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A note on the Sturm bound for Siegel modular forms of type (*k*, 2)

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Abstract
We study analogues of Sturm's bounds for vector valued Siegel modular forms of type $(k, 2)$, which was already studied by Sturm in the case of an elliptic modular form and by Choi–Choie–Kikuta, Poor–Yuen and Raum–Richter in the case of scalar valued Siegel modular forms.

Keywords Vector valued Siegel modular forms · Congruence property · Sturm bound · Witt operator

Mathematics Subject Classification 11F30 · 11F33 · 11F46

1 Introduction

In this paper, we consider a question about congruences for the Fourier coefficients of vector valued Siegel modular forms of type $(k, 2)$, which was answered by Sturm [11] in the case of an elliptic modular form and by Choi–Choie–Kikuta [2], Poor–Yuen [8] in the case of a scalar valued Siegel modular form and Raum–Richter [9] in the case of a scalar valued Siegel modular form and of some, but not of type $(k, 2)$, vector valued Siegel modular forms.

Let p be a prime number and $\mathbb{Z}_{(p)}$ be the local ring of the p-integral rational numbers. Suppose that $f = \sum_{n=0}^{\infty} a(n;f)q^n$ is an elliptic modular form of weight k with integral coefficients. In [11] Sturm proved that if $a(n;f) \equiv 0 \pmod{p}$ for $0 \le n \le \frac{k}{12}$, then $a(n;f) \equiv 0 \pmod{p}$ for every $n \ge 0$. This bound is called a Sturm bound. In this paper we study the Sturm bound of the vector-valued Siegel modular forms of type $(k, 2)$ and degree 2 such that all Fourier coefficients lie in $\mathrm{Sym}_2^*(\mathbb{Z}_{(p)}) := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}_{(p)}\}$. Here p is a prime with $p \ge 5$ and k is an even integer.

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We state our result more precisely. A Siegel modular form of type $(k, 2)$ is a holomorphic function f on the Siegel upper-half plane $\mathbb{H}_2$ with values in $\mathrm{Sym}_2(\mathbb{C})$, satisfying

$$f(M\langle Z\rangle) = \det(CZ+D)^k (CZ+D) f(Z)\, {}^t(CZ+D)$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the Siegel modular group $\Gamma_2 = Sp_2(\mathbb{Z})$ and for all $Z \in \mathbb{H}_2$. Here $(k, 2)$ comes from the fact that the automorphy factor is the one of representatives in the equivalence class of the representation $\det^k \otimes \mathrm{Sym}(2)$. We denote by $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ the module consisting of all such f whose Fourier coefficients are in $\mathrm{Sym}_2^*(\mathbb{Z}_{(p)})$. The following theorem is our main result.

Theorem 1 *For each even integer k and each prime $p \geq 5$, suppose that F is a Siegel modular form in $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ having the form*

$$F(\tau, \tau', \omega) = \sum_{\substack{m,n \in \mathbb{Z},\ r \in \frac{1}{2}\mathbb{Z} \\ m,n,mn-r^2 \geq 0}} A(m,n,r) q_\tau^m q_{\tau'}^n q_\omega^{2r}$$

with $q_\tau = e^{2\pi i \tau}, q_{\tau'} = e^{2\pi i \tau'}, q_\omega = e^{2\pi i \omega}$ and $\begin{pmatrix} \tau & \omega \\ \omega & \tau' \end{pmatrix}$ in the Siegel upper half space $\mathbb{H}_2$ of degree 2, where $\mathbb{H}_2 := \left\{ \begin{pmatrix} \tau & \omega \\ \omega & \tau' \end{pmatrix} \in M_2(\mathbb{C}) \mid Im \begin{pmatrix} \tau & \omega \\ \omega & \tau' \end{pmatrix} > 0 \right\}$. If $A(m,n,r) \equiv 0 \pmod p$, i.e. all the elements of $A(m, n, r)$ are congruent to 0 mod p, for every m, n such that

$$0 \leq m \leq \left[\frac{k}{10}\right], \quad 0 \leq n \leq \left[\frac{k}{10}\right],$$

then $F \equiv 0 \pmod p$.

The proof of the theorem is due to an inductive argument on the determinant weight k and our main tool is the Witt operator and Theorem 2 (Sect. 2.5).

2 Preliminary

2.1 Siegel modular forms of type $(k, 2)$ and degree 2

The Siegel upper-half space of degree 2 is defined as

$$\mathbb{H}_2 := \{ Z = X + iY \in \mathrm{Sym}_2(\mathbb{C}) \mid Y > 0 \text{ (positive definite) } \}.$$

The real symplectic group $\mathrm{Sp}_2(\mathbb{R})$ acts on $\mathbb{H}_2$ in the following way:

$$Z \longrightarrow M\langle Z\rangle := (AZ+B)(CZ+D)^{-1},$$
$$Z \in \mathbb{H}_2,\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R}).$$

A Siegel modular form of type $(k, 2)$ on Γ_2 with character ν is a holomorphic function f on $\mathbb{H}_2$ with values in $\mathrm{Sym}_2(\mathbb{C})$, satisfying

$$f(M\langle Z\rangle) = \nu(M)\det(CZ+D)^k(CZ+D)f(Z)^t(CZ+D)$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2$ and for all $Z \in \mathbb{H}_2$. We denote by $M_{k,2}(\Gamma_2,\nu)$ (resp. $S_{k,2}(\Gamma_2,\nu)$) the $\mathbb{C}$-vector space of Siegel modular forms (resp. cusp forms) of type $(k, 2)$ on Γ_2 with character ν.

2.2 Fourier expansions

Any $F(Z) \in M_{k,2}(\Gamma_2,\nu)$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \le T \in \frac{1}{2}\Lambda_2} a(T;F)\exp(2\pi i \mathrm{tr}(TZ)), \quad a(T;F) \in \mathrm{Sym}_2(\mathbb{C}),$$

where T runs over all positive semi-definite elements of $\frac{1}{2}\Lambda_2$ defined as

$$\Lambda_2 := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}.$$

Taking $q_\tau := \exp(2\pi i\tau)$, $q_\omega := \exp(2\pi i\omega)$ and $q_{\tau'} := \exp(2\pi i\tau')$ for $Z = \begin{pmatrix} \tau & \omega \\ \omega & \tau' \end{pmatrix} \in \mathbb{H}_2$, we can write

$$q^T := \exp(2\pi i \mathrm{tr}(TZ)) = q_\omega^{2t_{12}} q_\tau^{t_{11}} q_{\tau'}^{t_{22}}.$$

Using this notation, we have the generalized q-expansion:

$$\begin{aligned} F &= \sum_{0 \le T \in \frac{1}{2}\Lambda_2} a(T;F)q^T \\ &= \sum_{0 \le (t_{ij}) \in \frac{1}{2}\Lambda_2} (a(T;F)q_\omega^{2t_{12}})q_\tau^{t_{11}} q_{\tau'}^{t_{22}} \in \mathrm{Sym}_2(\mathbb{C})[q_\omega^{-\frac{1}{2}}, q_\omega^{\frac{1}{2}}][\![q_\tau^{\frac{1}{2}}, q_{\tau'}^{\frac{1}{2}}]\!]. \end{aligned}$$

For any subring R of $\mathbb{C}$, we denote by $M_{k,2}(\Gamma_2,\nu)_R$ the R-module consisting of those F in $M_{k,2}(\Gamma_2,\nu)$ for which $a(T;F)$ is in $\mathrm{Sym}_2^*(R)$ for every $T \in \frac{1}{2}\Lambda_2$ where

$$\mathrm{Sym}_2^*(R) := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{C}) \mid t_{ii}, 2t_{ij} \in R\}.$$

2.3 Generators of scalar valued Siegel modular forms

Let φ_4, φ_6, X_{10}, X_{12} be Igusa's generators over $\mathbb{Z}$ of weight 4, 6, 10, 12, respectively given in [5]. Let $M_k(\Gamma_2,\nu)$ (resp. $S_k(\Gamma_2,\nu)$) be the $\mathbb{C}$-vector space consisting of the scalar valued Siegel modular forms (resp. cusp forms) of weight k on Γ_2 with the character ν. We denote by $M_k(\Gamma_2,\nu)_{\mathbb{Z}_{(p)}}$ (resp. $S_k(\Gamma_2,\nu)_{\mathbb{Z}_{(p)}}$) the $\mathbb{Z}_{(p)}$-module consisting of the scalar valued Siegel modular forms in $M_k(\Gamma_2,\nu)$ (resp. cusp forms in $S_k(\Gamma_2,\nu)$) for which $a(T;F)$ is in $\mathbb{Z}_{(p)}$ for every $T \in \frac{1}{2}\Lambda_2$. In the case of scalar valued Siegel modular forms, we have

$$M_*^{\mathrm{ev}}(\mathbb{Z}_{(p)}) := \bigoplus_{k\in 2\mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \\ = \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{10}, X_{12}], \text{ if } p \geq 5.$$

Let $\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})$ be the commutator subgroup of Γ_2. Let $\chi : \Gamma_2 \to \{\pm 1\}$ be a non-trivial abelian character, which is basically a character of $\mathrm{Sp}_2(\mathbb{Z}/2\mathbb{Z}) \cong \Sigma_6$, the symmetric group of 6 letters. Any Siegel modular form of weight k on $\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})$ also has a Fourier expansion of the form

$$\sum_{0\leq T\in \frac{1}{2}\Lambda_2} b(T;F)\exp(2\pi i \mathrm{tr}(TZ)), \quad b(T;F) \in \mathbb{C}.$$

In this case Igusa [4] showed that

$$M_*(\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})) := \bigoplus_k M_k(\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})) \\ = \mathbb{C}[\varphi_4, \Delta_5, \varphi_6, X_{12}, \Delta_{30}],$$

where $\Delta_k \in M_k(\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})) = M_k(\Gamma_2, \chi)$ is constructed by theta series with Fourier coefficients

$$b\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}; \Delta_5\right) = b\left(\begin{pmatrix} \frac{3}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{2} \end{pmatrix}; \Delta_{30}\right) = 1.$$

The modular forms $\varphi_4, \Delta_5, \varphi_6, X_{12}$ are algebraically independent and $\Delta_5^2 = X_{10}$. We remark that there exists a unique relation among the generators;

$$\Delta_{30}^2 \in \mathbb{C}[\varphi_4, \Delta_5, \varphi_6, X_{12}].$$

2.4 *p*-order of modular forms

We shall define the p-order of modular forms. Let p be a prime with $p \geq 5$ and v_p the additive valuation on $\mathbb{Q}$ normalized as $v_p(p) = 1$.

Let F be a formal power series with bounded denominators of the form

$$F = \sum_{T\in \frac{1}{N}\Lambda_2} a(T;F)q^T, \quad a(T;F) \in \mathrm{Sym}_2(\mathbb{Q}).$$

for some positive integer N.

In the scalar valued case, let v_p be just as in Böcherer–Nagaoka [1] and elsewhere. Define a value v_p for F with $a(T;F) \in \mathrm{Sym}_2(\mathbb{Q})$ as

$$v_p(F) := \inf \left\{ v_p(a(T;F)) \mid T \in \mathrm{Sym}_2(\mathbb{Q}) \right\},$$

where $\nu_p(a(T;F)) := \min_{1\le i,j\le 2}(\nu_p(a_{ij}(T;F)))$ for $a(T;F) = \begin{pmatrix} a_{11}(T;F) & a_{12}(T;F) \\ a_{12}(T;F) & a_{22}(T;F) \end{pmatrix}$.

The following statement and its proof are due to Kikuta:

Lemma 1 (1) *Let* $f = \sum_{T\in\frac{1}{N}\Lambda_2} a(T;f)q^T$ *and* $g = \sum_{T\in\frac{1}{N}\Lambda_2} a(T;g)q^T$ *with* $a(T;f)$, $a(T;g) \in \mathbb{Q}$ *be formal power series with bounded denominators. Then we have* $\nu_p(fg) = \nu_p(f) + \nu_p(g)$.

(2) *Let* $F = \sum_{T\in\frac{1}{N}\Lambda_2} a(T;F)q^T$ *with* $a(T;F) \in \mathrm{Sym}_2(\mathbb{Q})$ *be a formal power series with bounded denominators and g be as in (1). Then we have* $\nu_p(Fg) = \nu_p(F) + \nu_p(g)$.

The proof of Lemma 1 is, for example, in [6].

We remark that, for a formal power series of the form

$$F = \sum_{T\in\frac{1}{N}\Lambda_2} a(T;F)q^T, \quad a(T;F) \in \mathrm{Sym}_2(\mathbb{Q}),$$

we have $a(T;F) \in \mathrm{Sym}_2^*(\mathbb{Z}_{(p)})$ for all $T \in \frac{1}{N}\Lambda_2$ if and only if $\nu_p(F) \ge 0$.

2.5 Generators of vector valued Siegel modular forms over $\mathbb{Z}_{(p)}$

Let R be a subring of $\mathbb{C}$ and N be 1 or 2. For a formal power series f of the form

$$f = \sum_{T\in\frac{1}{N}\Lambda_2} a(T;f)q^T \in R[q_\omega^{-\frac{1}{N}}, q_\omega^{\frac{1}{N}}][\![q_\tau^{\frac{1}{N}}, q_{\tau'}^{\frac{1}{N}}]\!],$$

the theta operator $\Theta^{[1]}$ is defined by

$$\Theta^{[1]}(f) = \sum_{T\in\frac{1}{N}\Lambda_2} T\cdot a(T;f)q^T \in \mathrm{Sym}_2^*(R)[q_\omega^{-\frac{1}{N}}, q_\omega^{\frac{1}{N}}][\![q_\tau^{\frac{1}{N}}, q_{\tau'}^{\frac{1}{N}}]\!].$$

Let Γ be either Γ_2 or $\mathcal{C}Sp_2(\mathbb{Z})$ and $f \in M_k(\Gamma)$ and $g \in M_j(\Gamma)$. We put

$$[f, g] := \frac{1}{j} f\Theta^{[1]}(g) - \frac{1}{k} g\Theta^{[1]}(f).$$

Then the results of Satoh [10] states that $[f, g] \in M_{k+j,2}(\Gamma)$.

Let φ_4, φ_6, X_{10}, X_{12} be Igusa's generators over $\mathbb{Z}$ of weight 4, 6, 10, 12, respectively given in [5]. It is known that the $M_*^{\mathrm{ev}}(\Gamma_2)$-module of Siegel modular forms of type $(k, 2)$ over $\mathbb{Z}_{(p)}$ has six generators:

Theorem 2 ([6])

For each even integer k and each prime $p \ge 5$, $\bigoplus_{k\in 2\mathbb{Z}} M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ *is a* $M_*^{\mathrm{ev}}(\mathbb{Z}_{(p)})$*-module generated by 6 elements whose weights are* 10, 14, 16, 16, 18, 22. *If we write them as* $\Phi_k \in M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} (k = 10, 14, 16, 18, 22)$ *and* $\Psi_{16} \in M_{16,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, *then we have (as a* $\mathbb{Z}_{(p)}$ *-module)*

$$M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} = M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}\boldsymbol{\Phi}_{10} \oplus M_{k-14}(\Gamma_2)_{\mathbb{Z}_{(p)}}\boldsymbol{\Phi}_{14} \oplus M_{k-16}(\Gamma_2)_{\mathbb{Z}_{(p)}}\boldsymbol{\Phi}_{16} \oplus V_{k-16}(\Gamma_2)_{\mathbb{Z}_{(p)}}\boldsymbol{\Psi}_{16} \oplus V_{k-18}(\Gamma_2)_{\mathbb{Z}_{(p)}}\boldsymbol{\Phi}_{18} \oplus W_{k-22}(\Gamma_2)_{\mathbb{Z}_{(p)}}\boldsymbol{\Phi}_{22},$$

where

$$V_k(\Gamma_2)_{\mathbb{Z}_{(p)}} = M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \cap \mathbb{Z}_{(p)}[\varphi_6, X_{10}, X_{12}], \quad W_k(\Gamma_2)_{\mathbb{Z}_{(p)}} = M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \cap \mathbb{Z}_{(p)}[X_{10}, X_{12}].$$

We construct $\boldsymbol{\Phi}_k$ ($k = 10, 14, 16, 18, 22$) and $\boldsymbol{\Psi}_{16}$ by taking constant multiples of these generators:

$$\boldsymbol{\Phi}_{10} = -\frac{1}{144}[\varphi_4, \varphi_6], \quad \boldsymbol{\Phi}_{14} = 10[\varphi_4, X_{10}], \boldsymbol{\Phi}_{16} = 12[\varphi_4, X_{12}],$$
$$\boldsymbol{\Psi}_{16} = 10[\varphi_6, X_{10}], \qquad \boldsymbol{\Phi}_{18} = 12[\varphi_6, X_{12}], \boldsymbol{\Phi}_{22} = -120[X_{10}, X_{12}].$$

Then we have

$$a\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \boldsymbol{\Phi}_{10}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}; \boldsymbol{\Phi}_{14}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}; \boldsymbol{\Phi}_{16}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix},$$
$$a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}; \boldsymbol{\Psi}_{16}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \quad a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}; \boldsymbol{\Phi}_{18}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, a\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \boldsymbol{\Phi}_{22}\right) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Moreover, we put also $\boldsymbol{\Phi}_9 := 10[\varphi_4, \Delta_5]$, $\boldsymbol{\Phi}_{11} := 10[\varphi_6, \Delta_5]$, $\boldsymbol{\Phi}_{17} := -120[\Delta_5, X_{12}]$. We will use them in the proof of our main theorem.

Proposition 1 ([6]) *Let p be a prime with $p \geq 5$. Then we have $v_p(\boldsymbol{\Psi}_{16}) \geq 0$ and $v_p(\boldsymbol{\Phi}_k) \geq 0$ for $k = 9, 10, 11, 14, 16, 17, 18, 22$.*

2.6 The Witt operator

Let F be a holomorphic function on $\mathbb{H}_2$. Then the Witt operator is defined by

$$W(F)(\tau, \tau') := F\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}, \quad (\tau, \tau') \in \mathbb{H}_1 \times \mathbb{H}_1.$$

This operator was first introduced in Witt [12]. We extend the Witt operator to the case of vector valued forms. Let $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} \in M_{k,2}(\Gamma_2, \nu)$ be a vector valued Siegel modular form of type $(k, 2)$ on Γ_2 with character ν, then we define

$$W(G)(\tau, \tau') := \begin{pmatrix} W(G_{11}) & W(G_{12}) \\ W(G_{12}) & W(G_{22}) \end{pmatrix}, \quad (\tau, \tau') \in \mathbb{H}_1 \times \mathbb{H}_1.$$

For later use, we introduce some examples:

$$W(\varphi_4)(\tau,\tau') = E_4(\tau)E_4(\tau'), \qquad W(\varphi_6)(\tau,\tau') = E_6(\tau)E_6(\tau'),$$
$$W(X_{10})(\tau,\tau') \equiv 0, \qquad W(X_{12})(\tau,\tau') = 12\Delta(\tau)\Delta(\tau'),$$
$$W(\Delta_5)(\tau,\tau') \equiv 0,$$
$$W(\Phi_{10}) = \begin{pmatrix} \Delta(\tau)E_4(\tau')E_6(\tau') & 0 \\ 0 & E_4(\tau)E_6(\tau)\Delta(\tau') \end{pmatrix},$$
$$W(\Phi_{16}) = 12\begin{pmatrix} E_6(\tau)\Delta(\tau)E_4(\tau')\Delta(\tau') & 0 \\ 0 & E_4(\tau)\Delta(\tau)E_6(\tau')\Delta(\tau') \end{pmatrix},$$
$$W(\Phi_{18}) = 12\begin{pmatrix} E_4(\tau)^2\Delta(\tau)E_6(\tau')\Delta(\tau') & 0 \\ 0 & E_6(\tau)\Delta(\tau)E_4(\tau')^2\Delta(\tau') \end{pmatrix},$$
$$W(\Phi_{14}) = W(\Psi_{16}) = W(\Phi_{22}) \equiv 0,$$
$$W(\Phi_9) = 2E_4(\tau)E_4(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 01 \\ 10 \end{pmatrix},$$
$$W(\Phi_{11}) = 2E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 01 \\ 10 \end{pmatrix},$$
$$W(\Phi_{17}) = 288\Delta(\tau)\Delta(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 01 \\ 10 \end{pmatrix},$$

where η is the usual Dedekind eta function defined as $\eta(\tau) = q_\tau^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q_\tau^m)$.

Theorem 3 (Freitag [3]) *If* $F \in M_k(\Gamma_2)$ *satisfies* $W(F) \equiv 0$, *then* $\frac{F}{X_{10}} \in M_{k-10}(\Gamma_2)$, *namely, F is divisible by* X_{10}.

Nagaoka's reasoning on page 416 of [7] proves the following lemma.

Lemma 2 (Nagaoka [7]) *Assume that* $p \geq 5$. *Let* $F \in \mathbb{Q}[\![q_\tau, q_{\tau'}]\!]$ *be a formal power series of the form*

$$F = \sum_{a,b,c\geq 0} \gamma_{abc} W(\varphi_4)^a W(\varphi_6)^b W(X_{12})^c, \quad \gamma_{abc} \in \mathbb{Q}.$$

If $\nu_p(F) \geq 0$, *then the* γ_{abc} *satisfy* $\nu_p(\gamma_{abc}) \geq 0$ *for all* $a, b, c \geq 0$.

From this lemma, we get the following corollary.

Corollary 1 *Assume that* $p \geq 5$. *Let* $F \in \mathbb{Q}[\![q_\tau, q_{\tau'}]\!]$ *be a formal power series of the form*

$$F = \sum_{a,b,c\geq 0} \gamma_{abc} W(\varphi_4)^a W(\varphi_6)^b W(X_{12})^c, \quad \gamma_{abc} \in \mathbb{Q}.$$

If $\nu_p(F) \geq 1$, *then the* γ_{abc} *satisfy* $\nu_p(\gamma_{abc}) \geq 1$ *for all* $a, b, c \geq 0$.

Proof Since $\nu_p(F) \geq 1$, we get $\nu_p(\frac{1}{p}F) \geq 0$. Hence from Lemma 2, we can take

$$\frac{1}{p}F = \sum_{a,b,c\geq 0} \frac{1}{p}\gamma_{abc} W(\varphi_4)^a W(\varphi_6)^b W(X_{12})^c, \quad \nu_p(\frac{1}{p}\gamma_{abc}) \geq 0.$$

Hence we can take $\nu_p(\gamma_{abc}) \geq 1$ for all $a, b, c \geq 0$. $\square$

3 Proof of the main theorem (Theorem 1)

We prove it by an inductive argument on the weight. By Theorem 2 (Sect. 2.5), for any $F \in M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, we can write F in the form

$$\begin{aligned} F = &(P_1 + X_{10}Q_1)\boldsymbol{\Phi}_{10} + (P_2 + X_{10}Q_2)\boldsymbol{\Phi}_{14} + (P_3 + X_{10}Q_3)\boldsymbol{\Phi}_{16} \\ &+ (P_4 + X_{10}Q_4)\boldsymbol{\Psi}_{16} + (P_5 + X_{10}Q_5)\boldsymbol{\Phi}_{18} + (P_6 + X_{10}Q_6)\boldsymbol{\Phi}_{22}, \end{aligned}$$

where $P_1 \in M_{k-10}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$, $Q_1 \in M_{k-20}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_2 \in M_{k-14}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$, $Q_2 \in M_{k-24}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_3 \in M_{k-16}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$, $Q_3 \in M_{k-26}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_4 \in V_{k-16}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_6, X_{12}]$, $Q_4 \in V_{k-26}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_5 \in V_{k-18}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_6, X_{12}]$, $Q_5 \in V_{k-28}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_6 \in W_{k-22}(\Gamma_2) \cap \mathbb{Z}_{(p)}[X_{12}]$, $Q_6 \in W_{k-32}(\Gamma_2)_{\mathbb{Z}_{(p)}}$.

Here we regard P_i as polynomials (with coefficients in $\mathbb{Z}_{(p)}$) $P_1 = P_1(\varphi_4, \varphi_6, X_{12})$, $P_2 = P_2(\varphi_4, \varphi_6, X_{12})$, $P_3 = P_3(\varphi_4, \varphi_6, X_{12})$, $P_4 = P_4(\varphi_6, X_{12})$, $P_5 = P_5(\varphi_6, X_{12})$, $P_6 = P_6(X_{12})$.

We apply the Witt operator to F. Since $W(X_{10}) = W(\boldsymbol{\Phi}_{14}) = W(\boldsymbol{\Psi}_{16}) = W(\boldsymbol{\Phi}_{22}) = 0$, we get

$$\begin{aligned} W(F) &= W(P_1)W(\boldsymbol{\Phi}_{10}) + W(P_3)W(\boldsymbol{\Phi}_{16}) + W(P_5)W(\boldsymbol{\Phi}_{18}) \\ &= \begin{pmatrix} ME_1 & 0 \\ 0 & ME_2 \end{pmatrix} \\ &:= \begin{pmatrix} \sum_{m,n \geq 0} B_{11}(m,n) q_\tau^m q_{\tau'}^n & 0 \\ 0 & \sum_{m,n \geq 0} B_{22}(m,n) q_\tau^m q_{\tau'}^n \end{pmatrix}, \end{aligned}$$

where

$$ME_1 = \sum_{\substack{12i+4j+6t=k+2 \\ 12i'+4j'+6t'=k \\ t,t'=0,1}} C_1(i,i') \Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t \Delta(\tau')^{i'} E_4(\tau')^{j'} E_6(\tau')^{t'}$$

$$ME_2 = \sum_{\substack{12i+4j+6t=k \\ 12i'+4j'+6t'=k+2 \\ t,t'=0,1}} C_2(i,i') \Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t \Delta(\tau')^{i'} E_4(\tau')^{j'} E_6(\tau')^{t'}.$$

The q_τ-expansion of $\Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t$ has the form

$$\Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t = q_\tau^i + \cdots.$$

The numbers j and t, where t is 0 or 1, are uniquely determined by choosing a value of i.

For each m, n such that $0 \leq m, n \leq \left[\frac{k}{10}\right]$, $A(m,n,r) \equiv 0 \pmod p$. We have that if $m \leq \left[\frac{k}{10}\right]$ and $n \leq \left[\frac{k}{10}\right]$, then $B_{11}(m,n) \equiv B_{22}(m,n) \equiv 0 \pmod p$. This implies that $C_1(i,i') \equiv C_2(i,i') \equiv 0 \pmod p$ for $i, i' \leq \left[\frac{k}{10}\right]$. Note that $i, i' \leq \left[\frac{k}{10}\right]$ since $12i + 4j + 6t = k$ or $k+2$ and $12i' + 4j' + 6t' = k$ or $k+2$ and $\left[\frac{k}{12}\right] \leq \left[\frac{k+2}{12}\right] \leq \left[\frac{k}{10}\right]$. Thus we have $W(F) \equiv 0 \pmod p$.

Lemma 3 $P_1, P_3, P_5 \equiv 0 \pmod{p}$.

Proof of Lemma 3 Using fact that $W(\varDelta_5) = 0$, we get

$$\begin{aligned}
W(F) &= W(P_1)W(\boldsymbol{\Phi}_{10}) + W(P_3)W(\boldsymbol{\Phi}_{16}) + W(P_5)W(\boldsymbol{\Phi}_{18}) \\
&= W(P_1)\begin{pmatrix} \varDelta(\tau)E_4(\tau')E_6(\tau') & 0 \\ 0 & E_4(\tau)E_6(\tau)\varDelta(\tau') \end{pmatrix} \\
&\quad + 12W(P_3)\begin{pmatrix} E_6(\tau)\varDelta(\tau)E_4(\tau')\varDelta(\tau') & 0 \\ 0 & E_4(\tau)\varDelta(\tau)E_6(\tau')\varDelta(\tau') \end{pmatrix} \\
&\quad + 12W(P_5)\begin{pmatrix} E_4(\tau)^2\varDelta(\tau)E_6(\tau')\varDelta(\tau') & 0 \\ 0 & E_6(\tau)\varDelta(\tau)E_4(\tau')^2\varDelta(\tau') \end{pmatrix} \\
&= \{W(P_1)\begin{pmatrix} E_4(\tau')E_6(\tau') & 0 \\ 0 & E_4(\tau)E_6(\tau) \end{pmatrix} \\
&\quad + 12W(P_3)\begin{pmatrix} E_6(\tau)E_4(\tau')\varDelta(\tau') & 0 \\ 0 & E_4(\tau)\varDelta(\tau)E_6(\tau') \end{pmatrix} \\
&\quad + 12W(P_5)\begin{pmatrix} E_4(\tau)^2E_6(\tau')\varDelta(\tau') & 0 \\ 0 & E_6(\tau)\varDelta(\tau)E_4(\tau')^2 \end{pmatrix}\} \begin{pmatrix} \varDelta(\tau) & 0 \\ 0 & \varDelta(\tau') \end{pmatrix} \\
&= \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}\begin{pmatrix} \varDelta(\tau) & 0 \\ 0 & \varDelta(\tau') \end{pmatrix}.
\end{aligned}$$

where the (1, 1)-component and (2, 2)-component of $W(F)$ are

$$\begin{aligned}
f_{11}\varDelta(\tau) &= (W(P_1)E_4(\tau')E_6(\tau') + 12W(P_3)E_6(\tau)E_4(\tau')\varDelta(\tau') \\
&\quad + 12W(P_5)E_4(\tau)^2E_6(\tau')\varDelta(\tau'))\varDelta(\tau), \\
f_{22}\varDelta(\tau') &= (W(P_1)E_4(\tau)E_6(\tau) + 12W(P_3)E_4(\tau)\varDelta(\tau)E_6(\tau') \\
&\quad + 12W(P_5)E_6(\tau)\varDelta(\tau)E_4(\tau')^2)\varDelta(\tau').
\end{aligned}$$

Since $\nu_p(W(F)) \geq 1$, and $\nu_p(\varDelta(\tau)) = \nu_p(\varDelta(\tau')) = 0$, we have $\nu_p(f_{11}) = \nu_p(f_{22}) \geq 1$. Then we get

$$\begin{aligned}
&\nu_p(-E_4(\tau)\varDelta(\tau)E_6(\tau')f_{11} + E_6(\tau)E_4(\tau')\varDelta(\tau')f_{22}) \\
&= \nu_p((E_4(\tau)^3\varDelta(\tau') - \varDelta(\tau)E_4(\tau')^3)(E_4(\tau)E_4(\tau')W(P_1) + 2^8 \cdot 3^4\varDelta(\tau)\varDelta(\tau')W(P_5))) \\
&\geq 1
\end{aligned}$$

and

$$\begin{aligned}
&\nu_p(E_4(\tau)E_6(\tau)f_{11} - E_4(\tau')E_6(\tau')f_{22}) \\
&\nu_p(2^2 \cdot 3(E_4(\tau)^3\varDelta(\tau') - \varDelta(\tau)E_4((\tau')^3))(E_4(\tau)E_4(\tau')W(P_3) + E_6(\tau)E_6(\tau')W(P_5))) \\
&\geq 1.
\end{aligned}$$

Since $\nu_p(E_4(\tau)^3\varDelta(\tau') - \varDelta(\tau)E_4((\tau')^3)) = 0$, we get

$$\nu_p(E_4(\tau)E_4(\tau')W(P_1) + 2^8 \cdot 3^4 \Delta(\tau)\Delta(\tau')W(P_5)) \geq 1, \tag{1}$$

$$\nu_p(E_4(\tau)E_4(\tau')W(P_3) + E_6(\tau)E_6(\tau')W(P_5)) \geq 1. \tag{2}$$

Case ($k \not\equiv 0 \pmod 6$)

We have $W(P_5) = 0$. Hence we have $\nu_p(W(P_1)) \geq 1$ and $\nu_p(W(P_3)) \geq 1$. We can write

$$W(P_1) = \sum_{4a+12b+12c=k-10} \gamma_{abc} W(\varphi_4)^a W(\varphi_6)^b W(X_{12})^c,$$
$$W(P_3) = \sum_{4a+12b+12c=k-16} \gamma'_{abc} W(\varphi_4)^a W(\varphi_6)^b W(X_{12})^c.$$

From Corollary 1, we have $\nu_p(\gamma_{abc}) \geq 1$ and $\nu_p(\gamma'_{abc}) \geq 1$. Using $W(P_1 - \sum_{4a+12b+12c=k-10} \gamma_{abc}\varphi_4^a\varphi_6^b X_{12}^c) = 0$, $W(P_3 - \sum_{4a+12b+12c=k-16} \gamma'_{abc}\varphi_4^a\varphi_6^b X_{12}^c) = 0$, we have $P_1 = \sum_{4a+12b+12c=k-10} \gamma_{abc}\varphi_4^a\varphi_6^b X_{12}^c$ and $P_3 = \sum_{4a+12b+12c=k-16} \gamma'_{abc}\varphi_4^a\varphi_6^b X_{12}^c$ because the Witt operator is injective on $\mathbb{C}[\varphi_4, \varphi_6, X_{12}]$ by Theorem 3 and the fact that Igusa's generators are algebraically independent over $\mathbb{C}$. Hence $\nu_p(P_1(\varphi_4, \varphi_6, X_{12})) \geq 1$ and $\nu_p(P_3(\varphi_4, \varphi_6, X_{12})) \geq 1$.

Case ($k \equiv 0 \pmod{12}$)

We can write

$$W(P_1) = E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 2 \pmod 3 \\ 4a+12b+12c=k-16}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_3) = \sum_{\substack{a\equiv 2 \pmod 3 \\ 4a+12b+12c=k-16}} \gamma'_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_5) = E_6(\tau)E_6(\tau') \sum_{12b+12c=k-24} \gamma''_{bc} W(\varphi_6^2)^b W(X_{12})^c.$$

Using these formulas, we can write

$$\begin{aligned} &E_4(\tau)E_4(\tau')W(P_1) + 2^8 \cdot 3^4 \Delta(\tau)\Delta(\tau')W(P_5) \\ &= E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 0 \pmod 3,\ a\geq 3 \\ 4a+12b+12c=k-12}} \gamma_{a-1bc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c \\ &\quad + 2^6 \cdot 3^3 E_6(\tau)E_6(\tau') \sum_{\substack{c\geq 1 \\ 12b+12c=k-12}} \gamma''_{bc-1} W(\varphi_6^2)^b W(X_{12})^c \\ &= E_6(\tau)E_6(\tau')\{ \sum_{\substack{a\equiv 0 \pmod 3,\ a\geq 3 \\ 4a+12b+12c=k-12}} \gamma_{a-1bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c \\ &\quad + 2^6 \cdot 3^3 \sum_{\substack{c\geq 1 \\ 12b+12c=k-12}} \gamma''_{bc-1} W(\varphi_6)^{2b} W(X_{12})^c \}. \end{aligned}$$

Since $v_p(\mathrm{LHS}) \geq 1$, we have $v_p(\mathrm{RHS}) \geq 1$ for both of two formulas above. From Corollary 1 and Theorem 3, we get $v_p(P_1(\varphi_4, \varphi_6, X_{12})) \geq 1$ and $v_p(P_5(\varphi_4, \varphi_6, X_{12})) \geq 1$. From the formula (2), $v_p(P_3(\varphi_4, \varphi_6, X_{12})) \geq 1$.

Case $k \equiv 6 \pmod{12}$($k \equiv 2 \pmod 4$ and $k \equiv 0 \pmod 6$): Similarly to the case of $k \equiv 0 \pmod{12}$, we can prove the assertion of Lemma 3. □

By the Lemma 3 above, we get

$$\begin{aligned} F &\equiv \Delta_5 \cdot \{\Delta_5(Q_1\boldsymbol{\Phi}_{10} + Q_3\boldsymbol{\Phi}_{16} + Q_5\boldsymbol{\Phi}_{18}) + (P_2 + X_{10}Q_2)\boldsymbol{\Phi}_9 + (P_4 + X_{10}Q_4)\boldsymbol{\Phi}_{11} \\ &\quad + (P_6 + X_{10}Q_6)\boldsymbol{\Phi}_{17}\} \\ &:= \Delta_5 \cdot G. \end{aligned}$$

It is known that $\Delta_5 \not\equiv 0 \pmod p$ and $q_\tau^{\frac{1}{2}} q_{\tau'}^{\frac{1}{2}} \mid \Delta_5$ but $q_\tau q_{\tau'} \nmid \Delta_5$. Next we apply the Witt operator to G.

$$\begin{aligned} W(G &= W(P_2)W(\boldsymbol{\Phi}_9) + W(P_4)W(\boldsymbol{\Phi}_{11}) + W(P_6)W(\boldsymbol{\Phi}_{17}) \\ &= \sum_{\substack{12i+4j+6t=k-10 \\ 12i'+4j'+6t'=k-10 \\ t,t'=0,1}} C_3(i,i')\Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t \Delta(\tau')^{i'} E_4(\tau')^{j'} E_6(\tau')^{t'} \eta(\tau)^{12} \eta_(\tau')^{12} \begin{pmatrix} 01 \\ 10 \end{pmatrix} \\ &= \sum_{m,n \geq 0} B_{12}(m,n) q_\tau^m q_{\tau'}^n \eta(\tau)^{12} \eta_(\tau')^{12} \begin{pmatrix} 01 \\ 10 \end{pmatrix}. \end{aligned}$$

It is known that $\eta(\tau)^{12} \not\equiv 0 \pmod p$ and $q_\tau^{\frac{1}{2}} q_{\tau'}^{\frac{1}{2}} \mid \eta(\tau)^{12}$ but $q_\tau q_{\tau'} \nmid \eta(\tau)^{12}$. Hence we have that if $m \leq \left[\frac{k}{10}\right] - 1$ and $n \leq \left[\frac{k}{10}\right] - 1$, then $B_{12}(m,n) \equiv 0 \pmod p$. This implies that $C_3(i,i') \equiv 0 \pmod p$ for $i, i' \leq \left[\frac{k}{10}\right] - 1$. Note that $i, i' \leq \left[\frac{k}{10}\right] - 1$ since $12i + 4j + 6t = k - 10$ and $12i' + 4j' + 6t' = k - 10$ and $\left[\frac{k-10}{12}\right] \leq \left[\frac{k}{10}\right] - 1$. Thus we have $W(G) \equiv 0 \pmod p$.

Lemma 4 $P_2, P_4, P_6 \equiv 0 \pmod p$.

Proof of Lemma 4 Using fact that $W(\Delta_5) = 0$, we get

$$\begin{aligned} W(G) &= W(P_2)W(\boldsymbol{\Phi}_9) + W(P_4)W(\boldsymbol{\Phi}_{11}) + W(P_6)W(\boldsymbol{\Phi}_{17}) \\ &= W(P_2)\left(2E_4(\tau)E_4(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 01 \\ 10 \end{pmatrix}\right) \\ &\quad + W(P_4)\left(2E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 01 \\ 10 \end{pmatrix}\right) \\ &\quad + W(P_6)\left(2^5 \cdot 3^2 \Delta(\tau)\Delta(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 01 \\ 10 \end{pmatrix}\right). \end{aligned}$$

Case $k \not\equiv 4 \pmod 6$: In this case we have $P_4 = P_6 = 0$ as polynomials. Therefore we get $v_p(W(P_2)) \geq 1$. From Corollary 1 and Theorem 3, we get $v_p(P_2(\varphi_4, \varphi_6, X_{12})) \geq 1$.

Case $k \equiv 4 \pmod{12}$: We can write

$$W(P_2) = E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 2 \pmod 3\\ 4a+12b+12c=k-20}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_4) = \sum_{12b+12c=k-16} \gamma'_{bc} W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_6) = 0.$$

Using these formulas, we can write as

$$W(G) = 2\Big(\sum_{\substack{a\equiv 0 \pmod 3,\ a\geq 3\\ 4a+12b+12c=k-16}} \gamma_{a-1bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c + \sum_{12b+12c=k-16} \gamma'_{bc} W(\varphi_6)^{2b} W(X_{12})^c \Big) E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12} \begin{pmatrix} 01\\ 10 \end{pmatrix}.$$

Again from Corollary 1, we have $v_p(\gamma_{a-1bc}) \geq 1$ and $v_p(\gamma'_{bc}) \geq 1$. These mean that, from Theorem 3, $v_p(P_2(\varphi_4, \varphi_6, X_{12})) \geq 1$ and $v_p(P_4(\varphi_6, X_{12})) \geq 1$.

Case$k \equiv 10 \pmod{12}$: We can write

$$W(P_2) = \sum_{\substack{a\equiv 2 \pmod 3\\ 4a+12b+12c=k-14}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_4) = E_6(\tau)E_6(\tau') \sum_{12b+12c=k-22} \gamma'_{bc} W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_6) = \gamma''_{\frac{k-22}{12}} W(X_{12})^{\frac{k-22}{12}}.$$

Using these formulas, we can write as

$$W(G) = \Big(2 \sum_{\substack{a\equiv 0 \pmod 3,\ a\geq 3\\ 4a+12b+12c=k-10}} \gamma_{a-3bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c + 2 \sum_{\substack{b\geq 1\\ 12b+12c=k-10}} \gamma'_{b-1c} W(\varphi_6)^{2b} W(X_{12})^c + 2^3 \cdot 3\gamma''_{\frac{k-22}{12}} W(X_{12})^{\frac{k-10}{12}} \Big)\eta(\tau)^{12}\eta(\tau')^{12} \begin{pmatrix} 01\\ 10 \end{pmatrix}.$$

Again from Corollary 1, we have $v_p(2\gamma_{a-3bc}) \geq 1$, $v_p(2\gamma'_{b-1c}) \geq 1$ and $v_p(2^3 \cdot 3\gamma''_{\frac{k-22}{12}}) \geq 1$. These mean that, from Theorem 3, $v_p(P_2(\varphi_4, \varphi_6, X_{12})) \geq 1$, $v_p(P_4(\varphi_6, X_{12})) \geq 1$ and $v_p(P_6(X_{12})) \geq 1$.

This completes the proof of Lemma 4. □

From Lemma 4, we get

$$F \equiv X_{10} \cdot (Q_1\Phi_{10} + Q_2\Phi_{14} + Q_3\Phi_{16} + Q_4\Psi_{16} + Q_5\Phi_{18} + Q_6\Phi_{22}) \pmod p.$$

Then $H_1 := Q_1\Phi_{10} + Q_2\Phi_{14} + Q_3\Phi_{16} + Q_4\Psi_{16} + Q_5\Phi_{18} + Q_6\Phi_{22} \in M_{k-10,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ and $A((m,n,r);H_1) \equiv 0 \pmod p$ for every m, n such that $0 \leq m \leq \left[\frac{k-10}{10}\right]$, $0 \leq n \leq \left[\frac{k-10}{10}\right]$. Moreover $v_p(F) = v_p(H_1)$ since $v_p(X_{10}) = 0$.

By repeating this argument, there exists the modular form H_t of weight $k-10t$ and t_0 such that

$$F \equiv H_t \cdot X_{10}^t \pmod{p}$$

where $1 \le t \le t_0$ and

$$A((m,n,r);H_t) \equiv 0 \pmod{p}$$

for every m, n such that $0 \le m \le \left[\frac{k-10t}{10}\right], \quad 0 \le n \le \left[\frac{k-10t}{10}\right]$. Thus we have

$$\nu_p(F) = \nu_p(H_{t_0}).$$

Since the weight of $H_{t_0} \le 22$, we should check the case $k \le 22$ directly.

Case ($k \equiv 0 \pmod{10}$)

$H_{t_0} \in M_{10,2}(\Gamma_2)$ and $t_0 = \frac{k-10}{10}$. Since

$$X_{10} = (-2 + q_\omega + q_\omega^{-1}) + \cdots,$$

we have that if $n \le \left[\frac{k}{10}\right] - t_0 = 1$ and $m \le \left[\frac{k}{10}\right] - t_0 = 1$, then $a((m,n,r);H_{t_0}) \equiv 0 \pmod{p}$. With $a_1 \in \mathbb{Z}_{(p)}$ we can write

$$H_{t_0} = a_1 \boldsymbol{\Phi}_{10} = \begin{pmatrix} a_1 0 \\ 0\,0 \end{pmatrix} q_\tau + \cdots.$$

Hence we have $a_1 \equiv 0 \pmod{p}$. Hence we have $F \equiv 0 \pmod{p}$.

Case ($k \equiv 4 \pmod{10}$)

$H_{t_0} \in M_{14,2}(\Gamma_2)$ and $t_0 = \frac{k-14}{10}$. Then we have that if $n \le \left[\frac{k}{10}\right] - t_0 = 1$ and $m \le \left[\frac{k}{10}\right] - t_0 = 1$, then $a((m,n,r);H_{t_0}) \equiv 0 \pmod{p}$. With $a_1, a_2 \in \mathbb{Z}_{(p)}$ we can write

$$\begin{aligned} H_{t_0} &= a_1 \varphi_4 \boldsymbol{\Phi}_{10} + a_2 \boldsymbol{\Phi}_{14} \\ &= \begin{pmatrix} a_1 0 \\ 0\,0 \end{pmatrix} q_\tau + \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix} q_{\tau'} \\ &\quad + \left\{ \begin{pmatrix} 30a_1 + 2a_2 & 0 \\ 0 & 30a_1 + 2a_2 \end{pmatrix} + \begin{pmatrix} -28a_1 - a_2 & -14a_1 - \frac{1}{2}a_2 \\ -14a_1 - \frac{1}{2}a_2 & -28a_1 - a_2 \end{pmatrix} q_\omega \right. \\ &\quad \left. + \begin{pmatrix} -28a_1 - a_2 & 14a_1 + \frac{1}{2}a_2 \\ 14a_1 + \frac{1}{2}a_2 & -28a_1 - a_2 \end{pmatrix} q_\omega^{-1} + \begin{pmatrix} a_1 a_1 \\ a_1 a_1 \end{pmatrix} q_\omega^2 + \begin{pmatrix} a_1 & -a_1 \\ -a_1 & a_1 \end{pmatrix} q_\omega^{-2} \right\} q_\tau q_{\tau'} + \cdots. \end{aligned}$$

Hence we have $a_1 \equiv 0 \pmod{p}$ and $-28a_1 - a_2 \equiv 0 \pmod{p}$. Hence we get $a_2 \equiv 0 \pmod{p}$. Hence we have $F \equiv 0 \pmod{p}$.

Case ($k \equiv 6 \pmod{10}$)

$H_{t_0} \in M_{16,2}(\Gamma_2)$ and $t_0 = \frac{k-16}{10}$. Then we have that if $n \le \left[\frac{k}{10}\right] - t_0 = 1$ and $m \le \left[\frac{k}{10}\right] - t_0 = 1$, then $a((m,n,r);H_{t_0}) \equiv 0 \pmod{p}$. With $a_1, a_2, a_3 \in \mathbb{Z}_{(p)}$ we can write

$$
\begin{aligned}
H_{t_0} &= a_1\varphi_6\boldsymbol{\Phi}_{10} + a_2\boldsymbol{\Phi}_{16} + a_3\boldsymbol{\Psi}_{16} \\
&= \begin{pmatrix} a_1 0 \\ 0\,0 \end{pmatrix} q_\tau + \begin{pmatrix} 0\ \ 0 \\ 0\ a_1 \end{pmatrix} q_{\tau'} \\
&\quad + \left\{ \begin{pmatrix} -714a_1 + 10a_2 - 2a_3 & 0 \\ 0 & -714a_1 + 10a_2 - 2a_3 \end{pmatrix} \right. \\
&\quad + \begin{pmatrix} -28a_1 + a_2 + a_3 & -14a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 \\ -14a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 & -28a_1 + a_2 + a_3 \end{pmatrix} q_\omega \\
&\quad \left. + \begin{pmatrix} -28a_1 + a_2 + a_3 & 14a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 \\ 14a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 & -28a_1 + a_2 + a_3 \end{pmatrix} q_\omega^{-1} + \begin{pmatrix} a_1 a_1 \\ a_1 a_1 \end{pmatrix} q_\omega^2 + \begin{pmatrix} a_1 & -a_1 \\ -a_1 & a_1 \end{pmatrix} q_\omega^{-2} \right\} q_\tau q_{\tau'} + \cdots .
\end{aligned}
$$

Hence we have $a_1 \equiv -714a_1 + 10a_2 - 2a_3 \equiv -28a_1 + a_2 + a_3 \equiv 0 \pmod p$. Hence we get $a_2 = \frac{1}{2^2\cdot 3}(10a_2 - 2a_3 + 2(a_2 + a_3)) \equiv 0 \pmod p$ and $a_3 = \frac{1}{2^2\cdot 3}(-(10a_2 - 2a_3) + 10(a_2 + a_3)) \equiv 0 \pmod p$. Hence we have $F \equiv 0 \pmod p$.

(*Case* $k \equiv 8 \pmod{10}$)

$H_{t_0} \in M_{18,2}(\Gamma_2)$ and $t_0 = \frac{k-18}{10}$. Then we have that if $n \leq \left[\frac{k}{10}\right] - t_0 = 1$ and $m \leq \left[\frac{k}{10}\right] - t_0 = 1$, then $a((m,n,r);H_{t_0}) \equiv 0 \pmod p$. With $a_1, a_2, a_3 \in \mathbb{Z}_{(p)}$ we can write

$$
\begin{aligned}
H_{t_0} &= a_1\varphi_4^2\boldsymbol{\Phi}_{10} + a_2\varphi_4\boldsymbol{\Phi}_{14} + a_3\boldsymbol{\Phi}_{18} \\
&= \begin{pmatrix} a_1 0 \\ 0\,0 \end{pmatrix} q_\tau + \begin{pmatrix} 0\ \ 0 \\ 0\ a_1 \end{pmatrix} q_{\tau'} \\
&\quad + \left\{ \begin{pmatrix} 270a_1 - 2a_2 + 10a_3 & 0 \\ 0 & 270a_1 - 2a_2 + 10a_3 \end{pmatrix} \right. \\
&\quad + \begin{pmatrix} -28a_1 + a_2 + a_3 & -14a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 \\ -14a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 & -28a_1 + a_2 + a_3 \end{pmatrix} q_\omega \\
&\quad \left. + \begin{pmatrix} -28a_1 + a_2 + a_3 & 14a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 \\ 14a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 & -28a_1 + a_2 + a_3 \end{pmatrix} q_\omega^{-1} + \begin{pmatrix} a_1 a_1 \\ a_1 a_1 \end{pmatrix} q_\omega^2 + \begin{pmatrix} a_1 & -a_1 \\ -a_1 & a_1 \end{pmatrix} q_\omega^{-2} \right\} q_\tau q_{\tau'} + \cdots .
\end{aligned}
$$

Hence we have $a_1 \equiv 270a_1 - 2a_2 + 10a_3 \equiv -28a_1 + a_2 + a_3 \equiv 0 \pmod p$. Hence we get $a_2 = \frac{1}{2^2\cdot 3}(-(-2a_2 + 10a_3) + 10(a_2 + a_3)) \equiv 0$ and $a_3 = \frac{1}{2^2\cdot 3}(-2a_2 + 10a_3 + 2(a_2 + a_3)) \equiv 0 \pmod p$. Hence we have $F \equiv 0 \pmod p$.

(*Case* $k \equiv 2 \pmod{10}$)

$H_{t_0} \in M_{22,2}(\Gamma_2)$ and $t_0 = \frac{k-22}{10}$. Then we have that if $n \leq \left[\frac{k}{10}\right] - t_0 = 2$ and $m \leq \left[\frac{k}{10}\right] - t_0 = 2$, then $a((m,n,r);H_{t_0}) \equiv 0 \pmod p$. With $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{Z}_{(p)}$ we can write

$$
\begin{aligned}
H_{t_0} &= (a_1\varphi_4^3 + a_2\varphi_6^2 + a_3X_{12})\boldsymbol{\Phi}_{10} + a_4\varphi_4^2\boldsymbol{\Phi}_{14} + a_5\varphi_6\boldsymbol{\Phi}_{16} + a_6\varphi_6\boldsymbol{\Psi}_{16} + a_7\boldsymbol{\Phi}_{22} \\
&= \begin{pmatrix} a_1+a_2 & 0 \\ 0 & 0 \end{pmatrix} q_\tau + \begin{pmatrix} 0 & 0 \\ 0 & a_1+a_2 \end{pmatrix} q_{\tau'} \\
&\quad + \left\{ \begin{pmatrix} 510a_1 - 1218a_2 - 2a_4 + 10a_5 - 2a_6 & 0 \\ 0 & 510a_1 - 1218a_2 - 2a_4 + 10a_5 - 2a_6 \end{pmatrix} \right. \\
&\quad + \begin{pmatrix} -28a_1 - 28a_2 + a_4 + a_5 + a_6 & -14a_1 - 14a_2 + \frac{1}{2}a_4 + \frac{1}{2}a_5 + \frac{1}{2}a_6 \\ -14a_1 - 14a_2 + \frac{1}{2}a_4 + \frac{1}{2}a_5 + \frac{1}{2}a_6 & -28a_1 - 28a_2 + a_4 + a_5 + a_6 \end{pmatrix} q_\omega \\
&\quad + \begin{pmatrix} -28a_1 - 28a_2 + a_4 + a_5 + a_6 & 14a_1 + 14a_2 - \frac{1}{2}a_4 - \frac{1}{2}a_5 - \frac{1}{2}a_6 \\ 14a_1 + 14a_2 - \frac{1}{2}a_4 - \frac{1}{2}a_5 - \frac{1}{2}a_6 & -28a_1 - 28a_2 + a_4 + a_5 + a_6 \end{pmatrix} q_\omega^{-1} \\
&\quad \left. + \begin{pmatrix} a_1+a_2 & a_1+a_2 \\ a_1+a_2 & a_1+a_2 \end{pmatrix} q_\omega^2 + \begin{pmatrix} a_1+a_2 & -a_1-a_2 \\ -a_1-a_2 & a_1+a_2 \end{pmatrix} q_\omega^{-2} \right\} q_\tau q_{\tau'} \\
&\quad + \begin{pmatrix} 696a_1 - 1032a_2 & 0 \\ 0 & 0 \end{pmatrix} q_\tau^2 + \begin{pmatrix} 0 & 0 \\ 0 & 696a_1 - 1032a_2 \end{pmatrix} q_{\tau'}^2 \\
&\quad + \left\{ \begin{pmatrix} -54324a_1 + 350028a_2 - 1404a_4 - 2772a_5 + 2052a_6 & 0 \\ 0 & 279432a_1 + 1088136a_2 + 10a_3 - 168a_4 - 10104a_5 + 408a_6 \end{pmatrix} \right. \\
&\quad + \begin{pmatrix} -46464a_1 + 1920a_2 + 704a_4 - 352a_5 - 1024a_6 & -23232a_1 + 960a_2 + 352a_4 - 176a_5 - 512a_6 \\ -23232a_1 + 960a_2 + 352a_4 - 176a_5 - 512a_6 & 17952a_1 + 114720a_2 + a_3 + 88a_4 - 1160a_5 - 200a_6 \end{pmatrix} q_\omega
\end{aligned}
$$

$$
\begin{aligned}
&+ \begin{pmatrix} -46464a_1 + 1920a_2 + 704a_4 - 352a_5 - 1024a_6 & 23232a_1 - 960a_2 - 352a_4 + 176a_5 + 512a_6 \\ 23232a_1 - 960a_2 - 352a_4 + 176a_5 + 512a_6 & 17952a_1 + 114720a_2 + a_3 + 88a_4 - 1160a_5 - 200a_6 \end{pmatrix} q_\omega^{-1} \\
&+ \begin{pmatrix} 510a_1 - 1218a_2 - 2a_4 + 10a_5 - 2a_6 & 510a_1 - 1218a_2 - 2a_4 + 10a_5 - 2a_6 \\ 510a_1 - 1218a_2 - 2a_4 + 10a_5 - 2a_6 & 1020a_1 - 2436a_2 - 4a_4 + 20a_5 - 4a_6 \end{pmatrix} q_\omega^2 \\
&\left. + \begin{pmatrix} 510a_1 - 1218a_2 - 2a_4 + 10a_5 - 2a_6 & -510a_1 + 1218a_2 + 2a_4 - 10a_5 + 2a_6 \\ -510a_1 + 1218a_2 + 2a_4 - 10a_5 + 2a_6 & 1020a_1 - 2436a_2 - 4a_4 + 20a_5 - 4a_6 \end{pmatrix} q_\omega^{-2} \right\} q_\tau q_{\tau'}^2 \\
&+ \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} + \begin{pmatrix} A_2 & C_2 \\ C_2 & B_2 \end{pmatrix} q_\omega + \begin{pmatrix} A_2 & -C_2 \\ -C_2 & B_2 \end{pmatrix} q_\omega^{-1} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \begin{pmatrix} -688416a_1 + 217056a_2 - 8a_3 + 3840a_4 - 1920a_5 + 19968a_6 + 2a_7 & -344208a_1 + 108528a_2 - 4a_3 + 1920a_4 - 960a_5 + 9984a_6 + a_7 \\ -344208a_1 + 108528a_2 - 4a_3 + 1920a_4 - 960a_5 + 9984a_6 + a_7 & -688416a_1 + 217056a_2 - 8a_3 + 3840a_4 - 1920a_5 + 19968a_6 + 2a_7 \end{pmatrix} q_\omega^2 \\
&+ \begin{pmatrix} -688416a_1 + 217056a_2 - 8a_3 + 3840a_4 - 1920a_5 + 19968a_6 + 2a_7 & 344208a_1 - 108528a_2 + 4a_3 - 1920a_4 + 960a_5 - 9984a_6 - a_7 \\ 344208a_1 - 108528a_2 + 4a_3 - 1920a_4 + 960a_5 - 9984a_6 - a_7 & -688416a_1 + 217056a_2 - 8a_3 + 3840a_4 - 1920a_5 + 19968a_6 + 2a_7 \end{pmatrix} q_\omega^{-2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \begin{pmatrix} 17952a_1 + 114720a_2 + a_3 + 88a_4 - 1160a_5 - 200a_6 & 41184a_1 + 113760a_2 + a_3 - 264a_4 - 984a_5 + 312a_6 \\ 41184a_1 + 113760a_2 + a_3 - 264a_4 - 984a_5 + +312a_6 & 17952a_1 + 114720a_2 + a_3 + 88a_4 - 1160a_5 - 200a_6 \end{pmatrix} q_\omega^3 \\
&+ \begin{pmatrix} 17952a_1 + 114720a_2 + a_3 + 88a_4 - 1160a_5 - 200a_6 & -41184a_1 - 113760a_2 - a_3 + 264a_4 + 984a_5 - 312a_6 \\ -41184a_1 - 113760a_2 - a_3 + 264a_4 + 984a_5 - 312a_6 & 17952a_1 + 114720a_2 + a_3 + 88a_4 - 1160a_5 - 200a_6 \end{pmatrix} q_\omega^{-3} \\
&\left. + \begin{pmatrix} 696a_1 - 1032a_2 & 696a_1 - 1032a_2 \\ 696a_1 - 1032a_2 & 696a_1 - 1032a_2 \end{pmatrix} q_\omega^4 + \begin{pmatrix} 696a_1 - 1032a_2 & -696a_1 + 1032a_2 \\ -696a_1 + 1032a_2 & 696a_1 - 1032a_2 \end{pmatrix} q_\omega^{-4} \right\} q_\tau^2 q_{\tau'}^2 + \cdots,
\end{aligned}
$$

where

$$A_1 = B_1 = -59095920a_1 - 257328624a_2 - 2288a_3 - 75776a_4 + 2537728a_5 - 329216a_6 - 36a_7$$
$$A_2 = B_2 = -20671008a_1 - 53005344a_2 - 577a_3 + 33960a_4 + 517512a_5 + 144840a_6 + 16a_7$$
$$C_2 = -24544032a_1 + 15272544a_2 - 139a_3 + 239160a_4 - 205656a_5 + 511608a_6 + 70a_7$$

Hence we get

$$a_1 = \frac{1}{2^6 \cdot 3^3}(1032(a_1 + a_2) + (696a_1 - 1032a_2)) \equiv 0 \pmod p,$$
$$a_2 = \frac{1}{2^6 \cdot 3^3}(696(a_1 + a_2) - (696a_1 - 1032a_2)) \equiv 0 \pmod p.$$

Then we get

$$a_5 = \frac{1}{2^2 \cdot 3}(2(a_4 + a_5 + a_6) + (-2a_4 + 10a_5 - 2a_6)) \equiv 0 \pmod p.$$

Hence we get $a_4 + a_6 \equiv 0 \pmod p$ and $704a_4 - 1024a_6 \equiv 0 \pmod p$. Hence we get

$$a_4 = \frac{1}{2^6 \cdot 3^3}(1024(a_4 + a_6) + (704a_4 - 1024a_6)) \equiv 0 \pmod p$$

and

$$a_6 = \frac{1}{2^6 \cdot 3^3}(704(a_4 + a_6) - (704a_4 - 1024a_6)) \equiv 0 \pmod p$$

. Hence we get $a_3 \equiv a_7 \equiv 0 \pmod p$. Hence we have $F \equiv 0 \pmod p$. □

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References

1. Böcherer, S., Nagaoka, S.: On mod *p* properties of Siegel modular forms. Math. Ann. **338**, 421–433 (2007)
2. Choi, D., Choie, Y., Kikuta, T.: Sturm type theorem for Siegel modular forms of genus 2 modulo *p*. Acta Arithmetica **158**, 129–139 (2013)
3. Freitag, E.: Zur Theorie der Modulformen zweiten Grades. Nachr. Akad. Wiss. Göttingen **61**, 151–157 (1965)
4. Igusa, J.-I.: On Siegel modular forms of genus two (I I). Am. J. Math. **86**, 392–412 (1964)
5. Igusa, J.-I.: On the ring of modular forms of degree two over **Z**. Am. J. Math. **101**, 149–183 (1979)
6. Kodama, H.: On certain vector valued Siegel modular forms of type $(k, 2)$ over $\mathbb{Z}_{(p)}$. Acta Arithmetica **188**, 83–98 (2019)
7. Nagaoka, S.: Note on mod *p* Siegel modular forms. Math. Z. **235**, 405–420 (2000)
8. Poor, C., Yuen, D.: Paramodular cusp forms. Math. Comput. **84**(293), 1401–1438 (2015)
9. Richter, O.K., Westerholt-Raum, M.: Sturm bounds for Siegel modular forms. Res. Number Theory **1**, Article 5 (2015)
10. Satoh, T.: On certain vector valued Siegel modular forms of degree two. Math. Ann. **274**, 335–352 (1986)
11. Sturm, J.: On the congruence of modular forms. In: Chudnovsky, D.V., Chudnovsky, G.V., Cohn, H., Nathanson, M.B. (eds.) Number Theory, pp. 275–280. Springer, Berlin (1987)
12. Witt, E.: Eine Identität zwischen Modulformen zweiten Grades. Abh. Math. Sem. Hansischen Univ. **14**, 323–337 (1941)