=1}^{N} v_k = p - 1$. In particular, it follows that $v_k \leq p - 1$ for all k, so the coefficients of each $H_{uv}(\Lambda)$ lie in $\mathbb{Q} \cap \mathbb{Z}_p$. Let $\bar{H}_{uv}(\Lambda) \in \mathbb{F}_p[\Lambda]$ be the reduction mod p of $H_{uv}(\Lambda)$ and let

$\bar{H}(\Lambda)$ be the reduction mod p of $H(\Lambda)$. Then $\bar{H}(\lambda)$ is the Hasse-Witt matrix of f_λ (relative to a certain basis: see Katz [8, Algorithm 2.3.7.14]).

Write $P_\lambda(t) = Q_\lambda(t)/R_\lambda(t)$, where $Q_\lambda(t), R_\lambda(t) \in 1 + \mathbb{Z}[t]$ are relatively prime polynomials. As a special case of [2, Theorem 1.4] we have the following result.

Proposition 1.1 *For all $\lambda \in \mathbb{F}_q^N$ we have $R_\lambda(t) \equiv 1 \pmod{q}$ and*

$$Q_\lambda(t) \equiv \det \left(I - t\bar{H}(\lambda^{p^{a-1}}) \bar{H}(\lambda^{p^{a-2}}) \cdots \bar{H}(\lambda) \right) \pmod{p}.$$

Proposition 1.1 implies that all the unit roots of $P_\lambda(t)$ occur in the numerator and that there are at most card(U) of them. The Hasse-Witt matrix is known to be generically invertible for a "sufficiently general" polynomial f_λ (Koblitz [9], Miller [10, 11]). We recall the precise version of that fact that we need.

We suppose the $\mathbf{a}_k$ to be ordered so that $U = \{\mathbf{a}_k\}_{k=1}^M$, $M = \binom{d-1}{n}$. We may then write the matrix $H(\Lambda)$ as $\left[H_{ij}(\Lambda) \right]_{i,j=1}^M$, where

$$H_{ij}(\Lambda) = \sum_{\substack{v \in \mathbb{N}^N \\ \sum_{k=1}^N v_k \mathbf{a}_k = p\mathbf{a}_i - \mathbf{a}_j}} \frac{\Lambda_1^{v_1} \cdots \Lambda_N^{v_N}}{v_1! \cdots v_N!} \in (\mathbb{Q} \cap \mathbb{Z}_p)[\Lambda_1, \dots, \Lambda_N]. \tag{1.3}$$

Consider the related matrix $B(\Lambda) = \left[B_{ij}(\Lambda) \right]_{i,j=1}^M$ of Laurent polynomials defined by

$$B_{ij}(\Lambda) = \Lambda_i^{-p} \Lambda_j H_{ij}(\Lambda) \in (\mathbb{Q} \cap \mathbb{Z}_p)[\Lambda_1, \dots, \Lambda_{i-1}, \Lambda_i^{-1}, \Lambda_{i+1}, \dots, \Lambda_N], \tag{1.4}$$

i. e., $B(\Lambda) = C(\Lambda^p)^{-1} H(\Lambda) C(\Lambda)$, where $C(\Lambda)$ is the diagonal matrix with entries $\Lambda_1, \dots, \Lambda_M$. Since $\lambda_k^{p^a} = \lambda_k$ for $\lambda_k \in \mathbb{F}_q$, Proposition 1.1 has the following corollary.

Corollary 1.1 *For all $\lambda \in (\mathbb{F}_q^\times)^M \times \mathbb{F}_q^{N-M}$,*

$$Q_\lambda(t) \equiv \det \left(I - t\bar{B}(\lambda^{p^{a-1}}) \bar{B}(\lambda^{p^{a-2}}) \cdots \bar{B}(\lambda) \right) \pmod{p}.$$

Put $D(\Lambda) = \det B(\Lambda)$. By [1, Proposition 2.11] we have the following result.

Proposition 1.2 *The Laurent polynomial $D(\Lambda)$ has constant term* 1.

Proposition 1.2 implies that neither $D(\Lambda)$ nor $\bar{D}(\Lambda)$ is the zero polynomial, so the matrix $\bar{B}(\lambda)$ is invertible for λ in a Zariski open subset of $(\mathbb{F}_q^\times)^M \times \mathbb{F}_q^{N-M}$, a nonempty set for q sufficiently large. It follows from Corollary 1.1 that for $\bar{D}(\lambda) \neq 0$, $P_\lambda(t)$ has M unit reciprocal roots. Let $\{\pi_j(\lambda)\}_{j=1}^M$ be these roots and put

$$\rho(\lambda, t) = \prod_{j=1}^M \left(1 - \pi_j(\lambda)t \right).$$

Our goal is to describe the polynomial $\rho(\lambda, t)$ in terms of the A-hypergeometric series introduced in [1, Equation (3.3)].

We begin by defining a ring that contains the desired series. For each $i = 1, \dots, M$, the Laurent polynomials $B_{ij}(\Lambda)$ have exponents lying in the set

$$L_i := \left\{ l = (l_1, \ldots, l_N) \in \mathbb{Z}^N \mid \sum_{k=1}^N l_k \mathbf{a}_k = \mathbf{0}, l_i \le 0, \text{ and } l_k \ge 0 \text{ for } k \ne i \right\}.$$

Let $\mathscr{C} \subseteq \mathbb{R}^N$ be the real cone generated by $\bigcup_{i=1}^M L_i$. By [1, Proposition 2.9], the cone $\mathscr{C}$ does not contain a line, hence it has a vertex at the origin. Put $E = \mathscr{C} \cap \mathbb{Z}^N$. Note that $(l_1, \ldots, l_N) \in E$ implies that $\sum_{k=1}^N l_k \mathbf{a}_k = \mathbf{0}$ and that $\sum_{k=1}^N l_k = 0$ since the last coordinate of each $\mathbf{a}_k$ equals 1. Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$ with absolute value $| \cdot |$ associated to the valuation ord, normalized by ord $p = 1$ and $|p| = p^{-1}$. Set

$$R_E = \left\{ \xi(\Lambda) = \sum_{l=(l_1, \ldots, l_N) \in E} c_l \Lambda_1^{l_1} \cdots \Lambda_N^{l_N} \mid c_l \in \mathbb{C}_p \text{ and } \{|c_l|\}_{l \in E} \text{ is bounded} \right\},$$

i. e., R_E is the set of Laurent series over $\mathbb{C}_p$ having exponents in E and bounded coefficients. Note that R_E is a ring (since $\mathscr{C}$ has a vertex at the origin) and that the entries of $B(\Lambda)$ all lie in R_E. By Proposition 1.2, the Laurent polynomial $D(\Lambda)$ is an invertible element of R_E, so $B(\Lambda)^{-1}$ has entries in R_E.

We define a matrix $F(\Lambda) = \big[F_{ij}(\Lambda)\big]_{i,j=1}^M$ with entries in R_E. For $i \ne j$, put

$$F_{ij}(\Lambda) = \sum_{\substack{(l_1, \ldots, l_N) \in L_i \\ l_j > 0}} (-1)^{-l_i-1} \frac{(-l_i-1)!}{(l_j-1)! \displaystyle\prod_{\substack{k=1 \\ k \ne i,j}}^N l_k!} \Lambda_1^{l_1} \cdots \Lambda_N^{l_N},$$

and for $i = j$ put

$$F_{ii}(\Lambda) = \sum_{(l_1,\ldots,l_N) \in L_i} (-1)^{-l_i} \frac{(-l_i)!}{\displaystyle\prod_{\substack{k=1 \\ k \ne i}}^N l_k!} \Lambda_1^{l_1} \cdots \Lambda_N^{l_N}.$$

The coefficients of these series are multinomial coefficients, hence lie in $\mathbb{Z}$, so these series lie in R_E. Note also that each $F_{ii}(\Lambda)$ has constant term 1, while the $F_{ij}(\Lambda)$ for $i \ne j$ have no constant term. It follows that $\det F(\Lambda)$ has constant term 1, hence $\det F(\Lambda)$ is an invertible element of R_E. We may therefore define a matrix $\mathscr{F}(\Lambda)$ with entries in R_E by the formula

$$\mathscr{F}(\Lambda) = F(\Lambda^p)^{-1} F(\Lambda). \tag{1.5}$$

The interpretation of the $F_{ij}(\Lambda)$ as A-hypergeometric series will be explained in Sect. 7 (see Equation (7.10)).

Put

$$\begin{aligned} \mathscr{D} = \{(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}_p^N \mid {}&|\lambda_k| = 1 \text{ for } k = 1, \ldots, M, \\ &|\lambda_k| \le 1 \text{ for } k = M+1, \ldots, N, \text{ and } |D(\lambda)| = 1\}. \end{aligned} \tag{1.6}$$

Our main result is the following.

Theorem 1.1 *The entries in the matrix $\mathscr{F}(\Lambda)$ are functions on $\mathscr{D}$. Let $\lambda \in (\mathbb{F}_q^\times)^M \times \mathbb{F}_q^{N-M}$ and let $\hat{\lambda} \in \mathbb{Q}_p(\zeta_{q-1})^N$ be its Teichmüller lifting. If $\bar{D}(\lambda) \neq 0$, then $\hat{\lambda}^{p^i} \in \mathscr{D}$ for $i = 0, \dots, a-1$ and*

$$\rho(\lambda, t) = \det\left(I - t\mathscr{F}(\hat{\lambda}^{p^{a-1}})\mathscr{F}(\hat{\lambda}^{p^{a-2}}) \cdots \mathscr{F}(\hat{\lambda})\right).$$

Remark 1 The first sentence of Theorem 1.1 will be made more precise in Sect. 2 (see Theorem 2.1). The assertion that $\hat{\lambda}^{p^i} \in \mathscr{D}$ for all i if $\bar{D}(\lambda) \neq 0$ follows immediately from the fact that $\bar{D}(\Lambda)$ has coefficients in $\mathbb{F}_p$.

Remark 2 The series $F_{ij}(\Lambda)$ are related to the Laurent polynomials $B_{ij}(\Lambda)$ by truncation (see [1, Proposition 3.8]). One can thus regard Theorem 1.1 as a refinement of Corollary 1.1, i. e., one may think of $\mathscr{F}(\Lambda)$ as a *p*-adic refinement of the Hasse-Witt matrix $B(\Lambda)$.

2 Rings of *p*-adic series

This section has two purposes. We need a ring R large enough to contain all the series that will be encountered in the proof of Theorem 1.1 and we need to identify a subring $R' \subseteq R$ whose elements define functions on $\mathscr{D}$.

Consider the $\mathbb{C}_p$-vector space

$$\mathbb{B} := \left\{ \xi(\Lambda) = \sum_{l \in \mathbb{Z}^N} c_l \Lambda^l \mid \{|c_l|\}_{l \in \mathbb{Z}^N} \text{ is bounded} \right\}$$

of all Laurent series with bounded coefficients. We define a norm on $\mathbb{B}$ by setting

$$|\xi(\Lambda)| = \sup_{l \in \mathbb{Z}^N} |c_l|.$$

The $\mathbb{C}_p$-vector space $\mathbb{B}$ is complete in this norm and R_E is a $\mathbb{C}_p$-subspace of $\mathbb{B}$ which is also complete.

We begin by defining a subring R'_E of the ring R_E whose elements define functions on $\mathscr{D}$. Let L_E denote the subring of R_E consisting of the Laurent polynomials that lie in R_E. Since $D(\Lambda)$ is an invertible element of R_E, we may define a subring of R_E

$$L_{E,D(\Lambda)} := \left\{ \frac{h(\Lambda)}{D(\Lambda)^k} \mid h(\Lambda) \in L_E \text{ and } k \in \mathbb{N} \right\}.$$

The definitions of R_E and $\mathscr{D}$ show that the elements of $L_{E,D(\Lambda)}$ are functions on $\mathscr{D}$. We define R'_E to be the completion of $L_{E,D(\Lambda)}$ under the norm on $\mathbb{B}$. Clearly R'_E is a subring of R_E. The next two lemmas will show that the elements of R'_E define functions on $\mathscr{D}$.

Lemma 2.1 *For $h(\Lambda) \in L_E$ and $k \in \mathbb{N}$, one has*

$$\left| \frac{h(\Lambda)}{D(\Lambda)^k} \right| = |h(\Lambda)|.$$

Proof Write

$$D(\Lambda)^{-k} = \sum_{l \in E} d_l \Lambda^l.$$

Since $D(\Lambda)$ has constant term 1 and p-integral coefficients, it follows that $d_0 = 1$ and the d_l are p-integral. Write

$$h(\Lambda) = \sum_{i=1}^{r} h_i \Lambda^{e_i}$$

where $e_i \in E$ for $i = 1, \dots, r$. The coefficient of Λ^l in the quotient $h(\Lambda)/D(\Lambda)^k$ is

$$\sum_{i=1}^{r} h_i d_{l-e_i}, \tag{2.1}$$

with the understanding that $d_{l-e_i} = 0$ if $l - e_i \notin E$. Since the d_{l-e_i} are p-integral

$$\left| \sum_{i=1}^{r} h_i d_{l-e_i} \right| \leq \sup_{i=1,\dots,r} |h_i|,$$

which implies that $|h(\Lambda)/D(\Lambda)^k| \leq |h(\Lambda)|$.

We must have $|h(\Lambda)| = |h_i|$ for some i. To fix ideas, suppose that

$$|h(\Lambda)| = |h_i| \quad \text{for } i = 1, \dots, s \tag{2.2}$$

and

$$|h(\Lambda)| > |h_i| \quad \text{for } i = s+1, \dots, r. \tag{2.3}$$

Since the cone $\mathscr{C}$ has a vertex at the origin, we can choose a vector $v \in \mathbb{R}^N$ such that the inner product $v \bullet w$ is ≥ 0 for all $w \in \mathscr{C}$ and $v \bullet w = 0$ only if $w = \mathbf{0}$. For some $i \in \{1, \dots, s\}$, the inner product $v \bullet e_i$ is minimal. To fix ideas, suppose that

$$v \bullet e_1 \leq v \bullet e_i \quad \text{for } i = 2, \dots, s. \tag{2.4}$$

From (2.1), the coefficient of Λ^{e_1} in $h(\Lambda)/D(\Lambda)^k$ is (using $d_0 = 1$)

$$h_1 + \sum_{i=2}^{r} h_i d_{e_1 - e_i}. \tag{2.5}$$

For $i = 2, \dots, s$ we have by (2.4) that $v \bullet (e_1 - e_i) \leq 0$. But $e_1 - e_i \neq \mathbf{0}$, so $e_1 - e_i \notin E$. This implies $d_{e_1 - e_i} = 0$, so (2.5) simplifies to

$$h_1 + \sum_{i=s+1}^{r} h_i d_{e_1 - e_i}.$$

Equations (2.2) and (2.3) now imply that

$$\left| h_1 + \sum_{i=s+1}^{r} h_i d_{e_1 - e_i} \right| = |h(\Lambda)|,$$

hence $|h(\Lambda)/D(\Lambda)^k| \geq |h(\Lambda)|$. $\square$

It follows from the definition of E that if $(l_1, \ldots, l_N) \in E$, then $l_i \geq 0$ for $i = M + 1, \ldots, N$. This implies that if $h(\Lambda) \in L_E$ and $\lambda \in \mathscr{D}$, then

$$|h(\lambda)| \leq |h(\Lambda)|. \tag{2.6}$$

Lemma 2.2 *Let $h_k(\Lambda) \in L_E$ for $k \geq 1$. We suppose the sequence $\{h_k(\Lambda)/D(\Lambda)^k\}_{k=1}^{\infty}$ converges to $\xi(\Lambda) \in R'_E$. Then for all $\lambda \in \mathscr{D}$, the sequence $\{h_k(\lambda)/D(\lambda)^k\}_{k=1}^{\infty}$ converges in $\mathbb{C}_p$.*

Proof For $k < k'$ we have

$$
\begin{aligned}
\left| \frac{h_k(\lambda)}{D^k(\lambda)} - \frac{h_{k'}(\lambda)}{D^{k'}(\lambda)} \right| &= \left| \frac{h_k(\lambda)D(\lambda)^{k'-k} - h_{k'}(\lambda)}{D^{k'}(\lambda)} \right| \\
&= |h_k(\lambda)D(\lambda)^{k'-k} - h_{k'}(\lambda)| \\
&\leq |h_k(\Lambda)D(\Lambda)^{k'-k} - h_{k'}(\Lambda)| \\
&= \left| \frac{h_k(\Lambda)D(\Lambda)^{k'-k} - h_{k'}(\Lambda)}{D^{k'}(\Lambda)} \right| \\
&= \left| \frac{h_k(\Lambda)}{D^k(\Lambda)} - \frac{h_{k'}(\Lambda)}{D^{k'}(\Lambda)} \right|,
\end{aligned}
$$

where the second line follows from the requirement that $|D(\lambda)| = 1$ for $\lambda \in \mathscr{D}$, the third line holds by (2.6), and the fourth line holds by Lemma 2.1.

Since the sequence $\{h_k(\Lambda)/D(\Lambda)^k\}_{k=1}^{\infty}$ converges, it is a Cauchy sequence. This inequality shows that the sequence $\{h_k(\lambda)/D(\lambda)^k\}_{k=1}^{\infty}$ is also a Cauchy sequence, which must converge because $\mathbb{C}_p$ is complete. $\square$

It is easy to see that the limit of the sequence $\{h_k(\lambda)/D(\lambda)^k\}_{k=1}^{\infty}$ is independent of the choice of sequence $\{h_k(\Lambda)/D(\Lambda)^k\}_{k=1}^{\infty}$ converging to $\xi(\Lambda)$. We may therefore define

$$\xi(\lambda) = \lim_{k \to \infty} \frac{h_k(\lambda)}{D^k(\lambda)}.$$

Using this definition, the elements of R'_E become functions on $\mathscr{D}$. Note that the proof of Lemma 2.2 shows that, as functions on $\mathscr{D}$, the sequence $\{h_k/D^k\}_{k=1}^{\infty}$ converges uniformly to ξ.

We shall need the following fact later.

Lemma 2.3 *If $\xi(\Lambda) \in R'_E$ and $\lambda \in \mathscr{D}$, then $|\xi(\lambda)| \leq |\xi(\Lambda)|$.*

Proof It follows from Lemma 2.1, Equation (2.6), and the fact that $|D(\lambda)| = 1$ for $\lambda \in \mathscr{D}$ that

$$\left| \frac{h(\lambda)}{D(\lambda)^k} \right| \leq \left| \frac{h(\Lambda)}{D(\Lambda)^k} \right| \tag{2.7}$$

for $h(\Lambda) \in L_E$, $k \in \mathbb{N}$, and $\lambda \in \mathscr{D}$. If we choose a sequence $\{h_k(\Lambda)/D(\Lambda)^k\}_{k=1}^{\infty}$, with the $h_k(\Lambda)$ in L_E, converging to $\xi(\Lambda) \in R'_E$, then for k sufficiently large we have

$$|\xi(\Lambda)| = \left| \frac{h_k(\Lambda)}{D(\Lambda)^k} \right|$$

and for $\lambda \in \mathscr{D}$ we have by the definition of $\xi(\lambda)$

$$|\xi(\lambda)| = \left| \frac{h_k(\lambda)}{D(\lambda)^k} \right|.$$

The assertion of the lemma now follows from (2.7).   □

The following result is our more precise version of Theorem 1.1.

Theorem 2.1 *The entries in the matrix $\mathscr{F}(\Lambda)$ lie in R'_E. Let $\lambda \in (\mathbb{F}_q^\times)^M \times \mathbb{F}_q^{N-M}$ and let $\hat{\lambda} \in \mathbb{Q}_p(\zeta_{q-1})^N$ be its Teichmüller lifting. If $\bar{D}(\lambda) \neq 0$, then $\hat{\lambda}^{p^i} \in \mathscr{D}$ for $i = 0, \ldots, a-1$ and*

$$\rho(\lambda, t) = \det \left(I - t \mathscr{F}(\hat{\lambda}^{p^{a-1}}) \mathscr{F}(\hat{\lambda}^{p^{a-2}}) \cdots \mathscr{F}(\hat{\lambda}) \right).$$

Most of this paper is devoted to proving the first sentence of Theorem 2.1. The remaining assertion will then follow by an application of Dwork's p-adic cohomology theory.

Unfortunately the rings R_E and R'_E are not large enough to contain all the relevant series we shall encounter and therefore need to be enlarged. Let $\tilde{R} = R_E[\Lambda_1^{\pm 1}, \ldots, \Lambda_N^{\pm 1}]$ and let

$$\tilde{R}' = R'_E[\Lambda_1^{\pm 1}, \ldots, \Lambda_M^{\pm 1}, \Lambda_{M+1}, \ldots, \Lambda_N].$$

It is clear from the definition of $\mathscr{D}$ that the elements of $\tilde{R}'$ are functions on $\mathscr{D}$. We define R (resp. R') to be the completion of $\tilde{R}$ (resp. $\tilde{R}'$) under the norm on $\mathbb{B}$. Note that the elements of R' define functions on $\mathscr{D}$. It follows from Lemma 2.3 that

$$|\xi(\lambda)| \leq |\xi(\Lambda)| \quad \text{for } \xi(\Lambda) \in R' \text{ and } \lambda \in \mathscr{D}. \tag{2.8}$$

Remark One can show using the argument of Dwork [6, Lemma 1.2] that for $\xi(\Lambda) \in R'$

$$\sup_{\lambda \in \mathscr{D}} |\xi(\lambda)| = |\xi(\Lambda)|.$$

Let $\mathscr{M} \subseteq \mathbb{Z}^{n+2}$ be the abelian group generated by A. We define an $\mathscr{M}$-grading on R and R'. Every $\xi(\Lambda) \in R$ can be written as a series $\xi(\Lambda) = \sum_{l \in \mathbb{Z}^N} c_l \Lambda^l$. We say that $\xi(\Lambda)$ *has degree* $u \in \mathscr{M}$ if $\sum_{k=1}^N l_k \mathbf{a}_k = u$ for all $l \in \mathbb{Z}^N$ for which $c_l \neq 0$. We denote by R_u the $\mathbb{C}_p$-subspace of R consisting of all elements of degree u. Each R_u is complete in the norm and is a module over $R_{\mathbf{0}}$, which is a ring. Identical remarks apply to the induced grading on R'.

3 p-adic estimates

In this section we recall some estimates from [3, Sect. 3] to be applied later. Let $AH(t) = \exp(\sum_{i=0}^\infty t^{p^i}/p^i)$ be the Artin-Hasse series, a power series in t with p-integral coefficients, let γ_0 be a zero of the series $\sum_{i=0}^\infty t^{p^i}/p^i$ having ord $\gamma_0 = 1/(p-1)$, and set

$$\theta(t) = \mathrm{AH}(\gamma_0 t) = \sum_{i=0}^{\infty} \theta_i t^i.$$

We then have

$$\mathrm{ord}\,\theta_i \geq \frac{i}{p-1}. \tag{3.1}$$

We define $\hat{\theta}(t) = \prod_{j=0}^{\infty} \theta(t^{p^j})$, which gives $\theta(t) = \hat{\theta}(t)/\hat{\theta}(t^p)$. If we set

$$\gamma_j = \sum_{i=0}^{j} \frac{\gamma_0^{p^i}}{p^i}, \tag{3.2}$$

then

$$\hat{\theta}(t) = \exp\left(\sum_{j=0}^{\infty} \gamma_j t^{p^j}\right) = \prod_{j=0}^{\infty} \exp(\gamma_j t^{p^j}). \tag{3.3}$$

If we write $\hat{\theta}(t) = \sum_{i=0}^{\infty} \hat{\theta}_i (\gamma_0 t)^i/i!$, then by [3, Equation (3.8)] we have

$$\mathrm{ord}\,\hat{\theta}_i \geq 0. \tag{3.4}$$

We shall also need the series

$$\hat{\theta}_1(t) := \prod_{j=1}^{\infty} \exp(\gamma_j t^{p^j}) =: \sum_{i=0}^{\infty} \frac{\hat{\theta}_{1,i}}{i!} (\gamma_0 t)^i. \tag{3.5}$$

Note that $\hat{\theta}(t) = \exp(\gamma_0 t)\hat{\theta}_1(t)$. By [3, Equation (3.10)]

$$\mathrm{ord}\,\hat{\theta}_{1,i} \geq \frac{i(p-1)}{p}. \tag{3.6}$$

Define the series $\hat{\theta}_1(\Lambda, x)$ by the formula

$$\hat{\theta}_1(\Lambda, x) = \prod_{k=1}^{N} \hat{\theta}_1(\Lambda_k x^{\mathbf{a}_k}). \tag{3.7}$$

Put $\mathbb{N}A = \{\sum_{j=1}^{N} l_j \mathbf{a}_j \mid l_j \in \mathbb{N}\}$. Expanding the product (3.7) according to powers of x we get

$$\hat{\theta}_1(\Lambda, x) = \sum_{u=(u_0,\ldots,u_{n+1})\in\mathbb{N}A} \hat{\theta}_{1,u}(\Lambda)\gamma_0^{u_{n+1}} x^u, \tag{3.8}$$

where

$$\hat{\theta}_{1,u}(\Lambda) = \sum_{\substack{k_1,\ldots,k_N \in \mathbb{N} \\ \sum_{j=1}^{N} k_j \mathbf{a}_j = u}} \left(\prod_{j=1}^{N} \frac{\hat{\theta}_{1,k_j}}{k_j!}\right) \Lambda_1^{k_1} \cdots \Lambda_N^{k_N}. \tag{3.9}$$

We have similar results for the reciprocal power series

$$\hat{\theta}_1(t)^{-1} = \prod_{j=1}^{\infty} \exp(-\gamma_j t^{p^j}).$$

If we write

$$\hat{\theta}_1(t)^{-1} = \sum_{i=0}^{\infty} \frac{\hat{\theta}'_{1,i}}{i!} (\gamma_0 t)^i, \tag{3.10}$$

then the coefficients satisfy

$$\operatorname{ord} \hat{\theta}'_{1,i} \geq \frac{i(p-1)}{p}. \tag{3.11}$$

We also have

$$\hat{\theta}_1(\Lambda, x)^{-1} = \prod_{k=1}^{N} \hat{\theta}_1(\Lambda_k x^{\mathbf{a}_k})^{-1}, \tag{3.12}$$

which we again expand in powers of x as

$$\hat{\theta}_1(\Lambda, x)^{-1} = \sum_{u=(u_0,\ldots,u_{n+1}) \in \mathbb{N}A} \hat{\theta}'_{1,u}(\Lambda) \gamma_0^{u_{n+1}} x^u \tag{3.13}$$

with

$$\hat{\theta}'_{1,u}(\Lambda) = \sum_{\substack{k_1,\ldots,k_N \in \mathbb{N} \\ \sum_{j=1}^{N} k_j \mathbf{a}_j = u}} \left(\prod_{j=1}^{N} \frac{\hat{\theta}'_{1,k_j}}{k_j!} \right) \Lambda_1^{k_1} \cdots \Lambda_N^{k_N}. \tag{3.14}$$

We also define

$$\theta(\Lambda, x) = \prod_{k=1}^{N} \theta(\Lambda_k x^{\mathbf{a}_k}). \tag{3.15}$$

Expanding the right-hand side in powers of x, we have

$$\theta(\Lambda, x) = \sum_{u \in \mathbb{N}A} \theta_u(\Lambda) x^u, \tag{3.16}$$

where

$$\theta_u(\Lambda) = \sum_{v \in \mathbb{N}^N} \theta_v^{(u)} \Lambda^v \tag{3.17}$$

and

$$\theta_v^{(u)} = \begin{cases} \prod_{k=1}^{N} \theta_{v_k} & \text{if } \sum_{k=1}^{N} v_k \mathbf{a}_k = u, \\ 0 & \text{if } \sum_{k=1}^{N} v_k \mathbf{a}_k \neq u, \end{cases} \tag{3.18}$$

so $\theta_u(\Lambda)$ is homogeneous of degree u. The equation $\sum_{k=1}^{N} v_k \mathbf{a}_k = u$ has only finitely many solutions $v \in \mathbb{N}^N$, so $\theta_u(\Lambda)$ is a polynomial in the Λ_k. Equations (3.1) and (3.18) show that

$$\text{ord}_p \, \theta_v^{(u)} \geq \frac{\sum_{j=1}^N v_j}{p-1} = \frac{u_{n+1}}{p-1}. \tag{3.19}$$

4 The Dwork-Frobenius operator

We define the spaces S and S' and Dwork's Frobenius operator on those spaces.

Note that the abelian group $\mathcal{M}$ generated by A lies in the hyperplane $\sum_{i=0}^n u_i = d u_{n+1}$ in $\mathbb{R}^{n+2}$. Set $\mathcal{M}_- = \mathcal{M} \cap (\mathbb{Z}_{<0})^{n+2}$. We denote by δ_- the truncation operator on formal Laurent series in variables $x_0, \ldots, x_{n+1}$ that preserves only those terms having all exponents negative:

$$\delta_- \left(\sum_{k \in \mathbb{Z}^{n+2}} c_k x^k \right) = \sum_{k \in (\mathbb{Z}_{<0})^{n+2}} c_k x^k.$$

We use the same notation for formal Laurent series in a single variable t:

$$\delta_- \left(\sum_{k=-\infty}^{\infty} c_k t^k \right) = \sum_{k=-\infty}^{-1} c_k t^k.$$

It is straightforward to check that if ξ_1 and ξ_2 are two series for which the product $\xi_1 \xi_2$ is defined and if no monomial in ξ_2 has a negative exponent, then

$$\delta_- \big(\delta_-(\xi_1) \xi_2 \big) = \delta_-(\xi_1 \xi_2). \tag{4.1}$$

Define S to be the $\mathbb{C}_p$-vector space of formal series

$$S = \left\{ \xi(\Lambda, x) = \sum_{u \in \mathcal{M}_-} \xi_u(\Lambda) \gamma_0^{u_{n+1}} x^u \mid \xi_u(\Lambda) \in R_u \text{ and } \{|\xi_u|\}_u \text{ is bounded} \right\}.$$

Let S' be defined analogously with the condition "$\xi_u(\Lambda) \in R_u$" being replaced by "$\xi_u(\Lambda) \in R'_u$". Define a norm on S by setting

$$|\xi(\Lambda, x)| = \sup_{u \in \mathcal{M}_-} \{|\xi_u|\}.$$

Both S and S' are complete under this norm.

Let

$$\xi(\Lambda, x) = \sum_{v \in \mathcal{M}_-} \xi_v(\Lambda) \gamma_0^{v_{n+1}} x^v \in S.$$

We show that the product $\theta(\Lambda, x) \xi(\Lambda^p, x^p)$ is well-defined as a formal series in x. We have formally

$$\theta(\Lambda, x) \xi(\Lambda^p, x^p) = \sum_{\rho \in \mathcal{M}} \zeta_\rho(\Lambda) x^\rho,$$

where

$$\zeta_\rho(\Lambda) = \sum_{\substack{u \in \mathbb{N}A, \, v \in \mathscr{M}_- \\ u + pv = \rho}} \gamma_0^{v_{n+1}} \theta_u(\Lambda) \xi_v(\Lambda^p). \tag{4.2}$$

Since $\theta_u(\Lambda)$ is a polynomial, the product $\theta_u(\Lambda)\xi_v(\Lambda^p)$ is a well-defined element of R_ρ. It follows from (3.17), (3.19), and the equality $u + pv = \rho$ that the coefficients of $\gamma_0^{v_{n+1}}\theta_u(\Lambda)$ all have p-ordinal at least $(\rho_{n+1}/(p-1)) - v_{n+1}$. Since $|\xi_v(\Lambda)|$ is bounded independently of v and there are only finitely many terms on the right-hand side of (4.2) with a given value of v_{n+1}, the series (4.2) converges to an element of R_ρ (because $-v_{n+1} \to \infty$ as $v \to \infty$). This estimate also shows that if $\xi(\Lambda, x) \in S'$, then $\zeta_\rho(\Lambda) \in R'_\rho$.

Define for $\xi(\Lambda, x) \in S$

$$\alpha^*\big(\xi(\Lambda, x)\big) = \delta_-\big(\theta(\Lambda, x)\xi(\Lambda^p, x^p)\big)$$
$$= \sum_{\rho \in \mathscr{M}_-} \zeta_\rho(\Lambda)x^\rho.$$

For $\rho \in \mathscr{M}_-$, put $\eta_\rho(\Lambda) = \gamma_0^{-\rho_{n+1}}\zeta_\rho(\Lambda)$, so that

$$\alpha^*\big(\xi(\Lambda, x)\big) = \sum_{\rho \in \mathscr{M}_-} \eta_\rho(\Lambda)\gamma_0^{\rho_{n+1}}x^\rho \tag{4.3}$$

with (by (4.2))

$$\eta_\rho(\Lambda) = \sum_{\substack{u \in \mathbb{N}A, \, v \in \mathscr{M}_- \\ u + pv = \rho}} \gamma_0^{-\rho_{n+1}+v_{n+1}} \theta_u(\Lambda) \xi_v(\Lambda^p). \tag{4.4}$$

Proposition 4.1 *The map α^* is an endomorphism of S and of S', and for $\xi(\Lambda, x) \in S$ we have*

$$|\alpha^*\big(\xi(\Lambda, x)\big)| \leq |p\xi(\Lambda, x)|. \tag{4.5}$$

Proof By (4.3), the proposition will follow from the estimate

$$|\eta_\rho(\Lambda)| \leq |p\xi(\Lambda, x)| \quad \text{for all } \rho \in \mathscr{M}_-.$$

Using (4.4), we see that this estimate will follow in turn from the estimate

$$|\gamma_0^{-\rho_{n+1}+v_{n+1}} \theta_u(\Lambda)| \leq |p|$$

for all $u \in \mathbb{N}A$, $v \in \mathscr{M}_-$, with $u + pv = \rho$. From (3.17) and (3.19) we see that all coefficients of $\gamma_0^{-\rho_{n+1}+v_{n+1}}\theta_u(\Lambda)$ have p-ordinal greater than or equal to

$$\frac{-\rho_{n+1} + v_{n+1} + u_{n+1}}{p-1}.$$

Since $u + pv = \rho$, this expression simplifies to $-v_{n+1}$, so

$$|\gamma_0^{-\rho_{n+1}+v_{n+1}} \theta_u(\Lambda)| \leq |p|^{-v_{n+1}} \tag{4.6}$$

and $-v_{n+1} \geq 1$ since $v \in \mathscr{M}_-$. $\qquad\square$

Note that the equality $-v_{n+1} = 1$ occurs for only M points $v \in \mathcal{M}_-$, namely, $v = -\mathbf{a}_1, \ldots, -\mathbf{a}_M$. The following corollary is then an immediate consequence of the proof of Proposition 4.1.

Corollary 4.1 *If* $\xi_v(\Lambda) = 0$ *for* $v = -\mathbf{a}_1, \ldots, -\mathbf{a}_M$, *then* $|\alpha^*(\xi(\Lambda, x))| \leq |p^2\xi(\Lambda, x)|$.

5 A technical lemma

The action of α^* is extended to S^M componentwise: if

$$\xi(\Lambda, x) = (\xi^{(1)}(\Lambda, x), \ldots, \xi^{(M)}(\Lambda, x)) \in S^M, \tag{5.1}$$

then

$$\alpha^*\big(\xi(\Lambda, x)\big) = \big(\alpha^*\big(\xi^{(1)}(\Lambda, x)\big), \ldots, \alpha^*\big(\xi^{(M)}(\Lambda, x)\big)\big). \tag{5.2}$$

Let $\xi(\Lambda, x)$ be as in (5.1) with

$$\xi^{(i)}(\Lambda, x) = \sum_{v \in \mathcal{M}_-} \xi^{(i)}_v(\Lambda) \gamma_0^{v_{n+1}} x^v. \tag{5.3}$$

Since $\xi^{(i)}_{-\mathbf{a}_j}(\Lambda) \in R_{-\mathbf{a}_j}$, the matrix

$$\mathbb{M}\big(\xi(\Lambda, x)\big) := \big(\Lambda_j \xi^{(i)}_{-\mathbf{a}_j}(\Lambda)\big)_{i,j=1}^M$$

has entries in $R_{\mathbf{0}}$. If $\xi(\Lambda, x) \in (S')^M$, then $\mathbb{M}\big(\xi(\Lambda, x)\big)$ has entries in $R'_{\mathbf{0}}$. The map $\xi(\Lambda, x) \mapsto \mathbb{M}(\xi(\Lambda, x))$ is a $\mathbb{C}_p$-linear map from S^M to the $\mathbb{C}_p$-vector space of $(M \times M)$-matrices with entries in $R_{\mathbf{0}}$. Note that if $Y(\Lambda)$ is an $(M \times M)$-matrix with entries in $R_{\mathbf{0}}$, then

$$\mathbb{M}\big(Y(\Lambda)\xi(\Lambda, x)\big) = Y(\Lambda)\mathbb{M}\big(\xi(\Lambda, x)\big), \tag{5.4}$$

where we regard $\xi(\Lambda, x)$ as a column vector for the purpose of matrix multiplication. In particular, if $\mathbb{M}\big(\xi(\Lambda, x)\big)$ is an invertible matrix, then

$$\mathbb{M}\left(\mathbb{M}\big(\xi(\Lambda, x)\big)^{-1}\xi(\Lambda, x)\right) = I. \tag{5.5}$$

We extend the norm on S to S^M and the norm on R to $\mathrm{Mat}_M(R)$, the $(M \times M)$-matrices with entries in R. For $\xi(\Lambda, x)$ as in (5.1), we define

$$|\xi(\Lambda, x)| = \max\{|\xi^{(i)}(\Lambda, x)|\}_{i=1}^M$$

and for $Y(\Lambda) = \big(Y_{ij}(\Lambda)\big)_{i,j=1}^M$ with $Y_{ij}(\Lambda) \in R$ we define

$$|Y(\Lambda)| = \max\{|Y_{ij}(\Lambda)|\}_{i,j=1}^M.$$

Note that the matrix $\mathbb{M}\big(\xi(\Lambda, x)\big)$ has an inverse with entries in $R_{\mathbf{0}}$ if and only if $\det \mathbb{M}\big(\xi(\Lambda, x)\big)$ is an invertible element of $R_{\mathbf{0}}$. Likewise, if $\xi(\Lambda, x) \in (S')^M$, this matrix has an inverse with entries in $R'_{\mathbf{0}}$ if and only if its determinant is an invertible element of $R'_{\mathbf{0}}$. The main result of this section is the following assertion.

Lemma 5.1 *Let* $\xi(\Lambda, x) \in S^M$ *with* $\mathbb{M}\big(\xi(\Lambda, x)\big)$ *invertible,* $\big|\mathbb{M}\big(\xi(\Lambda, x)\big)\big| = |\xi(\Lambda, x)|$, *and* $\big|\det \mathbb{M}\big(\xi(\Lambda, x)\big)\big| = |\xi(\Lambda, x)|^M$. *Then* $\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)$ *is invertible,*

$$\big|\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)\big| = \big|\alpha^*\big(\xi(\Lambda, x)\big)\big| = |p\xi(\Lambda, x)|, \tag{5.6}$$

and

$$\big|\det \mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)\big| = |p\xi(\Lambda, x)|^M. \tag{5.7}$$

Remark To prove Lemma 5.1, it suffices to show that $\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)$ is invertible and that (5.7) holds: from Proposition 4.1 we get

$$\big|\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)\big| \leq \big|\alpha^*\big(\xi(\Lambda, x)\big)\big| \leq |p\xi(\Lambda, x)|.$$

If any of these inequalities were strict, the usual formula for the determinant of a matrix in terms of its entries would imply that

$$\big|\det \mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)\big| < |p\xi(\Lambda, x)|^M.$$

Similar reasoning applies to the inverse of $\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)$. By (5.7) we have

$$\big|\det \mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)^{-1}\big| = |p\xi(\Lambda, x)|^{-M}. \tag{5.8}$$

The usual formula for the inverse of a matrix in terms of its cofactors implies that

$$\big|\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)^{-1}\big| \leq \frac{1}{|p\xi(\Lambda, x)|}.$$

But if this inequality were strict it would lead to a contradiction of (5.8), so we have the following corollary.

Corollary 5.1 *Under the hypotheses of Lemma 5.1,*

$$\big|\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)^{-1}\big| = \frac{1}{|p\xi(\Lambda, x)|}.$$

The proof of Lemma 5.1 will require several steps. We first consider a matrix constructed from $\theta(\Lambda, x)$ (Equation (3.16)):

$$\mathbb{M}_\theta(\Lambda) = \big(\theta_{p\mathbf{a}_i - \mathbf{a}_j}(\Lambda)\big)_{i,j=1}^M.$$

From (3.17) and (3.18) we have explicit formulas for these matrix entries:

$$\theta_{p\mathbf{a}_i - \mathbf{a}_j}(\Lambda) = \sum_{\substack{\nu \in \mathbb{N}^N \\ \sum_{k=1}^N \nu_k \mathbf{a}_k = p\mathbf{a}_i - \mathbf{a}_j}} \left(\prod_{k=1}^N \theta_{\nu_k}\right) \Lambda^\nu.$$

Furthermore, since the last coordinate of $p\mathbf{a}_i - \mathbf{a}_j$ equals $p - 1$, each ν_k in this summation is $\leq p - 1$, so from their definition $\theta_{\nu_k} = \gamma_0^{\nu_k}/\nu_k!$. Since $\sum_{k=1}^N \nu_k = p - 1$, we have

$$\theta_{p\mathbf{a}_i - \mathbf{a}_j}(\Lambda) = \sum_{\substack{v \in \mathbb{N}^N \\ \sum_{k=1}^N v_k \mathbf{a}_k = p\mathbf{a}_i - \mathbf{a}_j}} \frac{\gamma_0^{p-1} \Lambda^v}{v_1! \cdots v_N!}.$$

Thus

$$\mathbb{M}_\theta(\Lambda) = \gamma_0^{p-1} H(\Lambda), \tag{5.9}$$

where $H(\Lambda)$ is given in Equation (1.3), and

$$C(\Lambda)^{-p} \mathbb{M}_\theta(\Lambda) C(\Lambda) = \gamma_0^{p-1} B(\Lambda), \tag{5.10}$$

where $B(\Lambda)$ is given in Equation (1.4). It now follows from the discussion in Sect. 1 that $C(\Lambda)^{-p} \mathbb{M}_\theta(\Lambda) C(\Lambda)$ is an invertible element of $R'_{\mathbf{0}}$. Furthermore, we have

$$\det C(\Lambda)^{-p} \mathbb{M}_\theta(\Lambda) C(\Lambda) = \gamma_0^{(p-1)M} D(\Lambda). \tag{5.11}$$

And since $|D(\Lambda)| = 1$ (since $D(\Lambda)$ has constant term 1 and integral coefficients) we get

$$|\det C(\Lambda)^{-p} \mathbb{M}_\theta(\Lambda) C(\Lambda)| = |p^M|. \tag{5.12}$$

With $\xi(\Lambda, x)$ as in (5.1), we rewrite (5.2) as

$$\alpha^*\big(\xi(\Lambda, x)\big) = \big(\eta^{(1)}(\Lambda, x), \dots, \eta^{(M)}(\Lambda, x)\big),$$

where (see (4.3) and (4.4))

$$\eta^{(i)}(\Lambda, x) = \sum_{\rho \in \mathscr{M}_-} \eta_\rho^{(i)}(\Lambda) \gamma_0^{\rho_{n+1}} x^\rho \tag{5.13}$$

with

$$\eta_\rho^{(i)}(\Lambda) = \sum_{\substack{u \in \mathbb{N}A, \, v \in \mathscr{M}_- \\ u + pv = \rho}} \gamma_0^{-\rho_{n+1} + v_{n+1}} \theta_u(\Lambda) \xi_v^{(i)}(\Lambda^p). \tag{5.14}$$

We then have

$$\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big) = \big(\Lambda_j \eta_{-\mathbf{a}_j}^{(i)}(\Lambda)\big)_{i,j=1}^M. \tag{5.15}$$

If $v \ne -\mathbf{a}_1, \dots, -\mathbf{a}_M$, then $v_{n+1} \le -2$, so by (5.14) we may write

$$\eta_{-\mathbf{a}_j}^{(i)}(\Lambda) = \sum_{k=1}^M \theta_{p\mathbf{a}_k - \mathbf{a}_j}(\Lambda) \xi_{-\mathbf{a}_k}^{(i)}(\Lambda^p) + \sum_{\substack{u \in \mathbb{N}A, \, v \in \mathscr{M}_- \\ u + pv = -\mathbf{a}_j \\ v_{n+1} \le -2}} \gamma_0^{-\rho_{n+1} + v_{n+1}} \theta_u(\Lambda) \xi_v^{(i)}(\Lambda^p).$$

$$\tag{5.16}$$

We write $\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big) = \mathbb{M}^{(1)}(\Lambda) + \mathbb{M}^{(2)}(\Lambda)$, where

$$\mathbb{M}_{ij}^{(1)}(\Lambda) = \Lambda_j \sum_{k=1}^{M} \theta_{p\mathbf{a}_k - \mathbf{a}_j}(\Lambda) \xi_{-\mathbf{a}_k}^{(i)}(\Lambda^p) \tag{5.17}$$

and

$$\mathbb{M}_{ij}^{(2)}(\Lambda) = \Lambda_j \sum_{\substack{u \in \mathbb{N}A, \, v \in \mathscr{M}_- \\ u + pv = -\mathbf{a}_j \\ v_{n+1} \leq -2}} \gamma_0^{-\rho_{n+1} + v_{n+1}} \theta_u(\Lambda) \xi_v^{(i)}(\Lambda^p). \tag{5.18}$$

It follows from (5.17), (5.18), and a short calculation that

$$\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big) = \mathbb{M}\big(\xi(\Lambda^p, x)\big) C(\Lambda)^{-p} \mathbb{M}_\theta(\Lambda) C(\Lambda) + \mathbb{M}^{(2)}(\Lambda). \tag{5.19}$$

Proof of Lemma 5.1 For notational convenience set

$$\widetilde{\mathbb{M}}(\Lambda) = \mathbb{M}\big(\xi(\Lambda^p, x)\big) C(\Lambda)^{-p} \mathbb{M}_\theta(\Lambda) C(\Lambda)$$

so that (5.19) can be rewritten as

$$\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big) = \widetilde{\mathbb{M}}(\Lambda) + \mathbb{M}^{(2)}(\Lambda). \tag{5.20}$$

From (5.10) we get

$$|\widetilde{\mathbb{M}}(\Lambda)| \leq |p\mathbb{M}(\xi(\Lambda^p, x))| = |p\mathbb{M}(\xi(\Lambda, x))|$$

and from (5.11)

$$|\det \widetilde{\mathbb{M}}(\Lambda)| = |p^M \det \mathbb{M}(\xi(\Lambda^p, x))| = |p^M \det \mathbb{M}(\xi(\Lambda, x))|.$$

Applying our hypotheses gives

$$|\widetilde{\mathbb{M}}(\Lambda)| \leq |p\xi(\Lambda, x)| \tag{5.21}$$

and

$$|\det \widetilde{\mathbb{M}}(\Lambda)| = |p\xi(\Lambda, x)|^M. \tag{5.22}$$

And since the equality in (5.22) would fail if the inequality in (5.21) were strict, we must have

$$|\widetilde{\mathbb{M}}(\Lambda)| = |p\xi(\Lambda, x)|. \tag{5.23}$$

The matrix $\widetilde{\mathbb{M}}(\Lambda)$ is invertible by (5.10) and our hypotheses, and (5.22) implies

$$|\det \widetilde{\mathbb{M}}(\Lambda)^{-1}| = |p\xi(\Lambda, x)|^{-M}. \tag{5.24}$$

Arguing as in the derivation of Corollary 5.1 from (5.8) then gives

$$|\widetilde{\mathbb{M}}(\Lambda)^{-1}| = \frac{1}{|p\xi(\Lambda, x)|}. \tag{5.25}$$

Rewrite (5.20) as

$$\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big) = \widetilde{\mathbb{M}}(\Lambda)\big(I + \widetilde{\mathbb{M}}(\Lambda)^{-1}\mathbb{M}^{(2)}(\Lambda)\big). \tag{5.26}$$

Estimate (4.6) implies that

$$|\mathbb{M}^{(2)}(\Lambda)| \leq |p^2\xi(\Lambda, x)|, \tag{5.27}$$

so by (5.25)

$$|\widetilde{\mathbb{M}}(\Lambda)^{-1}\mathbb{M}^{(2)}(\Lambda)| \leq |p|. \tag{5.28}$$

It follows from (5.28) that $I + \widetilde{\mathbb{M}}(\Lambda)^{-1}\mathbb{M}^{(2)}(\Lambda)$ is invertible, so by (5.26) the invertibility of $\widetilde{\mathbb{M}}(\Lambda)$ implies the invertibility of $\mathbb{M}\big(\alpha^*(\xi(\Lambda, x))\big)$. Estimate (5.28) implies that

$$\big|\det\big(I + \widetilde{\mathbb{M}}(\Lambda)^{-1}\mathbb{M}^{(2)}(\Lambda)\big)\big| = 1, \tag{5.29}$$

so (5.7) follows from (5.26) and (5.22). $\qquad\square$

6 Contraction mapping

We use the Dwork-Frobenius operator to construct a contraction mapping on a subset of S^M. Finding the fixed point of this contraction mapping will be the crucial step in proving the first assertion of Theorem 2.1.

Put

$$T = \{\xi(\Lambda, x) \in S^M \mid \mathbb{M}(\xi(\Lambda, x)) = I \text{ and } |\xi(\Lambda, x)| = 1\}$$

and put $T' = T \cap (S')^M$. Note that T and T' are closed in the topology on S^M. Elements of T satisfy the hypotheses of Lemma 5.1. By that result, if $\xi(\Lambda, x) \in T$, then $\mathbb{M}(\alpha^*(\xi(\Lambda, x)))$ is invertible, so we may define

$$\phi\big(\xi(\Lambda, x)\big) := \mathbb{M}(\alpha^*(\xi(\Lambda, x)))^{-1}\alpha^*\big(\xi(\Lambda, x)\big),$$

where we regard $\alpha^*(\xi(\Lambda, x))$ on the right-hand side as a column vector for the purpose of matrix multiplication. By Equation (5.5) we have

$$\mathbb{M}\big(\phi(\xi(\Lambda, x))\big) = I. \tag{6.1}$$

Equation (4.5) and Corollary 5.1 imply that $|\phi(\xi(\Lambda, x))| \leq 1$, so in fact

$$|\phi(\xi(\Lambda, x))| = 1 \tag{6.2}$$

by (6.1). Equations (6.1) and (6.2) show that $\phi(T) \subseteq T$ and $\phi(T') \subseteq T'$.

Proposition 6.1 *The operator ϕ is a contraction mapping on T. More precisely, if* $\xi^{(1)}(\Lambda, x), \xi^{(2)}(\Lambda, x) \in T$, *then*

$$\big|\phi\big(\xi^{(1)}(\Lambda, x)\big) - \phi\big(\xi^{(2)}(\Lambda, x)\big)\big| \leq |p| \cdot |\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)|.$$

Proof For notational convenience we sometimes write $\eta^{(i)}(\Lambda, x) = \alpha^*(\xi^{(i)}(\Lambda, x))$ for $i = 1, 2$. Then

$$
\begin{aligned}
\phi\big(\xi^{(1)}(\Lambda, x)\big) &- \phi\big(\xi^{(2)}(\Lambda, x)\big) \\
&= \mathbb{M}\big(\eta^{(1)}(\Lambda, x)\big)^{-1}\eta^{(1)}(\Lambda, x) - \mathbb{M}\big(\eta^{(2)}(\Lambda, x)\big)^{-1}\eta^{(2)}(\Lambda, x) \\
&= \mathbb{M}\big(\eta^{(1)}(\Lambda, x)\big)^{-1}\bigg(\eta^{(1)}(\Lambda, x) - \eta^{(2)}(\Lambda, x)\bigg) \\
&\quad - \mathbb{M}\big(\eta^{(1)}(\Lambda, x)\big)^{-1}\bigg(\mathbb{M}\big(\eta^{(1)}(\Lambda, x) - \eta^{(2)}(\Lambda, x)\big)\bigg)\mathbb{M}\big(\eta^{(2)}(\Lambda, x)\big)^{-1}\eta^{(2)}(\Lambda, x).
\end{aligned}
\tag{6.3}
$$

The difference $\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)$ satisfies the hypothesis of Corollary 4.1, so

$$
\big|\alpha^*\big(\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)\big)\big| \le |p^2| \cdot |\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)|.
\tag{6.4}
$$

By Corollary 5.1 we have

$$
|\mathbb{M}\big(\alpha^*(\xi^{(1)}(\Lambda, x))\big)^{-1}| = |\mathbb{M}\big(\alpha^*(\xi^{(2)}(\Lambda, x))\big)^{-1}| = 1/|p|,
\tag{6.5}
$$

so

$$
\bigg|\mathbb{M}\big(\eta^{(1)}(\Lambda, x)\big)^{-1}\bigg(\eta^{(1)}(\Lambda, x) - \eta^{(2)}(\Lambda, x)\bigg)\bigg| \le |p| \cdot |\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)|.
\tag{6.6}
$$

By Proposition 4.1 we have

$$
\big|\alpha^*\big(\xi^{(2)}(\Lambda, x)\big)\big| \le |p|,
$$

and by (6.4) we have

$$
\big|\mathbb{M}\big(\alpha^*\big(\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)\big)\big)\big| \le |p^2| \cdot |\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)|,
$$

so

$$
\begin{aligned}
&\bigg|\mathbb{M}\big(\eta^{(1)}(\Lambda, x)\big)^{-1}\bigg(\mathbb{M}\big(\eta^{(1)}(\Lambda, x) - \eta^{(2)}(\Lambda, x)\big)\bigg)\mathbb{M}\big(\eta^{(2)}(\Lambda, x)\big)^{-1}\eta^{(2)}(\Lambda, x)\bigg| \\
&\le |p| \cdot |\xi^{(1)}(\Lambda, x) - \xi^{(2)}(\Lambda, x)|.
\end{aligned}
\tag{6.7}
$$

The assertion of the proposition now follows from (6.3), (6.6), and (6.7). $\qquad\square$

By a well-known theorem, the contraction mapping ϕ has a unique fixed point, and since ϕ is stable on T' that fixed point lies in T'. In the next section we discuss certain hypergeometric series and their p-adic relatives. These p-adic relatives will be used to describe explicitly this fixed point.

7 *A*-hypergeometric series

The entries of the matrix $F(\Lambda)$ are *A*-hypergeometric in nature. The purpose of this section is to recall their construction and to introduce some related series that satisfy better p-adic estimates.

Let $L \subseteq \mathbb{Z}^N$ be the lattice of relations on the set A:

$$L = \left\{ l = (l_1, \dots, l_N) \in \mathbb{Z}^N \mid \sum_{j=1}^{N} l_j \mathbf{a}_j = \mathbf{0} \right\}.$$

For each $l = (l_1, \dots, l_N) \in L$, we define a partial differential operator $\square_l$ in variables $\{\Lambda_j\}_{j=1}^{N}$ by

$$\square_l = \prod_{l_j > 0} \left(\frac{\partial}{\partial \Lambda_j} \right)^{l_j} - \prod_{l_j < 0} \left(\frac{\partial}{\partial \Lambda_j} \right)^{-l_j}. \tag{7.1}$$

For $\beta = (\beta_0, \beta_1, \dots, \beta_{n+1}) \in \mathbb{C}^{n+2}$, the corresponding Euler (or homogeneity) operators are defined by

$$Z_i = \sum_{j=1}^{N} a_{ij} \Lambda_j \frac{\partial}{\partial \Lambda_j} - \beta_i \tag{7.2}$$

for $i = 0, \dots, n+1$. The *A-hypergeometric system with parameter β* consists of Equations (7.1) for $l \in L$ and (7.2) for $i = 0, 1, \dots, n+1$.

Consider the formal series

$$q(t) = \sum_{l=0}^{\infty} (-1)^l l! t^{-l-1}.$$

Let $i \in \{1, \dots, M\}$. The *A*-hypergeometric series of interest arise as the coefficients of $\gamma_0^{u_{n+1}} x^u$ in the expression

$$F^{(i)}(\Lambda, x) := \delta_- \left(q(\gamma_0 \Lambda_i x^{\mathbf{a}_i}) \prod_{\substack{k=1 \\ k \neq i}}^{N} \exp(\gamma_0 \Lambda_k x^{\mathbf{a}_k}) \right). \tag{7.3}$$

For $u \in \mathcal{M}_-$, let $F_u^{(i)}(\Lambda)$ be the coefficient of $\gamma_0^{u_{n+1}} x^u$ in (7.3) and write

$$F^{(i)}(\Lambda, x) = \sum_{u \in \mathcal{M}_-} F_u^{(i)}(\Lambda) \gamma_0^{u_{n+1}} x^u. \tag{7.4}$$

If we set

$$L_{i,u} = \left\{ l = (l_1, \dots, l_N) \in \mathbb{Z}^N \mid \sum_{k=1}^{N} l_k \mathbf{a}_k = u, l_i \leq 0, \text{ and } l_k \geq 0 \text{ for } k \neq i \right\},$$

then from (7.3)

$$F_u^{(i)}(\Lambda) = \sum_{l=(l_1,\dots,l_N) \in L_{i,u}} \frac{(-1)^{-l_i-1}(-l_i-1)!}{\displaystyle\prod_{\substack{k=1 \\ k \neq i}}^{N} l_k!} \prod_{k=1}^{N} \Lambda_k^{l_k}. \tag{7.5}$$

It follows from the definition of $q(t)$ that for $i = 1, \dots, M$

$$\frac{\partial}{\partial \Lambda_i} q(\gamma_0 \Lambda_i x^{\mathbf{a}_i}) = \gamma_0 x^{\mathbf{a}_i} q(\gamma_0 \Lambda_i x^{\mathbf{a}_i}) - \frac{1}{\Lambda_i}.$$

A straightforward calculation from (7.3) gives

$$\frac{\partial}{\partial \Lambda_k} F^{(i)}(\Lambda, x) = \delta_- \left(\gamma_0 x^{\mathbf{a}_k} F^{(i)}(\Lambda, x) \right) \tag{7.6}$$

for $k = 1, \dots, N$. Equivalently, for $u \in \mathcal{M}_-$, we have by (7.6)

$$\frac{\partial}{\partial \Lambda_k} F_u^{(i)}(\Lambda) = F_{u - \mathbf{a}_k}^{(i)}(\Lambda). \tag{7.7}$$

More generally, if $l_1, \dots, l_N$ are nonnegative integers, it follows that

$$\prod_{k=1}^N \left(\frac{\partial}{\partial \Lambda_k} \right)^{l_k} F_u^{(i)}(\Lambda) = F_{u - \sum_{k=1}^N l_k \mathbf{a}_k}^{(i)}(\Lambda). \tag{7.8}$$

In particular, it follows from the definition of the box operators (7.1) that

$$\Box_l \left(F_u^{(i)}(\Lambda) \right) = 0 \quad \text{for all } l \in L, i = 1, \dots, M, \text{ and all } u \in \mathcal{M}_-. \tag{7.9}$$

The condition on the summation in (7.5) implies that $F_u^{(i)}(\Lambda)$ satisfies the Euler operators (7.2) with parameter $\beta = u$. In summary:

Lemma 7.1 *For $i = 1, \dots, M$ and $u \in \mathcal{M}_-$, the series $F_u^{(i)}(\Lambda)$ is a solution of the A-hypergeometric system with parameter $\beta = u$.*

Lemma 7.2 *For $i = 1, \dots, M$ and $u \in \mathcal{M}_-$, the series $F_u^{(i)}(\Lambda)$ lies in R_u.*

Proof The homogeneity condition is clear from (7.5). We prove that $F_u^{(i)}(\Lambda)$ lies in $\tilde{R}$.

Let $u \in \mathcal{M}_-$. By the definition of the set A, we can write $-u = \sum_{k=1}^N l_k \mathbf{a}_k$ with the l_k in $\mathbb{N}$. Suppose that $(l_1', \dots, l_N') \in L_{i,u}$. Then

$$\sum_{k=1}^N (l_k + l_k') \mathbf{a}_k = \mathbf{0}.$$

Furthermore, we have $l_k + l_k' \geq 0$ for $k \neq i$ and $l_i + l_i' \leq 0$, so

$$(l_1 + l_1', \dots, l_N + l_N') \in L_i.$$

It follows that $L_{i,u} \subseteq -(l_1, \dots, l_N) + L_i$, hence $F_u^{(i)}(\Lambda) \in \Lambda_1^{-l_1} \cdots \Lambda_N^{-l_N} R_E$. $\qquad \square$

Remark We note the relation between the $F_u^{(i)}(\Lambda)$ and the $F_{ij}(\Lambda)$ defined in the Introduction: for $i, j = 1, \dots, M$,

$$F_{ij}(\Lambda) = \Lambda_j F_{-\mathbf{a}_j}^{(i)}(\Lambda) = \mathbb{M} \left(F(\Lambda, x) \right). \tag{7.10}$$

Lemma 7.3 *For $i = 1, \dots, M$ and $u \in \mathcal{M}_-$, the coefficients of the series $F_u^{(i)}(\Lambda)$ are integers divisible by $(-u_{n+1} - 1)!$.*

Proof Since the last coordinate of each $\mathbf{a}_k$ equals 1, the condition on the summation in (7.5) implies that

$$\sum_{k=1}^{N} l_k = u_{n+1},$$

i. e.,

$$(-l_i - 1) = (-u_{n+1} - 1) + \sum_{\substack{k=1 \\ k \neq i}}^{N} l_k.$$

Applying this formula to the coefficients of the series in (7.5) implies the lemma. □

Although the series $F_u^{(i)}(\Lambda)$ are more natural to consider, we replace them in what follows by some related series $G_u^{(i)}(\Lambda)$ that are more closely tied to the Dwork-Frobenius operator α^*.

For $i = 1, \ldots, M$, define

$$G^{(i)}(\Lambda, x) = \delta_- \left(F^{(i)}(\Lambda, x)\hat{\theta}_1(\Lambda, x) \right)$$

$$= \delta_- \left(q(\gamma_0 \Lambda_i x^{\mathbf{a}_i})\hat{\theta}_1(\Lambda_i x^{\mathbf{a}_i}) \left(\prod_{\substack{k=1 \\ k \neq i}}^{N} \hat{\theta}(\Lambda_k x^{\mathbf{a}_k}) \right) \right), \quad (7.11)$$

where the second equality follows from (4.1). If we write

$$G^{(i)}(\Lambda, x) = \sum_{u \in \mathscr{M}_-} G_u^{(i)}(\Lambda)\gamma_0^{u_{n+1}} x^u, \quad (7.12)$$

then by (3.8) and (7.4) one gets from the first equality of (7.11)

$$G_u^{(i)}(\Lambda) = \sum_{\substack{u^{(1)} \in \mathscr{M}_-, u^{(2)} \in \mathbb{N}A \\ u^{(1)} + u^{(2)} = u}} F_{u^{(1)}}^{(i)}(\Lambda)\hat{\theta}_{1,u^{(2)}}(\Lambda). \quad (7.13)$$

Note that as a series in Λ, $G_u^{(i)}(\Lambda)$ has exponents in $L_{i,u}$.

We have an analogue of Lemma 7.3 for these series.

Lemma 7.4 *For $i = 1, \ldots, M$ and $u \in \mathscr{M}_-$, $G_u^{(i)}(\Lambda)/(-u_{n+1} - 1)!$ has p-integral coefficients.*

Proof Using (3.9) we can write (7.13) in the form

$$G_u^{(i)}(\Lambda) = \sum_{\substack{u^{(1)} \in \mathscr{M}_-, u^{(2)} \in \mathbb{N}A \\ u^{(1)} + u^{(2)} = u}} \frac{F_{u^{(1)}}^{(i)}(\Lambda)}{(-u_{n+1}^{(1)} - 1)!} \sum_{\substack{l_1, \ldots, l_N \in \mathbb{N} \\ \sum_{k=1}^{N} l_k \mathbf{a}_k = u^{(2)}}} \left(\prod_{k=1}^{N} \hat{\theta}_{1,l_k} \right) \frac{(-u_{n+1}^{(1)} - 1)!}{\prod_{k=1}^{N} l_k!} \Lambda_1^{l_1} \cdots \Lambda_N^{l_N}.$$

$$(7.14)$$

The series $F_{u^{(1)}}^{(i)}(\Lambda)/(-u_{n+1}^{(1)} - 1)!$ has integral coefficients by Lemma 7.3. The condition on the first summation on the right-hand side of (7.14) implies that

$$(-u_{n+1} - 1) + u_{n+1}^{(2)} = -u_{n+1}^{(1)} - 1, \tag{7.15}$$

where $-u_{n+1}, -u_{n+1}^{(1)} \geq 1$ since $u, u^{(1)} \in \mathcal{M}_-$. Since the last coordinate of each $\mathbf{a}_k$ equals 1, the condition on the second summation on the right-hand side of (7.14) implies that

$$u_{n+1}^{(2)} = \sum_{k=1}^{N} l_k,$$

so by (7.15)

$$(-u_{n+1} - 1) + \sum_{k=1}^{N} l_k = -u_{n+1}^{(1)} - 1.$$

It follows that the ratio $(-u_{n+1}^{(1)} - 1)! / \prod_{k=1}^{N} l_k$ appearing in the second summation on the right-hand side of (7.14) is an integer divisible by $(-u_{n+1} - 1)!$. For each N-tuple $(l_1, \dots, l_N)$ appearing in the second summation on the right-hand side of (7.14) we have

$$\operatorname{ord} \prod_{k=1}^{N} \hat{\theta}_{1,l_k} \geq \frac{\sum_{k=1}^{N} l_k(p-1)}{p} = \frac{u_{n+1}^{(2)}(p-1)}{p} \tag{7.16}$$

by (3.6). This implies that the series on the right-hand side of (7.14) converges to a series in the Λ_k with p-integral coefficients that remain p-integral when divided by $(-u_{n+1} - 1)!$. □

We also have an analogue of Lemma 7.2.

Lemma 7.5 *For $i = 1, \dots, M$ and $u \in \mathcal{M}_-$ the series $G_u^{(i)}(\Lambda)$ lies in R_u.*

Proof The homogeneity condition is clear. Since the second summation on the right-hand side of (7.14) is finite, the sum

$$\frac{F_{u^{(1)}}^{(i)}(\Lambda)}{(-u_{n+1}^{(1)} - 1)!} \sum_{\substack{l_1, \dots, l_N \in \mathbb{N} \\ \sum_{k=1}^{N} l_k \mathbf{a}_k = u^{(2)}}} \left(\prod_{k=1}^{N} \hat{\theta}_{1,l_k} \right) \frac{(-u_{n+1}^{(1)} - 1)!}{\prod_{k=1}^{N} l_k!} \Lambda_1^{l_1} \cdots \Lambda_N^{l_N} \tag{7.17}$$

lies in $\tilde{R}$ by the proof of Lemma 7.2. Furthermore, it follows from (7.16) that the expression (7.17) has norm bounded by $p^{-u_{n+1}^{(2)}(p-1)/p}$. In the first summation on the right-hand side of (7.14), a given $u^{(2)} \in \mathbb{N}A$ can appear at most once, i. e, $u^{(2)} \to \infty$ in this summation. This implies that the first summation on the right-hand side of (7.14) converges in the norm on $\mathbb{B}$. □

We define a matrix $G(\Lambda) = \left[G_{ij}(\Lambda) \right]_{i,j=1}^{M}$ corresponding to $F(\Lambda)$ by the analogue of (7.10):

$$G_{ij}(\Lambda) = \Lambda_j G_{-\mathbf{a}_j}^{(i)}(\Lambda) = \mathbb{M}\big(G(\Lambda, x) \big). \tag{7.18}$$

Like the $F_{ij}(\Lambda)$, the monomials in the $G_{ij}(\Lambda)$ have exponents in L_i. Furthermore, the $G_{ii}(\Lambda)$ have constant term 1, while the $G_{ij}(\Lambda)$ for $i \neq j$ have no constant term. This implies that $\det G(\Lambda)$ is an invertible element of R, hence $G(\Lambda)^{-1}$ is a matrix with entries in R. Since the

series $G_{ij}(\Lambda)$ have integral coefficients (Lemma 7.4) and the $G_{ii}(\Lambda)$ have constant term 1, it follows that

$$\big|G(\Lambda)\big| = \big|\det G(\Lambda)\big| = 1. \tag{7.19}$$

This implies in particular that

$$|G(\Lambda)^{-1}| \leq 1. \tag{7.20}$$

We simplify our notation: for $u, u^{(1)} \in \mathcal{M}_-, u \neq u^{(1)}$, set

$$C_{u,u^{(1)}} = \sum_{\substack{l_1,\ldots,l_N \in \mathbb{N} \\ \sum_{k=1}^{N} l_k \mathbf{a}_k = u - u^{(1)}}} \left(\prod_{k=1}^{N} \hat{\theta}_{1,l_k} \right) \frac{(-u_{n+1}^{(1)} - 1)!}{\prod_{k=1}^{N} l_k!} \Lambda_1^{l_1} \cdots \Lambda_N^{l_N}.$$

Note that this is a finite sum, $C_{u,u^{(1)}}$ is *p*-integral, and ord $C_{u,u^{(1)}} > 0$ by (7.16). Equation (7.14) then simplifies to

$$G_u^{(i)}(\Lambda) = F_u^{(i)}(\Lambda) + \sum_{\substack{u^{(1)} \in \mathcal{M}_- \\ u^{(1)} \neq u}} C_{u,u^{(1)}} \frac{F_{u^{(1)}}^{(i)}(\Lambda)}{(-u_{n+1}^{(1)} - 1)!}. \tag{7.21}$$

Furthermore, the estimate (7.16) implies that $C_{u,u^{(1)}} \to 0$ as $u^{(1)} \to \infty$ in the sense that for any $\kappa > 0$, the estimate ord $C_{u,u^{(1)}} > \kappa$ holds for all but finitely many $u^{(1)}$.

We need the analogue of (7.21) with the roles of F and G reversed. It follows from (7.11) and (4.1) that for $i = 1, \ldots, M$

$$F^{(i)}(\Lambda, x) = \delta_- \big(G^{(i)}(\Lambda, x) \hat{\theta}_1(\Lambda, x)^{-1} \big). \tag{7.22}$$

This leads to the analogue of (7.14):

$$F_u^{(i)}(\Lambda) = \sum_{\substack{u^{(1)} \in \mathcal{M}_-, u^{(2)} \in \mathbb{N}A \\ u^{(1)} + u^{(2)} = u}} \frac{G_{u^{(1)}}^{(i)}(\Lambda)}{(-u_{n+1}^{(1)} - 1)!} \sum_{\substack{l_1,\ldots,l_N \in \mathbb{N} \\ \sum_{k=1}^{N} l_k \mathbf{a}_k = u^{(2)}}} \left(\prod_{k=1}^{N} \hat{\theta}'_{1,l_k} \right) \frac{(-u_{n+1}^{(1)} - 1)!}{\prod_{k=1}^{N} l_k!} \Lambda_1^{l_1} \cdots \Lambda_N^{l_N}, \tag{7.23}$$

where the $\hat{\theta}'_{1,l_k}$ are defined by (3.10) and Lemma 7.4 tells us that the

$$G_{u^{(1)}}^{(i)}(\Lambda)/(-u_{n+1}^{(1)} - 1)!$$

have *p*-integral coefficients. We define for $u, u^{(1)} \in \mathcal{M}_-, u \neq u^{(1)}$,

$$C'_{u,u^{(1)}} = \sum_{\substack{l_1,\ldots,l_N \in \mathbb{N} \\ \sum_{k=1}^{N} l_k \mathbf{a}_k = u - u^{(1)}}} \left(\prod_{k=1}^{N} \hat{\theta}'_{1,l_k} \right) \frac{(-u_{n+1}^{(1)} - 1)!}{\prod_{k=1}^{N} l_k!} \Lambda_1^{l_1} \cdots \Lambda_N^{l_N}.$$

Since the $\hat{\theta}'_{1,l_k}$ also satisfy the estimate (7.16) (see (3.11)), we get that $C'_{u,u^{(1)}}$ is *p*-integral and ord $C'_{u,u^{(1)}} > 0$. In addition, $C'_{u,u^{(1)}} \to 0$ as $u^{(1)} \to \infty$. Substitution into (7.23) now gives the desired formula:

$$F_u^{(i)}(\Lambda) = G_u^{(i)}(\Lambda) + \sum_{\substack{u^{(1)} \in \mathscr{M}_- \\ u^{(1)} \neq u}} C'_{u,u^{(1)}} \frac{G_{u^{(1)}}^{(i)}(\Lambda)}{(-u_{n+1}^{(1)} - 1)!} \tag{7.24}$$

For future reference, we record estimates that were used in the proof of (7.21) and (7.24).

Proposition 7.1 *For $u, u^{(1)} \in \mathscr{M}_-$, $u \neq u^{(1)}$, we have $\operatorname{ord} C_{u,u^{(1)}} > 0$ (resp. $\operatorname{ord} C'_{u,u^{(1)}} > 0$) and $C_{u,u^{(1)}} \to 0$ (resp. $C'_{u,u^{(1)}} \to 0$) as $u^{(1)} \to \infty$.*

8 Eigenvectors of α^*

We use the series of Sect. 7 to construct eigenvectors of α^*. These eigenvectors will lead to the fixed point of the contraction mapping of Sect. 6. We begin by recalling some results from [3].

By [3, Lemma 6.1] the product $\hat{\theta}_1(t)q(\gamma_0 t)$ is well-defined so we may set

$$Q(t) = \delta_- \big(\hat{\theta}_1(t)q(\gamma_0 t) \big) = \sum_{i=1}^{\infty} Q_i i! \gamma_0^{-i-1} t^{-i-1}, \tag{8.1}$$

where the Q_i are defined by the second equality. The proof of [3, Lemma 6.1] shows that the Q_i are p-integral. By [3, Proposition 6.10] we have

$$\delta_- \big(\theta(t)Q(t^p) \big) = pQ(t). \tag{8.2}$$

It follows from (8.2) that

$$\theta(t)Q(t^p) = pQ(t) + A(t)$$

for some series $A(t)$ in nonnegative powers of t. Fix $i, 1 \leq i \leq M$, and replace t in this equation by $\Lambda_i x^{\mathbf{a}_i}$:

$$\theta(\Lambda_i x^{\mathbf{a}_i})Q(\Lambda_i^p x^{p\mathbf{a}_i}) = A(\Lambda_i x^{\mathbf{a}_i}) + pQ(\Lambda_i x^{\mathbf{a}_i}).$$

Since $\delta_-(Q(\Lambda_i x^{\mathbf{a}_i})) = Q(\Lambda_i x^{\mathbf{a}_i})$ and $\delta_-(A(\Lambda_i x^{\mathbf{a}_i})) = 0$, we get

$$\delta_- \left(\theta(\Lambda_i x^{\mathbf{a}_i})Q(\Lambda_i^p x^{p\mathbf{a}_i}) \right) = pQ(\Lambda_i x^{\mathbf{a}_i}). \tag{8.3}$$

These series are related to the $G^{(i)}(\Lambda, x)$ defined in Sect. 7.

Lemma 8.1 *We have*

$$G^{(i)}(\Lambda, x) = \delta_- \left(Q(\Lambda_i x^{\mathbf{a}_i}) \prod_{\substack{j=1 \\ j \neq i}}^{N} \hat{\theta}(\Lambda_j x^{\mathbf{a}_j}) \right).$$

Proof Combining (7.11), (7.3), and (3.7) and using (4.1) we get

$$G^{(i)}(\Lambda, x) = \delta_- \left(q(\gamma_0 \Lambda_i x^{\mathbf{a}_i}) \prod_{\substack{j=1 \\ j \neq i}}^N \exp(\gamma_0 \Lambda_j x^{\mathbf{a}_j}) \prod_{j=1}^N \hat{\theta}_1(\Lambda_j x^{\mathbf{a}_j}) \right).$$

Using (3.3) and (3.5), this may be rewritten as

$$G^{(i)}(\Lambda, x) = \delta_- \left(q(\gamma_0 \Lambda_i x^{\mathbf{a}_i}) \hat{\theta}_1(\Lambda_i x^{\mathbf{a}_i}) \prod_{\substack{j=1 \\ j \neq i}}^N \hat{\theta}(\Lambda_j x^{\mathbf{a}_j}) \right).$$

The assertion of the lemma now follows from (8.1) and (4.1). □

The following result is a key step in the proof of Theorem 2.1.

Theorem 8.1 *For $i = 1, \dots, M$ we have*

$$\alpha^* \left(G^{(i)}(\Lambda, x) \right) = p G^{(i)}(\Lambda, x).$$

Proof Since $\theta(t) = \hat{\theta}(t)/\hat{\theta}(t^p)$ we have

$$\prod_{\substack{j=1 \\ j \neq i}}^N \theta(\Lambda_j x^{\mathbf{a}_j}) \prod_{\substack{j=1 \\ j \neq i}}^N \hat{\theta}(\Lambda_j^p x^{p\mathbf{a}_j}) = \prod_{\substack{j=1 \\ j \neq i}}^N \hat{\theta}(\Lambda_j x^{\mathbf{a}_j}). \tag{8.4}$$

We now compute

$$\alpha^* \left(G^{(i)}(\Lambda, x) \right) = \delta_- \left(\prod_{j=1}^N \theta(\Lambda_j x^{\mathbf{a}_j}) \delta_- \left(Q(\Lambda_i^p x^{p\mathbf{a}_i}) \prod_{\substack{j=1 \\ j \neq i}}^N \hat{\theta}(\Lambda_j^p x^{p\mathbf{a}_j}) \right) \right)$$

$$= \delta_- \left(\theta(\Lambda_i x^{\mathbf{a}_i}) Q(\Lambda_i^p x^{p\mathbf{a}_i}) \prod_{\substack{j=1 \\ j \neq i}}^N \theta(\Lambda_j x^{\mathbf{a}_j}) \prod_{\substack{j=1 \\ j \neq i}}^N \hat{\theta}(\Lambda_j^p x^{p\mathbf{a}_j}) \right)$$

$$= p \delta_- \left(Q(\Lambda_i x^{\mathbf{a}_i}) \prod_{\substack{j=1 \\ j \neq i}}^N \hat{\theta}(\Lambda_j x^{\mathbf{a}_j}) \right)$$

$$= p G^{(i)}(\Lambda, x),$$

where the first equality follows from Lemma 8.1, the second follows from (4.1) and rearranging the product, the third follows from (8.3) and (8.4), and the fourth follows from Lemma 8.1. Note that the rearrangement of the product occuring in the second equality is not completely trivial. One of the series contains negative powers of the x_i, the rest contain positive powers of the x_i, so they do not all lie in a common commutative ring. One has to write out both sides to verify that they are equal. (See Sect. 5 of: Adolphson, A., Sperber,

S.: On the integrality of hypergeometric series whose coefficients are factorial ratios, available at arXiv:2001.03283, for more details on this calculation.) □

The action of α^* was extended to S^M coordinatewise, so if we set

$$G(\Lambda, x) = \left(G^{(1)}(\Lambda, x), \ldots, G^{(M)}(\Lambda, x)\right) \in S^M, \tag{8.5}$$

then we get the following result.

Corollary 8.1 *We have* $\alpha^*\left(G(\Lambda, x)\right) = pG(\Lambda, x)$.

We now verify that $G(\Lambda)^{-1}G(\Lambda, x) \in T$. First of all,

$$\mathbb{M}\left(G(\Lambda)^{-1}G(\Lambda, x)\right) = I$$

by (7.18) and (5.5). This implies in particular that

$$|G(\Lambda)^{-1}G(\Lambda, x)| \geq 1.$$

But $|G(\Lambda, x)| \leq 1$ by Lemma 7.4 and $|G(\Lambda)^{-1}| \leq 1$ by (7.20), so

$$|G(\Lambda)^{-1}G(\Lambda, x)| \leq 1$$

also. We thus conclude that

$$|G(\Lambda)^{-1}G(\Lambda, x)| = 1. \tag{8.6}$$

Corollary 8.2 *The product* $G(\Lambda)^{-1}G(\Lambda, x)$ *is the unique fixed point of* ϕ, *hence lies in* T'.

Proof From the definition of α^* and Corollary 8.1 we have

$$\alpha^*\left(G(\Lambda)^{-1}G(\Lambda, x)\right) = G(\Lambda^p)^{-1}\alpha^*\left(G(\Lambda, x)\right) = G(\Lambda^p)^{-1}pG(\Lambda, x). \tag{8.7}$$

This implies by (7.18) and (5.4) that

$$\mathbb{M}\left(\alpha^*\left(G(\Lambda)^{-1}G(\Lambda, x)\right)\right) = pG(\Lambda^p)^{-1}G(\Lambda), \tag{8.8}$$

hence

$$\phi\left(G(\Lambda)^{-1}G(\Lambda, x)\right) = G(\Lambda)^{-1}G(\Lambda, x).$$

□

Corollary 8.3 *The entries of* $G(\Lambda^p)^{-1}G(\Lambda)$ *lie in* R'_0.

Proof Corollary 8.2 implies that $G(\Lambda)^{-1}G(\Lambda, x)$ lies in $(S')^M$, and since α^* is stable on $(S')^M$ we also have $\alpha^*\left(G(\Lambda)^{-1}G(\Lambda, x)\right) \in (S')^M$. The assertion of the corollary now follows from (8.7). □

Put $\mathscr{G}(\Lambda) = G(\Lambda^p)^{-1}G(\Lambda)$. We note one more consequence of Theorem 8.1.

Proposition 8.1 *The reciprocal eigenvalues of $\mathscr{G}(\lambda)$ are p-adic units for all λ in $\mathscr{D}$.*

Proof We showed earlier that $G(\Lambda)^{-1}G(\Lambda, x) \in T$, so $G(\Lambda)^{-1}G(\Lambda, x)$ satisfies the hypotheses of Lemma 5.1. By (8.6) and (5.7) we have

$$\left| \det \mathbb{M}\left(\alpha^*\left(G(\Lambda)^{-1}G(\Lambda, x) \right) \right) \right| = |p|$$

and by (8.6) and Corollary 5.1 we have

$$\left| \det \mathbb{M}\left(\alpha^*\left(G(\Lambda)^{-1}G(\Lambda, x) \right) \right)^{-1} \right| = |1/p|.$$

Combining these equalities with (8.8) gives

$$\left| \det \left(G(\Lambda^p)^{-1}G(\Lambda) \right) \right| = \left| \det \left(G(\Lambda)^{-1}G(\Lambda^p) \right) \right| = 1.$$

By (2.8), we then have

$$|\det \mathscr{G}(\lambda)| \leq 1 \text{ and } |\det \mathscr{G}(\lambda)^{-1}| \leq 1 \text{ for all } \lambda \in \mathscr{D},$$

which implies that $\det \mathscr{G}(\lambda)$ assumes unit values on $\mathscr{D}$. Since $\mathscr{G}(\lambda)$ assumes p-integral values on $\mathscr{D}$, it follows that the roots of $\det(I - t\mathscr{G}(\lambda))$ are p-adic units. $\qquad\square$

We introduce some additional notation that will be useful later. Put

$$\mathscr{G}(\Lambda, x) = \left(\mathscr{G}^{(1)}(\Lambda, x), \ldots, \mathscr{G}^{(M)}(\Lambda, x) \right) = G(\Lambda)^{-1}G(\Lambda, x). \tag{8.9}$$

This is an element of T' by Corollary 8.2, so each $\mathscr{G}^{(i)}(\Lambda, x)$ lies in S' and we may write

$$\mathscr{G}^{(i)}(\Lambda, x) = \sum_{u \in \mathscr{M}_-} \mathscr{G}^{(i)}_u(\Lambda)\gamma_0^{u_{n+1}}x^u \tag{8.10}$$

with $\mathscr{G}^{(i)}_u(\Lambda)$ in R'_u for all i, u.

9 Relation between $\mathscr{F}(\Lambda)$ and $\mathscr{G}(\Lambda)$

In this section we prove the first assertion of Theorem 2.1 and show that the second assertion of Theorem 2.1 is equivalent to the same statement with $\mathscr{F}(\Lambda)$ replaced by $\mathscr{G}(\Lambda)$ (see Proposition 9.2).

Put $F(\Lambda, x) = (F^{(1)}(\Lambda, x), \ldots, F^{(M)}(\Lambda, x))$, where $F^{(i)}(\Lambda, x)$ is given by (7.3). We regard $F(\Lambda, x)$ and $G(\Lambda, x)$ (defined in (8.5)) as column vectors so they can be multiplied on the left by $(M \times M)$-matrices. The following result is a consequence of the fact that $G(\Lambda)^{-1}G(\Lambda, x)$ lies in $(S')^M$ (Corollary 8.2).

Proposition 9.1 *The vectors $F(\Lambda)^{-1}F(\Lambda, x)$, $G(\Lambda)^{-1}F(\Lambda, x)$, and $F(\Lambda)^{-1}G(\Lambda, x)$ lie in $(S')^M$.*

Proof We begin by showing that $G(\Lambda)^{-1}F(\Lambda, x) \in (S')^M$. Writing Equation (7.24) for $i = 1, \ldots, M$ and multiplying by $G(\Lambda)^{-1}$ gives the vector equation

$$
G(\Lambda)^{-1} \begin{bmatrix} F_u^{(1)}(\Lambda) \\ \vdots \\ F_u^{(M)}(\Lambda) \end{bmatrix}
$$

$$
= G(\Lambda)^{-1} \begin{bmatrix} G_u^{(1)}(\Lambda) \\ \vdots \\ G_u^{(M)}(\Lambda) \end{bmatrix} + \sum_{\substack{u^{(1)} \in \mathcal{M}_- \\ u^{(1)} \neq u}} \frac{C'_{u,u^{(1)}}}{(-u_{n+1}^{(1)} - 1)!} G(\Lambda)^{-1} \begin{bmatrix} G_{u^{(1)}}^{(1)}(\Lambda) \\ \vdots \\ G_{u^{(1)}}^{(M)}(\Lambda) \end{bmatrix}. \tag{9.1}
$$

We have $|G(\Lambda)^{-1}| \leq 1$ by (7.20). Lemma 7.4, Proposition 7.1, and Corollary 8.2 imply that the sum on the right-hand side of (9.1) converges to an element of $(R'_u)^M$. Since $u \in \mathcal{M}_-$ was arbitrary, this shows that $G(\Lambda)^{-1} F(\Lambda, x) \in (S')^M$.

Take successively $u = -\mathbf{a}_j$, $j = 1, \ldots, M$ in (9.1) and multiply the j-th equation by Λ_j. We combine the column vectors in the resulting M equations into matrices and apply (7.10) and (7.18). The resulting matrix equation is

$$
G(\Lambda)^{-1} F(\Lambda) = I + H(\Lambda), \tag{9.2}
$$

where we have written $H(\Lambda)$ for the $(M \times M)$-matrix whose j-th column is

$$
\sum_{\substack{u^{(1)} \in \mathcal{M}_- \\ u^{(1)} \neq -\mathbf{a}_j}} \frac{C'_{-\mathbf{a}_j, u^{(1)}} \Lambda_j}{(-u_{n+1}^{(1)} - 1)!} G(\Lambda)^{-1} \begin{bmatrix} G_{u^{(1)}}^{(1)}(\Lambda) \\ \vdots \\ G_{u^{(1)}}^{(M)}(\Lambda) \end{bmatrix}.
$$

The entries in the matrix $H(\Lambda)$ all lie in R'_0 by Corollary 8.2 and have norm < 1 by (7.20), Lemma 7.4, and Proposition 7.1. We can thus apply the usual geometric series formula to invert the right-hand side of (9.2). This proves that $F(\Lambda)^{-1} G(\Lambda)$ is a matrix with entries in R'_0, all entries having norm ≤ 1. Since the left-hand side of (9.1) lies in $(R'_u)^M$, we can now multiply it by $F(\Lambda)^{-1} G(\Lambda)$ to conclude that

$$
F(\Lambda)^{-1} \begin{bmatrix} F_u^{(1)}(\Lambda) \\ \vdots \\ F_u^{(M)}(\Lambda) \end{bmatrix} \in (R'_u)^M \quad \text{for all } u \in \mathcal{M}_-. \tag{9.3}
$$

This shows that $F(\Lambda)^{-1} F(\Lambda, x)$ lies in $(S')^M$.

The remaining assertion, that $F(\Lambda)^{-1} G(\Lambda, x)$ lies in $(S')^M$, can be proved similarly by reversing the roles of F and G and using (7.21) in place of (7.24). Since that result is not needed in what follows, we omit the details. $\square$

Proposition 9.2 *The matrix $\mathscr{F}(\Lambda)$ has entries in R'_0. For any $\lambda \in (\mathbb{F}_q^\times)^M \times \mathbb{F}_q^{N-M}$ with $\bar{D}(\lambda) \neq 0$ we have*

$$
\det \left(I - t\mathscr{F}(\hat{\lambda}^{p^{a-1}}) \mathscr{F}(\hat{\lambda}^{p^{a-2}}) \cdots \mathscr{F}(\hat{\lambda}) \right) = \det \left(I - t\mathscr{G}(\hat{\lambda}^{p^{a-1}}) \mathscr{G}(\hat{\lambda}^{p^{a-2}}) \cdots \mathscr{G}(\hat{\lambda}) \right)
$$

where $\hat{\lambda} \in \mathbb{Q}_p(\zeta_{q-1})^N$ denotes the Teichmüller lifting of λ.

Proof We showed in the proof of Proposition 9.1 that $\mathscr{H}(\Lambda) := F(\Lambda)^{-1} G(\Lambda)$ and its inverse $\mathscr{H}(\Lambda)^{-1} = G(\Lambda)^{-1} F(\Lambda)$ have entries in R'_0. We then have

$$\begin{aligned}\mathscr{F}(\Lambda) &= F(\Lambda^p)^{-1}F(\Lambda) = \mathscr{H}(\Lambda^p)\big(G(\Lambda^p)^{-1}G(\Lambda)\big)\mathscr{H}(\Lambda)^{-1} \\ &= \mathscr{H}(\Lambda^p)\mathscr{G}(\Lambda)\mathscr{H}(\Lambda)^{-1},\end{aligned}$$ (9.4)

which implies the first assertion of the proposition since $\mathscr{G}(\Lambda)$ has entries in R'_0 by Corollary 8.3. Equation (9.4) implies

$$\mathscr{H}(\Lambda^{p^a})^{-1}\mathscr{F}(\Lambda^{p^{a-1}})\mathscr{F}(\Lambda^{p^{a-2}})\cdots\mathscr{F}(\Lambda)\mathscr{H}(\Lambda) = \mathscr{G}(\Lambda^{p^{a-1}})\mathscr{G}(\Lambda^{p^{a-2}})\cdots\mathscr{G}(\Lambda).$$ (9.5)

The Teichmüller lifting $\hat{\lambda}$ satisfies $\hat{\lambda}^{p^a} = \hat{\lambda}$ and $\mathscr{H}(\Lambda)$ is a function on $\mathscr{D}$, so the second assertion of the proposition follows by evaluating (9.5) at $\Lambda = \hat{\lambda}$. □

The first assertion of Proposition 9.2 implies the first assertion of Theorem 2.1. The second assertion of Proposition 9.2 shows that the second assertion of Theorem 2.1 is equivalent to the following statement.

Theorem 9.1 *Let* $\lambda \in (\mathbb{F}_q^\times)^M \times \mathbb{F}_q^{N-M}$ *and let* $\hat{\lambda} \in \mathbb{Q}_p(\zeta_{q-1})^N$ *be its Teichmüller lifting. If* $\bar{D}(\lambda) \neq 0$, *then* $\hat{\lambda}^{p^i} \in \mathscr{D}$ *for* $i = 0, \ldots, a-1$ *and*

$$\rho(\lambda, t) = \det\big(I - t\mathscr{G}(\hat{\lambda}^{p^{a-1}})\mathscr{G}(\hat{\lambda}^{p^{a-2}})\cdots\mathscr{G}(\hat{\lambda})\big).$$

10 Proof of Theorem 9.1

We apply Dwork's *p*-adic cohomology theory to prove Theorem 9.1.

We begin by recalling the formula for the rational function $P_\lambda(t)$ that was proved in [3]. For a subset $I \subseteq \{0, 1, \ldots, n\}$, let

$$L^I = \bigg\{ \sum_{u \in \mathbb{N}^{n+2}} c_u \gamma_0^{pu_{n+1}} x^u \mid \sum_{i=0}^n u_i = du_{n+1}, u_i > 0 \text{ for } i \in I, c_u \in \mathbb{C}_p,$$
$$\text{and } \{c_u\} \text{ is bounded} \bigg\}.$$

Put $g_\lambda = x_{n+1}f_\lambda$, so that

$$g_\lambda(x_0, \ldots, x_{n+1}) = \sum_{k=1}^N \lambda_k x^{\mathbf{a}_k} \in \mathbb{F}_q[x_0, \ldots, x_{n+1}],$$

and let $\hat{g}_\lambda$ be its Teichmüller lifting:

$$\hat{g}_\lambda(x_0, \ldots, x_{n+1}) = \sum_{k=1}^N \hat{\lambda}_k x^{\mathbf{a}_k} \in \mathbb{Q}(\zeta_{q-1})[x_0, \ldots, x_{n+1}].$$

From (3.15) we have

$$\theta(\hat{\lambda}, x) = \prod_{j=1}^N \theta(\hat{\lambda}_j x^{\mathbf{a}_j}).$$ (10.1)

We also need the series $\theta_0(\hat{\lambda}, x)$ defined by

$$\theta_0(\hat{\lambda}, x) = \prod_{i=0}^{a-1} \prod_{j=1}^{N} \theta\big((\hat{\lambda}_j x^{\mathbf{a}_j})^{p^i}\big) = \prod_{i=0}^{a-1} \theta(\hat{\lambda}^{p^i}, x^{p^i}). \tag{10.2}$$

Define an operator ψ on formal power series by

$$\psi\bigg(\sum_{u \in \mathbb{N}^{n+2}} c_u x^u\bigg) = \sum_{u \in \mathbb{N}^{n+2}} c_{pu} x^u. \tag{10.3}$$

Denote by $\alpha_{\hat{\lambda}}$ the composition

$$\alpha_{\hat{\lambda}} := \psi^a \circ \text{``multiplication by } \theta_0(\hat{\lambda}, x).\varepsilon$$

The map $\alpha_{\hat{\lambda}}$ is stable on L^I for all I and we have ([3, Equation (7.12)])

$$P_\lambda(qt) = \prod_{I \subseteq \{0,1,\dots,n\}} \det(I - q^{n+1-|I|} t \alpha_{\hat{\lambda}} \mid L_0^I)^{(-1)^{n+1+|I|}}. \tag{10.4}$$

For notational convenience, put $\Gamma = \{0, 1, \dots, n\}$. The following lemma is an immediate consequence of [3, Proposition 7.13(b)] (note that since we are assuming $d \geq n + 1$, we have $\mu = 0$ in that proposition).

Lemma 10.1 *The unit reciprocal roots of $P_\lambda(t)$ are obtained from the reciprocal roots of* $\det(I - t\alpha_{\hat{\lambda}} \mid L_0^\Gamma)$ *of q-ordinal equal to 1 by division by q.*

We give an alternate description of $\det(I - t\alpha_{\hat{\lambda}} \mid L_0^\Gamma)$ ([3, Sect. 7]). Set

$$B = \bigg\{ \xi^* = \sum_{u \in (\mathbb{Z}_{<0})^{n+2}} c_u^* \gamma_0^{p u_{n+1}} x^u \mid c_u^* \to 0 \text{ as } u \to -\infty \bigg\},$$

a p-adic Banach space with norm $|\xi^*| = \sup_{u \in (\mathbb{Z}_{<0})^{n+2}} \{|c_u^*|\}$. Define a map Φ on formal power series by

$$\Phi\bigg(\sum_{u \in \mathbb{Z}^n} c_u x^u\bigg) = \sum_{u \in \mathbb{Z}^n} c_u x^{pu}.$$

Consider the formal composition $\alpha_{\hat{\lambda}}^* = \delta_- \circ \theta_0(\hat{\lambda}, x) \circ \Phi^a$. The following result is [3, Proposition 7.30].

Proposition 10.1 *The operator $\alpha_{\hat{\lambda}}^*$ is an endomorphism of B which is adjoint to* $\alpha_{\hat{\lambda}} : L_0^\Gamma \to L_0^\Gamma$.

From Proposition 10.1, it follows by Serre [12, Proposition 15] that

$$\det(I - t\alpha_{\hat{\lambda}} \mid L_0^\Gamma) = \det(I - t\alpha_{\hat{\lambda}}^* \mid B), \tag{10.5}$$

so Lemma 10.1 gives the following result.

Corollary 10.1 *The unit reciprocal roots of $P_\lambda(t)$ are obtained from the reciprocal roots of* $\det(I - t\alpha_{\hat{\lambda}}^* \mid B)$ *of q-ordinal equal to 1 by division by q.*

We saw in Sect. 1 that $P_\lambda(t)$ has exactly M unit roots when $\lambda \in (\mathbb{F}_q^\times)^M \times \mathbb{F}_q^{N-M}$ and $\bar{D}(\lambda) \neq 0$. By Proposition 8.1, the eigenvalues of the $(M \times M)$-matrix $\mathscr{G}(\lambda)$ are units for all $\lambda \in \mathscr{D}$. So to prove Theorem 9.1, it suffices to establish the following result.

Proposition 10.2 *Let* $\lambda \in (\mathbb{F}_q^\times)^M \times \mathbb{F}_q^{N-M}$ *and let* $\hat{\lambda} \in \mathbb{Q}_p(\zeta_{q-1})^N$ *be its Teichmüller lifting. Assume that* $\bar{D}(\lambda) \neq 0$. *Then* $\hat{\lambda}^{p^i} \in \mathscr{D}$ *for* $i = 0, \ldots, a - 1$ *and*

$$\det\left(I - qt\mathscr{G}(\hat{\lambda}^{p^{a-1}})\mathscr{G}(\hat{\lambda}^{p^{a-1}}) \cdots \mathscr{G}(\hat{\lambda})\right)$$

is a factor of $\det(I - t\alpha_{\hat{\lambda}}^* \mid B)$.

Proof Using the notation of (8.9), we have by (8.7)

$$\alpha^*\left(\mathscr{G}(\Lambda, x)\right) = p\mathscr{G}(\Lambda)\mathscr{G}(\Lambda, x). \tag{10.6}$$

Iterating this gives for all $m \geq 0$

$$(\alpha^*)^m\left(\mathscr{G}(\Lambda, x)\right) = p^m\mathscr{G}(\Lambda^{p^{m-1}})\mathscr{G}(\Lambda^{p^{m-2}}) \cdots \mathscr{G}(\Lambda)\mathscr{G}(\Lambda, x). \tag{10.7}$$

Evaluate $\mathscr{G}(\Lambda, x)$ at $\Lambda = \hat{\lambda}$:

$$\mathscr{G}(\hat{\lambda}, x) = \left(\mathscr{G}^{(1)}(\hat{\lambda}, x), \cdots, \mathscr{G}^{(M)}(\hat{\lambda}, x)\right),$$

where by (8.10)

$$\begin{aligned}
\mathscr{G}^{(i)}(\hat{\lambda}, x) &= \sum_{u \in \mathscr{M}_-} \mathscr{G}_u^{(i)}(\hat{\lambda})\gamma_0^{u_{n+1}} x^u \\
&= \sum_{u \in \mathscr{M}_-} \left(\gamma_0^{-(p-1)u_{n+1}}\mathscr{G}_u^{(i)}(\hat{\lambda})\right)\gamma_0^{pu_{n+1}} x^u.
\end{aligned}$$

Since $\gamma_0^{-(p-1)u_{n+1}} \to 0$ as $u \to \infty$, this expression lies in B. One checks that the specialization of the left-hand side of (10.7) with $m = a$ at $\Lambda = \hat{\lambda}$ is $\alpha_{\hat{\lambda}}^*(\mathscr{G}(\hat{\lambda}, x))$, so specializing (10.7) with $m = a$ and $\Lambda = \hat{\lambda}$ gives

$$\alpha_{\hat{\lambda}}^*\left(\mathscr{G}(\hat{\lambda}, x)\right) = q\mathscr{G}(\hat{\lambda}^{p^{a-1}})\mathscr{G}(\hat{\lambda}^{p^{a-2}}) \cdots \mathscr{G}(\hat{\lambda})\mathscr{G}(\hat{\lambda}, x). \tag{10.8}$$

We have proved that $\mathscr{G}(\hat{\lambda}, x)$ is a vector of M elements of B and that the action of $\alpha_{\hat{\lambda}}^*$ on these M elements is represented by the matrix

$$q\mathscr{G}(\hat{\lambda}^{p^{a-1}})\mathscr{G}(\hat{\lambda}^{p^{a-2}}) \cdots \mathscr{G}(\hat{\lambda}).$$

This implies the assertion of the proposition. $\qquad\square$

Declarations

Conflict of interest The authors declare that they have no conflicts of interest or competing interests.

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