

# Motives of derived equivalent K3 surfaces

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**Abstract** We observe that derived equivalent K3 surfaces have isomorphic Chow motives. The result holds more generally for arbitrary surfaces, as pointed out by Charles Vial.

**Keywords** Motives · K3 surfaces · Derived categories

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There are many examples of non-isomorphic K3 surfaces  $S$  and  $S'$  (over a field  $k$ ) with equivalent derived categories  $D^b(S) \simeq D^b(S')$  of coherent sheaves. It is known that in this case  $S'$  is isomorphic to a fine moduli space of slope-stable bundles on  $S$  and vice versa, cf. [11, 21, 23]. However, the precise relation between  $S$  and  $S'$  still eludes us and many basic questions are hard to answer. For example, one could ask whether  $S'$  contains infinitely many rational curves, expected for all K3 surfaces, if this is known already for  $S$ . Or, if  $k$  is a number field, does potential density for  $S$  imply potential density for  $S'$ ?

As a direct geometric approach to these questions seems difficult, one may wonder whether  $S$  and  $S'$  can be compared at least motivically. This could mean various things, e.g. one could study their classes  $[S]$  and  $[S']$  in the Grothendieck ring of varieties  $K_0(\text{Var}(k))$  or their associated motives  $\mathfrak{h}(S)$  and  $\mathfrak{h}(S')$  in the category of Chow motives  $\text{Mot}(k)$ . In the recent articles [8, 15, 17], examples in degree 8 and 12 have been studied for which  $[S] - [S']$  in  $K_0(\text{Var}(k))$  is annihilated by  $\ell := [\mathbb{A}^1]$ . The question whether the Chow motives  $\mathfrak{h}(S)$  and  $\mathfrak{h}(S')$  in  $\text{Mot}(k)$  are isomorphic was first addressed and answered affirmatively in special cases in [24]. Assuming finite-dimensionality of the motives the question was settled in [4].

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In this short note we point out that the available techniques in the theory of motives are enough to show that two derived equivalent K3 surfaces have indeed isomorphic Chow motives.

**Theorem 0.1** *Let  $S$  and  $S'$  be K3 surfaces over a field  $k$ . Assume that there exists an exact  $k$ -linear equivalence  $D^b(S) \simeq D^b(S')$  between their bounded derived categories of coherent sheaves. Then there is an isomorphism*

$$h(S) \simeq h(S')$$

in the category of Chow motives  $\text{Mot}(k)$ .

We had originally expected that the invariance of the Beauville–Voisin ring as proved in [12, 13] would be central to the argument. However, it turns out to have no bearing on the problem, but it implies that a distinguished decomposition of the motives in their algebraic and transcendental parts is preserved under derived equivalence. In fact, the result holds for arbitrary surfaces, see Sect. 2.4.

### 1 From derived categories to Chow motives

We follow the convention and notation in [2, 22] and denote by  $\text{Mot}(k)$  the pseudo-abelian category of Chow motives. Objects of this category are triples  $M = (X, p, m)$ , where  $X$  is a smooth projective  $k$ -variety of dimension  $d_X$ ,  $p \in \text{CH}^{d_X}(X \times X) \otimes \mathbb{Q}$  with  $p \circ p = p$ , and  $m \in \mathbb{Z}$ . Morphisms  $f: M = (X, p, m) \rightarrow N = (Y, q, n)$  are elements  $\alpha \in \text{CH}^{d_X+n-m}(X \times Y) \otimes \mathbb{Q}$  such that  $q \circ \alpha = \alpha = \alpha \circ p$ .

The Chow motive of a smooth projective variety  $X$  is denoted  $h(X) := (X, [\Delta_X]) := (X, [\Delta_X], 0)$ . The Lefschetz motive  $\mathbb{L} := (\mathbb{P}^1, \mathbb{P}^1 \times \text{pt}, 0) \simeq (\text{Spec}(k), \text{id}, -1)$  plays a distinguished role, e.g. the Chow groups of a motive  $M = (X, p, m)$  are defined to be  $\text{CH}_{\mathbb{Q}}^i(M) := \text{Mor}(\mathbb{L}^{\otimes i}, M) = p \circ \text{CH}^i(M) \otimes \mathbb{Q}$ .

1.1. The following result combines [5, Lemma 1] and [26, Lemma 3.1, Theorem 3.18], cf. [25, Lemma 4.3].

**Lemma 1.1** *Let  $f: M \rightarrow N$  be a morphism in the category  $\text{Mot}(k)$  of Chow motives over a field  $k$ . Then  $f$  is an isomorphism if and only if for all field extensions  $K/k$  the map  $f_{K*}: \text{CH}_{\mathbb{Q}}^*(M_K) \xrightarrow{\sim} \text{CH}_{\mathbb{Q}}^*(N_K)$  induced by the base change  $f_K: M_K \rightarrow N_K$  is bijective.* □

One direction is obvious and for the other one use Manin’s identity principle and induction over the dimension.

1.2. We start by describing Orlov’s approach [23]. Let  $\Phi: D^b(S) \xrightarrow{\sim} D^b(S')$  be an exact  $k$ -linear equivalence between the bounded derived categories of coherent sheaves on two K3 surfaces  $S$  and  $S'$ . Then  $\Phi$  is a Fourier–Mukai functor, i.e.  $\Phi \simeq \Phi_{\mathcal{E}}: F \mapsto p_*(q^*F \otimes \mathcal{E})$  for some  $\mathcal{E} \in D^b(S \times S')$  (unique up to isomorphism), see [24]. The Mukai vector

$$v(\mathcal{E}) := \text{ch}(\mathcal{E}) \cdot \sqrt{\text{td}(S \times S')} \in \text{CH}^*(S \times S') \otimes \mathbb{Q}$$

induces an (ungraded) isomorphism  $v(\mathcal{E})_*: \text{CH}^*(S) \xrightarrow{\sim} \text{CH}^*(S')$ , cf. [14, Ch. 16].<sup>1</sup>

<sup>1</sup> For K3 surfaces the isomorphism is indeed between integral Chow groups, but this is not true for other types of varieties.

The inverse of  $\Phi$  can be described as  $\Phi^{-1} \simeq \Phi_{\mathcal{E}^*[2]}$ , where  $\mathcal{E}^*$  is the derived dual of  $\mathcal{E}$ . In particular,  $v(\mathcal{E}^*) \circ v(\mathcal{E}) = [\Delta_S]$  and  $v(\mathcal{E}) \circ v(\mathcal{E}^*) = [\Delta_{S'}]$ . Thus, if  $v(\mathcal{E})$  is purely of degree two, i.e.  $v(\mathcal{E}) = v_2(\mathcal{E}) \in \text{CH}^2(S \times S') \otimes \mathbb{Q}$ , then  $v(\mathcal{E})$  can be considered as a morphism in  $\text{Mot}(k)$  and defines an isomorphism

$$v(\mathcal{E}) : \mathfrak{h}(S) \xrightarrow{\sim} \mathfrak{h}(S').$$

However, it is not expected that for two derived equivalent K3 surfaces  $S$  and  $S'$  there always exists an equivalence  $\Phi_{\mathcal{E}} : D^b(S) \xrightarrow{\sim} D^b(S')$  with  $v(\mathcal{E}) = v_2(\mathcal{E})$ . In fact, when  $S'$  is viewed as a moduli space of slope-stable bundles the universal family  $\mathcal{E}$  will certainly not have this property, for  $v_0(\mathcal{E}) = \text{rk}(\mathcal{E})$ .

Next, one could try to just work with the degree two part  $v_2 := v_2(\mathcal{E}) \in \text{CH}^2(S \times S') \otimes \mathbb{Q}$  of  $v(\mathcal{E})$ . It defines a morphism  $\mathfrak{h}(S) \rightarrow \mathfrak{h}(S')$  in  $\text{Mot}(k)$ , which, however, will usually not be an isomorphism, for the induced action on cohomology  $v_{2*}^H : H^*(S) \rightarrow H^*(S')$  is often not bijective, e.g.  $v_{2*}^H$  for the spherical twist  $T_{\mathcal{O}} : D^b(S) \xrightarrow{\sim} D^b(S)$  is trivial on  $H^0$ .

1.3. Instead of producing an isomorphism  $\mathfrak{h}(S) \xrightarrow{\sim} \mathfrak{h}(S')$  directly, we shall first decompose both motives with respect to their degree. The decompositions depend on the choice of additional cycles of degree one  $c \in \text{CH}^2(S) \otimes \mathbb{Q}$  and  $c' \in \text{CH}^2(S') \otimes \mathbb{Q}$ , respectively, cf. [2, 22]. For  $S$  it reads

$$\mathfrak{h}(S) \simeq \mathfrak{h}^0(S) \oplus \mathfrak{h}^2(S) \oplus \mathfrak{h}^4(S), \tag{1.1}$$

where  $\mathfrak{h}^0(S) := (S, p_0 := [c \times S])$ ,  $\mathfrak{h}^4(S) := (S, p_4 := [S \times c])$ , and  $\mathfrak{h}^2(S) := (S, p_2 := [\Delta_S] - p_0 - p_4)$ . Note that  $\mathfrak{h}^0(S) \simeq \mathfrak{h}(\text{Spec}(k)) \simeq \mathbb{L}^0$  and  $\mathfrak{h}^4(S) \simeq \mathbb{L}^2$ . Furthermore,

$$\begin{aligned} \text{CH}_{\mathbb{Q}}^*(\mathfrak{h}^0(S)) &= \text{CH}^0(S) \otimes \mathbb{Q}, \quad \text{CH}_{\mathbb{Q}}^*(\mathfrak{h}^4(S)) = \mathbb{Q} \cdot c \subset \text{CH}^2(S) \otimes \mathbb{Q}, \\ \text{and } \text{CH}_{\mathbb{Q}}^*(\mathfrak{h}^2(S)) &= (\text{CH}^1(S) \oplus \text{CH}^2(S)_0) \otimes \mathbb{Q}, \end{aligned}$$

where  $\text{CH}^2(S)_0 := \text{Ker}(\text{deg} : \text{CH}^2(S) \rightarrow \mathbb{Z})$ .

Following [16], the degree two part  $\mathfrak{h}^2(S)$  can be further decomposed into its algebraic and transcendental parts

$$\mathfrak{h}^2(S) \simeq \mathfrak{h}_{\text{alg}}^2(S) \oplus \mathfrak{h}_{\text{tr}}^2(S). \tag{1.2}$$

Here,  $\mathfrak{h}_{\text{alg}}^2(S) = (S, p_{2\text{alg}})$  and  $\mathfrak{h}_{\text{tr}}^2(S) = (S, p_{\text{tr}})$  with  $p_{2\text{alg}} := \sum 1/(\ell_i)^2 \ell_i \times \ell_i$ , for an orthogonal basis  $\{\ell_i\}$  of  $\text{NS}(S) \otimes \mathbb{Q}$ , and  $p_{\text{tr}} := [\Delta_S] - p_{2\text{alg}} - p_0 - p_4$ . Note that

$$\text{CH}_{\mathbb{Q}}^*(\mathfrak{h}_{\text{tr}}^2(S)) = \text{CH}^2(S)_0 \otimes \mathbb{Q} \text{ and } \text{CH}_{\mathbb{Q}}^*(\mathfrak{h}_{\text{alg}}^2(S)) = \text{CH}^1(S) \otimes \mathbb{Q} \simeq \text{NS}(S) \otimes \mathbb{Q}$$

and, in fact,  $\mathfrak{h}_{\text{alg}}^2(S) \simeq \mathbb{L}^{\oplus \rho(S)}$ .

We shall call  $\mathfrak{h}_{\text{alg}}(S) := \mathfrak{h}^0(S) \oplus \mathfrak{h}_{\text{alg}}^2(S) \oplus \mathfrak{h}^4(S) = (S, p_{\text{alg}} := p_0 + p_4 + p_{2\text{alg}})$  the algebraic motive of  $S$ . Then

$$\mathfrak{h}_{\text{alg}}(S) \simeq \mathbb{L}^0 \oplus \mathbb{L}^{\oplus \rho(S)} \oplus \mathbb{L}^2 \tag{1.3}$$

and

$$\mathfrak{h}(S) \simeq \mathfrak{h}_{\text{alg}}(S) \oplus \mathfrak{h}_{\text{tr}}^2(S).$$

Next, under a field extension  $K/k$  satisfying  $\rho(S) = \rho(S_K)$  (which is the case whenever  $k$  is algebraically closed), one has  $\mathfrak{h}_{\text{alg}}(S)_K \simeq \mathfrak{h}_{\text{alg}}(S_K)$  and, therefore,  $\mathfrak{h}_{\text{tr}}^2(S)_K \simeq \mathfrak{h}_{\text{tr}}^2(S_K)$ . In particular,  $\text{CH}^*(\mathfrak{h}_{\text{tr}}^2(S)_K) = \text{CH}^2(S_K)_0 \otimes \mathbb{Q}$ . Without this assumption,  $\mathfrak{h}_{\text{alg}}(S_K) \simeq \mathfrak{h}_{\text{alg}}(S)_K \oplus \mathbb{L}^{\oplus \rho(S_K) - \rho(S)}$  and  $\mathfrak{h}_{\text{tr}}^2(S)_K \simeq \mathfrak{h}_{\text{tr}}^2(S_K) \oplus \mathbb{L}^{\oplus \rho(S_K) - \rho(S)}$ . In this case,

$\text{CH}^*(\mathfrak{h}_{\text{tr}}^2(S)_K) = (\text{CH}^2(S_K)_0 \oplus \text{NS}(S)^\perp) \otimes \mathbb{Q}$  with the orthogonal complement taken in  $\text{NS}(S_K) \otimes \mathbb{Q}$ . Note that the Picard rank is invariant under derived equivalence, as it can be read off from the numerical Grothendieck group which is a derived invariant, cf. [14, Ch. 16].

1.4. As derived equivalent K3 surfaces have equal Picard numbers, by (1.3) their algebraic motives are abstractly (i.e. without choosing an actual equivalence) isomorphic:

$$D^b(S) \simeq D^b(S') \Rightarrow \mathfrak{h}_{\text{alg}}(S) \simeq \mathfrak{h}_{\text{alg}}(S').$$

Thus, in order to prove Theorem 0.1, it suffices to show the following.

**Proposition 1.2** *If  $S$  and  $S'$  are derived equivalent K3 surfaces, then there exists an isomorphism*

$$\mathfrak{h}_{\text{tr}}^2(S) \xrightarrow{\sim} \mathfrak{h}_{\text{tr}}^2(S').$$

*Proof* Pick an equivalence  $\Phi_{\mathcal{E}} : D^b(S) \xrightarrow{\sim} D^b(S')$  and consider the Mukai vector  $v := v(\mathcal{E}) = \oplus v_i(\mathcal{E}) \in \bigoplus \text{CH}^i(S \times S') \otimes \mathbb{Q}$  of its kernel  $\mathcal{E} \in D^b(S \times S')$ . The degree two part  $v_2(\mathcal{E})$  induces a morphism  $f : \mathfrak{h}(S) \rightarrow \mathfrak{h}(S')$ , which decomposes as

$$f = \begin{pmatrix} f_{\text{tr}}^{\text{tr}} & f_{\text{alg}}^{\text{tr}} \\ f_{\text{tr}}^{\text{alg}} & f_{\text{alg}}^{\text{alg}} \end{pmatrix} : \mathfrak{h}_{\text{tr}}^2(S) \oplus \mathfrak{h}_{\text{alg}}(S) \rightarrow \mathfrak{h}_{\text{tr}}^2(S') \oplus \mathfrak{h}_{\text{alg}}(S').$$

We show that  $f_{\text{tr}}^{\text{tr}} : \mathfrak{h}_{\text{tr}}^2(S) \rightarrow \mathfrak{h}_{\text{tr}}^2(S')$  is an isomorphism. According to Lemma 1.1, it suffices to show that for all field extensions  $K/k$  the induced map

$$f_{\text{tr}K*}^{\text{tr}} : \text{CH}^*(\mathfrak{h}_{\text{tr}}^2(S)_K) \xrightarrow{\sim} \text{CH}^*(\mathfrak{h}_{\text{tr}}^2(S')_K) \tag{1.4}$$

is bijective. Let us for simplicity first assume that  $k$  is algebraically closed or, at least, that  $\rho(S) = \rho(S_{\bar{k}})$ . Then the two sides of (1.4) are the cohomologically trivial parts of  $\text{CH}^*(S_K) \otimes \mathbb{Q}$  and  $\text{CH}^*(S'_K) \otimes \mathbb{Q}$ , respectively, and hence  $f_{\text{tr}K*}^{\text{tr}} = f_{K*} = v_2(\mathcal{E}_K)_*$ . However, as the cohomologically trivial part of the Chow ring of a K3 surface is concentrated in degree two, the action of  $v_2(\mathcal{E}_K)$  on  $\text{CH}^2(S_K)_0$  coincides with the action of  $v(\mathcal{E}_K)$  on it. As  $\mathcal{E}_K$  is the base change of the Fourier–Mukai kernel of an equivalence, it again describes an equivalence and, therefore, induces an ungraded isomorphism  $\text{CH}^*(S_K) \otimes \mathbb{Q} \xrightarrow{\sim} \text{CH}^*(S'_K) \otimes \mathbb{Q}$ . This shows the bijectivity of  $f_{\text{tr}K*}^{\text{tr}}$ . Without any assumption on the field  $k$ , the left hand side in (1.4) decomposes into the cohomologically trivial part  $\text{CH}^2(S_K)_0 \otimes \mathbb{Q}$  and  $\text{NS}(S)^\perp \otimes \mathbb{Q}$  ( $\subset \text{NS}(S_K) \otimes \mathbb{Q}$ ) and similarly for the right hand side. Now, as above, the action of  $v_2(\mathcal{E}_K)$  on the cohomologically trivial part coincides with the action of  $v(\mathcal{E})$  and so is bijective. The image of  $v_2(\mathcal{E}_K)$  on the newly created part  $\text{NS}(S)^\perp \subset \text{NS}(S_K)$  may differ from the action of  $v(\mathcal{E}_K)$  by components in  $\text{CH}^2(S'_K)_0 \otimes \mathbb{Q}$ . However, as  $v(\mathcal{E}_K)$  induces an isomorphism  $\text{NS}(S)^\perp \otimes \mathbb{Q} \xrightarrow{\sim} \text{NS}(S')^\perp \otimes \mathbb{Q}$ , one still concludes that (1.4) is bijective.

*Remark 1.3* Assume the classes  $c \in \text{CH}^2(S) \otimes \mathbb{Q}$  and  $c' \in \text{CH}^2(S') \otimes \mathbb{Q}$  are the Beauville–Voisin classes of  $S$  and  $S'$ , respectively. Then, using [14], we know that  $v(\mathcal{E})$  induces an ungraded isomorphism  $R(S) \simeq R(S')$  (of groups) between the Beauville–Voisin rings. Here,  $R(S) \subset \text{CH}^*(S) \otimes \mathbb{Q}$  is the  $\mathbb{Q}$ -subalgebra generated by  $\text{CH}^1(S)$ . We show that this implies that  $f$  is in fact diagonal

$$f = \begin{pmatrix} f_{\text{tr}} & 0 \\ 0 & f_{\text{alg}} \end{pmatrix},$$

i.e.  $f_{\text{tr}}^{\text{alg}} = 0$  and  $f_{\text{alg}}^{\text{tr}} = 0$ . Clearly,  $f_{\text{alg}K^*}^{\text{tr}} : \text{CH}^*(S_K)_0 \otimes \mathbb{Q} \rightarrow R(S'_K) \hookrightarrow H^*(S')$  is trivial and, using [14], also  $f_{\text{tr}K^*}^{\text{alg}} : R(S) \rightarrow \text{CH}^2(S_K)_0$  vanishes. However, to conclude one needs to show that  $p'_{\text{alg}} \circ v_2 \circ p_{\text{tr}} = 0 = p'_{\text{tr}} \circ v_2 \circ p_{\text{alg}}$  and both are implied by  $p'_{\text{alg}} \circ v_2 = v_2 \circ p_{\text{alg}}$ . The latter holds in  $H^*(S \times S')$  and as both sides are contained in  $R(S) \otimes R(S')$  which injects into  $H^*(S \times S')$ , this suffices to conclude.

## 2 Applications and further comments

We conclude by mentioning a few consequences and possible generalizations of the result.

2.1. Assume the two derived equivalent K3 surfaces  $S$  and  $S'$  are defined over a finite field  $\mathbb{F}_q$ . Then due to [19], using crystalline cohomology, and to [14, Proposition 16.4.6], using étale cohomology, the Zeta functions of  $S$  and  $S'$  are known to coincide,  $Z(S, t) = Z(S', t)$ . In particular,

$$D^b(S) \simeq D^b(S') \Rightarrow |S(\mathbb{F}_q)| = |S'(\mathbb{F}_q)|.$$

This result can now also be seen as an immediate consequence of Theorem 0.1.

2.2. Another approach to capture the motivic nature of a K3 surface  $S$  is to consider its class  $[S] \in K_0(\text{Var}(k))$  in the Grothendieck ring of varieties. To compare  $[S]$  with the Chow motive  $\mathfrak{h}(S) \in \text{Mot}(k)$  or its class  $[\mathfrak{h}(S)]$  in the Grothendieck ring

$$K(\text{Mot}(k)) := \mathbb{Z}[\text{Mot}(k)] / ([M \oplus N] - [M] - [N]),$$

one uses the motivic Euler characteristic with compact support  $\chi_{\text{mot}}^c$ , see [6, 7] or [3] for an easier proof in characteristic zero. This is a ring homomorphism

$$\chi_{\text{mot}}^c : K_0(\text{Var}(k)) \rightarrow K(\text{Mot}(k)),$$

satisfying  $\chi_{\text{mot}}^c([X]) = [\mathfrak{h}(X)]$  for any smooth projective variety  $X$  and  $\chi_{\text{mot}}^c(\ell) = [\mathbb{L}]$ , where  $\ell := [\mathbb{A}^1]$ .

Theorem 0.1 shows that  $\mathfrak{h}(S) \simeq \mathfrak{h}(S')$  in  $\text{Mot}(k)$  for derived equivalent K3 surfaces  $S$  and  $S'$  and, therefore,  $[\mathfrak{h}(S)] = [\mathfrak{h}(S')]$  in  $K(\text{Mot}(k))$ . This immediately yields

**Corollary 2.1** *Let  $S$  and  $S'$  be two derived equivalent K3 surfaces and let  $[S]$  and  $[S']$  be their classes in  $K_0(\text{Var}(k))$ . Then*

$$\chi_{\text{mot}}^c([S]) = \chi_{\text{mot}}^c([S']) \in K(\text{Mot}(k)).$$

□

As  $\chi_{\text{mot}}^c$  factorizes via the localization  $K_0(\text{Var}(k))[\ell^{-1}]$ , the corollary would also follow from  $[S] = [S']$  in  $K_0(\text{Var}(k))[\ell^{-1}]$ , i.e.  $([S] - [S']) \cdot \ell^n = 0$  in  $K_0(\text{Var}(k))$  for some  $n > 0$ . In fact, in the examples studied in [8, 15, 17], dealing with K3 surfaces of degree 8 and 12,  $([S] - [S']) \cdot \ell = 0$ . Note that  $[S] = [S']$  in  $K_0(\text{Var}(k))^2$  if and only if  $S \simeq S'$ , see [20]. In particular, the kernel of

$$K_0(\text{Var}(k)) \rightarrow K(\text{Mot}(k))$$

<sup>2</sup> or, weaker, in  $K_0(\text{Var}(k))/(\ell)$  which in characteristic zero is isomorphic to the Grothendieck ring of stable birational classes of smooth projective varieties  $\mathbb{Z}[\text{SB}(k)]$ , see [18].

is certainly non-trivial. However, to the best of my knowledge, nothing is known about the kernel of

$$K_0(\mathrm{Var}(k))[\ell^{-1}] \rightarrow K(\mathrm{Mot}(k)).$$

2.3. Theorem 0.1 clearly implies that for two derived equivalent K3 surfaces  $S$  and  $S'$ , one is finite-dimensional (in the sense of Kimura–O’Sullivan) if and only if the other one is. This was already observed in [4, Proposition 1.5]. In fact, in this paper Theorem 0.1 was shown assuming finite-dimensionality of  $\mathfrak{h}(S)$ , which however is only known in very few cases.

2.4. As Charles Vial explained to me, Theorem 0.1 can be generalized to arbitrary surfaces. Indeed, it can be shown that the odd degree part  $\mathfrak{h}^1 \oplus \mathfrak{h}^3$  is a derived invariant by considering Abel–Jacobi trivial cycles rather than homologically trivial cycles. For details see [9] and in particular [1], where the more complicated situation of derived equivalent threefolds is studied.

He also explained to me that Lemma 1.1 can be avoided by showing that the isomorphism of ungraded Chow motives induced by  $v(\mathcal{E})$  necessarily maps  $\mathfrak{h}_{\mathrm{tr}}^2(S)$  into  $\mathfrak{h}_{\mathrm{tr}}^2(S')$ . The inverse is then given by  $v(\mathcal{E}^*)$ .

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