Abh. Math. Semin. Univ. Hambg. (2021) 91:117–135
https://doi.org/10.1007/s12188-021-00234-2

Infinite order linear differential equation satisfied by *p*-adic Hurwitz-type Euler zeta functions

Su Hu[1] · Min-Soo Kim[2]

Received: 4 September 2020 / Accepted: 19 February 2021 / Published online: 17 March 2021

Abstract

In 1900, at the international congress of mathematicians, Hilbert claimed that the Riemann zeta function $\zeta(s)$ is not the solution of any algebraic ordinary differential equations on its region of analyticity. In 2015, Van Gorder (J Number Theory 147:778–788, 2015) considered the question of whether $\zeta(s)$ satisfies a non-algebraic differential equation and showed that it *formally* satisfies an infinite order linear differential equation. Recently, Prado and Klinger-Logan (J Number Theory 217:422–442, 2020) extended Van Gorder's result to show that the Hurwitz zeta function $\zeta(s,a)$ is also *formally* satisfies a similar differential equation

$$T\left[\zeta(s,a)-\frac{1}{a^s}\right]=\frac{1}{(s-1)a^{s-1}}.$$

But unfortunately in the same paper they proved that the operator T applied to Hurwitz zeta function $\zeta(s,a)$ does not converge at any point in the complex plane $\mathbb{C}$. In this paper, by defining T_p^a, a p-adic analogue of Van Gorder's operator T, we establish an analogue of Prado and Klinger-Logan's differential equation satisfied by $\zeta_{p,E}(s,a)$ which is the p-adic analogue of the Hurwitz-type Euler zeta functions

$$\zeta_E(s,a)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^s}.$$

In contrast with the complex case, due to the non-archimedean property, the operator T_p^a applied to the p-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s,a)$ is convergent p-adically in

Dedicated to the memory of Prof. David Goss (1952–2017)

✉ Min-Soo Kim
mskim@kyungnam.ac.kr

Su Hu
mahusu@scut.edu.cn

[1] Department of Mathematics, South China University of Technology, Guangzhou 510640, Guangdong, China

[2] Department of Mathematics Education, Kyungnam University, Changwon, Gyeongnam 51767, Republic of Korea

the area of $s \in \mathbb{Z}_p$ with $s \neq 1$ and $a \in K$ with $|a|_p > 1$, where K is any finite extension of $\mathbb{Q}_p$ with ramification index over $\mathbb{Q}_p$ less than $p-1$.

Keywords p-adic Hurwitz-type Euler zeta function · Differential equation

Mathematics Subject Classification 11M35 · 11B68

1 Introduction

Throughout this paper we shall use the following notations.

$\mathbb{C}$ – the field of complex numbers.
p – an odd rational prime number.
$\mathbb{Z}_p$ – the ring of p-adic integers.
$\mathbb{Q}_p$ – the field of fractions of $\mathbb{Z}_p$.
$\mathbb{C}_p$ – the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

The Riemann zeta function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1, \tag{1.1}$$

it can be analytically continued to the whole complex plane except for a single pole at $s = 1$ with residue 1. In 1900, at the international congress of mathematicians, David Hilbert [5] claimed that $\zeta(s)$ is not the solution of any algebraic ordinary differential equations on its region of analyticity. In 2015, Van Gorder [19] considered the question of whether $\zeta(s)$ satisfies a non-algebraic differential equation and showed that it *formally* satisfies an infinite order linear differential equation. In fact, he established the differential equation

$$T[\zeta(s) - 1] = \frac{1}{s-1} \tag{1.2}$$

formally, where

$$T = \sum_{n=0}^{\infty} L_n \tag{1.3}$$

and

$$\begin{aligned} L_n &:= p_n(s)\exp(nD), \\ p_n(s) &:= \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{(n+1)!}\prod_{j=0}^{n-1}(s+j) & \text{if } n > 0, \end{cases} \\ \exp(nD) &:= id + \sum_{k=1}^{\infty} \frac{n^k}{k!} D_s^k \end{aligned}$$

for $D_s^k := \frac{\partial^k}{\partial s^k}$.

For $0 < a \leq 1$, $\mathrm{Re}(s) > 1$, in 1882 Hurwitz [4] defined the partial zeta functions

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, \tag{1.4}$$

which generalized (1.1). As (1.1), this function can also be analytically continued to a meromorphic function in the complex plane with a simple pole at $s=1$. Recently, Prado and Klinger-Logan [15] extended Van Gorder's result to show that the Hurwitz zeta function $\zeta(s,a)$ also *formally* satisfies a similar differential equation

$$T\left[\zeta(s,a)-\frac{1}{a^s}\right]=\frac{1}{(s-1)a^{s-1}} \tag{1.5}$$

for $s\in\mathbb{C}$ satisfying $s+n\neq 1$ for all $n\in\mathbb{Z}_{\geq 0}$, where T is the Van Gorder's operator defined as in (1.3) (see [15, Corollary 4]). But unfortunately, in the same paper they proved that

$$T\left[\zeta(s,a)-\frac{1}{a^s}\right]=\sum_{n=0}^{\infty}p_n(s)\exp(nD)\left[\zeta(s,a)-\frac{1}{a^s}\right],$$

the operator T applied to Hurwitz zeta function, does not converge at any point in the complex plane $\mathbb{C}$ (see [15, Theorem 8]). Then they defined a generalized operator G instead of T. That is, let $\mathcal{M}$ be the collection of meromorphic functions on $\mathbb{C}$ and $f\in\mathcal{M}$, define $G:\mathcal{M}\to\mathcal{M}$ by

$$G[f](s)=\sum_{n=0}^{\infty}p_n(s)f(s+n). \tag{1.6}$$

Under this linear operator, we have a convergent difference equation

$$G\left[\zeta(s,a)-\frac{1}{a^s}\right]=\frac{1}{(s-1)a^{s-1}}. \tag{1.7}$$

But it needs to mention that G is not a differential operator.

For $\mathrm{Re}(s)>0$, the Euler zeta function (also called alternative series or Dirichlet eta function) is defined by

$$\zeta_E(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}. \tag{1.8}$$

This function can be analytically continued to the complex plane without any pole. For $\mathrm{Re}(s)>0$, (1.1) and (1.8) are connected by the following equation

$$\zeta_E(s)=(1-2^{1-s})\zeta(s). \tag{1.9}$$

By Weil's history [21, p. 273–276] (also see a survey by Goss [3, Sect. 2]), Euler used (1.8) to "prove"

$$\frac{\zeta_E(1-s)}{\zeta_E(s)}=\frac{-\Gamma(s)(2^s-1)\cos(\pi s/2)}{(2^{s-1}-1)\pi^s}, \tag{1.10}$$

which leads to the functional equation of $\zeta(s)$.

For $s\in\mathbb{C}$ and $a\neq 0,-1,-2,\ldots$, the Hurwitz-type Euler zeta function is defined as the Hurwitz zeta function (1.4) twisted by $(-1)^n$

$$\zeta_E(s,a)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^s}. \tag{1.11}$$

This function can also be analytically continued to the complex plane without any pole. It represents a partial zeta function of cyclotomic fields in one version of Stark's conjectures in algebraic number theory (see [11, p. 4249, (6.13)]). Recently, several interesting properties for the function $\zeta_E(s,a)$ have been studied, including its Fourier expansion and several integral representations [7], special values and power series expansions [6], convexity properties [2], etc.

In [10], using the fermionic p-adic integral (see (2.6) below), we defined $\zeta_{p,E}(s,a)$, the p-adic analogue of Hurwitz-type Euler zeta functions (1.11), which interpolates (1.11) at non-positive integers (see Theorem 2.4 below), so called the p-adic Hurwitz-type Euler zeta functions. In the same paper, we also proved several properties of $\zeta_{p,E}(s,a)$, including the analyticity, the convergent Laurent series expansion, the distribution formula, the difference equation, the reflection functional equation, the derivative formula and the p-adic Raabe formula.

In this note, we define a p-adic analogue of the operator T, denoted by T_p^a (see (2.10) below). Under this operator, the p-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s,a)$ satisfies an infinite order linear differential equation

$$T_p^a\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right]=\frac{1}{s-1}\left(\langle a-1\rangle^{1-s}-\langle a\rangle^{1-s}\right) \tag{1.12}$$

(see Theorem 3.5). In contrast with the complex case, the left hand side of the above equation is convergent everywhere for $s\in\mathbb{Z}_p$ with $s\neq 1$ and $a\in K$ with $|a|_p>1$, where K is any finite extension of $\mathbb{Q}_p$ with ramification index over $\mathbb{Q}_p$ less than $p-1$ (see Corollary 3.8 and Remarks 3.7 and 3.9 below).

2 Preliminaries

2.1 *p*-adic Teichmüller character

To our purpose, in this subsection, we recall some notions from p-adic analysis, including the p-adic Teichmüller character $\omega_v(a)$ and the projection function $\langle a\rangle$ for $a\in\mathbb{C}_p^\times$. Our approach follows Tangedal and Young in [18] closely.

Given $a\in\mathbb{Z}_p, p\nmid a$ and $p>2$, there exists a unique $(p-1)$th root of unity $\omega(a)\in\mathbb{Z}_p$ such that

$$a\equiv\omega(a)\pmod{p},$$

where ω is the Teichmüller character. Let $\langle a\rangle=\omega^{-1}(a)a$, so $\langle a\rangle\equiv 1\pmod{p}$.

In what follows we extend the definition domain of the projection function $\langle a\rangle$ from $\mathbb{Z}_p$ to $\mathbb{C}_p$. Fixed an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$, denote the image of the set of positive real rational powers of p under this embedding in $\mathbb{C}_p^\times$ by $p^{\mathbb{Q}}$, and the group of roots of unity with order not divisible by p in $\mathbb{C}_p^\times$ by μ. Given $a\in\mathbb{C}_p$ with $|a|_p=1$, there exists a unique element $\hat{a}\in\mu$ such that

$$|a-\hat{a}|_p<1, \tag{2.1}$$

which is also named the Teichmüller representative of a; it can also be defined from $\hat{a} = \lim_{n\to\infty} a^{p^{n!}}$. Then we extend this definition to $a \in \mathbb{C}_p^\times$ by

$$\hat{a} := (\widehat{a/p^{v_p(a)}}), \tag{2.2}$$

that is, we define $\hat{a} = \hat{u}$ if $a = p^r u$ with $p^r \in p^{\mathbb{Q}}$ and $|u|_p = 1$, then we define the function $\langle \cdot \rangle$ on $\mathbb{C}_p^\times$ by

$$\langle a \rangle = p^{-v_p(a)} a/\hat{a}.$$

Now we define $\omega_v(\cdot)$ on $\mathbb{C}_p^\times$ by

$$\omega_v(a) = \frac{a}{\langle a \rangle} = p^{v_p(a)}\hat{a}. \tag{2.3}$$

From this we get an internal product decomposition of multiplicative groups

$$\mathbb{C}_p^\times \simeq p^{\mathbb{Q}} \times \mu \times D, \tag{2.4}$$

where $D = \{a \in \mathbb{C}_p : |a-1|_p < 1\}$, given by

$$a = p^{v_p(a)} \cdot \hat{a} \cdot \langle a \rangle \mapsto (p^{v_p(a)}, \hat{a}, \langle a \rangle). \tag{2.5}$$

As remarked by Tangedal and Young in [18], this decomposition of $\mathbb{C}_p^\times$ depends on the choice of embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$; the projections $p^{v_p(a)}, \hat{a}, \langle a \rangle$ are uniquely determined up to roots of unity. However for $a \in \mathbb{Q}_p^\times$ the projections $p^{v_p(a)}, \hat{a}, \langle a \rangle$ are uniquely determined and do not depend on the choice of the embedding. Notice that the projections $a \mapsto p^{v_p(a)}$ and $a \mapsto \hat{a}$ are constant on discs of the form $\{a \in \mathbb{C}_p : |a-y|_p < |y|_p\}$ and therefore have derivative zero whereas the projections $a \mapsto \langle a \rangle$ has derivative $\frac{d}{da}\langle a \rangle = \langle a \rangle / a$.

2.2 The fermionic *p*-adic integral and the *p*-adic Hurwitz-type Euler zeta functions

In this subsection, we recall the definition of the p-adic Hurwitz-type Euler zeta functions $\zeta_{p,E}(s,a)$ from the fermionic p-adic integral. For details, we refer to [10].

Let $UD(\mathbb{Z}_p)$ be the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$ (see [1, §11.1.2]). The fermionic p-adic integral $I_{-1}(f)$ on $\mathbb{Z}_p$ of a function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(t)d\mu_{-1}(t) = \lim_{r\to\infty} \sum_{k=0}^{p^r-1} f(k)(-1)^k. \tag{2.6}$$

The fermionic p-adic integral (2.6) was independently found by Katz [8, p. 486] (in Katz's notation, the $\mu^{(2)}$-measure), Shiratani and Yamamoto [17], Osipov [14], Lang [12] (in Lang's notation, the $E_{1,2}$-measure), Kim [9] from very different viewpoints.

For $a \in \mathbb{C}_p^\times$ and $s \in \mathbb{C}_p$, the two-variable function $\langle a \rangle^s$ ([16, p. 141]) is defined by

$$\langle a \rangle^s = \sum_{n=0}^{\infty} \binom{s}{n} (\langle a \rangle - 1)^n, \tag{2.7}$$

when this sum is convergence. The analytic property of $\langle a \rangle^s$ is stated in the following proposition.

Proposition 2.1 (see Tangedal and Young [18]) *For any $a \in \mathbb{C}_p^\times$ the function $s \mapsto \langle a \rangle^s$ is a C^∞ function of s on $\mathbb{Z}_p$ and is analytic on a disc of positive radius about $s = 0$; on this disc it is locally analytic as a function of a and independent of the choice made to define the $\langle \cdot \rangle$ function. If a lies in a finite extension K of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$ is less than $p-1$ then $s \mapsto \langle a \rangle^s$ is analytic for $|s|_p < |\pi|_p^{-1} p^{-1/(p-1)}$, where (π) is the maximal ideal of the ring of integers O_K of K. If $s \in \mathbb{Z}_p$, the function $a \mapsto \langle a \rangle^s$ is an analytic function of a on any disc of the form $\{a \in \mathbb{C}_p : |a-y|_p < |y|_p\}$.*

Now we are at the position to recall the definition for the p-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s,a)$.

Definition 2.2 (see [10, Definition 3.3]) For $a \in \mathbb{C}_p \backslash \mathbb{Z}_p$, we define the p-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s,a)$ by the formula

$$\zeta_{p,E}(s,a) = \int_{\mathbb{Z}_p} \langle a+t \rangle^{1-s} d\mu_{-1}(t). \tag{2.8}$$

The following theorem summarize the analytic property of $\zeta_{p,E}(s,x)$ and Tangedal and Young proved a similar result for p-adic multiple zeta functions (see [18, Theorem 3.1]).

Theorem 2.3 (see [10, Theorem 3.4]) *For any choice of $a \in \mathbb{C}_p \backslash \mathbb{Z}_p$ the function $\zeta_{p,E}(s,a)$ is a C^∞ function of s on $\mathbb{Z}_p$, and is an analytic function of s on a disc of positive radius about $s = 0$; on this disc it is locally analytic as a function of a and independent of the choice made to define the $\langle \cdot \rangle$ function. If a is so chosen to lie in a finite extension K of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$ is less than $p-1$ then $\zeta_{p,E}(s,a)$ is analytic for $|s|_p < |\pi|_p^{-1} p^{-1/(p-1)}$. If $s \in \mathbb{Z}_p$, the function $\zeta_{p,E}(s,a)$ is locally analytic as a function of a on $\mathbb{C}_p \backslash \mathbb{Z}_p$.*

It needs to mention that the p-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s,a)$ interpolates its complex counterpart $\zeta_E(s,a)$ (1.11) p-adically, that is,

Theorem 2.4 (see [10, Theorem 3.8]) *Suppose that $a \in \mathbb{C}_p$ and $|a|_p > 1$. For $m \in \mathbb{N}$,*

$$\zeta_{p,E}(1-m,a) = \frac{1}{\omega_v^m(a)} E_m(a) = \frac{1}{\omega_v^m(a)} \zeta_E(-m,a),$$

where the Euler polynomials $E_m(x)$ is defined by the generating function

$$\frac{2e^{xz}}{e^z+1} = \sum_{m=0}^{\infty} E_m(x) \frac{z^m}{n!}, \quad |z| < \pi. \tag{2.9}$$

2.3 The p-adic operator T_p^a

In this subsection, we give a definition of T_p^a, the p-adic analogue of the operator T (see (1.3)). Let $E = \{x \in \mathbb{C}_p : |x|_p < p^{-\frac{1}{p-1}}\}$ be the region of convergence of the power series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. The p-adic exponential function is given by

$$\exp_p(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad (x \in E)$$

(see [16, p. 70]) and the p-adic Van Gorder's operator is defined as follows

$$T_p^a = \sum_{n=0}^{\infty} L_{p,n}^a, \tag{2.10}$$

where

$$\begin{aligned} L_{p,n}^a &:= P_{p,n}^a(s) \exp_p(nD), \\ P_{p,n}^a(s) &:= \begin{cases} \frac{2}{s-1} & \text{if } n = 0 \\ \frac{1}{\omega_v(a)} & \text{if } n = 1 \\ \frac{1}{n!\omega_v^n(a)} \prod_{j=1}^{n-1}(s-1+j) & \text{if } n \geq 2, \end{cases} \\ \exp_p(nD) &:= id + \sum_{k=1}^{\infty} \frac{n^k}{k!} D_s^k \end{aligned} \tag{2.11}$$

for $D_s^k := \frac{\partial^k}{\partial s^k}$.

3 Main results

In this section, we shall prove (1.12). First we need to establish the following identity for $\zeta_{p,E}(s,a)$.

Lemma 3.1 *Let $\zeta_{p,E}(s,a)$ be the p-adic Hurwitz-type Euler zeta function. Then, for $s \in \mathbb{Z}_p$ with $s \neq 1$ and $a \in \mathbb{C}_p$ with $|a|_p > 1$, we have that*

$$\begin{aligned} &\frac{2}{s-1}\left(\zeta_{p,E}(s,a) - \langle a \rangle^{1-s}\right) + \frac{1}{\omega_v(a)}\left(\zeta_{p,E}(s+1,a) - \langle a \rangle^{1-(s+1)}\right) \\ &\quad + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\zeta_{p,E}(s+n,a) - \langle a \rangle^{1-(s+n)}\right) \\ &= \frac{1}{s-1}\left(\langle a-1 \rangle^{1-s} - \langle a \rangle^{1-s}\right). \end{aligned} \tag{3.1}$$

Remark 3.2 This is a p-adic analogue of complex identities for the Hurwitz zeta function $\zeta(s,a)$ (see [15, Lemma 1]) and for the Riemann zeta function $\zeta(s)$ (see [15, (3.3)]).

Proof of Lemma 3.1 Fix $s \in \mathbb{Z}_p$ and $a \in \mathbb{C}_p$ with $|a|_p > 1$. For any $r \in \mathbb{N}$ we have

$$
\begin{aligned}
-\frac{1}{(s-1)\langle a\rangle^{s-1}} &= \frac{1}{s-1}\left(\sum_{k=0}^{p^r-1}\frac{(-1)^{k+1}}{\langle k+a\rangle^{s-1}}+\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}}\right)\\
&= \frac{1}{s-1}\left(\sum_{k=1}^{p^r}\frac{(-1)^{k}}{\langle k-1+a\rangle^{s-1}}+\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}}\right)\\
&= \frac{1}{s-1}\left(\sum_{k=1}^{p^r-1}\frac{(-1)^{k}}{\langle k-1+a\rangle^{s-1}}+\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}}+\frac{(-1)^{p^r}}{\langle p^r-1+a\rangle^{s-1}}\right)\\
&= \frac{1}{s-1}\sum_{k=1}^{p^r-1}\frac{(-1)^{k}}{\langle k+a\rangle^{s-1}}\left(\left\langle\frac{k+a}{k-1+a}\right\rangle^{s-1}+1\right)\\
&\quad -\frac{1}{s-1}\frac{1}{\langle p^r-1+a\rangle^{s-1}}\\
&\quad \text{(since } p \text{ is an odd prime).}
\end{aligned}
\tag{3.2}
$$

Since $|a|_p > 1$, for $k \in \mathbb{N}$, we have $|k+a|_p > 1$, thus

$$\left|1-\frac{1}{k+a}\right|_p = 1$$

and

$$\left|\frac{1}{1-\frac{1}{k+a}}-1\right|_p = \left|\frac{\frac{1}{k+a}}{1-\frac{1}{k+a}}\right|_p < 1.$$

Then from (2.1) we see that

$$\widehat{\frac{1}{1-\frac{1}{k+a}}} = 1$$

and by (2.3)

$$\omega_v\left(\frac{1}{1-\frac{1}{k+a}}\right) = 1.$$

Again by (2.3), we have

$$\begin{aligned}\left\langle\frac{k+a}{k-1+a}\right\rangle=&\left\langle\frac{1}{1-\frac{1}{k+a}}\right\rangle\\=&\omega_v^{-1}\left(\frac{1}{1-\frac{1}{k+a}}\right)\left(\frac{1}{1-\frac{1}{k+a}}\right)\\=&\frac{1}{1-\frac{1}{k+a}}\\=&\left(1-\frac{1}{k+a}\right)^{-1}.\end{aligned}\tag{3.3}$$

From [16, p.140, Lemma 47.6], for $s \in \mathbb{Z}_p$ we have the expansion

$$(1+x)^s=\sum_{n=0}^{\infty}\binom{s}{n}x^n,\quad |x|_p<1.$$

Thus by (3.3) we get

$$\begin{aligned}\left\langle\frac{k+a}{k-1+a}\right\rangle^{s-1}=&\left(1-\frac{1}{k+a}\right)^{1-s}=\sum_{n=0}^{\infty}\binom{1-s}{n}\frac{(-1)^n}{(k+a)^n}\\=&1+\sum_{n=1}^{\infty}\frac{\prod_{j=0}^{n-1}(s-1+j)}{n!}\frac{1}{(k+a)^n}.\end{aligned}\tag{3.4}$$

Substituting the above expansion into (3.2), we have

$$
\begin{aligned}
&-\frac{1}{(s-1)\langle a\rangle^{s-1}} \\
&=\frac{1}{s-1}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}}\left(\left(1+\sum_{n=1}^{\infty}\frac{\prod_{j=0}^{n-1}(s-1+j)}{n!}\frac{1}{(k+a)^n}\right)+1\right) \\
&\quad-\frac{1}{s-1}\frac{1}{\langle p^r-1+a\rangle^{s-1}} \\
&=\frac{2}{s-1}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}} \\
&\quad+\frac{1}{s-1}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}}\sum_{n=1}^{\infty}\frac{\prod_{j=0}^{n-1}(s-1+j)}{n!}\frac{1}{(k+a)^n} \\
&\quad-\frac{1}{s-1}\frac{1}{\langle p^r-1+a\rangle^{s-1}} \\
&=\frac{2}{s-1}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}} \\
&\quad+\frac{1}{s-1}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}}\left(\frac{s-1}{k+a}+\sum_{n=2}^{\infty}\frac{\prod_{j=0}^{n-1}(s-1+j)}{n!}\frac{1}{(k+a)^n}\right) \\
&\quad-\frac{1}{s-1}\frac{1}{\langle p^r-1+a\rangle^{s-1}}.
\end{aligned} \tag{3.5}
$$

Since $|a|_p>1$ and $k\in\mathbb{N}$, by (2.3) we have

$$\omega_v(k+a)=\omega_v(a)$$

and

$$k+a=\omega_v(k+a)\langle k+a\rangle=\omega_v(a)\langle k+a\rangle.$$

Substituting the above identity into (3.5), we get

$$
\begin{aligned}
-\frac{1}{(s-1)\langle a\rangle^{s-1}}=&\frac{2}{s-1}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}}+\frac{1}{\omega_v(a)}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s}} \\
&+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}} \\
&-\frac{1}{s-1}\frac{1}{\langle p^r-1+a\rangle^{s-1}}.
\end{aligned} \tag{3.6}
$$

Taking $r\to\infty$ in the above equality, by the continuity of the p-adic function $\langle a\rangle^s$ in a (see the last sentence of Proposition 2.1), we have

$$\lim_{r\to\infty}\langle p^r-1+a\rangle^{s-1}=\langle a-1\rangle^{s-1}$$

and

$$
\begin{aligned}
-\frac{1}{(s-1)\langle a\rangle^{s-1}} =& \frac{2}{s-1}\lim_{r\to\infty}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}} + \frac{1}{\omega_v(a)}\lim_{r\to\infty}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s}} \\
&+ \sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\lim_{r\to\infty}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}} \\
&\text{(see Proposition 3.3)} \\
&- \frac{1}{s-1}\frac{1}{\langle a-1\rangle^{s-1}} \\
=& \frac{2}{s-1}\left(\lim_{r\to\infty}\sum_{k=0}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s-1}} - \frac{1}{\langle a\rangle^{s-1}}\right) \\
&+ \frac{1}{\omega_v(a)}\left(\lim_{r\to\infty}\sum_{k=0}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s}} - \frac{1}{\langle a\rangle^{s}}\right) \\
&+ \sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\lim_{r\to\infty}\sum_{k=0}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}} - \frac{1}{\langle a\rangle^{s+n-1}}\right) \\
&- \frac{1}{s-1}\frac{1}{\langle a-1\rangle^{s-1}}.
\end{aligned}
\tag{3.7}
$$

Then by the definitions of the fermionic p-adic integral (2.7) and the p-adic Hurwitz-type zeta function $\zeta_{p,E}(s,a)$ (2.8), we have

$$
\begin{aligned}
-\frac{1}{(s-1)\langle a\rangle^{s-1}} =& \frac{2}{s-1}\left(\int_{\mathbb{Z}_p}\langle k+a\rangle^{1-s}d\mu_{-1}(a) - \frac{1}{\langle a\rangle^{s-1}}\right) \\
&+ \frac{1}{\omega_v(a)}\left(\int_{\mathbb{Z}_p}\langle k+a\rangle^{-s}d\mu_{-1}(a) - \frac{1}{\langle a\rangle^{s}}\right) \\
&+ \sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\int_{\mathbb{Z}_p}\langle k+a\rangle^{1-(s+n)}d\mu_{-1}(a) - \frac{1}{\langle a\rangle^{s+n-1}}\right) \\
&- \frac{1}{s-1}\frac{1}{\langle a-1\rangle^{s-1}} \\
=& \frac{2}{s-1}\left(\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}\right) \\
&+ \frac{1}{\omega_v(a)}\left(\zeta_{p,E}(s+1,a) - \langle a\rangle^{1-(s+1)}\right) \\
&+ \sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\zeta_{p,E}(s+n,a) - \langle a\rangle^{1-(s+n)}\right) \\
&- \frac{1}{s-1}\frac{1}{\langle a-1\rangle^{s-1}},
\end{aligned}
\tag{3.8}
$$

which is equivalent to

$$\begin{aligned}&\frac{2}{s-1}\left(\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right)+\frac{1}{\omega_v(a)}\left(\zeta_{p,E}(s+1,a)-\langle a\rangle^{1-(s+1)}\right)\\&\quad+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\zeta_{p,E}(s+n,a)-\langle a\rangle^{1-(s+n)}\right)\\&=\frac{1}{s-1}\left(\langle a-1\rangle^{1-s}-\langle a\rangle^{1-s}\right).\end{aligned}\tag{3.9}$$

This completes the proof. □

As pointed out by the referee, in order to move the limit to the inside of the summation $\sum_{n=2}^{\infty}$ in (3.7) of the above lemma, we need to show that the convergence of the inner limit is uniform for $r \in \mathbb{N}$. To this end, we add the following proposition.

Proposition 3.3 *For $s \in \mathbb{Z}_p$ and $a \in \mathbb{C}_p$ with $|a|_p > 1$, the series*

$$\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}}$$

converges uniformly for $r \in \mathbb{N}$ and

$$\begin{aligned}&\lim_{r\to\infty}\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}}\\&\quad=\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\lim_{r\to\infty}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}}.\end{aligned}\tag{3.10}$$

Proof For $n \geq 2$ we have

$$\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!}=\binom{s+n-2}{n-1}\frac{1}{n}.$$

By [16, p. 138, Proposition 47.2(v)], for $s \in \mathbb{Z}_p$,

$$\left|\binom{s+n-2}{n-1}\right|_p \leq 1.$$

Since $|n|_p = \left(\frac{1}{p}\right)^{v_p(n)}$, we have

$$\left|\frac{1}{n}\right|_p = p^{v_p(n)} \leq n,$$

thus for $n \geq 2$,

$$\left|\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!}\right|_p=\left|\binom{s+n-2}{n-1}\frac{1}{n}\right|_p \leq n.\tag{3.11}$$

Since for any $a \in \mathbb{C}_p^{\times}$, $\hat{a} \in \mu$ is a root of unity, we have $|\hat{a}|_p = 1$ and by (2.3)

$$|\omega_v(a)|_p = |p^{v_p(a)}\hat{a}|_p = |p|_p^{v_p(a)} = \left(\frac{1}{p}\right)^{v_p(a)},$$

thus

$$\left|\frac{1}{\omega_v^n(a)}\right|_p = p^{nv_p(a)}. \tag{3.12}$$

Combining (3.11) and (3.12), for any fixed $a \in \mathbb{C}_p$ with $|a|_p > 1$ we have

$$\left|\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\right|_p \leq p^{nv_p(a)} \cdot n. \tag{3.13}$$

Now fix $a \in \mathbb{C}_p$ with $|a|_p > 1$, we know that $\log_p\langle y+a\rangle$ is a continuous function in $y \in \mathbb{Z}_p$. Since $\mathbb{Z}_p$ is compact in the p-adic topology, by the Weierstrass maximum value theorem ([13, p. 61, Theorem 4.3]) there exists a $y_0 \in \mathbb{Z}_p$ such that

$$|\log_p\langle y+a\rangle|_p \leq |\log_p\langle y_0+a\rangle|_p \tag{3.14}$$

for all $y \in \mathbb{Z}_p$. Since $\langle y_0+a\rangle - 1 \in (p)$, we have

$$|\langle y_0+a\rangle - 1|_p \leq p^{-1} < p^{-1/(p-1)} \tag{3.15}$$

and by [20, p. 51, Lemma 5.5],

$$|\log_p\langle y_0+a\rangle|_p = |\langle y_0+a\rangle - 1|_p \leq p^{-1}. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we see that

$$|\log_p\langle y+a\rangle|_p \leq p^{-1} \tag{3.17}$$

for all $y \in \mathbb{Z}_p$. By [18, p. 1245, (2.22)], for $(x,s) \in \mathbb{C}_p^\times \times \mathbb{C}_p$ satisfying $|s|_p < p^{-1/(p-1)}|\log_p\langle x\rangle|_p^{-1}$, we have

$$\langle x\rangle^s = \exp_p(s\log_p\langle x\rangle). \tag{3.18}$$

Let $D = \mathbb{Z}_p \times \mathbb{Z}_p$. For $(y,s) \in D$, at first we have $|s|_p \leq 1$ and by (3.17), we see that $p^{-1/(p-1)}|\log_p\langle y+a\rangle|_p^{-1} \geq p^{-1/(p-1)} \cdot p = p^{\frac{p-2}{p-1}} > 1$, thus $|s|_p < p^{-1/(p-1)}|\log_p\langle y+a\rangle|_p^{-1}$. Then by (3.18) we have

$$\langle y+a\rangle^s = \exp_p(s\log_p\langle y+a\rangle) \tag{3.19}$$

for $(y,s) \in D$. Hence the two variable function $f(y,s) = \langle y+a\rangle^s$ is continuous on the domain D. Since $D = \mathbb{Z}_p \times \mathbb{Z}_p$ is compact in the p-adic topology, for any fixed $a \in \mathbb{C}_p$ with $|a|_p > 1$ it is bounded as a function for $(y,s) \in D$, so there exists a positive constant N_a such that for any $k \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\begin{aligned}\left|\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}}\right|_p &=\left|(-1)^k\langle k+a\rangle^{1-s-n}\right|_p \\ &=\left|(-1)^k f(k,1-s-n)\right|_p \\ &\leq N_a\end{aligned} \tag{3.20}$$

and by the non-archimedean property, for any $r \in \mathbb{N}$,

$$\left|\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}}\right|_p \leq N_a. \tag{3.21}$$

Then combining (3.13) and (3.21), for any fixed $a \in \mathbb{C}_p$ with $|a|_p > 1$ and for any $r \in \mathbb{N}$ we have

$$\left|\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}}\right|_p \leq N_a \cdot p^{nv_p(a)} \cdot n. \tag{3.22}$$

Since $|a|_p > 1$, i.e., $v_p(a) < 0$, we have $\lim_{n\to\infty} N_a \cdot p^{nv_p(a)} \cdot n = 0$, which implies the series

$$N_a\sum_{n=2}^{\infty}p^{nv_p(a)}n \tag{3.23}$$

is convergent. Finally by (3.22), (3.23) and the Weierstrass test (see [13, p. 230, Theorem 5.1]), we see that the series

$$\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\sum_{k=1}^{p^r-1}\frac{(-1)^k}{\langle k+a\rangle^{s+n-1}}$$

converges uniformly for $r \in \mathbb{N}$. Then applying [13, p. 185, Theorem 3.5] we conclude that the limit $r \to \infty$ can be moved to the inside of the above series, which is the desired result. □

The following result ensures the convergence of (3.1), which is a p-adic analogue of [15, Lemma 2].

Lemma 3.4 *The left hand side of* (3.1) *in Lemma* 3.1 *converges p-adically for $s \in \mathbb{Z}_p$ with $s \neq 1$ and $a \in \mathbb{C}_p$ with $|a|_p > 1$.*

Proof By Proposition 2.1 and Theorem 2.3, for $a \in \mathbb{C}_p$ with $|a|_p > 1$, $\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}$ is a C^∞ function of s on $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is compact in the p-adic topology, for any fixed $a \in \mathbb{C}_p$ with $|a|_p > 1$ it is bounded as a function for $s \in \mathbb{Z}_p$, i.e., there exists a positive constant M_a such that

$$\left|\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}\right|_p \leq M_a. \tag{3.24}$$

Then combining (3.13) and (3.24), for any fixed $a \in \mathbb{C}_p$ with $|a|_p > 1$ we have

$$\left|\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\zeta_{p,E}(s+n,a)-\langle a\rangle^{1-(s+n)}\right)\right|_p \leq M_a \cdot p^{nv_p(a)} \cdot n. \tag{3.25}$$

Since $|a|_p > 1$, i.e., $v_p(a) < 0$, we have $\lim_{n\to\infty} M_a \cdot p^{nv_p(a)} \cdot n = 0$, which implies

$$\lim_{n\to\infty}\left|\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\zeta_{p,E}(s+n,a)-\langle a\rangle^{1-(s+n)}\right)\right|_p = 0,$$

thus the series

$$\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\zeta_{p,E}(s+n,a)-\langle a\rangle^{1-(s+n)}\right)$$

is convergent under the p-adic topology. □

The above result implies the following theorem.

Theorem 3.5 *Let T_p^a be as defined in (2.10). Then $\zeta_{p,E}(s,a)$ formally satisfies the following differential equation*

$$T_p^a\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right] = \frac{1}{s-1}\left(\langle a-1\rangle^{1-s}-\langle a\rangle^{1-s}\right) \tag{3.26}$$

for $s \in \mathbb{Z}_p$ with $s \neq 1$ and $a \in \mathbb{C}_p$ with $|a|_p > 1$.

Proof Denote by $D_s^k := \frac{\partial^k}{\partial s^k}$. For any analytic function $f(s)$ on $\mathbb{Z}_p$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}\exp_p(nD)f(s) =& \left(id + \sum_{k=1}^{\infty}\frac{n^k}{k!}D_s^k\right)f(s)\\ =& f(s) + \sum_{k=1}^{\infty}\frac{n^k}{k!}\frac{\partial^k f(s)}{\partial s^k}\\ =& f(s+n),\end{aligned} \tag{3.27}$$

which maybe interpreted operationally through its formal Taylor expansion in n. By Proposition 2.1 and Theorem 2.3, for $a \in \mathbb{C}_p$ with $|a|_p > 1$, the function $\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}$ is analytic for $s \in \mathbb{Z}_p$. Thus from (3.27) we get

$$\begin{aligned}L_{p,n}^a\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right] =& P_{p,n}^a(s)\exp_p(nD)\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right]\\ =& P_{p,n}^a(s)\left(\zeta_{p,E}(s+n,a)-\langle a\rangle^{1-(s+n)}\right)\end{aligned} \tag{3.28}$$

for $n \geq 0$ and by the definition of T_p^a (2.10) and Lemma 3.1

$$
\begin{aligned}
&T_p^a\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right]\\
&=\sum_{n=0}^{\infty}L_{p,n}^a\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right]\\
&=\sum_{n=0}^{\infty}P_{p,n}^a(s)\left(\zeta_{p,E}(s+n,a)-\langle a\rangle^{1-(s+n)}\right)\\
&=\frac{2}{s-1}\left(\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right)+\frac{1}{\omega_v(a)}\left(\zeta_{p,E}(s+1,a)-\langle a\rangle^{1-(s+1)}\right)\\
&\quad+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\zeta_{p,E}(s+n,a)-\langle a\rangle^{1-(s+n)}\right)\\
&=\frac{1}{s-1}\left(\langle a-1\rangle^{1-s}-\langle a\rangle^{1-s}\right),
\end{aligned}
\tag{3.29}
$$

which is the desired result. □

In what follows, we shall investigate the area of convergence for Theorem 3.5 and show that the operator T_p^a applied to the p-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s,a)$ is convergent in certain area of the p-adic plane. First we need to prove the following proposition.

Proposition 3.6 *Let K be a finite extension of $\mathbb{Q}_p$ with ramification index e over $\mathbb{Q}_p$ less than $p-1$. Let $s\in\mathbb{C}_p$ with $|s|_p<r_p:=p^{\frac{1}{e}-\frac{1}{p-1}}$, and $a\in K\backslash\mathbb{Z}_p$. For any $n\geq 2$ the series*

$$
\exp_p(nD)\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right]=\sum_{k=0}^{\infty}\frac{D_s^k\left(\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right)}{k!}n^k
$$

converges.

Remark 3.7 This is mainly because the non-archimedean property of the p-adic metric and it is quite different from the complex situation for Hurwitz zeta functions. In that case, by [15, Proposition 5], we have "for any $s\in\mathbb{C}$, we can find some $N\geq 0$ so that the series

$$
\exp(ND)\left[\zeta(s,a)-\frac{1}{a^s}\right]=\sum_{k=0}^{\infty}\frac{D_s^k\left(\zeta(s,a)-\frac{1}{a^s}\right)}{k!}N^k
$$

diverges."

Proof of Propsoition 3.6 Let (π) be the maximal ideal of the ring of integers O_K of K. Then

$$
|\pi|_p=|p|_p^{\frac{1}{e}}=\left(\frac{1}{p}\right)^{\frac{1}{e}}.
$$

By Proposition 2.1 and Theorem 2.3, given $a\in K\backslash\mathbb{Z}_p$, the function $\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}$ is analytic for

$$
|s|_p<r_p:=|\pi|_p^{-1}p^{-1/(p-1)}=p^{\frac{1}{e}-\frac{1}{p-1}}.
$$

Fix $s_0\in\mathbb{C}_p$ with $|s_0|_p<r_p$. For any $s\in\mathbb{C}_p$ with $|s-s_0|_p<r_p$, we have

$$|s|_p \leq \max\{|s-s_0|_p, |s_0|_p\} < r_p,$$

so the disc $\{s : |s-s_0| < r_p\}$ is contained in the disc $\{s : |s| < r_p\}$. In fact,

$$\{s : |s-s_0| < r_p\} = \{s : |s| < r_p\}.$$

Thus $\zeta_{p,E}(s,a)$ can be expanded as a power series around s_0 with the radius of convergence equal to r_p.

Since $e < p-1$ as the assumption, we have $r_p > 1$ and for any $n \in \mathbb{N}$, we have $|(s_0+n)-s_0|_p = |n|_p \leq 1 < r_p$. From the discussion above, we have the following convergent power series expansion of $\zeta_{p,E}(s,a)$ at s_0

$$\zeta_{p,E}(s_0+n,a) - \langle a\rangle^{1-(s_0+n)} = \sum_{k=0}^{\infty} \frac{D_s^k|_{s=s_0}\left(\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}\right)}{k!} n^k.$$

Then by the definition of $\exp_p(nD)$ (2.10), we see that

$$\begin{aligned}\zeta_{p,E}(s_0+n,a) - \langle a\rangle^{1-(s_0+n)} &= \sum_{k=0}^{\infty} \frac{D_s^k|_{s=s_0}\left(\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}\right)}{k!} n^k \\ &= \exp_p(nD)|_{s=s_0}\left[\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}\right],\end{aligned} \tag{3.30}$$

which is the desired result. □

From the above proposition we have the following result which asserts that the operator T_p^a applied to the p-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s,a)$ is convergent in the p-adic topology.

Corollary 3.8 *Let K be stated as in the Proposition 3.6. Then*

$$T_p^a\left[\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}\right] = \sum_{n=1}^{\infty} P_{p,n}^a(s)\exp_p(nD)\left[\zeta_{p,E}(s,a) - \langle a\rangle^{1-s}\right] \tag{3.31}$$

converges for $s \in \mathbb{Z}_p$ with $s \neq 1$ and $a \in K$ with $|a|_p > 1$.

Remark 3.9 Notice that in the complex situation, we have

$$T\left[\zeta(s,a) - \frac{1}{a^s}\right] = \sum_{n=0}^{\infty} p_n(s)\exp(nD)\left[\zeta(s,a) - \frac{1}{a^s}\right]$$

diverges for all complex numbers $s \in \mathbb{C}$ (see [15, Theorem 8]).

Proof of Corollary 3.8 By (3.29) we have

$$
\begin{aligned}
T_p^a&\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right]\\
&=\sum_{n=0}^{\infty}P_{p,n}^a(s)\exp_p(nD)\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right]\\
&=\frac{2}{s-1}\left(\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right)+\frac{1}{\omega_v(a)}\left(\zeta_{p,E}(s+1,a)-\langle a\rangle^{1-(s+1)}\right)\\
&\quad+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\left(\zeta_{p,E}(s+n,a)-\langle a\rangle^{1-(s+n)}\right).
\end{aligned}
\tag{3.32}
$$

Suppose that $s\in\mathbb{Z}_p$ with $s\neq 1$ and $a\in K$ with $|a|_p>1$. By (2.11) and (3.13), for $n\geq 2$ we have

$$
|P_{p,n}^a(s)|_p=\left|\frac{\prod_{j=1}^{n-1}(s-1+j)}{n!\omega_v^n(a)}\right|_p\leq p^{nv_p(a)}\cdot n
$$

and $\lim_{n\to\infty}P_{p,n}^a(s)=0$. Then combining the conclusions of Proposition 3.6 and Lemma 3.4, for each $n\geq 2$, both the series

$$
\exp_p(nD)\left[\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right]=\sum_{k=0}^{\infty}\frac{D_s^k\left(\zeta_{p,E}(s,a)-\langle a\rangle^{1-s}\right)}{k!}n^k
$$

and the right hand side of (3.32) converge, which have established our result. □

Acknowledgements The authors are enormously grateful to the anonymous referee for his/her very careful reading of this paper, and for his/her many valuable and detailed suggestions. We also thank Professor Lawrence C. Washington for pointing out a gap in the proof of Lemma 3.1 of the original manuscript and for his helpful suggestions. Su Hu is supported by the Natural Science Foundation of Guangdong Province, China (No. 2020A1515010170). Min-Soo Kim is supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2019R1F1A1062499).

References

1. Cohen, H.: *Number Theory Vol. II: Analytic and Modern Tools*, Graduate Texts in Mathematics, 240, Springer, New York (2007)
2. Cvijović, D.: *A note on convexity properties of functions related to the Hurwitz zeta and alternating Hurwitz zeta function*. J. Math. Anal. Appl. **487**(1), 123972, 8 (2020)
3. Goss, D.: Zeroes of L-series in characteristic p. Int. J. Appl. Math. Stat. **11**(N07), 69–80 (2007)
4. Hurwitz, A.: Einige Eigenschaften der Dirichletschen Funktionen $F(s)=\sum(\frac{D}{n})\cdot\frac{1}{n^s}$, die bei der Bestimmung der Klassenanzahlen Binärer quadratischer Formen auftreten. Z. für Math. und Physik **27**, 86–101 (1882)
5. Hilbert, D.: *Mathematische Probleme, in: Die Hilbertschen Probleme*, Akademische Verlagsgesellschadt Geest & Portig, Leipzig (1971) pp. 23–80
6. Hu, S., Kim, M.-S.: On Dirichlet's lambda function. J. Math. Anal. Appl. **478**(2), 952–972 (2019)
7. Hu, S., Kim, D., Kim, M.-S.: Special values and integral representations for the Hurwitz-type Euler zeta functions. J. Korean Math. Soc. **55**(1), 185–210 (2018)
8. Katz, N.M.: p-adic L-functions via moduli of elliptic curves. Algebraic geometry. In: Proc. Sympos. Pure Math., Humboldt State Univ., Arcata, Calif., 1974, vol. 29, pp. 479–506. Amer. Math. Soc., Providence, RI (1975)
9. Kim, T.: On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$. J. Math. Anal. Appl. **331**, 779–792 (2007)

10. Kim, M.-S., Hu, S.: On *p*-adic Hurwitz-type Euler zeta functions. J. Number Theory **132**, 2977–3015 (2012)
11. Kim, M.-S., Hu, S.: On *p*-adic Diamond-Euler Log Gamma functions. J. Number Theory **133**, 4233–4250 (2013)
12. Lang, S.: Cyclotomic Fields I and II, Combined, 2nd edn. Springer-Verlag, New York (1990)
13. Lang, S.: Undergraduate analysis, 2nd edn. Undergraduate Texts in Mathematics. Springer-Verlag, New York (1997)
14. Osipov, Ju.V.: *p*-adic zeta functions. Uspekhi Mat. Nauk **34**, 209–210 (1979)
15. Prado, B.B., Klinger-Logan, K.: Linear Operators, the Hurwitz Zeta Function and Dirichlet *L*-Functions. J. Number Theory **217**, 422–442 (2020)
16. Schikhof, W.H.: Ultrametric Calculus. An Introduction to *p-Adic Analysis*. Cambridge University Press, London (1984)
17. Shiratani, K., Yamamoto, S.: On a *p*-adic interpolation function for the Euler numbers and its derivatives. Mem. Fac. Sci. Kyushu Univ. Ser. A **39**, 113–125 (1985)
18. Tangedal, B.A., Young, P.T.: On *p*-adic multiple zeta and log gamma functions. J. Number Theory **131**, 1240–1257 (2011)
19. Van Gorder, R.A.: Does the Riemann zeta function satisfy a differential equation? J. Number Theory **147**, 778–788 (2015)
20. Washington, L.C.: Introduction to Cyclotomic Fields. Graduate Texts in Mathematics, vol. 83, 2nd edn. Springer, New York (1997)
21. Weil, A.: Number theory. An approach through history, From Hammurapi to Legendre, Birkhäuser Boston Inc, Boston, MA (1984)