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Variants of Hörmander's theorem on q-convex manifolds by a technique of infinitely many weights

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Abstract
By introducing a new approximation technique in the L^2 theory of the $\bar{\partial}$-operator, Hörmander's L^2 variant of Andreotti-Grauert's finiteness theorem is extended and refined on q-convex manifolds and weakly 1-complete manifolds. As an application, a question on the L^2 cohomology suggested by a theory of Ueda (Tohoku Math J (2) 31(1):81–90, 1979), Ueda (J Math Kyoto Univ 22(4):583–607, 1982/83) is solved.

Keywords q-Convex manifolds · Weakly 1-complete manifolds · L^2 · Harmonic representation · Approximation · Andreotti-Grauert's finiteness theorem · Ueda theory

Mathematics Subject Classification Primary 32E40 · Secondary 32T05

1 Introduction

This article is a continuation of [34, 43]. A variant of Andreotti-Grauert's finiteness theorem on weakly 1-complete manifolds was obtained in [34] (see also [1, 30]) and it was recalled in [43] to invoke its connection to an extension problem from submanifolds with semipositive normal bundles. This connection was suggested by Serre's celebrated works [50, 51] on algebraic sheaves that translated the ideas of Oka-Cartan's theory in several complex variables into algebraic geometry. Hörmander's method in [23] was employed in

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[34] to explore an analytic aspect of the sheaf cohomology on weakly 1-complete manifolds. The point is that Andreotti-Grauert's finiteness theorem does not say anything directly about the effect of twisting the sheaves by line bundles, which was the main interest of [50, 51], but Hörmander's theorem 3.4.9 in [23] does, although this advantage was not so explicitly stated there. Therefore, it may be worthwhile to pursue it further to look for a more general principle.

For that purpose, we shall revisit here Hörmander's theorem on the finite-dimensionality and harmonic representation of the $\bar\partial$-cohomology groups on q-convex manifolds and strengthen the result in a way required in some geometric questions. It has been done to some extent in [46], but we would like to extend the method in a different way so that we can apply it to study a class of L^2 $\bar\partial$-cohomology of certain 1-convex surfaces. Our specific interest is in a compact complex surface S containing a smooth divisor C whose self-intersection number is zero. Such a situation arises in the classification of the compactifications of $\mathbb{C}^* \times \mathbb{C}^*$ (cf. [56]), for instance. A general result by Ueda says that $S \setminus C$ is 1-convex if the embedding $C \hookrightarrow S$ is of finite type (cf. [57] or section 5 in this article). Ueda showed moreover that $S \setminus C$ admits no plurisubharmonic exhaustion function which is of logarithmic growth if $S \setminus C \hookrightarrow S$ is of finite type. Therefore it might be of some interest whether or not one can extend the following result to this case.

Theorem 0.1 (cf. [43], Theorem 1.4[1]) *Let M be a compact complex manifold of dimension n, let $E \to M$ be a holomorphic vector bundle and let D be an effective divisor on M. If the line bundle $[D]$ associated to D is semipositive and $E|_{|D|}$ is Nakano positive, then there exists a positive number μ_0 such that*

$$H^0(M, \mathscr{O}_M(K_M \otimes E \otimes [D]^\mu)) \to H^0(|D|, \mathscr{O}_D(K_M \otimes E \otimes [D]^\mu))$$

is surjective if $\mu \geq \mu_0$ and

$$H^{n,k}(M, E \otimes [D]^\mu) \cong H^{n,k}(M \setminus |D|, E)$$

if $k \geq 1$ and $\mu \geq \mu_0$. Here H^k and $H^{j,k}$ stand for the k-th sheaf cohomology and the $\bar\partial$-cohomology of type (j, k), respectively, K_M denotes the canonical line bundle of M, $|D|$ the support of D and $\mathscr{O}_X(\cdot)$ ($X = M$ or $|D|$) the sheaf of the germs of holomorphic sections.[2]

A remarkable fact is that, for any embedding $C \hookrightarrow S$ of finite type, the bundle $[C]$ is never semipositive on S although $S \setminus C$ is 1-convex. Moreover, for any embedding of C with topologically trivial normal bundle, it follows immediately from Ueda's classification and Siu's solution [52] of the Grauert-Riemenschneider conjecture on the characterization of Moishezon manifolds that $[C]$ is semipositive if and only if $[C]$ is $U(1)$-flat on some neighborhood of C. See the remark at the end of §5 and also Koike's recent paper [27] for the relation between the semipositivity and $U(1)$-flatness in higher dimensional cases. [27] also gives an example of nef, big and non semipositive line bundle on a nonsingular projective surface. (See also [14], Example 5.2 for higer dimensional cases.)

[1] For the validity of the consequence of Theorem 1.2 in [43], it suffices to assume that M is weakly 1-complete and E is Nakano positive outside a compact subset of $M \setminus |D|$ as long as $|D|$ is compact.

[2] That $\varinjlim_\mu H^{n,0}(M, E \otimes [D]^\mu) \to H^{n,0}(M \setminus |D|, E)$ is dense in $H^{n,0}(M \setminus |D|, E)$ is also contained, although it is not stated in the statement, in the proof of Theorem 0.1.

In order to extend Theorem 0.1 to cover Ueda's case, we shall first recall Hörmander's theorem with a modification of an approximation argument (cf. Lemma 1.1) to deduce the finite-dimensionality and harmonic representation. Although the result itself is essentially included in [23], the method contains a new technique which will be explored further in the spirit of [46] (cf. Theorems 2.1–2.3).

The main goal of the present paper is to show the following.

Theorem 0.2 *Let (X, g) be a complete Hermitian manifold of dimension n and let (E, h) be a holomorphic Hermitian vector bundle over X such that (X, g, E, h) is q-elliptic at infinity (see Sect. 1). Assume that X is equipped with a positive C^∞ exhaustion function Φ satisfying*

$$\sup\{\Phi(x);(\partial\bar\partial\Phi)_q(x)<0\}<\infty$$

and

$$\lim_{c\to\infty}\inf\{(\partial\bar\partial\Phi-\Phi^{-1}\partial\Phi\bar\partial\Phi)_q(x);\Phi(x)>c\}\geq 0.$$

(For the definition of $(\cdot)_q$ see Sect. 1.) Then the following (a) and (b) hold.

(a) *The E-valued L^2 $\bar\partial$-cohomology group $H^{n,k}_{(2),\Phi}(X,E)$ of X with respect to $(g, he^{-\Phi})$ is mapped for all $k\geq q$ bijectively onto $H^{n,k}(X,E)$ by the homomorphism induced from the inclusion. Moreover, the map $H^{n,q-1}_{(2),\Phi}(X,E)\to H^{n,q-1}(X,E)$ has a dense image.*

(b) *If moreover*

$$\lim_{c\to\infty}\inf\{\Phi(x)^{1+\epsilon}(\partial\bar\partial\log\Phi)_q(x);\Phi(x)>c\}\geq 0 \tag{0.1}$$

holds for some $\epsilon>0$, then the L^2 cohomology groups $H^{n,k}_{(2),\mu\log\Phi}(X,E)$ are isomorphic to $H^{n,k}(X,E)$ for $k\geq q$ if μ is sufficiently large, and the map

$$\varinjlim_{\mu} H^{n,q-1}_{(2),\mu\log\Phi}(X,E)\to H^{n,q-1}(X,E)$$

has a dense image. Here μ runs through $\mathbb{N}$.

The proof of Theorem 0.2 is based on a method of approximation which will be introduced in the proof of Theorem 1.1 (cf. Lemma 1.1). Theorem 1.1 is substantially due to Hörmander [23] so that nothing is new in the statement itself. But a new trick is added in the proof to conclude some part more directly. Its general principle will be summarized in Theorem 4.1.

By using a similar method, Theorem 0.1 will be generalized as follows.

Theorem 0.3 *Let M be a weakly 1-complete manifold of dimension n, let $E\to M$ be a holomorphic vector bundle and let D be an effective divisor with compact support. Assume that $[D]|_{|D|}$ is semipositive and $E|_{M\setminus K}$ is Nakano positive for some compact subset K of $M\setminus|D|$. Then multiplication by a canonical section of $[D]$ induces isomorphisms between $H^{n,k}(M,E\otimes[D]^{\mu-1})$ and $H^{n,k}(M,E\otimes[D]^{\mu})$ $(k\geq 1)$ for sufficiently large μ. In particular*

$$H^k(M,\mathscr{O}_M(K_M\otimes E\otimes[D]^\mu))\to H^k(|D|,\mathscr{O}_D(K_M\otimes E\otimes[D]^\mu))$$

is surjective for sufficiently large μ and for all k. Moreover, if D is a pseudoconcave divisor of order >1 (see Sect. 5 for the definition), then

$$H^{n,k}(M, E \otimes [D]^{\mu}) \cong H^{n,k}(M \setminus |D|, E), \quad k \geq 1$$

hold for sufficiently large μ and the set of meromorphic sections of $K_M \otimes E$ with poles (at most) along $|D|$ is dense in $H^0(M \setminus |D|, \mathscr{O}_M(K_M \otimes E))$.

Corollary 0.1 *Let M be a connected compact complex manifold. If there exist an effective divisor $D \neq 0$ on M and a holomorphic line bundle $B \to M$ such that $[D]|_{|D|}$ is semipositive and $B|_{|D|}$ is positive. Then M is a Moishezon manifold.*

Corollary 0.1 was proved in [43] under a stronger assumption that $[D]$ is semipositive on M.

Corollary 0.2 *Let M be a connected weakly 1-complete Kähler manifold, let (E, h) be a Nakano semipositive vector bundle over M and let $D(\neq 0)$ be a pseudoconcave divisor on M of order >1 such that $|D|$ is compact and the curvature form of h is Nakano positive on $M \setminus K$ for some compact set $K \subset M \setminus |D|$. Then $H^{n,k}(M, E \otimes [D]^{\mu}) = 0$ $(k \geq 1)$ for sufficiently large μ.*

Corollary 0.2 extends vanishing theorems by Grauert and Riemenschneider [18, 19]. We note that Takegoshi [54] has shown a vanishing theorem on weakly 1-complete Kähler manifolds saying in particular that $H^{n,k}(M, E) = 0$ $(k \geq 1)$ holds in the situation of Corollary 0.2 for $D = 0$.

For the question arising from Ueda's theory, the following is an answer.

Theorem 0.4 *Let S be a compact complex surface and let $C \subset S$ be a complex curve such that* $\deg([C]|C) \geq 0$. *Then, for any holomorphic vector bundle $E \to S$ such that $E|_C$ is positive, there exists a positive number μ_0 such that*

$$H^k(S, \mathscr{O}_S(K_S \otimes E \otimes [C]^{\mu-1})) \cong H^k(S, \mathscr{O}_S(K_S \otimes E \otimes [C]^{\mu}))$$

canonically if $\mu \geq \mu_0$. In particular, the maps

$$H^0(S, \mathscr{O}_S(K_S \otimes E \otimes [C]^{\mu})) \to H^0(C, \mathscr{O}_C(K_S \otimes E \otimes [C]^{\mu})) \ \ (k = 0, 1)$$

are surjective if $\mu \geq \mu_0$. If moreover the embedding $C \hookrightarrow S$ is of finite type, then

$$H^{2,k}(S, E \otimes [C]^{\mu}) \cong H^{2,k}(S \setminus C, E) \ \ (k = 1, 2)$$

holds for sufficiently large μ and the set of meromorphic sections of $K_S \otimes E$ with poles along C is dense in $H^0(S \setminus C, \mathscr{O}(K_S \otimes E))$.

Corollary 0.3 *Let S be a connected compact complex surface, let $C \subset S$ be a smooth complex curve of finite type and let $L \to S$ be a holomorphic line bundle such that $L|_C$ is positive. Then L is big. Moreover the following holds.*

1. *If* $\deg(L|_C) \geq 1$, *then for sufficiently large m one can find*

$s_0, s_1, s_2, s_3, s_4, s_5 \in H^0(S, K_S \otimes L \otimes [C]^m)$ *such that* $\bigcap_{k=0}^5 s_k^{-1}(0) \cap C$ *is a finite set and* $(s_0 : s_1 : s_2 : s_3 : s_4 : s_5)$ *embeds* $S \setminus (C \cup \bigcap_{k=0}^5 s_k^{-1}(0))$ *into* $\mathbb{CP}^5$.

2. *If* $\deg(L|_C) \geq 2$*, then for sufficiently large* m *one can find* $s_0, s_1, s_2, s_3, s_4, s_5 \in H^0(S, K_S \otimes L \otimes [C]^m)$ *with* $\bigcap_{k=0}^5 s_k^{-1}(0) \cap C = \emptyset$ *such that* $(s_0 : s_1 : s_2 : s_3 : s_4 : s_5)$ *embeds* $S \setminus (C \cup \bigcap_{k=0}^5 s_k^{-1}(0))$ *into* $\mathbb{CP}^5$.
3. *If* $\deg(L|_C) \geq 3$*, then for sufficiently large* m *one can find* $s_0, s_1, s_2, s_3, s_4, s_5 \in H^0(S, K_S \otimes L \otimes [C]^m)$ *such that* $(s_0 : s_1 : s_2 : s_3 : s_4 : s_5)$ *embeds* $S \setminus \bigcap_{k=0}^5 s_k^{-1}(0)$ *into* $\mathbb{CP}^5$.

Corollary 0.4 *A connected compact complex surface is projective algebraic if and only if it contains a smooth curve of genus* ≥ 2 *with semipositive normal bundle.*

Since S is projective algebraic in the situation of Corollary 0.3 (Chow-Kodaira's theorem), it is naturally expected that the assertion has an algebraic proof.

2 Hörmander's theorem revisited

As a preliminary to the proof of Theorem 0.2, we shall recall Hörmander's isomorphism and approximation theorem (Theorem 3.4.9 in [23]) with a little modification in the presentation and proof. Since its background materials are not so popular as they used to be, we shall recall them briefly at first for the convenience of the reader who are not so familar with the L^2 method in the sheaf theory.

In complex geometry, the sheaf cohomology groups are the most important biholomorphic invariants of complex analytic spaces. There are many formulas described in terms of the dimension of cohomology groups. In some circumstances, analytic sheaf cohomology classes are natual generalization of holomorphic functions (cf. [25, 42]). Andreotti and Grauert [3] established finiteness theorems for the sheaf cohomology on the spaces with certain convexity or concavity properties. Among other things, they generalized Oka-Cartan's theory on the existence and approximation for holomorphic functions, from Stein spaces to q-convex spaces. The 1-convex case was settled earlier by Grauert [15] by extending the method of Oka [47] for the domains over $\mathbb{C}^n$ to complex manifolds. Let us recall that a real-valued C^2 function φ on a complex manifold X of dimension n is called q-convex on $K \subset X$ if its Levi form (or complex Hessian) $\partial\bar{\partial}\varphi$ has everywhere at least $n - q + 1$ positive eigenvalues on K, where the eigenvalues of $\partial\bar{\partial}\varphi$ are defined with respect to any Hermitian metric on the manifold. A complex manifold X of dimension n is called ***q*-convex** if X is equipped with a q-convex exhaustion function, i.e. if there exists a function $\varphi : X \to \mathbb{R}$ of class C^2 such that its sublevel sets $X_c := \{x \in X; \varphi(x) < c\}$ are relatively compact and φ is q-convex on $X \setminus X_{c_0}$ for some c_0. X is said to be ***weakly q-complete*** (resp. ***weakly q-convex***) if there exists a C^∞ exhaustion function whose Levi form has at most $q - 1$ negative eigenvalues everywhere on X (resp. outside a compact subset of X). It is obvious that q-convex manifolds are weakly q-complete. It is also quite easy to see that weakly q-convex manifolds are weakly q-complete. For simplicity, given a q-convex manifold (X, φ) we shall assume that φ is of class C^∞. q-convexity is naturally generalized to complex spaces. Modifications of Stein spaces along compact subsets are characterized by Grauert as 1-convex spaces (cf. [15, 16, 31]). Interesting examples of q-convex manifolds are the complements of compact complex submanifolds with positive normal bundles in compact complex manifolds (cf. [17, 45]).

By Dolbeault's isomorphism theorem, the sheaf cohomology group of a paracompact complex manifold with coefficients in a locally free analytic sheaf is canonically isomorphic to the $\bar{\partial}$-cohomology group with coefficients in a holomorphic vector bundle. Accordingly, it is a natural question to represent analytic cohomology classes by harmonic forms in various geometric situations. Kodaira's vanishing theorem for positive line bundles over compact manifolds arose in this context (cf. [26]). Let us recall that the method of Kodaira is based on Hodge's theorem on the harmonic representatives of cohomology classes and a variant of Bochner's technique on the Laplace-Beltrami operator. It was extended by Akizuki and Nakano [2] who showed that the $\bar{\partial}$-cohomology groups $H^{p,q}(X,B)$ vanish for all p and q with $p+q>n$ if X is compact and B is a holomorphic Hermitian line bundle over X whose curvature form is positive. For noncompact manifolds, Andreotti and Vesentini [5, 6] first generalized Kodaira's method to prove basic existence theorems on Stein manifolds. The point in this sophistication is an observation that the solvability of the $\bar{\partial}$-equation with L^2 norm estimate follows from an estimate for the $\bar{\partial}$-operator and its adjoint $\bar{\partial}^*$ acting on the set of compactly supported C^∞ bundle-valued differential forms, if the L^2 norms are measured with respect to a complete Hermitian metric.

Hörmander [23] strengthened this approach independently by establishing an isomorphism between the $\bar{\partial}$-cohomology and the L^2 $\bar{\partial}$-cohomology on smoothly bounded domains and extended it to q-convex manifolds by approximation, which is essentially the original form of Theorem 1.1 below. The formulation is modified here in order to make the presentation of its refinements easier.

Theorem 1.1 *Let (X,φ) be a q-convex manifold of dimension n such that φ is q-convex on $X\setminus X_0$ and let $E\to X$ be a holomorphic vector bundle with a C^∞ fiber metric h. Then the following assertions hold.*

1. *There exist a complete Hermitian metric g on X, a C^∞ increasing function $\lambda:\mathbb{R}\to\mathbb{R}$ and a constant $c_0>0$ such that, for any C^∞ function $\psi: X\to\mathbb{R}$ and for any $k\ge q$, the inequality*

$$c_0\|u\|^2+((\partial\bar{\partial}\psi)_q u,u)\le\|\bar{\partial}u\|^2+\|\bar{\partial}^*u\|^2 \tag{1.1}$$

holds for any E-valued C^∞ (n,k)-form u on X whose support is compact and contained in $X\setminus X_0$. Here $\|\cdot\|$ denotes the L^2 norm measured by g and $he^{-\lambda(\varphi)-\psi}$, $(\partial\bar{\partial}\psi)_q$ is defined by $(\partial\bar{\partial}\psi)_q(x)=\sum_{j=1}^q e_j(x)$ for the eigenvalues $e_1(x)\le e_2(x)\le\cdots\le e_n(x)$ of $\partial\bar{\partial}\psi$ with respect to g at x, and $(\cdot,\cdot)$ stands for the inner product associated to $\|\cdot\|$.
2. *If (1.1) holds for all the above u and ψ with respect to $(X,\varphi,E,g,h,\lambda)$, then $\dim H^{n,k}(X,E)<\infty$ for $k\ge q$. Moreover, the L^2 $\bar{\partial}$-cohomology groups $H^{n,k}_{(2)}(X,E)(=H^{n,k}_{(2)}(X,E)_{\lambda,\psi})$ with respect to $(g,he^{-\lambda(\varphi)-\psi})$ are finite dimensional for all $k\ge q$ if $(\partial\bar{\partial}\psi)_q\ge c-c_0$ holds on $X\setminus X_0$ for some positive number c. Furthermore, if λ satisfies $\inf_{X\setminus X_0}(\partial\bar{\partial}\lambda(\varphi))_q>0$, then the homomorphism $H^{n,k}_{(2)}(X,E)_{\mu\lambda,0}\to H^{n,k}(X,E)$ $(k\ge q)$ induced from the inclusion is bijective for sufficiently large μ and the homomorphism*

$$\varinjlim_{\mu} H^{n,q-1}_{(2)}(X,E)_{\mu\lambda,0}\to H^{n.q-1}(X,E)$$

has a dense image.
3. *If 0 is not a critical value of φ, then the L^2 $\bar{\partial}$-cohomology groups $H^{n,k}_{(2)}(X_0,E)$ of X_0 with respect to the restrictions of the metrics g and h to X_0 and $E|_{X_0}$ are finite dimensional for*

$k \geq q$. Moreover, the restriction homomorphism $H^{n,k}(X, E) \to H^{n,k}_{(2)}(X_0, E)$ is bijective if $k \geq q$ and has a dense image if $k = q - 1$.

Here the part (1) was added in order to describe the decisive consequence of the calculation in Kodaira's method. For the detail of its derivation from the q-convexity assumption, see [9, 11, 37, 44] as well as [7, 23, 58]. The estimate (1.1) is particularly important in order to state the assumptions of the refinements of Theorem 1.1. We shall use the definition of $(\partial\bar{\partial}\psi)_q$ by naturally extending it to Θ_q for any Hermitian form Θ along the fibers of holomorphic tangent bundle of M. We shall say that (X, g, E, h) is ***q-elliptic at infinity*** if there exists a compact set $K \subset X$ and $c_0 > 0$ such that (1.1) for $k \geq q$ with $\lambda = 0$ holds for any compactly supported C^∞ E-valued (n, k)-form u with supp$u \subset X \setminus K$ and ψ as above. If (1.1) holds in the form of Theorem 1.1 with respect to $(X, \varphi, E, g, h, \lambda)$, we shall also say shortly that $(X, \varphi, E, g, h, \lambda)$ is ***q-elliptic modulo X_0***[3]. We shall call c_0 a ***q-ellipticity constant of*** $(X, \varphi, E, g, h, \lambda)$).

It may not be too exaggerating to say that the origin of (2) is Abel's theorem on the convergence of the Taylor series of holomorphic functions. More apparently, (3) is a cohomological counterpart of Oka-Weil's generalization of Runge's approximation theorem.

We could have removed from (3) the regularity assumption on ∂X_0 in view of [9, 11, 37, 44], but the original form is kept here for simplicity.

The following is a direct consequence of Theorem 1.1 (combined with the Serre duality). The proof may well be skipped because it is routine and the result is not used in the sequel. However, the observation was actually the prototype of Theorem 0.1. (See Corollary 3.2, too.)

Theorem 1.2 *Let M be a compact complex manifold of dimension n and let $A \subset M$ be a closed complex submanifold of codimension q whose normal bundle is positive in the sense of Griffiths (see [22] for the definition). Then, for any holomorphic vector bundle $E \to M$, $\dim H^{n,k}(M \setminus A, E) < \infty$ for $k \geq q$ and there exists a positive integer μ_0 such that $H^j(M, E \otimes \mathscr{I}_A^{\mu}) \cong H^j(M, E \otimes \mathscr{I}_A^{\mu_0})$ for all $j \leq n - q$ and $\mu \geq \mu_0$. Here $\mathscr{I}_A$ denotes the ideal sheaf of A and E is identified with the associated sheaf of the germs of its holomorphic sections.*

The crucial part of the proof of Theorem 1.1 is done by contradiction whose original form was presented as the proof of 3) (cf.Proposition 3.4.5 in [23]). Since we shall modify this argument to prove 2), let us outline the original proof of 3) for the convenience of the reader.

Outline of the proof of 3): That $\dim H^{n,k}_{(2)}(X_0, E) < \infty$ for $k \geq q$ follows from an estimate similar to (1.1) that holds for compactly supported E-valued C^∞ forms on $\overline{X_0} \setminus K$ for some compact set $K \subset X_0$ satisfying the boundary condition for $\bar{\partial}^*$, based on the fact that a Hilbert space with relatively compact unit ball must be finite dimensional (cf. [23], Theorem 3.3.1)[4]. To prove the injectivity of the restriction homomorphism $H^{n,k}(X, E) \to H^{n,k}(X_0, E)$ $(k \geq q)$, it suffices to show that the homomorphisms

[3] In [5–7, 58], E is said to be $W^{p,q}$-elliptic if there exist metrics g and h such that (1.1) holds for some c_0 with $X_0 = \emptyset$ and $\psi = 0$ for C^∞ compactly supported E-valued (p, q)-forms u on X. The estimate still holds for u lying in the domains of $\bar{\partial}$ and $\bar{\partial}^*$, provided that g is complete.

[4] That ∂X_0 is smooth is used here to derive a formula by integration by parts and to let an approximation argument needed here to work.

$H^{n,k}(X,E)_{\mu\rho\circ\lambda,0} \to H^{n,k}(X_0,E)$ are injective for sufficiently large μ for any C^∞ convex increasing function ρ satisfying $\rho|_{(-\infty,0]} = 0$ and $\rho|_{(0,\infty)} > 0$. This assertion immediately follows from the existence of $C > 0$ and $\mu_0 \in \mathbb{N}$ such that

$$\|u\| \leq C(\|\bar{\partial}u\| + \|\bar{\partial}^*u\|)$$

holds for E-valued (n, k)-forms u in $\mathrm{Dom}\bar{\partial} \cap \mathrm{Dom}\bar{\partial}^*$ with respect to $(g, he^{-\mu\rho(\lambda(\varphi))})$ for all $k \geq q$ and $\mu \geq \mu_0$, provided that $u|_{X_0}$ is orthogonal to the kernels of $\bar{\partial}$ and its adjoint on X_0. Existence of C and μ_0 is a direct consequence of the argument by contradiction.

Surjectivity of $H^{n,k}(X,E) \to H^{n,k}(X_0,E)$ $k \geq q$ and the denseness of the image of $H^{n,q-1}(X,E) \to H^{n,q-1}(X_0,E)$ follows also from the above estimate. □

Although we shall give an alternate proof for the part 2), there is no control for those μ which satisfy $H^{n,k}_{(2)}(X,E)_{\mu\lambda,0} \cong H^{n,k}(X,E)$ $(k \geq q)$. In fact, existence of such μ is approved only after a limiting argument. Of course the bound for μ depends on the geometry of (X, g, E, h) inside X_0, but we do not know precisely how they do. Nevertheless, we shall show later that the L^2 $\bar{\partial}$-cohomology with respect to $(g, he^{-\varphi})$ coincides with the ordinary $\bar{\partial}$ -cohomology if (X, g, E, h) is q-elliptic at infinity and φ satisfies certain growth condition besides the q-convexity.

Proof of Theorem 1.1 We shall only prove the part (2) here by an argument whose crucial part (i.e. Lemma 1.1) does not seem to be in the literature. □

By the q-ellipticity modulo X_0, one can find $\epsilon > 0$ and $C > 0$ such that, for any C^∞ function ψ satisfying $(\partial\bar{\partial}\psi)_q \geq 0$ everhwhere,

$$\|u\|^2 \leq C\left(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \int_{X_{-\epsilon}} e^{-\lambda(\varphi)-\psi}|u|^2 dV_g\right) \tag{1.2}$$

holds for any C^∞ E-valued (n, k)-form u on X with compact support, if $k \geq q$. Here $|\cdot|$ denotes the pointwise length with respect to (g, h) and dV_g denotes the volume form of g. Hence, by the completeness of g and Rellich's lemma, Hörmander's criterion (see Theorem1.1.2~Theorem 1.1.4 in [23]) implies that $\dim H^{n,k}_{(2)}(X,E) < \infty$ for $k \geq q$. That $\dim H^{n,k}(X,E) < \infty$ $(k \geq q)$ can be seen similarly as in [36, Proof of Theorem 1, 10]. For the convenience of the reader we shall recall the argument below.

Given any sequence of $\bar{\partial}$-closed locally square integrable E-valued (n, k)-forms u_m $(m = 1, 2, \dots)$ on X, one can find a C^∞ convex increasing function τ on $(-\infty, \sup\lambda)$ such that u_m are all square integrable with respect to the metrics g and $he^{-\tau(\lambda(\varphi))}$. In particular, the image of $H^{n,k}_{(2)}(X,E)_{\lambda,\psi} \to H^{n,k}(X,E)$ contains the linear span of $\{u_m\}$ if $\psi = \tau(\lambda(\varphi))$. Since $\dim H^{n,k}(X,E)_{\lambda,\tau(\lambda(\varphi))} < \infty$, this implies that $H^{n,k}(X,E)$ must be finite dimensional for $k \geq q$. In particular, we can choose τ so that the map $H^{n,k}_{(2)}(X,E)_{\lambda,\tau(\lambda(\varphi))} \to H^{n,k}(X,E)$ is surjective.

From now on we assume that τ was chosen in such a way that

$$\lim_{t\to\sup\lambda} \frac{\tau(\lambda(t))}{\mu\lambda(t)} = \infty \tag{1.3}$$

holds for all $\mu \geq 1$. By (1.3) it makes sense for a square integrable E-valued (n, k)-form u with respect to $(g, he^{-\mu\lambda(\varphi)})$ to be orthogonal to a given form on X with respect to $(g, he^{-\lambda(\varphi)-\tau(\lambda(\varphi))})$. By using $\tau(\lambda(\varphi))$ as an auxiliary weight, we shall show that the maps $H^{n,k}_{(2)}(X,E)_{\mu\lambda,0} \to H^{n,k}(X,E)$ are bijective for sufficiently large μ. For that we shall show at first the following.

Lemma 1.1 *Let the situation be as above and let* $d = \sup \lambda \in (-\infty, +\infty]$. *Then, for any convex increasing function* τ *on* $(-\infty, d)$ *and for any sequence* $\mu_m \in \mathbb{N}$ $(m = 1, 2, \dots)$ *with* $\lim_{m\to\infty} \mu_m = \infty$, *there exist a subsequence* μ_{m_j} $(j = 1, 2, \dots)$, *a strictly increasing sequence* t_j *of real numbers with* $\lim_{j\to\infty} t_j = d$ *and a sequence of convex increasing functions* τ_j *satisfying* $\tau_j(t) = \tau(t)$ *for* $t \le t_j$ *and* $\tau_j'(t) = \mu_{m_j}$ *for* $t \ge t_{j+1}$, *such that there exist a constant* $C > 0$ *and* $j_0 \in \mathbb{N}$ *for which the estimate*

$$\|u\| \le C(\|\bar\partial u\| + \|\bar\partial^* u\|) \tag{1.4}$$

holds with respect to $(g, he^{-\lambda(\varphi)-\tau_j(\lambda(\varphi))})$ *for any locally square integrable* E*-valued* (n, k)*-form* u *with* $k \ge q$ *lying in the domains of* $\bar\partial$ *and* $\bar\partial^*$, *provided that* $j \ge j_0$ *and* u *is orthogonal to the space* $\mathrm{Ker}\bar\partial \cap \mathrm{Ker}\bar\partial^*$ *with respect to* $(g, he^{-\lambda(\varphi)-\tau(\lambda(\varphi))})$.

Proof In the above situation, one can find a subsequence μ_{m_j} of μ_m, an increasing sequence of convex increasing functions τ_j and increasing sequences of positive numbers t_j and A_j such that $\lim_{j\to\infty} t_j = d$, $(\partial\bar\partial\lambda(\varphi))_q > 0$ on $X \setminus X_{t_1}$, $\tau_j(t) = \tau(t)$ for $t < t_j$, $\tau_j(t_j) = \tau(t_j)$ and $\tau_j(t) = \mu_{m_j} t + A_j$ for $t > t_{j+1}$.

In this circumstance, suppose that there exist no C and j_0 as above. Then one can find a subsequence of μ_{m_j}, say $\mu_{m_{j_r}}$ $(r = 1, 2, \dots)$ and E-valued (n, k)-forms u_r such that $\|u_r\| = 1$, $\|\bar\partial u_r\| < \frac{1}{r}$ and $\|\bar\partial^* u_r\| < \frac{1}{r}$ hold with respect to $(g, he^{-\lambda(\varphi)-\tau_{j_r}(\lambda(\varphi))})$ and that $u_r \perp (\mathrm{Ker}\bar\partial \cap \mathrm{Ker}\bar\partial^*)$ hold with respect to $(g, he^{-\lambda(\varphi)-\tau(\lambda(\varphi))})$. Then one can find a locally strongly convergent subsequence of u_r with respect to $(g, he^{-\lambda(\varphi)-\tau(\lambda(\varphi))})$ whose limit is not zero because of (1.2) but obviously belongs to $\mathrm{Ker}\bar\partial \cap \mathrm{Ker}\bar\partial^* \cap (\mathrm{Ker}\bar\partial \cap \mathrm{Ker}\bar\partial^*)^\perp = \{0\}$, which is a contradiction. □

Lemma 1.1 implies that the map $H^{n,k}_{(2)}(X, E)_{\mu\lambda,0} \to H^{n,k}(X, E)$ $(k \ge q)$ is injective for sufficiently large μ. The surjectivity will follow from the denseness of the images of

$$\varinjlim_{\mu} H^{n,k}_{(2)}(X, E)_{\mu\lambda,0} \to H^{n,k}(X, E)$$

for $k \ge q - 1$. But this also follows from Lemma 1.1 because the map

$$\varinjlim_{\sigma} H^{n,k}_{(2)}(X, E)_{\lambda,\sigma(\varphi)} \to H^{n,k}(X, E)$$

is surjective for all $k \ge 0$, where σ runs through the convex increasing functions on $\mathbb{R}$. This completes the proof of the part 2) of Theorem 1.1. □

Let us recall that the L^2 $\bar\partial$-cohomology groups are canonically isomorphic to the spaces of L^2 harmonic forms if the images of the operator $\bar\partial$ are closed. We recall also that, since $\bar\partial$ is a closed operator, the image of $\bar\partial$ is closed if the L^2 $\bar\partial$-cohomology group is finite dimensional. So Theorem 1.1 essentially deals with harmonic representation and approximation. Since the geometric structures of X and E are reflected in the algebra of differential operators on X through curvature and symmetry, so that in the harmonic forms as well, Theorem 1.1 is not only a quantitative reformulation of the Andreotti-Grauert theory but also has a rich potential applicability in complex geometry. Noncompact variants of the Hodge theory and Kodaira's embedding theorem may be regarded as prototypes of such applications (cf. [29, 7, 9, 34–37, 39, 40, 41, 46] and [53]). In the next section we shall recall a result in [46] which is a refinement of Theorem 1.1 meant for that purpose.

3 Generalizations of a precise harmonic representation

The main result of [46] is much more specific than Theorem 1.1 and stated as follows.

Theorem 2.1 *Let (X, φ) be a q-convex manifold of dimension n and let (E, h) be a Hermitian holomorphic vector bundle on X whose curvature form is identically zero outside some compact set. Assume that X admits a complete Hermitian metric g which is Kählerian outside a compact set, that the eigenvalues $\gamma_1(x) \geq \cdots \geq \gamma_n(x)$ of $\partial\bar{\partial}\varphi$ at $x \in X$ with respect to g satisfy $\lim_{d\to\infty} \sup_{X\setminus X_d}\gamma_1 = 1$ and $\lim_{d\to\infty} \inf_{X\setminus X_d}\gamma_{n-q+1} = 1$ and that the least eigenvalue of $\partial\bar{\partial}\varphi - 8\partial\varphi\bar{\partial}\varphi$ on $X \setminus X_d$ is estimated from below by $\frac{-1}{100n}$ as $d \to \infty$. Then*

$$\dim H^{j,k}_{(2)}(X, E) < \infty \quad \text{and} \quad H^{j,k}(X, E) \cong H^{j,k}_{(2)}(X, E)$$

hold for $j + k \geq n + q$.

The proof of Theorem 2.1 is similar as that of Theorem 1.1. The point is that, expressing in our terminology here, replacing φ by $\varphi - d$ if necessary for some $d \in \mathbb{R}$, $(X, \varphi, E \otimes (\bigwedge^j T^*_X) \otimes K^*_X, g, h \otimes g_{(j)} \otimes \det g, \lambda)$ is q-elliptic modulo X_0 for some *bounded* increasing function λ, where T^*_X denotes the holomorphic cotangent bundle of X, K^*_X denotes the dual bundle of K_X and $g_{(j)}$ denotes the fiber metric of $\bigwedge^j T^*_X$ of X induced from g. This follows from the assumption on $\partial\bar{\partial}\varphi - 8\partial\varphi\bar{\partial}\varphi$ and those on g and h near the infinity, by virtue of a calculation found by Donnelly and Fefferman [12]. For the detail, see [46].

Note that the condition on φ is satisfied in many cases. For instance, one can take the logarithm of the Bergman kernel as φ on bounded domains in $\mathbb{C}^n$ which are strictly pseudoconvex or homogeneous (cf. [12] and [24]). Recently it turned out that Theorem 2.1 has an interesting application to the Chern forms on strongly pseudoconvex CR manifolds (cf. [55]).

We are going to present some variants of Theorem 1.1 which lie between Theorem 1.1 and Theorem 2.1, in the sense that the situation is less general than Theorem 1.1 but the fiber metrics $he^{-\mu\lambda(\varphi)}$ for unbounded λ are also taken into account. First of all we shall refine Theorem 1.1 to the following.

Theorem 2.2 *Let (X, φ, E, h) be as in Theorem 1.1, let g be a complete Hermitian metric on X and let $\lambda : \mathbb{R} \to \mathbb{R}$ be a C^∞ increasing function such that $(X, \varphi, E, g, h, \lambda)$ is q-elliptic modulo X_0. Assume moreover that $c_1 := \inf_{X\setminus X_0} (\partial\bar{\partial}\lambda(\varphi))_q > 0$ and that there exists a sequence of bounded increasing functions κ_μ ($\mu \in \mathbb{N}$) and a positive number δ such that κ_μ converges locally uniformly to λ and $(\partial\bar{\partial}\kappa_\mu(\varphi))_q(x) \geq (\delta - 1)(c_0 + c_1)$ for all $x \in X$ and $\mu \in \mathbb{N}$. Then, for any $b \geq 0$, the E-valued L^2 $\bar{\partial}$-cohomology groups $H^{n,k}_{(2)}(X, E)_b$ of X with respect to g and $he^{-(1+b)\lambda(\varphi)}$ satisfy the following.*

- (i) $\dim H^{n,k}_{(2)}(X, E)_b < \infty$ *and* $H^{n,k}_{(2)}(X, E)_b \cong H^{n,k}(X, E)$ *for all* $k \geq q$.
- (ii) *The homomorphism* $H^{n,q-1}_{(2)}(X, E)_b \to H^{n,q-1}(X, E)$ *has a dense image.*
- (iii) *If 0 is not a critical value of φ, the restriction homomorphisms $H^{n,k}_{(2)}(X, E)_b \longrightarrow H^{n,k}(X_0, E)$ are isomorphisms for all $k \geq q$ and the image of $H^{n,q-1}_{(2)}(X, E)_b \to H^{n,q-1}(X_0, E)$ is dense.*

The proof of Theorem 2.2 is a simplified modification of that of Theorem 1.1 in the spirit of Theorem 2.1. The role of τ_j will be played by a family of bounded functions.

Proof of Theorem 2.2 By the q-ellipticity of $(X, \varphi, E, g, h, \lambda)$ modulo X_0, that $\dim H^{n,k}_{(2)}(X,E)_0 < \infty$ for $k \geq q$ is contained in Theorem 1.1. We shall show that $H^{n,k}_{(2)}(X,E)_0 \cong H^{n,k}(X,E)$ for all $k \geq q$. For that, let us take κ_μ as in the assumption. Since κ_μ are bounded, the L^2 cohomology groups $H^{n,k}_{(2)}(X,E)_{\lambda,\kappa_\mu(\varphi)}$ are isomorphic to each other for all μ. Moreover one knows from (1) of Theorem 1.1 that for all μ the estimate

$$\delta c_0 \|u\|^2 \leq \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 \tag{2.1}$$

with respect to $(g, he^{-\lambda(\varphi)-\kappa_\mu(\varphi)})$ holds for any compactly supported C^∞ E-valued (n,k)-form u with $k \geq q$ on X satisfying $\mathrm{supp} u \subset X \setminus X_0$. Hence, similarly as in Theorem 1.1.2, there exist estimates of type (1.4) for the orthocomplement of the space of harmonic forms on X with respect to $(g, he^{-2\lambda(\varphi)})$. Hence the map $H^{n,k}_{(2)}(X,E)_0 \to H^{n,k}_{(2)}(X,E)_1 (= H^{n,k}_{(2)}(X,E)_{2\lambda,0})$ is injective. Similarly one has the injectivity of $H^{n,k}_{(2)}(X,E)_0 \to H^{n,k}_{(2)}(X,E)_b$ for all $b \geq 0$ and $k \geq q$.

To prove the surjectivity and ii), let $k \geq q$ and take any locally square integrable $\bar{\partial}$-closed E-valued $(n, k-1)$-form on X, say v, and any compact set $K \subset X$. Then one can find $d > 0$ such that $K \subset X_d$. Replacing φ by $\varphi - d$ if necessary, we may assume that $d = 0$ in advance. By Theorem 1.1, it suffices to approximate v by assuming that v is square integrable with respect to $(g, he^{-\mu_0\lambda(\varphi)})$ for some $\mu_0 \geq 1$. In this situation, one can solve the $\bar{\partial}$-equation $\bar{\partial} u = \bar{\partial}\chi(\varphi) \wedge v$ for a C^∞ function $\chi : \mathbb{R} \to [0,1]$ satisfying $\mathrm{supp}\chi \subset (-\infty, 1]$ and $\chi|_{(-\infty,0]} = 1$ with uniform (in μ) L^2 norm estimates with respect to $(g, he^{-\mu_0\lambda(\varphi)-\kappa_\mu(\varphi)})$, for sufficiently large μ.

Hence, given any $\epsilon > 0$, one can choose χ so that there exists a locally square integrable E-valued $(n, k-1)$-form u_μ on X satisfying $\bar{\partial}(\chi(\varphi)v - u_\mu) = 0$ and $\int_X e^{-\lambda(\varphi)-\kappa_\mu(\varphi)}|u_\mu|^2 dV_g < \epsilon$. Therefore the image of the map $H^{n,k-1}_{(2)}(X,E)_{\mu_0-1} \to H^{n,k-1}(X,E)$ is dense, so that one eventually arrives at the denseness of the image of $H^{n,k-1}_{(2)}(X,E)_0 \to H^{n,k-1}(X,E)$, too. Thus we obtain ii) in particular and i) holds since it is true for sufficiently large b. iii) follows from Theorem 1.1.3 because of i) and ii). □

By the way, it is quite easy to see that the method in the proof of Proposition 3.4.5 in [23], as was outlined after Theorem 1.2, can be applied to show the following.

Theorem 2.3 *Let (X, φ), (E, h), g and λ be as in Theorem 2.2. Assume moreover that there exists an increasing sequence of C^∞ bounded increasing functions κ_μ on $\mathbb{R}$ such that $\kappa_\mu|_{(-\infty,0]} \equiv 0$, $\lim_{\mu\to\infty} \kappa_\mu(t) = \infty$ for all $t > 0$ and that the eigenvalues of $\kappa''_\mu(\varphi)\partial\varphi\bar{\partial}\varphi$ with respect to g are bounded from below by a constant which does not depend on μ. Then the conclusions $(i) \sim (iii)$ of Theorem 2.2 hold.*

The proof may well be left to the reader.

4 A comparison theorem

Now we shall examine the setting where $X = M \setminus |D|$ for some compact complex manifold M and an effective divisor D on M. We assume that E extends to a holomorphic vector bundle over M, which we shall denote also by E by an abuse of notation.

As a preliminary to Theorem 0.2 and Theorem 0.3, we observe that one can deduce the following comparison theorem from Theorem 1.1.

Theorem 3.1 *Let M be a compact complex manifold of dimension n and let D be an effective divisor on M such that the complement of its support $|D|$ admits a C^∞ exhaustion function Φ satisfying the following conditions* (1) *and* (2).

1. $\Phi + \log|s|$ *is a bounded function on* $M \setminus |D|$ *for a canonical section s of* $[D]$ *where* $|s|$ *denotes the length of s with respect to some fiber metric of* $[D]$.
2. $\liminf_{x\to|D|} (\partial\bar\partial\Phi)_q(x) > 0$ *holds with respect to some Hermitian metric on M.*

Then there exists a neighborhood W of $|D|$ such that, for any holomorphic vector bundle $E \to M$, there exists a positive integer μ_0 such that the natural restriction homomorphisms

$$H^{n,k}(M, E\otimes[D]^\mu) \longrightarrow H^{n,k}(M\setminus|D|, E) \tag{3.1}$$

and

$$H^{n,k}(M\setminus|D|, E) \longrightarrow H^{n,k}(M\setminus\overline{W}, E) \tag{3.2}$$

are bijective for $k \geq q$ and $\mu \geq \mu_0$.

Proof Let $\Phi : M\setminus|D| \to \mathbb{R}$ be a C^∞ exhaustion function satisfying (1) and (2) with respect to a Hermitian metric g_M on M. Since $|D|$ is analytic and of codimension one, there exists a complete Hermitian metric on $M\setminus|D|$ of the form $g_M + \partial\bar\partial\kappa$ for some bounded C^∞ function κ on $M\setminus|D|$ such that the eigenvalues $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ of $\partial\bar\partial\kappa$ with respect to g_M satisfy $\lim_{x\to|D|}\gamma_j(x) = 0$ for $1 \leq j \leq n-1$ and $\lim_{x\to|D|}\gamma_n(x) = \infty$. (One may put $\kappa = \frac{1}{\log\log|s|^{-1}}$ near $|D|$ for instance.) Hence, since $\liminf_{x\to|D|}(\partial\bar\partial\Phi)_q(x) > 0$ by the assumption, for any Hermitian holomorphic vector bundle (E, h_E) over M one can find $m > 0$ such that $(M\setminus|D|, g_M + \epsilon^2\partial\bar\partial\kappa, E, h_E e^{-m(\Phi+\epsilon\kappa)})$ is q-elliptic at infinity if ϵ is a sufficiently small positive number. Hence the conclusions (3.1) and (3.2) are obtained from (2) and (3) of Theorem 1.1, respectively, by letting $\varphi = m(\Phi + \epsilon\kappa)$, $g = g_M + \epsilon^2\partial\bar\partial\kappa$, $h = h_E e^{-m(\Phi+\epsilon\kappa)}$ and $\lambda(t) = t$, by taking into account the equivalence between the Čech cohomology and $\bar\partial$-cohomology with L^2 conditions as in [39] and [28]. □

Corollary 3.1 *In the above situation, the natural restriction homomorphism*

$$H^j(M, E\otimes[D]^\mu) \longrightarrow H^j(|D|, E\otimes[D]^\mu))$$

is surjective for $j \geq q-1$ and $\mu \geq \mu_0 + 1$.

Corollary 3.2 *Let M be as above and let $A \subset M$ be a closed complex submanifold of codimension q whose normal bundle is positive in the sense of Griffiths. Let $\pi : \tilde M \to M$ be the blow up along A and put $D = \pi^{-1}(A)$. Then, for any holomorphic vector bundle $E \to \tilde M$, one can find μ_0 such that*

$$H^j(\tilde M, E\otimes[D]^\mu) \longrightarrow H^j(\tilde M\setminus D, E)$$

is bijective if $j \geq q$ and $\mu \geq \mu_0$.

We note that the case $D = 0$ ($|D| = \emptyset$) is trivially allowed in the above statements. On the other hand, if $D \neq 0$ one has the following because (1) and (2) are obviously satisfied for $D \neq 0$ if $q = n$. Recall that $M \setminus |D|$ has then the vanishing top cohomology for all coherent analytic sheaves since it is n-complete by Greene-Wu's theorem in [20] (see also [38] and [10]).

Corollary 3.3 *For any connected complex manifold M of dimension n, for any holomorphic vector bundle $E \to M$ and for any effective divisor $D \neq 0$ on M, $H^{n,n}(M, E \otimes [D]^{\mu}) = 0$ holds for sufficiently large μ.*

Remark 3.3 In the case where M is projective algebraic, Theorem 3.1 is essentially contained in Okonek's generalization in [48] of Serre's GAGA principle in [51]. See also ([49], Lemma 15). Corollary 3.3 is a special case of a vanishing theorem of Griffiths obtained in [21] by a different method, up to the existence of a fiber metric of $[D]$ whose scalar curvature is everywhere positive, which is quite elementary .

5 Proof of Theorem 0.2

Proof of a) Since (X, g, E, h) is q-elliptic at infinity and Φ is a C^{∞} exhaustion function on X satisfying

$$\sup\{\Phi(x); (\partial\bar{\partial}\Phi)_q(x) < 0\} < \infty,$$

we may assume in advance that $(X, \Phi, E, g, h, \lambda(t) = t)$ is q-elliptic modulo X_0. For any $\mu \in \mathbb{N}$ we put $k_{\mu}(t) = t$ for $t < \mu$ and

$$k_{\mu}(t) = -2\mu \log\left(\frac{1}{t} + \frac{1}{\mu}\right) - 2\mu \log\frac{\mu}{2} + \mu \quad \text{for} \quad t \geq \mu.$$

Then k_{μ} is of class C^1,

$$(\partial\bar{\partial}k_{\mu}(\Phi))_q = (\partial\bar{\partial}\Phi)_q \geq 0$$

at $x \in X$ if $\Phi(x) < \mu$ and

$$\partial\bar{\partial}k_{\mu}(\Phi) = \frac{2\mu^2}{\mu\Phi + \Phi^2}\left\{\partial\bar{\partial}\Phi - \left(\frac{1}{\mu + \Phi} + \frac{1}{\Phi}\right)\partial\Phi\bar{\partial}\Phi\right\}$$

$$\left(= \frac{2\mu^2}{\mu\Phi + \Phi^2}\left(\partial\bar{\partial}\Phi - \frac{\mu + 2\Phi}{\mu\Phi + \Phi^2}\partial\Phi\bar{\partial}\Phi\right)\right)$$

at $x \in X$ if $\Phi(x) > \mu$.

Hence it is easy to see that

$$\liminf_{\mu\to\infty}\{\partial\bar{\partial}(2\Phi + k_{\mu}(\Phi))_q(x); \Phi(x) > \mu\} \geq 0$$

holds, since

$$\liminf_{c\to\infty}\{(\partial\bar\partial\Phi-\Phi^{-1}\partial\Phi\bar\partial\Phi)_q(x);\Phi(x)>c)\}\geq 0$$

by assumption. Therefore, since $(X,\Phi-\mu_0,E,g,he^{-\gamma(\Phi+k_\mu(\Phi))},t)$ becomes q-elliptic modulo X_0 for all $\mu\geq\mu_0$ for sufficiently large μ_0 and for all $\gamma\in[0,\frac{1}{2}]$ with a common q-ellipticity constant, the approximation argument works similarly as in Lemma 1.1 and Theorem 2.2.i to show that the map $H^{n,k}_{(2),\Phi}(X,E)\to H^{n,k}_{(2),(1+\Gamma)\Phi}(X,E)$ $(k\geq q)$ is bijective for any $\Gamma\in[0,1]$ and that so is $H^{n,k}_{(2),a\Phi}(X,E)\to H^{n,k}_{(2),(1+\Gamma)a\Phi}(X,E)$ $(k\geq q)$ a fortiori for any $\Gamma\in[0,1]$ and $a>1$.

Hence, eventually one has $H^{n,k}_{(2),\Phi}(X,E)\cong H^{n,k}(X,E)$ $(k\geq q)$, since $H^{n,k}_{(2),a\Phi}(X,E)\cong H^{n,k}(X,E)$ $(k\geq q)$ holds for sufficiently large a by Theorem 1.1.2. We note that $\sup\{\Phi(x);(\partial\bar\partial\Phi)_q(x)<0\}<\infty$ cannot be weakened to $\lim_{c\to\infty}\inf\{(\partial\bar\partial\Phi)_q(x);\Phi(x)>c\}\geq 0$ at the last point. The denseness of the image of $H^{n,q-1}_{(2),\Phi}(X,E)$ in $H^{n,q-1}(X,E)$ holds by a similar reason. □

Proof of b) By a), it suffices to show that

$$H^{n,k}_{(2),\mu\log\Phi}(X,E)\cong H^{n,k}_{(2),\Phi}(X,E)\ \ (k\geq q)$$

holds for sufficiently large μ. For that we set

$$\ell_\mu(t)=t$$

for $t<\mu$ and

$$\ell_\mu(t)=\mu\log t-\mu\log\mu+\mu$$

for $t\geq\mu$. Then, by the assumption (0.1), the approximation argument works similarly as above to conclude that $H^{n,k}_{(2),\mu\log\Phi}(X,E)$ are isomorphic to $H^{n,k}_{(2),\Phi}(X,E)(\cong H^{n,k}(X,E))$ for $k\geq q$ if μ is sufficiently large and the map

$$\varinjlim_{\mu} H^{n,q-1}_{(2),\mu\log\Phi}(X,E)\to H^{n,q-1}(X,E)(=\overline{H^{n,q-1}_{(2),\Phi}(X,E)})$$

has a dense image. □

That's all, at least at the moment, for the generality of isomorphism and approximation by the technique of infinitely many weights originated from Proposition 3.4.5 in [23]. A new aspect of the method of approximation by infinitely many weights is summarized as follows if one does not stick so much to the precise control of the weights for the harmonic representation. The proof may well be left to the reader as a quite easy exercise.

Theorem 4.1 *Let (X,g) be a complete Hermitian manifold of dimension n and let (E,h) be a Hermitian holomorphic vector bundle such that (X,g,E,h) is q-elliptic at infinity for some $q\in\mathbb{N}$ and there exists a C^∞ exhaustion function $\varphi: X\to\mathbb{R}$ satisfying*

$$\lim_{c\to\infty}\inf\{(\partial\bar\partial\varphi)_q(x);\varphi(x)>c\}\geq 0.$$

Then $\dim H^{n,k}_{(2),\mu\varphi}(X,E)<\infty$ $(k\geq q)$ for all $\mu\geq 0$ and there exists a strictly convex increasing function $\lambda:\mathbb{R}\to\mathbb{R}$ with $\lim\limits_{t\to\infty}\lambda'(t)=\infty$ such that $\dim H^{n,k}_{(2),\lambda(\varphi)}(X,E)<\infty$ $(k\geq q)$, the maps

$$H^{n,k}_{(2),\mu\varphi}(X,E) \to H^{n,k}_{(2),\lambda(\varphi)}(X,E) \ \ (k \geq q)$$

are bijective for sufficiently large μ *and the map*

$$\varinjlim_{\mu} H^{n,q-1}_{(2),\mu\varphi}(X,E) \to H^{n,q-1}_{(2),\lambda(\varphi)}(X,E)$$

has a dense image. If

$$\lim_{c\to\infty} \inf\{(\partial\bar{\partial}\varphi)_q(x); \varphi(x) > c\} > 0,$$

then one can choose λ *so that* $H^{n,k}_{(2),\lambda(\varphi)}(X,E) \cong H^{n,k}(X,E)\,(k \geq q)$ *and the map*

$$H^{n,q-1}_{(2),\lambda(\varphi)}(X,E) \to H^{n,q-1}(X,E)$$

has a dense image.

We shall proceed to apply Theorem 0.2 by restricting ourselves to geometrically special cases.

6 Extension, comparison and vanishing on weakly 1-complete manifolds

In this section, we shall prove Theorem 0.3 and Theorem 0.4 after recalling the basic notion of (semi-) positivity for vector bundles and introducing the notion of pseudoconcavity of effective divisors.

From now on, let M be a weakly 1-complete manifold of dimension n equipped with a C^∞ plurisubharmonic exhaustion function ρ and let (E, h) be a Hermitian holomorphic vector bundle over M. We also fix a Hermitian metric g_M on M. We recall that the curvature form of h, denoted by Θ_h, is defined as the $(E^* \otimes E)$-valued (1, 1)-form whose exterior multiplication from the left hand side coincides with $(\partial_h + \bar{\partial})^2$, where ∂_h is defined as $h^{-1}\circ\partial\circ h$ by identifying h with a map transforming E-valued forms to $\overline{E}^*$-valued ones. By identifying $h\circ\Theta_h$ with a section of $(T_M \otimes E)^* \otimes \overline{(T_M \otimes E)^*}$, it is naturally regarded as a Hermitian form on the fibers of $T_M \otimes E$. (E, h) is said to be *Nakano positive* (resp. *Nakano semipositive*) if $h\Theta_h$ is fiberwise positive (resp. semipositive) in this sense. By an abuse of language, we shall also say that Θ_h is Nakano positive (resp. Nakano semipositive) in this case. Nakano will not be referred to if the rank of E or the dimension of M is one.

For any (complex) analytic set $A \subset M$, we say that $E|_A$ is Nakano (semi-) positive if E admits a fiber metric $\tilde{h}$ such that $\tilde{h}\Theta_{\tilde{h}}$ is (semi-) positive on the fibers of $T_A \otimes E$, where T_A denotes the set of Zariski tangent vectors of A.

Let us recall that an effective divisor on M is by definition a locally finite formal linear combination $\sum m_j D_j$ of irreducible analytic sets $D_j \subset M$ of codimension one with positive integral coefficients m_j. By an abuse of language, 0 is admitted to be effective. $\bigcup_j D_j$ is called the support of D and denoted by $|D|$. It is easy to see that $M \setminus |D|$ is weakly 1-complete if the line bundle $[D]$ associated to D is seimipositive. In fact, letting b be a fiber metric of $[D]$ whose curvature form is semipositive on M, letting s be a canonical section of $[D]$ and letting $|s|_b$ be the pointwise length of s with respect to b,

$-\log|s|_b + \lambda(\rho)$ will become a plurisubharmonic exhaustion function on $M \setminus |D|$ for a sufficiently rapidly growing convex increasing function λ.

In what follows we shall restrict ourselves to the case where $|D|$ is nonempty and compact. Instead of the semipositivity of $[D]$ we shall only assume that $[D]|_{|D|}$ is semipositive. In this situation it is easy to see that

$$\liminf_{x\to|D|}(|s|_b^{-2+\epsilon}\partial\bar\partial\log|s|_b^{-1})_1(x) \geq 0 \tag{5.1}$$

holds for any $\epsilon > 0$ with respect to the metric $g := g_M + \partial\bar\partial(e^\rho + \frac{\delta}{\log((\log|s|_b)^2+1)})$ for some $\delta > 0$ and for some fiber metric b of $[D]$. For any positive number a, we shall say that D is ***pseudoconcave of order a*** if there exists a fiber metric $\tilde b$ $(=\tilde b(a))$ of $[D]$ and a neighborhood U of $|D|$ such that $|s|_{\tilde b}^{-a}$ is plurisubharmonic on $U \setminus |D|$. $|s|_{\tilde b}^{-a}$ will be called then a ***canonical exhaustion of order a***.

Since

$$\partial\bar\partial\Phi = \Phi\partial\bar\partial\log\Phi + \Phi^{-1}\partial\Phi\bar\partial\Phi$$

holds for any positive C^∞ function Φ, (5.1) implies that for any canonical exhaustion Ψ of order $a < 2$,

$$\liminf_{x\to|D|}(\partial\bar\partial\Psi - \Psi^{-1}\partial\Psi\bar\partial\Psi)_1(x) \geq 0$$

holds with respect to g.

Proof of Theorem 0.3 By the assumption on E, we may assume that g_M is Kählerian on a neighborhood of $|D|$ and outside a compact subset of M. By such a choice of g_M, it is routine that one can find a fiber metric h of E and positive numbers C and δ in such a way that $(M \setminus |D|, g, E, he^{-\delta(\log(\log|s|_b+C))^{-1}})$ is 1-elliptic at infinity. Moreover, since M is weakly 1-complete and $E|_{M\setminus K}$ is Nakano positive for some compact set $K \subset M$, one may choose h in advance so that

$$H^{n,k}_{(2),-2\mu\log|s|_b}(M \setminus |D|, E) \cong H^{n,k}(M, E \otimes [D]^\mu)$$

canonically for all $\mu \geq 0$ and $k \geq 0$ (see the proof of Theorem 1.1 and the remark at the end of the proof of Theorem 3.1). Note that δ can be chosen arbitarily small. On the other hand, by (5.1), which holds for $\varphi = \log|s|_b^{-1}$, one can infer from Theorem 4.1 that the sequence $\dim H^{n,k}(M, E \otimes [D]^\mu)$ $(\mu = 1, 2, \dots)$ stabilizes for sufficiently large μ. This is the end of the proof of the first part of the assertion.

To see the validity of the second part, instead of Theorem 4.1 we appeal to Theorem 0.2 by setting $\Phi = |s|_b^{-a}$ for some $1 < a < 2$ and some fiber metric b of $[D]$ so that Φ is plurisubharmonic. Then as we have seen above, Φ satisfies the assumptions of Theorem 0.2.*a* for $q = 1$. Moreover, (0.1) holds for $q = 1$ because of (5.1). Hence the desired conclusion is an immediate consequence of Theorem 0.2.*b*. □

We note that the second assertion in Theorem 0.3 contains something new about the $\bar\partial$ -cohomology of weakly 1-complete manifolds. For instance, let M be a weakly 1-complete Kähler manifold of dimension n and let $E \to M$ be a holomorphic vector bundle which admits a fiber metric whose curvature form is Nakano positive outside a compact subset say K of M. In this situation, Takegoshi [54] showed that $H^{n,k}(M, E) = 0$ for $k \geq 1$. If there exists a compact nonzero effective divisor D on $M \setminus K$ which is pseudoconcave of order >1, then although we do not know whether or not $E \otimes [D]^{\mu}$ has the positivity property outside some compact set of M, we have $H^{n,k}(M, E \otimes [D]^{\mu}) = 0$ $(k \geq 1)$ for sufficiently large μ. Indeed, since $H^{n,k}(M \setminus |D|, E) = 0$ $(k \geq 1)$ holds by Takegoshi's theorem by the Nakano positivity of $E|_{M\setminus K}$ and the weakly 1-completeness of $M \setminus |D|$, by the comparison assertion of Theorem 0.3 one has Corollary 0.2.

Definition 5.1 An effective divisor D is said to be **of finite type** $\boldsymbol{\nu}$ if $[D]|_{|D|}$ is topologically trivial and $[D]|_A$ is not equivalent to any unitary flat line bundle, where $A = (|D|, \mathscr{O}_M/\mathscr{I}_{|D|}^{\nu+1})$.

Ueda proved the following in ([57], §3).

Lemma 5.1 *If D is of finite type ν and $|D|$ is a compact smooth curve, then νD is pseudoconcave of order a for any $a > 1$.*

Combining Lemma 5.1 with Theorem 0.3, we obtain Theorem 0.4.

Remark 5.2 Given a compact complex curve C smoothly embedded into a complex surface S, in [57] it is also proved that $[C]$ is not semipositive if C is of finite type (cf. [57], Theorem 2). If $C \cdot C = 0$ and C is not of finite type, it is not known whether or not $[C]$ is semipositive. Nevertheless, if S is compact and $[C]$ is semipositive, the curvature form of $[C]$ must degenerate everywhere, by virtue of Siu's solution [52] of Grauert-Riemenschneider's conjecture. Hence, as was recently shown by Koike in [27], $[C]$ is $U(1)$-flat on a neighborhood of C.

7 Supplementary remarks

In the context of the Levi problem, the L^2 method on q-convex manifolds has been applied to show basic function theoretic properties of their cycle spaces (cf. [33, 42]). In the same vein, Theorem 3.1 can be applied to prove their algebraicity in the following form.

Theorem 6.1 *Let M be a compact Kähler manifold of dimension n and let $A \subset M$ be a closed complex submanifold of codimension q whose normal bundle is positive in the sense of Griffiths. Then the Barlet space of $(q-1)$-dimensional cycles in $M \setminus A$ is holomorphically convex. Moreover, its irreducible components are equivalent to affine algebraic varieties up to modifications along compact sets.*

We recall that the study of embeddings with positive normal bundles was motivated by the rigidity problem, of Nirenberg and Spencer [32], which goes as follows: Given a germ of embedding $A \subset M$ ($\dim A \geq 2$) with positive normal bundle, is M determined by a finite neighborhood of A? Griffiths [21] answerted this question affirmatively, provided that the normal bundle of A in M is *sufficiently positive*. Theorem 6.1 is contained in Fujiki's result in [13] if $q = n$. If $q = 1$, it is contained essentially in [15] and stated more explicitly in [18]. See also [4, 33] and [8].

References

1. Abdelkader, O.: Généralisation d'un théorémede finitude. C. R. Acad. Sci. Paris Séér I. Math. **293**(14), 629–632 (1981)
2. Akizuki, Y., Nakano, S.: Note on Kodaira–Spencer's proof of Lefschetz theorems. Proc. Jpn. Acad. **30**, 266–272 (1954)
3. Andreotti, A., Grauert, H.: Théoréme de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. France **90**, 193–259 (1962)
4. Andreotti, A., Norguet, F.: Problème de Levi et convexité holomorphe pour les classes de cohomologie. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **20**, 197–241 (1966)
5. Andreotti, A., Vesentini, E.: Sopra un teorema di Kodaira. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **15**, 283–309 (1961)
6. Andreotti, A.: Vesentini, E. Les théorémes fondamentaux de la théorie des espaces holomorphiquement complets, 1963 Topologie et géométrie différentielle (Séminaire C. Ehresmann), Vol. IV (1962–63), Cahier, pp. 1–31, Institut H. Poincaré, Paris
7. Andreotti, A., Vesentini, E.: Carleman estimates for the Laplace-Beltrami equation on complex manifolds. Inst. Hautes Études Sci. Publ. Math. No. **25**, 81–130 (1965). (**Erratum: Inst. Hautes ÉÉtudes Sci. Publ. Math. No. 27 (1965), 153–155**)
8. Barlet, D.: Convexité de l'espace des cycles. Bull. Soc. Math. France **106**(4), 373–397 (1978)
9. Demailly, J.-P.: Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète. Ann. Sci. École Norm. Sup. (4) **15**(3), 457–511 (1982)
10. Demailly, J.-P.: Cohomology of q-convex spaces in top degrees. Math. Z. **204**(2), 283–295 (1990)
11. Demailly, J.-P.: Analytic methods in algebraic geometry, Surveys of Modern Mathematics. **1**. International Press, Somerville, MA, Higher Education Press, Beijing, viii+231 pp (2012)
12. Donnelly, H., Fefferman, C.: L^2-cohomology and index theorem for the Bergman metric. Ann. Math. (2) **118**(3), 593–618 (1983)
13. Fujiki, A.: Projectivity of the space of divisors on a normal compact complex space. Publ. Res. Inst. Math. Sci. **18**(3), 1163–1173 (1982)
14. Fujino, O.: A transcendental approach to Kollár's injectivity theorem II. J. Reine Angew. Math. **681**, 149–174 (2013)
15. Grauert, H.: On Levi's problem and the imbedding of real-analytic manifolds. Ann. Math. (2) **68**, 460–472 (1958)
16. Grauert, H.: Über Modifikationen und exzeptionelle analytische Mengen. Math. Ann. **146**, 331–368 (1962)
17. Grauert, H.: Theory of q-convexity and q-concavity, pp. 259–284. Springer, Several complex variables VII (1994)
18. Grauert, H.: Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen. Invent. Math. **11**, 263–292 (1970)
19. Grauert, H.: Kählersche Mannigfaltigkeiten mit hyper- q-konvexem Rand, Problems in analysis (Lectures Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), pp. 61–79. Princeton Univ. Press, Princeton (1970)
20. Greene, R. E., Wu, H.: Embedding of open Riemannian manifolds by harmonic functions. Ann. Inst. Fourier (Grenoble) **25**(1), vii, 215–235 (1975)
21. Griffiths, P.-A.: The extension problem in complex analysis II. Embeddings with positive normal bundle. Am. J. Math. **88**, 366–446 (1966)
22. Griffiths, P.-A.: Hermitian differential geometry, Chern classes, and positive vector bundles, 1969 Global Analysis (Papers in Honor of K. Kodaira) pp. 185–251 Univ. Tokyo Press, Tokyo
23. Hörmander, L.: L^2 estimates and existence theorems for the $\bar{\partial}$-operator. Acta Math. **113**, 89–152 (1965)
24. Kai, C., Ohsawa, T.: A note on the Bergman metric of bounded homogeneous domains. Nagoya Math. J. **186**, 157–163 (2007)
25. Kaneko, A.: Introduction to hyperfunctions, Translated from the Japanese by Y. Yamamoto. Mathematics and its Applications (Japanese Series), 3. Kluwer Academic Publishers Group, Dordrecht; SCIPRESS, Tokyo, 1988. xiv+458 pp
26. Kodaira, K.: On a differential-geometric method in the theory of analytic stacks. Proc. Natl. Acad. Sci. U.S.A. **39**, 1268–1273 (1953)
27. Koike, T.: Linearlization of transition functions of a semi-positive line bundle along a certain submanifold. arXiv:2002.07830 to appear in Ann. Inst. Fourier
28. Matsumura, S.: A vanishing theorem of Kollár-Ohsawa type. Math. Ann. **366**(3–4), 1451–1465 (2016)
29. Nakano, S.: Vanishing theorems for weakly 1-complete manifolds, Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, Kinokuniya, Tokyo, pp. 169–179 (1973)

30. Nakano, S., Rhai, T.-S.: Vector bundle version of Ohsawa's finiteness theorems. Math. Japan. **24**(6):657–664 (1979/80)
31. Narasimhan, R.: The Levi problem for complex spaces. II. Math. Ann. **146**, 195–216 (1962)
32. Nirenberg, L., Spencer, D. C.: On rigidity of holomorphic imbeddings, 1960 Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960) pp. 133–137 Tata Institute of Fundamental Research, Bombay
33. Norguet, F., Siu, Y.-T.: Holomorphic convexity of spaces of analytic cycles. Bull. Soc. Math. France **105**(2), 191–223 (1977)
34. Ohsawa, T.: Finiteness theorems on weakly 1-complete manifolds. Publ. Res. Inst. Math. Sci. **15**(3), 853–870 (1979)
35. Ohsawa, T.: A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds. Invent. Math. **63**(2), 335–354 (1981). (**Addendum: Invent. Math. 66 (1982), No. 3, 391–393**)
36. Ohsawa, T.: On $H^{p,q}(X, B)$ of weakly 1-complete manifolds. Publ. RIMS, Kyoto Univ. **17**, 113–126 (1981)
37. Ohsawa, T.: Isomorphism theorems for cohomology groups of weakly 1-complete manifolds. Publ. Res. Inst. Math. Sci. **18**(1), 191–232 (1982)
38. Ohsawa, T.: Completeness of noncompact analytic spaces. Publ. Res. Inst. Math. Sci. **20**(3), 683–692 (1984)
39. Ohsawa, T.: Vanishing theorems on complete Kähler manifolds. Publ. Res. Inst. Math. Sci. **20**(1), 21–38 (1984)
40. Ohsawa, T.: Hodge spectral sequence on compact Kähler spaces. Publ. Res. Inst. Math. Sci. **23**(2), 265–274 (1987). (**Supplement: Publ. Res. Inst. Math. Sci. 27 (1991), No. 3, 505–507**)
41. Ohsawa, T.: Hodge spectral sequence and symmetry on compact Kähler spaces. Publ. Res. Inst. Math. Sci. **23**(4), 613–625 (1987)
42. Ohsawa, T.: An interpolation theorem on cycle spaces for functions arising as integrals of $\bar{\partial}$-closed forms. Publ. Res. Inst. Math. Sci. **43**(4), 911–922 (2007)
43. Ohsawa, T.: A remark on Hörmander's isomorphism, Complex analysis and geometry. 273–280, Springer Proc. Math. Stat., **144**, Springer, Tokyo (2015)
44. Ohsawa, T.: L^2 approaches in several complex variables. Towards the Oka-Cartan theory with precise bounds, Second edition. Springer Monographs in Mathematics. Springer, Tokyo, 2018. xi+258
45. Ohsawa, T., Pawlaschyk, T.: A survey on q-convexity. submitted for publication
46. Ohsawa, T., Takegoshi, K.: Hodge spectral sequence on pseudoconvex domains. Math. Z. **197**(1), 1–12 (1988)
47. Oka, K.: Sur les fonctions analytiques de plusieurs variables VI. Domaines pseudoconvexes. Tôhoku Math. J. **49**, 15–52 (1942)
48. Okonek, C.: A comparison theorem. Math. Ann. **273**(2), 271–276 (1986)
49. Peternell, M.: Algebraic and analytic cohomology of quasiprojective varieties. Math. Ann. **286**(1–3), 511–528 (1990)
50. Serre, J.-P.: Faisceaux algébriques cohérents. Ann. Math. (2) **61**, 197–227 (1955)
51. Serre, J.-P.: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier (Grenoble) **6**:1–42 (1955/56)
52. Siu, Y.-T.: Some recent results in complex manifold theory related to vanishing theorems for the semipositive case. Workshop Bonn 1984 (Bonn, 1984), 169–192, Lecture Notes in Math., 1111, Springer, Berlin (1985)
53. Takayama, S.: Adjoint linear series on weakly 1-complete Kähler manifolds. I. Global projective embedding. Math. Ann. **311**(3), 501–531 (1998)
54. Takegoshi, K.: A generalization of vanishing theorems for weakly 1-complete manifolds. Publ. Res. Inst. Math. Sci. **17**(1), 311–330 (1981)
55. Takeuchi, Y.: A constraint on Chern classes of strictly pseudoconvex CR manifolds. SIGMA Symmetry Integr. Geom. Methods Appl. **16**(005), 5 (2020)
56. Ueda, T.: Compactifications of $\mathbb{C} \times \mathbb{C}^*$ and $(\mathbb{C}^*)^2$. Tohoku Math. J. (2) **31**(1), 81–90 (1979)
57. Ueda, T.: On the neighborhood of a compact complex curve with topologically trivial normal bundle. J. Math. Kyoto Univ. **22**(4), 583–607 (1982/83)
58. Vesentini, E.: Lectures on Levi convexity of complex manifolds and cohomology vanishing theorems, Notes by M. S. Raghunathan. Tata Institute of Fundamental Research Lectures on Mathematics, No. **39** Tata Institute of Fundamental Research, Bombay 1967 iii+131+vi pp