Abh. Math. Semin. Univ. Hambg. (2021) 91:287–296
https://doi.org/10.1007/s12188-021-00242-2

A geometric splitting theorem for actions of semisimple Lie groups

José Rosales-Ortega[1]

Received: 4 December 2020 / Published online: 7 June 2021

Abstract
Let M be a compact connected smooth pseudo-Riemannian manifold that admits a topologically transitive G-action by isometries, where $G = G_1 \dots G_l$ is a connected semisimple Lie group without compact factors whose Lie algebra is $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$. If m_0, n_0, n_0^i are the dimensions of the maximal lightlike subspaces tangent to M, G, G_i, respectively, then we study G-actions that satisfy the condition $m_0 = n_0^1 + \cdots + n_0^l$. This condition implies that the orbits are non-degenerate for the pseudo Riemannian metric on M and this allows us to consider the normal bundle to the orbits. Using the properties of the normal bundle to the G-orbits we obtain an isometric splitting of M by considering natural metrics on each G_i.

Keywords Bi-invariant metric · Pseudo-Riemannian · Semisimple Lie group · Topologically transitive action

Mathematics Subject Classification 53C05 · 53C10

1 Introduction

A fundamental problem in geometry is to understand the actions of a noncompact connected semisimple Lie group G on pseudo-Riemannian manifolds. This is particularly interesting when one of these G-actions preserves a geometric structure on a manifold M.

An example to consider is Gromov's centralizer theorem which proves that for a noncompact semisimple Lie group G acting analytically on a manifold M preserving a finite volume and either a connection or a geometric structure of finite type, there is a nontrivial space of globally defined Killing vector fields on the universal cover $\widetilde{M}$ that centralizes the action of G (see [16] for more about the Gromov-Zimmer machinery).

It was proved in [2] that for a compact pseudo-Riemannian manifold M and a connected noncompact simple Lie group G acting on M by isometries, then some covering

Communicated by Vicente Cortés.

✉ José Rosales-Ortega
jose.rosales@ucr.ac.cr

[1] School of Mathematics, UCR-ITCR, San José, Costa Rica

$\widetilde{M}$ splits as a product of a set S and G. In this paper we improve that result using our own work (see for example [14–16]) and some ideas established in [1, 12, 17].

It is well known that interesting examples of pseudo-Riemannian manifolds appear when we consider Lie groups with bi-invariant pseudo-Riemannian metrics. In addition, this class of Lie groups that support bi-invariant pseudo-Riemannian metrics is quite large. In [6], Milnor discussed the problem of determining the real simple Lie groups admitting left invariant Riemannian metrics. Using results established in [3] (see also [10]), about simple real Lie algebras $\mathfrak{g}$, in terms of their complexifications $\mathfrak{g}^{\mathbb{C}}$, we found in [14] a classification of the bi-invariant pseudo-Riemannian metric on G, a Lie group with real semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$.

If the G-action is topologically transitive then the G-orbits have the same dimension, and because of that the G-orbits on M define a smooth foliation $\mathcal{O}$, whose tangent bundle $T(\mathcal{O})$ can be trivialized by the G action. In [15] we proved that the normal bundle $T\mathcal{O}^{\perp}$ is integrable, and from now on we call $\mathfrak{F}$ the foliation associated with the normal bundle. Furthermore, we prove that the leaves of the foliation $\mathfrak{F}$ are totally geodesic.

Under certain conditions it can be proven that a manifold M is isometric to a product of two manifolds, and this is known as a splitting theorem, see [11]. Our main goal in this paper is to prove an isometric splitting theorem for the metric of the pseudo-Riemannian manifold M on which a semisimple Lie group G acts.

The organization of this article is as follows. In Sect. 2 we collect some basic results about bi-invariant pseudo-Riemannian metrics on a Lie group G. Also we give the classification of the $\mathrm{Ad}\,(G)$-invariant bilinear forms on a semisimple Lie algebra. This is mentioned in [2], but the generalization to semisimple Lie groups is new, also see [14]. As a consequence we give the classification of the bi-invariant pseudo-Riemannian metrics on G. In Sect. 3 we give some results about the foliation given by the normal bundle of the tangent space of the G-orbits. In Sect. 4 we show the main result of this work.

I would like to thank Michael Josephy for useful comments that allowed us to simplify the exposition of this work.

2 Bi-invariant metrics on a semi-simple Lie group

We are going to study the geometry of the orbits and the normal bundle in the case where a semi-simple Lie group G acts on a pseudo-Riemannian manifold M, with the aim of obtaining a description of the pseudo-Riemannian manifold on which the Lie group acts.

The next lemma, although trivial, will be of great importance in the subsequent results, and therefore we present it for the sake of completeness of the work. The proof is essentially contained in [9, Lemma 3, Chapter 11].

Lemma 1 *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $F : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a symmetric bilinear form such that $F([X,Y],Z) = -F(Y,[X,Z])$ for all $X, Y, Z \in \mathfrak{g}$. Then F is $\mathrm{Ad}(G)$-invariant.*

The classification of the $\mathrm{Ad}(G)$-invariant bilinear forms on a simple Lie algebra can be found in [14]. The next result gives the classification of the $\mathrm{Ad}(G)$-invariant bilinear forms on a real semisimple Lie algebra.

Theorem 1 *Let G be a connected semisimple Lie group such that* $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$, *where each $\mathfrak{g}_i$ is a simple ideal of the Lie algebra $\mathfrak{g}$. We shall suppose the following*:

- *The complexification of each $\mathfrak{g}_i$, for $i = 1, \dots, k$ is simple; and*
- *The complexification of each $\mathfrak{g}_i$, for $i = k+1, \dots, l$ is not simple and so there exists a complex structure J_i for each $\mathfrak{g}_i$.*

Then every $\mathrm{Ad}(G)$-*invariant bilinear form B on $\mathfrak{g}$ is given by*

$$B = \lambda_1 B_{\mathfrak{g}_1} + \cdots + \lambda_k B_{\mathfrak{g}_k} + \left(\mu_1^{k+1} B_{\mathfrak{g}_{k+1}} + \mu_2^{k+1} B_{\mathfrak{g}_{k+1}}^{J_{k+1}}\right) + \cdots + \left(\mu_1^{l} B_{\mathfrak{g}_l} + \mu_2^{l} B_{\mathfrak{g}_l}^{J_l}\right),$$

where each $B_{\mathfrak{g}_i}$ is the Killing–Cartan form on $\mathfrak{g}_i$, for $i = 1, \dots, l$, all λ_i and μ_i^j are real numbers, and $B_{\mathfrak{g}_j}^{J_i}(X, Y) = B_{\mathfrak{g}_j}(X, J_i Y)$.

Proof By the previous lemma it follows that $B(X, [Y, Z]) = B([X, Y], Z)$, for all $X, Y, Z \in \mathfrak{g}$.

On the other hand, it is easy to show the following properties: $[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$ for all $i \neq j$, and $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$, for all i.

We have for $Y \in \mathfrak{g}_j$ there exists $Z, W \in \mathfrak{g}_j$ such that $Y = [Z, W]$, and for $X \in \mathfrak{g}_i$ we conclude $B(X, Y) = B([X, Z], W) = 0$.

Therefore $\mathfrak{g}_i \perp \mathfrak{g}_j$ for all $i \neq j$ with respect to B. From this it follows that $B = B_{\mathfrak{g}_1} \oplus \cdots \oplus B_{\mathfrak{g}_l}$.

Now we use the classification of $\mathrm{Ad}(G)$-invariants bilinear forms on a simple Lie algebra given in [14]. □

Using [9, Proposition 9, Chapter 11] and the previous results we can give a classification of the bi-invariant metrics on semisimple Lie groups.

Theorem 2 *Let G and M be as in Theorem 1. Then every bi-invariant pseudo-Riemannian metric ϕ on G is given by*

$$\phi = \lambda_1 B_{\mathfrak{g}_1} + \cdots + \lambda_k B_{\mathfrak{g}_k} + \left(\mu_1^{k+1} B_{\mathfrak{g}_{k+1}} + \mu_2^{k+1} B_{\mathfrak{g}_{k+1}}^{J_{k+1}}\right) + \cdots + \left(\mu_1^{l} B_{\mathfrak{g}_l} + \mu_2^{l} B_{\mathfrak{g}_l}^{J_l}\right),$$

where each $B_{\mathfrak{g}_i}$ is the Killing–Cartan form on $\mathfrak{g}_i$, for $i = 1, \dots, l$, all λ_i and μ_i^j are real numbers, and $B_{\mathfrak{g}_j}^{J_i}(X, Y) = B_{\mathfrak{g}_j}(X, J_i Y)$.

3 Properties of the foliation given by a *G*-action

From now on $G = G_1 \dots G_l$ will be a connected semisimple Lie group without compact factors and with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$.

We shall suppose the following:

- The complexification of each $\mathfrak{g}_i$, for $i = 1, \dots, k$ is simple; and

- The complexification of each $\mathfrak{g}_i$, for $i = k+1, \dots, l$ is not simple and so there exists a complex structure J_i for each $\mathfrak{g}_i$.

The above assumptions are possible due to [3, Proposition 1.5, Chapter X].

In this work M will denote a compact pseudo-Riemannian manifold, unless otherwise stated.

We always assume that the action of G on M is analytic, faithful, and preserves the pseudo-Riemannian metric.

Definition 1 The dimension of maximal lightlike tangent subspaces for M will be denoted by $m_0 = \min\{m_1, m_2\}$, where (m_1, m_2) represents the signature of M, i.e., that m_1 corresponds to the dimension of maximal timelike tangent subspaces and m_2 corresponds to the dimension of the maximal spacelike tangent subspaces.

The group G itself can be considered as a pseudo-Riemannian manifold. In fact, Gromov remarked in [2] that if (n_1, n_2) is the signature of the metric given by the Killing-Cartan form on $\mathfrak{g}$, then any other bi-invariant pseudo-Riemannian metric on G has signature given by either (n_1, n_2) or (n_2, n_1). We extended that context in the previous section when we proved that any bi-invariant pseudo-Riemannian metric on G can be described in terms of the Killing-Cartan form.

Definition 2 The dimension of maximal lightlike tangent subspaces for G_i, $i = 1, \dots, l$, will be denoted by $n_0^i = \min\{n_1^i, n_2^i\}$, where (n_1^i, n_2^i) represents the signature of G_i.

We are interested in comparing the numbers m_0 and $n_0^1 + \cdots + n_0^l$. We obtain a geometric property of the G-orbits on M when the condition $n_0^1 + \cdots + n_0^l = m_0$ is satisfied.

When the G-action is locally free on M we obtain a foliation $\mathcal{O}$ of M whose leaves are the orbits $G \cdot p$ of the action, where $p \in M$. The results established in [17] guarantee that if the action of G on M is topologically transitive then the G-action is locally free on M.

We will denote by $T\mathcal{O}$ the tangent bundle to the orbits of the G-action on M. If $X \in \mathfrak{g}$, we define the infinitesimal generator X^* as the vector field on M induced by X. This new vector field is given by

$$X_p^* = \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot p.$$

It is clear that X^* is a Killing vector field, $X_x^* \in T_x(G \cdot x)$, for $x \in M$, and the following relation holds for every $X, Y \in \mathfrak{g}$: $[X^*, Y^*] = -[X, Y]^*$, see [5, Proposition 4.1, Chapter I]. Furthermore, since the G-action is locally free on M, the condition $X_x^* = 0$ for some $x \in M$ implies $X = 0$.

We will use the following map $\varphi_x : \mathfrak{g} \to T_x(Gx)$, given by $\varphi_x(X) = X_x^*$, where $x \in M$. We refer to [12] for further details about this map.

Theorem 3 *Suppose G is a connected semisimple Lie group without compact factors acting topologically transitively, i.e. there is a dense G-orbit, on a pseudo-Riemannian manifold M preserving its pseudo-Riemannian metric. If $n_0^1 + \cdots + n_0^l = m_0$, then the G-orbits are nondegenerate with respect to the metric on M.*

Proof We observe that the G-action on M is everywhere locally free by the results in [17] and so the G-orbits define a smooth foliation $\mathcal{O}$ on M.

The condition for G-orbits to be nondegenerate means that there is a G-invariant open subset U of M so that the G-orbit of every point in U is nondegenerate.

We will prove that for a G-invariant open subset U of M the G-orbit of every point in U is nondegenerate.

We consider the well known smooth map $\Psi : M \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ given by $\Psi_x = h_x(\varphi_x \cdot, \varphi_x \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, for h the metric on M.

This map is G-equivariant and the G-action is tame on $\mathfrak{g} \times \mathfrak{g}$, then using results established in [18] we conclude that this map is constant on the support of almost every ergodic component of M. If N is the support of one such ergodic component, then there is an $\mathrm{Ad}(G)$-invariant bilinear form B_N on $\mathfrak{g}$ such that the metric induced by M on $T_N(\mathcal{O})$ is almost everywhere given by B_N on each fiber.

Using Lemma 1, with B_N, it is easy to see that its kernel, K, is an ideal of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$.

If $K = \mathfrak{g}$, then the tangent bundle restricted to N: $T_N(\mathcal{O})$ is lightlike which implies $\dim(\mathfrak{g}) \le m_0$. On the other hand, for each $i = 1 \dots l$ it follows that $\dim \mathfrak{g}_i \ge n_0^i$ and $\dim \mathfrak{g} = \dim \mathfrak{g}_1 + \cdots + \dim \mathfrak{g}_l$, and so $n_0^1 + \cdots + n_0^l < m_0$. But this contradicts the condition $n_0^1 + \cdots + n_0^l = m_0$. If $K = \bigoplus_{i \in J} \mathfrak{g}_j$, where $J \subset \{1, \dots, l\}$, then we have a subspace of null vectors in the tangent bundle to the G-orbits which has a dimension equal to $\dim \bigoplus_{j \in J} \mathfrak{g}_j + n_0^{j_1} + \cdots + n_0^{j_s}$. We conclude as before that $m_0 \ge \dim \bigoplus_{j \in J} \mathfrak{g}_j + n_0^{j_1} + \cdots + n_0^{j_s} > n_0^1 + \cdots n_0^l$. If K is trivial then B_N is nondegenerate, and so almost every G-orbit contained in N is nondegenerate. In particular, the set U is conull and so not empty, because almost every G-orbit is nondegenerate.

There is a G-orbit $\mathcal{O}_0$ which is dense in M and so $\mathcal{O}_0 \cap U \neq \emptyset$. Using the G-invariancy of U, it is easy to show that $\mathcal{O}_0 \subset U$. The above shows that $\Psi(\mathcal{O}_0) = B_0$, where B_0 is the nondegenerate bilinear form on $\mathfrak{g}$ obtained when the metric on M is restricted to $\mathcal{O}_0$. Using the continuity of Ψ and the density of $\mathcal{O}_0$ it follows that $\Psi(M) = B_0$. We conclude that all G-orbits are nondegenerate. ☐

We will consider the so-called smooth normal bundle $T\mathcal{O}^{\perp}$. In a previous paper, [15], we proved that the foliation on M associated with this normal bundle is integrable.

It is well known that a transverse Riemannian structure defines a Riemannian metric on the quotient bundle. This metric remains invariant when we move along the leaves on the manifold. Considering suitable Riemannian metrics on the manifold that carry the foliation, we can construct transverse Riemannian structures for a foliation. These suitable metrics are the bundle-like metrics. This is a well known concept whose further discussion can be found in [7].

Theorem 4 *Suppose G is a connected semisimple Lie group without compact factors acting topologically transitively on a manifold M preserving its pseudo-Riemannian metric and satisfying $n_0^1 + \cdots + n_0^l = m_0$. Then the foliation $\mathfrak{F}$ on M associated to $T\mathcal{O}^{\perp}$ is totally geodesic.*

Proof First, the normal bundle $T\mathcal{O}^{\perp}$ is integrable. By Frobenius's theorem there exists an induced foliation $\mathfrak{F}$ on M.

We show that its leaves are totally geodesic submanifolds of M, i.e, the second fundamental form II of the leaves of $\mathfrak{F}$ is equal to zero.

By h we will denote the metric on M preserved by G. If $X \in \mathfrak{g}$, we define X^*, the infinitesimal generator, as the vector field on M induced by X. If Y, Z are horizontal vector fields or local sections of $T\mathcal{O}^\perp$ that preserve the foliation $\mathfrak{F}$, then $[X^*, Y]$ and $[X^*, Z]$ are vertical vector fields or local sections of $T\mathcal{O}$. Hence, $X^*(h(Y,Z)) = h([X^*, Y], Z) + h(Y, [X^*, Z]) = 0$
.

Now note that the function $h(Y, Z)$ is constant along the G-orbits because $X^*(h(Y,Z)) = 0$. We conclude that the metric h is a bundle-like metric for the foliation $\mathcal{O}$ whose leaves are the G orbits. See [7] for further details about bundle-like metrics.

By results in [7] we have a transverse metric to the foliation $\mathcal{O}$ from h, and at every point of M we can also obtain a pseudo-Riemannian submersion $\pi : U \to B$, where U is a open set in M, such that the fibers of π define the foliation $\mathcal{O}$ restricted to U.

Let A be the associated fundamental tensor defined in [8]. The second fundamental tensor Π for the leaves of the foliation $\mathfrak{F}$ is given by $A_X Y$, for X, Y tangent vector fields to $\mathfrak{F}$.

On the other hand, by [4, Lemma 1.2] we have, for X, Y tangent to $\mathfrak{F}$, that $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$, where $\mathcal{V}[X, Y]$ denotes the projection of $[X, Y]$ on $T\mathcal{O}$. We conclude that $A_X Y$ takes values in $T\mathcal{O}$.

We know that $T\mathcal{O}^\perp$ is integrable and hence A vanishes on vertical vector fields to $\mathfrak{F}$, and therefore the leaves of the foliation $\mathfrak{F}$ are totally geodesic. □

A remarkable property of Riemannian foliations is that, with respect to compatible bundle-like metrics, geodesics which start perpendicular to a leaf of the foliation stay perpendicular to all leaves. The proof of this result can be found in [7].

The next lemma is fundamental to obtain an isometric splitting.

Lemma 2 *If G is a connected semisimple Lie group without compact factors acting topologically transitively on M preserving its pseudo-Riemannian metric and satisfying $n_0^1 + \cdots + n_0^l = m_0$, then the leaves of the foliation defined by $T\mathcal{O}^\perp$ are complete for the metric induced by M.*

Proof We know that $T\mathcal{O}^\perp$ is either Riemannian or antiRiemannian, see [14]. Hence, the foliation by G-orbits on M carries a Riemannian or antiRiemannian structure obtained from $T\mathcal{O}^\perp$.

On the other hand, using the compactness of M it follows that geodesic completeness is satisfied for geodesics orthogonal to the G-orbits, then we get the completeness for leaves of the foliation given by $T\mathcal{O}^\perp$, see [7]. □

Using Lemma 2 we can obtain an isometric covering map in the case of a compact pseudo-Riemannian manifold M.

Theorem 5 *Suppose G is a semisimple Lie group without compact factors acting topologically transitively and by isometries on a compact manifold M and satisfying $n_0^1 + \cdots + n_0^l = m_0$. Let N be a leaf of the foliation defined by $T\mathcal{O}^\perp$, and consider it as a pseudo-Riemannian manifold with the metric inherited from M. Then the map $G \times N \to M$,obtained by restricting the G-action to N, is a G-equivariant pseudo-Riemannian covering map, when we fix on G a bi-invariant pseudo-Riemannian metric induced by the metric inherited by M. Also, we obtain a G-equivariant pseudo-Riemannian covering map $G \times \widetilde{N} \to M$, where $\widetilde{N}$ is the universal covering space of N.*

Proof By Lemma 2 we have that N is a complete manifold. It is known that G is complete, see chapter II of [3]. Hence $G \times N$ is a complete pseudo-Riemannian manifold. As $G \times N \to M$ is a local isometry it follows (see [8, Corollary 29, Chapter 7]) that $G \times N \to M$ is a pseudo-Riemannian covering map. The G-invariancy follows from this, $h \cdot (g, n) = (hg, n)$.

Let Φ be the map $G \times N \to M$, obtained by restricting the G-action to N. Then we have a local isometry $\psi : G \times \widetilde{N} \to M$ if we define $\psi(g, \widetilde{n}) = \Phi(g, \pi(\widetilde{n}))$, where $\pi : \widetilde{N} \to N$.

The map ψ is G-equivariant because of the following:

$$\begin{aligned}\psi\big(g_1 \cdot (g, \widetilde{n})\big) &= \psi\big(g_1 g, \widetilde{n}\big)\\ &= \Phi\big(g_1 g, \pi(\widetilde{n})\big)\\ &= g_1 \cdot \Phi(g, \pi(\widetilde{n}))\\ &= g_1 \cdot \psi(g, \widetilde{n}).\end{aligned}$$

□

4 An isometric splitting

In this section we want to improve on the previous result, Theorem 5.

Theorem 6 *With the hypothesis of Theorem 4, the pseudo-Riemannian metric h* on *M restricted to the orbits defines a leafwise pseudo-Riemannian metric which is given by*

$$\sum_{i=1}^{k} f_i B_{\mathfrak{g}_i} + \sum_{j=k+1}^{l} \left(f_{1,j} B_{\mathfrak{g}_j} + f_{2,j} B^{\mathfrak{g}_j, J_j} \right),$$

where $f_i, f_{1,j}, f_{2,j} : \mathbb{R} \to \mathbb{R}$ are *G-invariant smooth functions, for all $i = 1, \dots, k$, and $j = k + 1, \dots, l$.*

Proof As in the proof of Theorem 3, we consider the following map $\Psi : M \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ given by the bilinear symmetric map $\Psi_x = h_x(\varphi_x \cdot, \varphi_x \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. We show that Ψ_x is $\mathrm{Ad}(G)$-invariant, and Ψ_x is independent of x.

If $x \in M$, $X, Y \in \mathfrak{g}$, and $g \in G$, then because G preserves the metric, we have $\Psi_{gx}(X, Y) = h_x(dg^{-1}_{gx} X^*_{gx}, dg^{-1}_{gx} Y^*_{gx})$. On the other hand the fact that $dg^{-1}(X^*) = \mathrm{Ad}(g)X^*$ implies $h_x(dg^{-1}_{gx} X^*_{gx}, dg^{-1}_{gx} Y^*_{gx}) = h_x(\mathrm{Ad}(g^{-1})X^*_x, \mathrm{Ad}(g^{-1})Y^*_x)$.

Based on the previous result, we conclude that the map Ψ satisfies $\Psi_{gx}(X, Y) = \Psi_x(\mathrm{Ad}(g^{-1})X, \mathrm{Ad}(g^{-1})Y)$, which means that Ψ is G-equivariant. Therefore Ψ is G-invariant, and the result follows by [17, Proposition 4.3]. Therefore $\Psi_x = \Psi_{gx}$ for all $g \in G$ and $x \in M$,

By a result in [14] the action is locally free everywhere and by Theorem 3 each orbit is nondegenerate. Then the metric h on M restricted to each orbit is nondegenerate, therefore Ψ_x is the metric induced on G. If this metric is bi-invariant, then we can use the classification of bi-invariant pseudo-Riemannian metrics for semisimple groups and the claim follows.

Given that $\Psi_x = \Psi_{gx}$ for all $g \in G$ and $x \in M$, it follows that $\Psi_x(X, Y) = \Psi_x(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y)$ for all $X, Y \in \mathfrak{g}$ showing that Ψ_x is an Ad-invariant bilinear symmetric form for all $x \in M$. Hence the result follows by Theorem 2. □

The next corollary is an easy consequence of the above and it will be useful in the main theorem of this work.

Corollary 1 *With the same hypothesis as Theorem 6, there is a G-equivariant isometric immersion from G to M when G is considered with a bi-invariant pseudo-Riemannian metric.*

In the previous theorem we can obtain a local result on an open conull subset S of M using the fact that there is an isomorphism between Killing vector fields that vanish at a specific point and elements of the Lie algebra of G. For that end we need to state the result due to Gromov [2, 13] whose proof for the case of semisimple Lie groups can be found in [16]. It is true without the assumption that the G-action is topologically transitive.

Definition 3 For any given pseudo-Riemannian manifold M we will denote by $\mathrm{Kill}(M, x)$ the Lie algebra of germs at x of local Killing vector field defined in a neighborhood U_x of x.

Theorem 7 *Let M be a smooth pseudo-Riemannian manifold of finite volume, and G a connected semisimple Lie group without compact factors and acting smoothly on M by isometries. If any normal subgroup acts nontrivially on M and G has a finite center, then there is a dense conull subset $S \subset M$ such that, for every $x \in S$ there exist an open set U_x of x, a Lie algebra $\mathrm{Kill}(M, x)$ and a homomorphism $\rho_x : \mathfrak{g} \to \mathrm{Kill}\,(M, x)$ satisfying*

- *all elements of $\mathrm{Kill}(M, x)$ vanish at x.*
- *ρ_x is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.*
- *$[\rho_x(\mathfrak{g}), \mathcal{Y}] \subset \mathcal{Y}$, where $\mathcal{Y} = \{Y^* : Y \in \mathfrak{g}\}$.*

The next theorem is needed to describe the geometry of the normal bundle to the G-orbits.

Theorem 8 *For G and M with the hypothesis of Theorem 4 suppose G acts topologically transitively on M preserving its pseudo-Riemannian metric, h. If $n_0^1 + \cdots + n_0^l = m_0$, then with the metric induced by M, the leaves of the normal foliation $\mathfrak{F}$ lying in a fixed component of M have isometric universal coverings.*

Proof As the orbits are nondegenerate then $dL_g\left(T_x(Gx)^\perp\right) \subset T_x(Gx)^\perp$, for every $x \in M$, and $g \in G$. In fact, for every $x \in M$, we have $T_x(M) = T_x(Gx) \oplus T_x(Gx)^\perp$. If $v^\perp \in T_x(Gx)^\perp$, then $dL_g(v^\perp) = w^\perp + w$, where $w \in T_x(Gx)$. Therefore, $h(dL_g(v^\perp), v) = h(w, v)$. On the other hand, it follows that $h(v^\perp, dL_{g^{-1}} v) = 0$, and then $dL_g(v^\perp) \in T_x(Gx)^\perp$. This proves that the action preserves the normal bundle $T\mathcal{O}^\perp$.

We now prove that GL is a connected component of M if L is a leaf of the foliation $T\mathcal{O}^\perp$. Using the same argument that appears in corollary 2.8 in [1], we define an equivalence relation on the leaves of $T\mathcal{O}^\perp$ of M by saying $L_1 \sim L_2$ if $L_2 = gL_1$ for some $g \in G$. It is easy to prove that $GL = [L]$, the equivalence class of L. Therefore GL is open in M, since $GL = \phi(G \times L)$, where the local diffeomorphism $\phi : G \times L \to M$ is the restricion of the G-action map to the leaf L. Also, GL is closed because it is the complement of the union of open sets. Therefore GL is a connected component of M. The theorem is now a direct consequence of the above. ☐

The next result is the main theorem of this work, and will give us a kind of splitting geometric theorem on the manifold M.

We observe this theorem is, in spirit, similar to [1, Theorem A] and [12, Theorem A].

Theorem 9 *Let M be a compact connected smooth pseudo-Riemannian manifold, and G a connected semisimple Lie group without compact factors, acting smoothly and topologically transitively on M by isometries. If $n_0^1 + \cdots + n_0^l = m_0$, and L is a leaf of the normal foliation $\mathcal{F}$* in *M, then there is a $\widetilde{G}$-equivariant map $\widetilde{G} \times \widetilde{L} \to M$ which is an isometric covering, where $\widetilde{G} \times \widetilde{L}$ is furnished with a metric k given as follows:*

$$k = \sum_{i=1}^{k} f_i B_{\mathfrak{g}_i} + \sum_{j=k+1}^{l} \left(f_{1,j} B_{\mathfrak{g}_j} + f_{2,j} B^{\mathfrak{g}_j, J_j} \right) + h^{\mathcal{F}},$$

where $f_i, f_{1,j}, f_{2,j} : \widetilde{L} \to \mathbb{R}$ are smooth functions, for all $i = 1, \ldots, k$, and $j = k+1, \ldots, l$ and $h^{\mathcal{F}}$ is the metric on the common universal cover of the leaves of $\mathcal{F}$, induced by the restriction of the metric h on M to the leaves of the foliation $\mathcal{F}$.

Proof By Theorem 3 it follows that G acts locally freely everywhere with nondegenerate orbits.

Let $\psi = \widetilde{G} \times \widetilde{M} \to \widetilde{M}$ be the lifted action. We consider $\widehat{L}$ a leaf of the normal bundle to the orbits in $\widetilde{M}$ that is mapped onto L by the covering map $\pi : \widetilde{M} \to M$.

Let $\phi = \psi|_{\widetilde{G} \times \widehat{L}}$ denote the restriction of ψ to $\widetilde{G} \times \widehat{L}$. It is an easy consequence of Theorem 6 and Theorem 8 that ϕ is an isometric immersion of $\widetilde{G} \times \widehat{L}$ to $\widetilde{M}$, where $\widetilde{G} \times \widehat{L}$ is considered as a pseudo-Riemannian manifold with metric, k, given by

$$k = \sum_{i=1}^{k} f_i B_{\mathfrak{g}_i} + \sum_{j=k+1}^{l} \left(f_{1,j} B_{\mathfrak{g}_j} + f_{2,j} B^{\mathfrak{g}_j, J_j} \right) + h^{\mathcal{F}}.$$

We will show that ϕ is injective and that the universal cover of L is equal to $\widehat{L}$.

Using Theorem 4 we conclude that the leaves of the normal foliation $\mathcal{F}$ on M and $\widetilde{M}$ are totally geodesic.

For the rest of the proof we are going to use some ideas established in [1, Theorem A].

In the proof of Theorem 8 we obtained that $GL = M$. From this it follows there are open sets $V_1 \subset G, V_2 \subset L, V_x \subset M$, for every $x \in M$, such that the map $H_x : V_1 \times V_2 \to V_x$ is a diffeomorphism.

If we denote by $\pi_2 : V_1 \times V_2 \to V_2$, the projection on the second factor, then we obtain a submersion $\pi_2 \circ H_x^{-1} : V_x \to V_2$. This submersion locally defines the foliation on M given by the G-orbits. Note that $\pi_2 \circ H_x^{-1}$ is a pseudo-Riemannian submersion.

We can obtain $\{U_\alpha\}_\alpha$, an open covering of M for which we have pseudo-Riemannian submersions $H_\alpha : U_\alpha \to L$. For each α, the open set U_α is connected, evenly covered by the universal covering $\pi : \widetilde{M} \to M$, and $H_\alpha(U_\alpha)$ contained in an open set evenly covered by $\pi_1 : \widehat{L} \to L$.

There exists a pseudo-Riemannian submersion $H : \widetilde{M} \to \widetilde{L}$ such that $\pi_1 \circ H|_{\widetilde{U}_{\alpha,k}} = H_\alpha \circ \pi$, for every α, and $\pi^{-1}(U_\alpha) = \cup_k \widetilde{U}_{\alpha,k}$. The proof of this is based on known arguments of algebraic topology.

We conclude that the foliation defined by the submersion H is the foliation defined by the orbits of the action of $\widetilde{G}$ on $\widetilde{M}$. In particular, H is a local isometry when it is restricted to every leaf $\widehat{L}_x$ of the foliation in $\widetilde{L}$ given by $\mathcal{F}$. Moreover, $H|_{\widehat{L}_x} : \widehat{L}_x \to \widetilde{L}$ is a bijection. This is proven by seeing that for each $x \in \widehat{L}_x$, every geodesic $\widehat{\sigma} : [0,1] \to \widehat{L}$ with $\widehat{\sigma}(0) = H(w)$ can be lifted a geodesic $\widetilde{\sigma} : [0,1] \to \hat{L}_x$ with $\widetilde{\sigma}(0) = w$. Therefore, $H|_{\widehat{L}_x} : \widehat{L}_x \to \widetilde{L}$ is a covering map, and $\widehat{L}_x$ is the universal cover of L_x for every $x \in M$.

If $\phi(g_1, x_1) = \phi(g_2, x_2)$, then $g_1x_1 = g_2x_2$. By using the fact that H is G-invariant, it follows that $H(x_1) = H(g_1x_1) = H(g_2x_2) = H(x_2)$, therefore $x_1 = x_2$.

On the other hand, it is easy to see that $g = g_1^{-1}g_2 \in \text{Stab}\,(x_1)$ and we can consider $V_{x_1} \subset \hat{L}$ a normal neighborhood of x_1. Let σ be a geodesic from x_1 to x, where $x \in V_{x_1}$. Then $H(\sigma(1)) = H(g\sigma(1))$, so we obtain $\sigma(1) = g\sigma(1) \in \widetilde{L}$. Therefore $g \in \text{Stab}\,(\sigma(1))$ and g fixes V_{x_1}. By using that the action is analytic it follows that g fixes $\widehat{L}$.

Let $\text{Stab}\,(\widehat{L})$ denote the subgroup of G that fixes the points in $\widehat{L}$. The map $\Phi : \widetilde{G}/\,\text{Stab}\,(\widehat{L}) \times \widetilde{L} \to \widetilde{M}$ given by $\Pi\Big(g + \text{Stab}\,(\widehat{L}), x\Big) = \phi(g, x)$ is a diffeomorphism. It follows that $\text{Stab}\,(\widehat{L}) = \{e\}$ because $\widetilde{M}$ is simply connected, so we obtain $g_1 = g_2$. Therefore, ϕ is injective.

We obtain the main part of the theorem by performing the following composition: $\pi \circ \phi : \widetilde{G} \times \widetilde{M} \to M$. □

References

1. Blumenthal, R.A., Hebda, J.J.: De Rham decomposition theorems for foliated manifolds. Ann. Inst. Fourier **33**, 183–198 (1983)
2. Gromov, M.: Rigid Transformations Groups, Géométrie Différentielle, Travaux en Cours 33. Hermann, Paris (1988)
3. Helgason, S.: Differential Geometry, Lie Groups and Symmetric Spaces. Academic Press, New York (1978)
4. Johnson, D.L., Whitt, L.B.: Totally geodesic foliations. J. Differ. Geom. **15**, 225–235 (1980)
5. Kobayashi, S., Nommizu, K.: Foundations on Differential Geometry, vol. 1. Wiley, New York (1963)
6. Milnor, J.: Curvatures of left invariant metrics on Lie groups. Adv. Math. **21**, 293–329 (1976)
7. Molino, P.: Riemannian Foliations. Progress in Mathematics, vol. 73. Birkhauser, Boston (1988)
8. O'Neill, B.: The fundamental equations of a submersion. Mich. Math. J. **13**, 459–469 (1966)
9. O'Neill, B.: Semi-Riemannian Geometry. Academic Press, New York (1983)
10. Onishchik, A.: Lectures on Real Semisimple Lie Algebras and Their Representations. European Mathematical Society, Germany (2004)
11. Petersen, P.: Riemannian Geometry. Springer, New York (2016)
12. Quiroga-Barranco, R.: Isometric actions of simple Lie groups on pseudo Riemannian manifolds. Ann. Math. **164**(3), 941–969 (2006)
13. Quiroga-Barranco, R.: Local and global rigidity for isometric actions of simple lie groups on pseudo-Riemannian manifolds. J. Lie Theory **30**(2), 565–586 (2020)
14. Rosales-Ortega, J.: The signature in actions of semisimple Lie groups on pseudo-Riemannian manifolds. Proyecciones (Antofagasta) **31**(1), 51–63 (2012)
15. Rosales-Ortega, J.: Foliations by *G*-actions. Bol. Soc. Paran. Mat. (3s) **33**(2), 79–87 (2015)
16. Rosales-Ortega, J.: The Gromov's centralizer theorem for semisimple Lie group actions. Bol. Soc. Mat. Mex. **23**, 825–845 (2017)
17. Szaro, J.: Isotropy of semisimple group actions on manifolds with geometric structures. Am. J. Math. **120**, 129–158 (1998)
18. Zimmer, R.J.: Ergodic Theory and Semisimple Lie Groups. Birkhäuser, Boston (1984)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.