Cryptography and Communications (2023) 15:131–144
https://doi.org/10.1007/s12095-022-00595-2

Reconstruction of permutations distorted by single Kendall τ-errors

Xiang Wang[1]

Received: 14 December 2020 / Accepted: 31 May 2022 / Published online: 25 June 2022

Abstract
Levenshtein first put forward the sequence reconstruction problem in 2001. This problem sets a model in which a sequence from some set is transmitted over multiple channels, and the decoder receives the different outputs. In this model, the sequence reconstruction problem is to find the minimum number of channels required to exactly recover the transmitted sequence. In the combinatorial context, the problem is equivalent to determining the maximum intersection between two balls of radius r, where the distance between their centers is at least d. The sequence reconstruction problem was studied for strings, permutations and so on. In this paper, we extend the study by Konstantinova et al. for reconstruction of permutations distorted by single Kendall τ-errors. While they solved the case where the transmitted permutation can be arbitrary and the erroneous patterns are distorted by at most two Kendall τ-errors, we study the setup where the transmitted permutation belongs to a permutation code of length n and the erroneous patterns are distorted by at most three Kendall τ-errors. In this scenario, it is shown that $n^2 - n + 1$ erroneous patterns are required in order to reconstruct an unknown permutation from some permutation code of minimum Kendall τ-distance 2 or an arbitrary unknown permutation for any $n \geq 3$.

Keywords Erroneous patterns · Kendall τ-distance · Permutation codes · Sequence reconstruction

Mathematics Subject Classification (2010) 68P30 · 94A15

1 Introduction

Levenshtein [7] first proposed the sequence reconstruction problem in 2001. In this scene, a sequence is transmitted through multiple channels and a decoder receives all the distinct outputs. Levenshtein [7, 8] determined the minimum number of transmission channels required to exactly recover the transmitted sequence. We denote by V and $\rho : V \times V \to \mathbb{N}$ a set of all sequences and a distance metric in V, respectively. In this model, Levenshtein [7]

✉ Xiang Wang
xqwang@mail.nankai.edu.cn

1 National Computer Network Emergency Response Technical Team/Coordination Center of China (CNCERT/CC), Beijing 100029, China

proved that the minimum number of transmission channels has to be greater than the largest intersection of two balls where their centers belong to V,

$$N(n, d, r) = \max_{x_1,x_2 \in V, \rho(x_1,x_2) \geq d} \{ |B_r(x_1) \cap B_r(x_2)| \}, \tag{1}$$

where $B_r(x) = \{y \in V \mid \rho(x,y) \leq r\}$ is a ball of radius r centered in x and n is the length of sequences in V. We refer to this problem of determining the value of $N(n,d,r)$ as the *sequence reconstruction problem*.

Levenshtein [7] studied the sequence reconstruction problem for several channels with some distances such as the Hamming distance, the Johnson graphs and the other metric distances. Later, this problem was discussed in the context of permutations [4–6, 12] and some other general error graphs [9, 10]. The deletion/insertion case was also explored in [11] for insertions and in [1, 2, 13] for deletions.

The sequence reconstruction problem over permutations has received a considerable attention in the literature. In particular, Konstantinova in [4, 5] solved this problem over permutations with reversal errors. The reconstruction problem for permutations with transposition errors was discussed in [6, 10]. The reconstruction problem over permutations under the Kendall τ-distance was studied for some special cases of d and r in [6, 12]. Specifically, Konstantinova et al. in [6] proved that $N(n,1,1) = 2$ and $N(n,1,2) = 2(n-1)$. Yaakobi et al. in [12] solved that $N(n, 2r, r) = \binom{2r}{r}$ for $r \leq n/4$ and presented some properties of $N(n,d,r)$.

In this paper, we discuss the reconstruction problem for permutations on n elements from their erroneous patterns which are distorted by at most three Kendall τ-errors. First, we present some upper bounds on the values of $\max_{x_1,x_2 \in V, \rho(x_1,x_2)=d} \{ |B_r(x_1) \cap B_r(x_2)| \}$ for $d = 2,3,4,5,6$ and $r = 3$. Next, we determine that $N(n,1,3) = N(n,2,3) = n^2 - n$ for any $n \geq 3$.

The rest of this paper is organized as follows. In Section 2, we formally give the definitions of the sequence reconstruction problem and permutations under the Kendall τ-metric. In Section 3, we find the exact values of $N(n,d,3)$ when $d = 1$ and 2. Section 4 concludes this paper.

2 Preliminaries

In this section, we present some definitions and notations of the sequence reconstruction problem and permutations with the Kendall τ-errors mentioned in [12] and [16].

We denote by $\mathcal{S}_n$ the set of all permutations over $[n] = \{1,2,...,n-1,n\}$. Let $\pi \in \mathcal{S}_n$ be a *permutation* and $\pi := [\pi(1),\pi(2),...,\pi(n)]$. For two permutations $\sigma, \pi \in \mathcal{S}_n$, their multiplication $\pi \circ \sigma$ is defined as the composition of σ on π, that is, $\pi \circ \sigma(i) = \sigma(\pi(i))$ for all $i \in [n]$. Thus, $\mathcal{S}_n$ under this definition of multiplication is a noncommutative *group* of size $|\mathcal{S}_n| = n!$. Assume $B \subset \mathcal{S}_n$ and $\alpha \in \mathcal{S}_n$, let $\alpha \circ B = \{\alpha \circ \beta | \beta \in B\}$ and $B \circ \alpha = \{\beta \circ \alpha | \beta \in B\}$. We denote by $\epsilon_n := [1,2,...,n]$ and π^{-1} the identity element of $\mathcal{S}_n$ and the *inverse* element of π, respectively. For an unordered pair of distinct numbers $i,j \in [n]$, this pair (i,j) is called an *inversion* in a permutation π if $i < j$ and simultaneously $\pi^{-1}(i) > \pi^{-1}(j)$. For convenience, we denote by $Iv(\pi)$ the set of all inversions in π. For example, let $\pi = [3,1,2]$, then $Iv(\pi) = \{(1,3),(2,3)\}$.

For any permutation $\pi = [\pi(1), \pi(2), ..., \pi(i), \pi(i+1), ...\pi(n)] \in \mathcal{S}_n$, an adjacent transposition is an exchange of two adjacent elements $\pi(i),\pi(i+1)$, resulting in the permutation $[\pi(1),\pi(2),...,\pi(i+1),\pi(i),...\pi(n)]$ for some $1 \leq i \leq n-1$. For any two permutations

$\sigma, \pi \in \mathcal{S}_n$, the Kendall τ-distance between two permutations π,σ, defined as $d_K(\pi,\sigma)$, is the minimum number of adjacent transpositions required to obtain the permutation σ from π. The expression for $d_K(\pi,\sigma)$ [3] is as follows:

$$d_K(\sigma, \pi) = |\{(i,j) : \sigma^{-1}(i) < \sigma^{-1}(j) \wedge \pi^{-1}(i) > \pi^{-1}(j)\}|. \quad (2)$$

The Kendall τ-weight of $\pi \in \mathcal{S}_n$, denoted by $w_K(\pi)$, is defined as the Kendall τ-distance between π and the identity permutation ϵ_n. For example, the Kendall τ-distance between ϵ_3 and $\pi = [3,1,2]$ is 2, since we can do the adjacent transpositions $[1,2,3] \rightarrow [1,3,2] \rightarrow [3,1,2]$. The Kendall τ-metric is right invariant [15], that is, for every three permutations $\alpha, \beta, \sigma \in \mathcal{S}_n$, we have

$$d_K(\alpha \circ \sigma, \beta \circ \sigma) = d_K(\alpha, \beta). \quad (3)$$

Given any permutation $\pi \in \mathcal{S}_n$, we denote by $B^n_K(\pi, r) := \{\sigma \in \mathcal{S}_n | d_K(\sigma, \pi) \le r\}$ and $S^n_K(\pi, r) := \{\sigma \in \mathcal{S}_n | d_K(\sigma, \pi) = r\}$ the Kendall τ-ball and the Kendall τ-sphere of radius r centered at π, respectively. The size of the Kendall τ-ball or the τ-sphere of radius r does not depend on the center of the ball or sphere under the Kendall τ-metric. Thus, we denote by $B^n_K(r)$ and $S^n_K(r)$ the size of $B^n_K(\pi, r)$ and $S^n_K(\pi, r)$, respectively.

For two integers d and r, let $I(n,d,r)$ be the size of the largest intersection of two balls of radius r and distance d between their centers. That is,

$$I(n, d, r) = \max_{\pi,\sigma \in \mathcal{S}_n, d_K(\pi,\sigma)=d} |B^n_K(\sigma, r) \cap B^n_K(\pi, r)|. \quad (4)$$

Similarly, let $N(n,d,r)$ be the size of the maximum intersection two balls of radius r and distance at least d between their centers. That is,

$$N(n, d, r) = \max_{l \ge d} I(n, l, r). \quad (5)$$

Assume that a permutation $\pi \in C$ is transmitted over N channels, where $C \subset \mathcal{S}_n$ and $d_K(\pi,\beta) \ge d$ for any two distinct π,$\beta \in C$, there are at most r errors on each channel, and all the channel outputs are different from each other. Then, Levenshtein [7] proved that the minimum number of channels that guarantees the existence of a decoder that will successfully decode any transmitted codeword is given by $N(n,d,r) + 1$, where the distance between any two distinct codewords is at least d. It was shown in [6] that $N(n,1,1) = 2$ and $N(n,1,2) = 2(n - 1)$.

Based on the above definitions and notations, we will determine the exact values of $N(n,d,3)$ for $d = 1,2$ in the following section.

3 The exact values of *N*(*n*,*d*,3) for *d* = 1,2

In this section, we will give some upper bounds on the values of $I(n,d,3)$ for $d \in [6]$ and determine the exact values of $N(n,d,3)$ for $d = 1,2$. In order to obtain these results, we need some lemmas as follows. The values of $I(n,1,r)$ have been determined by Yaakobi et al. in [12] in the following lemma.

Lemma 1 [12, Theorem 5] For $r \ge 2$, the values of $I(n,1,r)$ satisfy the following recursive formula

$$I(n,1,r) = 2B_K^n(r-1) - I(n,1,r-1), \quad (6)$$

where $I(n,1,1) = 2$.

Yaakobi et al. in [12] also proposed some properties of $I(n,2,r)$. Given two permutations $\alpha, \beta \in \mathcal{S}_n$ such that $d_K(\alpha,\beta) = 2$, there are two options:

1) There is only a single permutation σ such that $d_K(\alpha,\sigma) = d_K(\beta,\sigma) = 1$.
2) There are two distinct permutations σ,π such that $d_K(\alpha,\sigma) = d_K(\beta,\sigma) = d_K(\alpha,\pi) = d_K(\beta,\pi) = 1$.

If the first option holds then we say that α and β are of type I, and otherwise, we say they are of type II. When α and β are of type II, Yaakobi et al. in [12] gave the following result.

Lemma 2 [12, Lemma 8] Assume that α,β satisfy $d_K(\alpha,\beta) = 2$ and they are of type II. Then, $|B_K^n(\alpha,r) \cap B_K^n(\beta,r)| = I(n,1,r)$.

By Lemma 2 and (5), we clearly obtain the following lemma.

Lemma 3 For any $r \geq 1$, we have $N(n,1,r) = N(n,2,r)$.

Proof By Lemma 2, we have $N(n,2,r) \geq I(n,1,r)$ for any $r \geq 1$. Hence, we can obtain that $N(n,1,r) = N(n,2,r)$ for any $r \geq 1$. □

Since $N(n,1,1) = 2$ and $N(n,1,2) = 2(n-1)$ in [6], then we have that $N(n,2,1) = 2$ and $N(n,2,2) = 2(n-1)$. In the following, we will determine the exact values of $N(n,d,3)$ for $d = 1,2$.

Wang et al. in [14] gave the recursive formula of $B_K^n(r)$ and the exact values of $B_K^n(r)$ for $r = 2,3$ as follows. Here, we have $B_K^n(0) = 1$ and $B_K^n(1) = n$.

Lemma 4 [14, Theorem 1] For all $n \geq 3$, we have $B_K^n(2) = \frac{n^2+n-2}{2}$ and $B_K^n(3) = \frac{(n+1)(n^2+2n-6)}{6}$.

By Lemmas 1 and 4, we obtain

$$\begin{aligned} I(n,1,1) &= 2, \\ I(n,1,2) &= 2B_K^n(1) - I(n,1,1) = 2n-2, \\ I(n,1,3) &= 2B_K^n(2) - I(n,1,2) = n^2-n. \end{aligned} \quad (7)$$

When α,β satisfy $d_K(\alpha,\beta) = 2$ and they are of type II, we have

$$|B_K^n(\alpha,3) \cap B_K^n(\beta,3)| = I(n,1,3) = n^2-n. \quad (8)$$

To obtain the values of $I(n,2,3)$, we now compute the exact value of $|B_K^n(\alpha,3) \cap B_K^n(\beta,3)|$ when α,β satisfy $d_K(\alpha,\beta) = 2$ and they are of type I. We start in the next lemma.

Lemma 5 Let n,r,d be integers and $\alpha,\beta \in \mathcal{S}_n$ such that $d_K(\alpha,\beta) = d$. Then there exists some permutation $\gamma \in \mathcal{S}_n$ of weight $w_K(\gamma) = d$ such that $|B_K^n(\alpha,r) \cap B_K^n(\beta,r)| = |B_K^n(\epsilon_n,r) \cap B_K^n(\gamma,r)|$.

Proof Let $\gamma = \beta \circ \alpha^{-1}$. If $\sigma \in B_K^n(\alpha, r) \cap B_K^n(\beta, r)$, then we have $d_K(\alpha,\sigma) \le r$ and $d_K(\beta,\sigma) \le r$. Since the Kendall τ-metric is right invariant, then $d_K(\epsilon_n,\sigma \circ \alpha^{-1}) = d_K(\alpha,\sigma) \le r$ and $d_K(\beta\circ\alpha^{-1},\sigma \circ \alpha^{-1}) = d_K(\beta,\sigma) \le r$. Hence, $\sigma\circ\alpha^{-1} \in B_K^n(\epsilon_n, r) \cap B_K^n(\gamma, r)$ and $|B_K^n(\alpha, r) \cap B_K^n(\beta, r)| \le |B_K^n(\epsilon_n, r) \cap B_K^n(\gamma, r)|$. Similarly, we also have $|B_K^n(\alpha, r) \cap B_K^n(\beta, r)| \ge |B_K^n(\epsilon_n, r) \cap B_K^n(\gamma, r)|$. So, we clearly obtain $|B_K^n(\alpha, r) \cap B_K^n(\beta, r)| = |B_K^n(\epsilon_n, r) \cap B_K^n(\gamma, r)|$. □

To compute the value of $|B_K^n(\alpha, 3) \cap B_K^n(\beta, 3)|$, by Lemma 5, we only consider $\alpha = \epsilon_n$ and $d_K(\epsilon_n,\beta) = 2$ such that ϵ_n,β are of type I. For convenience, the i-th adjacent transposition, denoted by e_i, exchanges the elements in positions i and $i + 1$ and keeps all other elements fixed for any $1 \le i \le n - 1$.

Lemma 6 Let $\beta \in \mathcal{S}_n$. Then ϵ_n,β are of type II if and only if $\beta = e_i \circ e_j$ for some $i,j \in [n - 1]$ and $|i - j| \ge 2$. Moreover, ϵ_n,β are of type I if and only if $\beta = e_i \circ e_{i+1}$ or $e_{i+1} \circ e_i$ for some $i \in [n - 2]$.

Proof When ϵ_n,β are of type II, we have $d_K(\epsilon_n,\beta) = 2$. Thus, $\beta = e_i \circ e_j$ for $i \ne j$ and $i,j \in [n - 1]$. If $|i - j| \ge 2$, it follows that ϵ_n,β are of type II. When $|i - j| = 1$, without loss of generality, we let $j = i + 1$. Then, we only have a permutation e_i such that $d_K(\epsilon_n,e_i) = d_K(e_i \circ e_{i+1},e_{i+1}) = 1$. Hence, ϵ_n and β are not of type II. So, ϵ_n,β are of type II if and only if $\beta = e_i \circ e_j$ for some $i,j \in [n - 1]$ and $|i - j| \ge 2$. Similarly, ϵ_n,β are of type I if and only if $\beta = e_i \circ e_{i+1}$ or $e_{i+1} \circ e_i$ for some $i \in [n - 2]$. □

When $d_K(\epsilon_n,\beta) = 2$ and they are of type I, by Lemma 6, then $\beta = e_i \circ e_{i+1}$ or $e_{i+1} \circ e_i$ for some $i \in [n - 2]$. For convenience, we call this kind of permutation the type I for $i \in [n - 2]$. Now, we give some lemmas in the following.

Lemma 7 Let $\alpha \in \mathcal{S}_n$. Then, it follows that

$$w_K(\alpha) = |Iv(\alpha)|. \tag{9}$$

Proof By the definition of $w_K(\alpha)$, we have that $w_K(\alpha) = d_K(\epsilon_n,\alpha) = |\{(i,j) : i < j \wedge \alpha^{-1}(i) > \alpha^{-1}(j)\}|$. Further, due to the definition of $Iv(\alpha)$, it follows that $Iv(\alpha) = \{(i,j) : i < j \wedge \alpha^{-1}(i) > \alpha^{-1}(j)\}$. Hence, we have $w_K(\alpha) = |Iv(\alpha)|$. □

For example, let $\pi = [3,1,2]$. Then we have that $Iv(\pi) = \{(1,3),(2,3)\}$. Hence, $w_K(\pi) = |Iv(\pi)| = 2$.

Lemma 8 Let $\alpha, \beta \in \mathcal{S}_n$. Then, we have that

$$\begin{aligned} d_K(\alpha, \beta) &= |(Iv(\alpha) \cup Iv(\beta))\setminus(Iv(\alpha) \cap Iv(\beta))| \\ &= |Iv(\alpha) \cup Iv(\beta)| - |Iv(\alpha) \cap Iv(\beta)| \\ &= |Iv(\alpha)| + |Iv(\beta)| - 2|Iv(\alpha) \cap Iv(\beta)|. \end{aligned} \tag{10}$$

Proof By the definition of $d_K(\alpha,\beta)$, it follows that $d_K(\alpha, \beta) = |\{(i,j) : \alpha^{-1}(i) < \alpha^{-1}(j) \wedge \beta^{-1}(i) > \beta^{-1}(j)\}|$. Let $(i_1,j_1) \in \{(i,j) : \alpha^{-1}(i) < \alpha^{-1}(j) \wedge \beta^{-1}(i) > \beta^{-1}(j)\}$. Then, $\alpha^{-1}(i_1) < \alpha^{-1}(j_1)$ and $\beta^{-1}(i_1) > \beta^{-1}(j_1)$. If $i_1 < j_1$, then it follows that $(i_1,j_1) \in Iv(\beta)$ and $(i_1,j_1) \notin Iv(\alpha)$. If $i_1 > j_1$, then we have that $(i_1,j_1) \in Iv(\alpha)$ and $(i_1,j_1) \notin Iv(\beta)$. Thus, $(i_1,j_1) \in (Iv(\alpha) \cup Iv(\beta))\setminus(Iv(\alpha) \cap Iv(\beta))$. Similarly, let $(i_1,j_1) \in (Iv(\alpha) \cup Iv(\beta))\setminus(Iv(\alpha) \cap Iv(\beta))$, we also have that $(i_1,j_1) \in \{(i,j) : \alpha^{-1}(i) < \alpha^{-1}(j) \wedge \beta^{-1}(i) > \beta^{-1}(j)\}$. Hence, it follows that

$$\{(i,j) : \alpha^{-1}(i) < \alpha^{-1}(j) \wedge \beta^{-1}(i) > \beta^{-1}(j)\} = (Iv(\alpha) \cup Iv(\beta)) \setminus (Iv(\alpha) \cap Iv(\beta)).$$

Further, we obtain that $d_K(\alpha,\beta) = |(Iv(\alpha)\cup Iv(\beta))\setminus(Iv(\alpha)\cap Iv(\beta))| = |Iv(\alpha)\cup Iv(\beta)|-|Iv(\alpha)\cap Iv(\beta)| = |Iv(\alpha)|+|Iv(\beta)|-2|Iv(\alpha)\cap Iv(\beta)|$. □

For example, let $\alpha = [3,1,2]$ and $\beta = [2,1,3]$. Then it follows that $Iv(\alpha) = \{(1,3),(2,3)\}$ and $Iv(\beta) = \{(1,2)\}$. It is easily obtained that $Iv(\alpha) \cap Iv(\beta) = \varnothing$. Hence, $d_K(\alpha,\beta) = |Iv(\alpha)| + |Iv(\beta)| - 2|Iv(\alpha) \cap Iv(\beta)| = 2 + 1 = 3$.

Lemma 9 For $n \geq r \geq 3$, we have $|B_K^n(\epsilon_n, r) \cap B_K^n(e_i \circ e_{i+1}, r)| = |B_K^n(\epsilon_n, r) \cap B_K^n(e_{i+1} \circ e_i, r)| = |B_K^n(e_i, r) \cap B_K^n(e_{i+1}, r)| = |B_K^n(\epsilon_n, r-1) \cup B_K^n(e_i \circ e_{i+1} \circ e_i, r-2)|$.

Proof Since the Kendall τ-metric is right invariant, we have $|B_K^n(\epsilon_n, r) \cap B_K^n(e_i \circ e_{i+1}, r)| = |B_K^n(\epsilon_n, r) \cap B_K^n(e_{i+1} \circ e_i, r)| = |B_K^n(e_i, r) \cap B_K^n(e_{i+1}, r)|$. Next, we prove that $|B_K^n(e_i, r) \cap B_K^n(e_{i+1}, r)| = |B_K^n(\epsilon_n, r-1) \cup B_K^n(e_i \circ e_{i+1} \circ e_i, r-2)|$. First, we discuss $(B_K^n(e_i, r) \cap B_K^n(e_{i+1}, r)) \subset (B_K^n(\epsilon, r-1) \cup B_K^n(e_i \circ e_{i+1} \circ e_i, r-2))$. Let $\gamma \in B_K^n(e_i, r) \cap B_K^n(e_{i+1}, r)$, and assume that $\gamma \notin B_K^n(\epsilon_n, r-1)$. By (10), we have

$$d_K(\gamma, e_i) = \begin{cases} |Iv(\gamma)| - 1, & \text{if } (i, i+1) \in Iv(\gamma), \\ |Iv(\gamma)| + 1, & \text{if } (i, i+1) \notin Iv(\gamma). \end{cases} \tag{11}$$

Since $d_K(\gamma,e_i) \leq r$ and $|Iv(\gamma)| \geq r$ by assumption, it follows that $(i,i+1) \in Iv(\gamma)$. Similarly, we also have that $(i+1,i+2) \in Iv(\gamma)$. By the definition of $Iv(\gamma)$ and $(i,i+1),(i+1,i+2) \in Iv(\gamma)$, we have that $\gamma^{-1}(i) > \gamma^{-1}(i+1)$ and $\gamma^{-1}(i+1) > \gamma^{-1}(i+2)$. Hence, $\gamma^{-1}(i) > \gamma^{-1}(i+2)$. So, $(i,i+2) \in Iv(\gamma)$. Therefore, $(i,i+1),(i+1,i+2),(i,i+2) \in Iv(\gamma)$. It is easily verified that $Iv(e_i \circ e_{i+1} \circ e_i) = \{(i,i+1),(i+1,i+2),(i,i+2)\}$. By (10), it follows that

$$d_K(\gamma, e_i \circ e_{i+1} \circ e_i) = |Iv(\gamma)| + 3 - 6 = |Iv(\gamma)| - 3. \tag{12}$$

By $(i,i+1) \in Iv(\gamma)$ and $\gamma \in B_K^n(e_i, r)$, we have $d_K(\gamma,e_i) = |Iv(\gamma)|-1 \leq r$. Hence, $|Iv(\gamma)| \leq r+1$. By (12), it follows that $d_K(\gamma,e_i \circ e_{i+1} \circ e_i) \leq r-2$. So, we have $\gamma \in B_K^n(\epsilon_n, r-1) \cup B_K^n(e_i \circ e_{i+1} \circ e_i, r-2)$. Similarly, when $\gamma \in B_K^n(\epsilon_n, r-1) \cup B_K^n(e_i \circ e_{i+1} \circ e_i, r-2)$, we also obtain that $\gamma \in B_K^n(e_i, r) \cap B_K^n(e_{i+1}, r)$. Therefore, we prove that $|B_K^n(e_i, r) \cap B_K^n(e_{i+1}, r)| = |B_K^n(\epsilon_n, r-1) \cup B_K^n(e_i \circ e_{i+1} \circ e_i, r-2)|$. □

When β is the type I, by Lemma 9, we can find the value of $|B_K^n(\epsilon_n, 3) \cap B_K^n(\beta, 3)|$.

Lemma 10 When β is the type I, for $n \geq r \geq 3$, we have

$$|B_K^n(\epsilon_n, 3) \cap B_K^n(\beta, 3)| = B_K^n(2) + B_K^n(1) - 2 = \frac{n^2 + 3n - 6}{2}. \tag{13}$$

Proof By Lemma 9, we have $|B_K^n(\epsilon_n, 3) \cap B_K^n(\beta, 3)| = |B_K^n(\epsilon_n, 2)| + |B_K^n(e_i \circ e_{i+1} \circ e_i, 1)| - |B_K^n(\epsilon_n, 2) \cap B_K^n(e_i \circ e_{i+1} \circ e_i, 1)|$ for some $i \in [n-2]$. For all $i \in [n-2]$, it is easily verified that $B_K^n(\epsilon_n, 2) \cap B_K^n(e_i \circ e_{i+1} \circ e_i, 1) = \{e_i \circ e_{i+1}, e_{i+1} \circ e_i\}$. Then, we have $|B_K^n(\epsilon_n, 2) \cap B_K^n(e_i \circ e_{i+1} \circ e_i, 1)| = 2$. Thus, we can obtain that $|B_K^n(\epsilon_n, 3) \cap B_K^n(\beta, 3)| = B_K^n(2) + B_K^n(1) - 2$. By Lemma 4, we have $B_K^n(2) = \frac{n^2+n-2}{2}$ and $B_K^n(1) = n$. So, we have $|B_K^n(\epsilon_n, 3) \cap B_K^n(\beta, 3)| = B_K^n(2) + B_K^n(1) - 2 = \frac{n^2+3n-6}{2}$. □

By (8) and (13), we can obtain the following theorem.

Theorem 1 For $n \geq 3$, we have

$$I(n, 2, 3) = n^2 - n. \tag{14}$$

Proof When α,β satisfy $d_K(\alpha,\beta) = 2$ and they are of type II, by (8), we have $|B_K^n(\alpha, 3) \cap B_K^n(\beta, 3)| = n^2 - n$. When α,β satisfy $d_K(\alpha,\beta) = 2$ and they are of type I, by Lemma 5 and (13), we can obtain that $|B_K^n(\alpha, 3) \cap B_K^n(\beta, 3)| = \frac{n^2+3n-6}{2}$. Hence, we have $I(n, 2, 3) = \max\{\frac{n^2+3n-6}{2}, n^2 - n\} = n^2 - n$ for all $n \geq 3$. □

We will present the upper bounds of $I(n,d,3)$ for $d = 3,4,5,6$ as follows. To get these properties, we need some lemmas.

When $w_K(\beta) = 2$ or 3, the forms of β and the inversions of β are given in the following lemma.

Lemma 11 Let $n \geq 3$ and $\beta \in \mathcal{S}_n$. When $w_K(\beta) = 2$, then $\beta = e_i \circ e_{i+1}$ or $e_{i+1} \circ e_i$ for some $i \in [n-2]$, or $e_j \circ e_i$ for some $i,j \in [n-1]$ and $|i - j| \geq 2$. When $w_K(\beta) = 3$, then $\beta = e_k \circ e_i \circ e_{i+1}$ for some $k \in [n-1]$, $i \in [n-2]$, and $k \neq i$, or $e_k \circ e_{i+1} \circ e_i$ for some $k \in [n-1]$, $i \in [n-2]$, and $k \neq i+1$, or $e_k \circ e_j \circ e_i$ for some $i,j,k \in [n-1]$, $|i - j| \geq 2$, and $k \neq i$ or j. Moveover, when $w_K(\beta) = 2$, the inversions of β may be $(i,i+1)$ or $(i,i+2)$ for some $i \in [n-1]$. When $w_K(\beta) = 3$, the inversions of β may be $(i,i+1)$, $(i,i+2)$, or $(i,i+3)$ for some $i \in [n-1]$.

Proof When $w_K(\beta) = 2$, if β is of type I, then $\beta = e_i \circ e_{i+1}$ or $e_{i+1} \circ e_i$ for some $i \in [n-2]$. Hence, the inversions of β are $(i,i+1)$ and $(i,i+2)$, or $(i,i+2)$ and $(i+1,i+2)$. If β is of type II, then $\beta = e_j \circ e_i$ for some $|i - j| \geq 2$. Thus, the inversions of β are $(i,i+1)$ and $(j,j+1)$. So, when $w_K(\beta) = 2$, then $\beta = e_i \circ e_{i+1}$ or $e_{i+1} \circ e_i$ for some $i \in [n-2]$, or $e_j \circ e_i$ for some $i,j \in [n-1]$ and $|i - j| \geq 2$. Moreover, the inversions of β may be $(i,i+1)$ or $(i,i+2)$ for some $i \in [n-1]$.

Similarly, when $w_K(\beta) = 3$, by using the forms of the permutation of weight 2, then $\beta = e_k \circ e_i \circ e_{i+1}$ for some $k \in [n-1]$, $i \in [n-2]$, and $k \neq i$, or $e_k \circ e_{i+1} \circ e_i$ for some $k \in [n-1]$, $i \in [n-2]$, and $k \neq i+1$, or $e_k \circ e_j \circ e_i$ for some $i,j,k \in [n-1]$, $|i - j| \geq 2$, and $k \neq i$ or j. When $\beta = e_k \circ e_i \circ e_{i+1}$ for some $k \in [n-1]$, $i \in [n-2]$, and $k \neq i$, then $(i,i+2),(i+1,i+2) \in Iv(\beta)$. If $k = i+2$, then $Iv(\beta) = \{(i,i+2),(i+1,i+2),(i+1,i+3)\}$. If $k = i+1$, then $Iv(\beta) = \{(i,i+1),(i,i+2),(i+1,i+2)\}$. If $k = i-1$, then $Iv(\beta) = \{(i,i+2),(i+1,i+2),(i-1,i+2)\}$. If $k \neq i-1,i,$or $i+2$, then $Iv(\beta) = \{(i,i+2),(i+1,i+2),(k,k+1)\}$.

When $\beta = e_k \circ e_{i+1} \circ e_i$ for some $k \in [n-1]$, $i \in [n-2]$, and $k \neq i+1$, then $(i,i+1),(i,i+2) \in Iv(\beta)$. If $k = i+2$, then $Iv(\beta) = \{(i,i+1),(i,i+2),(i,i+3)\}$. If $k = i$, then $Iv(\beta) = \{(i,i+2),(i+1,i+2),(i,i+1)\}$. If $k = i-1$, then $Iv(\beta) = \{(i,i+1),(i,i+2),(i-1,i+1)\}$. If $k \neq i-1,i+1,$or $i+2$, then $Iv(\beta) = \{(i,i+1),(i,i+2),(k,k+1)\}$.

When $\beta = e_k \circ e_j \circ e_i$ for some $i,j,k \in [n-1]$, $|i - j| \geq 2$, and $k \neq i$ or j, then $(i,i+1),(j,j+1) \in Iv(\beta)$. It is easily verified that $\beta = e_k \circ e_j \circ e_i = e_k \circ e_i \circ e_j$. For convenience, let $i < j$. If $j - i = 2$ and $k = (i+j)/2$, then $Iv(\beta) = \{(i,i+1),(j,j+1),(k-1,k+2)\}$. If $|k - i| \geq 2$ and $|k - j| \geq 2$, then $Iv(\beta) = \{(i,i+1),(j,j+1),(k,k+1)\}$. If $k = i-1$, then $\beta = e_k \circ e_j \circ e_i = e_{i-1} \circ e_i \circ e_j$ and $Iv(\beta) = \{(i-1,i+1),(i,i+1),(j,j+1)\}$. If $k = j+1$, then $\beta = e_k \circ e_j \circ e_i = e_{j+1} \circ e_j \circ e_i$ and $Iv(\beta) = \{(i,i+1),(j,j+1),(j,j+2)\}$. If $k = i+1$ and $j - i > 2$, then $\beta = e_k \circ e_j \circ e_i = e_{i+1} \circ e_i \circ e_j$ and $Iv(\beta) = \{(i,i+1),(j,j+1),(i,i+2)\}$. If $k = j-1$ and $j - i > 2$, then $\beta = e_k \circ e_j \circ e_i = e_{j-1} \circ e_j \circ e_i$ and $Iv(\beta) = \{(i,i+1),(j,j+1),(j-1,j+1)\}$.

Therefore, when $w_K(\beta) = 3$, the inversions of β may be $(i,i+1)$, $(i,i+2)$, or $(i,i+3)$ for some $i \in [n-1]$. □

Now, given two distinct inversions I_1,I_2, we estimate the size of $S^n_{(3,I_1,I_2)} := \{\beta \in \mathcal{S}_n | w_K(\beta) = 3, I_i \in Iv(\beta) \text{ for all } i \in [2]\}$ and the size of $S^n_{(2,I_1)} := \{\beta \in \mathcal{S}_n | w_K(\beta) = 2, I_1 \in Iv(\beta)\}$, respectively.

Lemma 12 Assume that $S^n_{(3,I_1,I_2)}$ and $S^n_{(2,I_1)}$ are defined as above. Then, we have

$$|S^n_{(3,I_1,I_2)}| \le (n-2) \text{ for all } n \ge 4, \tag{15}$$

$$|S^n_{(2,I_1)}| \le (n-2) \text{ for all } n \ge 4. \tag{16}$$

Proof By Lemma 11, we will discuss the value of $|S^n_{(3,I_1,I_2)}|$ according to the forms of I_1,I_2. Let $\beta \in S^n_{(3,I_1,I_2)}$. Then $I_1,I_2 \in Iv(\beta)$ and $|Iv(\beta)| = 3$. If I_1,I_2 has one form of $(i,i+3)$, by Lemma 11, then β must be $e_i \circ e_{i+1} \circ e_{i+2}$, $e_{i+2} \circ e_{i+1} \circ e_i$, or $e_{i+1} \circ e_i \circ e_{i+2}$. Hence, when I_1,I_2 has one form of $(i,i+3)$, we have that

$$|S^n_{(3,I_1,I_2)}| \le 3. \tag{17}$$

If I_1,I_2 has one form of $(i,i+2)$, then $(i,i+2) \in Iv(\beta)$. For convenience, let $I_1 = (i,i+2)$. By Lemma 11, $Iv(\beta)$ must contain two inversions $(i,i+1),(i,i+2)$ or $(i,i+2),(i+1,i+2)$. When $\{I_1,I_2\} = \{(i,i+1),(i,i+2)\}$, by the proof of Lemma 11, it follows that $\beta = e_k \circ e_{i+1} \circ e_i$ for some $k \in [n-1]$, $i \in [n-2]$, and $k \ne i+1$. Since $k \in [n-1]$ and $k \ne i+1$, the number of this kind of β is $n-2$. Hence, $|S^n_{(3,I_1,I_2)}| = n-2$. Similarly, when $\{I_1,I_2\} = \{(i,i+2),(i+1,i+2)\}$, then $\beta = e_k \circ e_i \circ e_{i+1}$ for some $k \in [n-1]$, $i \in [n-2]$, and $k \ne i$. Thus, we also have that $|S^n_{(3,I_1,I_2)}| = n-2$. Then, when $\{I_1,I_2\} = \{(i,i+1),(i,i+2)\}$ or $\{(i,i+2),(i+1,i+2)\}$, it follows that

$$|S^n_{(3,I_1,I_2)}| = n-2. \tag{18}$$

When $\{I_1,I_2\} \ne \{(i,i+1),(i,i+2)\}$ or $\{(i,i+2),(i+1,i+2)\}$, then $I_2 \ne (i,i+1)$ or $(i+1,i+2)$. Since $I_1,I_2 \in Iv(\beta)$ and $Iv(\beta)$ must contain two inversions $(i,i+1),(i,i+2)$ or $(i,i+2),(i+1,i+2)$, it follows that $Iv(\beta)$ must be $\{(i,i+1),(i,i+2),I_2\}$ or $\{(i+1,i+2),(i,i+2),I_2\}$. Hence, we have that

$$|S^n_{(3,I_1,I_2)}| \le 2. \tag{19}$$

If $I_1 = (i,i+1)$ and $I_2 = (j,j+1)$, then $(i,i+1),(j,j+1) \in Iv(\beta)$. For convenience, let $j > i$. When $j - i = 1$, we have that $(i,i+1),(i+1,i+2) \in Iv(\beta)$. By Lemma 11, then $(i,i+2) \in Iv(\beta)$. Hence, we have that

$$|S^n_{(3,I_1,I_2)}| = 1. \tag{20}$$

When $j - i \ge 2$, by the proof of Lemma 11, it follows that $\beta = e_k \circ e_j \circ e_i$ for some $k \in [n-1]$ and $k \ne i$ or j. Thus, it follows that

$$|S^n_{(3,I_1,I_2)}| = n-3. \tag{21}$$

By the above discussion, we obtain that

$$|S^n_{(3,I_1,I_2)}| \leq (n-2) \ \text{ for all } n \geq 4.$$

Next, we will discuss the value of discuss the value of $|S^n_{(2,I_1)}|$ according to the forms of I_1. Let $\beta \in S^n_{(2,I_1)}$. Then $I_1 \in Iv(\beta)$ and $|Iv(\beta)| = 2$. If $I_1 = (i,i+2)$, by Lemma 11, then $\beta = e_i \circ e_{i+1}$ or $\circ\ e_{i+1} \circ e_i$. Hence, when $I_1 = (i,i+2)$, we have that $|S^n_{(2,I_1)}| = 2$. If $I_1 = (i,i+1)$, by Lemma 11, then $\beta = e_i \circ e_j$ for some $j \in [n-1]$ and $j \neq i$. Hence, $|S^n_{(2,I_1)}| = n-2$. So, we also have that

$$|S^n_{(2,I_1)}| \leq (n-2) \ \text{ for all } n \geq 4.$$

□

By Lemma 12, we can obtain the following lemma.

Lemma 13 For $n \geq 6$, we have

$$I(n,3,3) \leq 6n-7. \tag{22}$$

Proof Let $\beta \in \mathcal{S}_n$ be a permutation such that $w_K(\beta) = 3$ and $I(n,3,3) = |B^n_K(\epsilon_n,3) \cap B^n_K(\beta,3)|$. For convenience, let $Iv(\beta) = \{I_1,I_2,I_3\}$. Obviously, we have $B^n_K(\epsilon_n,3) \cap B^n_K(\beta,3) = \cup_{i=0}^{3}\{\gamma | w_K(\gamma) = i, d_K(\gamma,\beta) \leq 3\}$. We estimate the value of $|\{\gamma | w_K(\gamma) = i, d_K(\gamma,\beta) \leq 3\}|$ for each $0 \leq i \leq 3$ as follows.

First, we consider the value of $|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}|$. Suppose $Iv(\gamma) = \{a,b,c\}$. By Lemma 8, then we have $d_K(\gamma,\beta) = |Iv(\gamma)| + |Iv(\beta)| - 2|(Iv(\beta) \cap Iv(\gamma))|$. Hence, we get $d_K(\gamma,\beta) = 6 - 2|(Iv(\beta) \cap Iv(\gamma))|$. When $d_K(\gamma,\beta) \leq 3$, then we have $|(Iv(\beta) \cap Iv(\gamma))| = 2$ or 3. If $|(Iv(\beta) \cap Iv(\gamma))| = 3$, the number of γ is 1. If $|(Iv(\beta) \cap Iv(\gamma))| = 2$, then $Iv(\gamma)$ contains two elements of $\{I_1,I_2,I_3\}$. By Lemma 12, the number of this kind of γ is at most $\binom{3}{2}(n-2)$. Since $\beta \in \{\gamma | w_K(\gamma) = 3, Iv(\gamma)$ contains two elements of $\{I_1,I_2,I_3\}\}$, then we have

$$|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}| \leq 3(n-2) - 2 = 3n-8. \tag{23}$$

Next, we compute $|\{\gamma | w_K(\gamma) = 2, d_K(\gamma,\beta) \leq 3\}|$. Suppose $Iv(\gamma) = \{a,b\}$. When $d_K(\gamma,\beta) \leq 3$, then we get $|(Iv(\beta) \cap Iv(\gamma))| = 1$ or 2. If $|(Iv(\beta) \cap Iv(\gamma))| = 2$, the number of γ is at most $\binom{3}{2} = 3$. If $|(Iv(\beta) \cap Iv(\gamma))| = 1$, then $Iv(\gamma)$ contains one element of $\{I_1,I_2,I_3\}$. By Lemma 12, the number of this kind of γ is at most $\binom{3}{1}(n-2)$. Hence, we have

$$|\{\gamma | w_K(\gamma) = 2, d_K(\gamma,\beta) \leq 3\}| \leq 3(n-2) + 3 = 3n-3. \tag{24}$$

Finally, we clearly obtain that $|\{\gamma | w_K(\gamma) = 1, d_K(\gamma,\beta) \leq 3\}| \leq 3$ and $|\{\gamma | w_K(\gamma) = 0, d_K(\gamma,\beta) \leq 3\}| \leq 1$. Therefore, by (23)-(24), we have

$$I(n,3,3) = \sum_{i=0}^{3} |\{\gamma | w_K(\gamma) = i, d_K(\gamma,\beta) \leq 3\}| \leq 6n-7.$$

□

Lemma 14 For $n \geq 6$, we have

$$I(n,4,3) \leq 6n-8. \tag{25}$$

Proof Let $\beta \in \mathcal{S}_n$ be a permutation such that $w_K(\beta) = 4$ and $I(n,4,3) = |B_K^n(\epsilon_n,3) \cap B_K^n(\beta,3)|$. For convenience, let $Iv(\beta) = \{I_1, I_2, I_3, I_4\}$. Clearly, we have $B_K^n(\epsilon_n,3) \cap B_K^n(\beta,3) = \cup_{i=0}^{3}\{\gamma | w_K(\gamma) = i, d_K(\gamma,\beta) \leq 3\}$. Next, we estimate the size of $|\{\gamma | w_K(\gamma) = i, d_K(\gamma,\beta) \leq 3\}|$ for each $0 \leq i \leq 3$ by the different kind of $Iv(\beta)$. Suppose $I_1 = (i_1, j_1) \in Iv(\beta)$ is an inversion with the maximum value of $|i_1 - j_1|$.

Now, we consider the value of $|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}|$. For convenience, $Iv(\gamma) = \{a, b, c\}$. By Lemma 8, then we have $d_K(\gamma,\beta) = |Iv(\gamma)| + |Iv(\beta)| - 2|(Iv(\beta) \cap Iv(\gamma))|$. Hence, we get $d_K(\gamma,\beta) = 7 - 2|(Iv(\beta) \cap Iv(\gamma))|$. When $d_K(\gamma,\beta) \leq 3$, we have that $|(Iv(\beta) \cap Iv(\gamma))| = 2$ or 3. Thus, $Iv(\gamma)$ contains at least two elements of $\{I_1, I_2, I_3, I_4\}$. If $|j_1 - i_1| \geq 4$, then $I_1 \notin Iv(\gamma)$. By Lemma 12, the number of this kind of γ is at most $\binom{3}{2}(n-2)$. Hence, we have that

$$|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}| \leq \binom{3}{2}(n-2) = 3n - 6. \tag{26}$$

If $|j_1 - i_1| = 3$, then I_1 may be an element of $Iv(\gamma)$. When $I_1 \in Iv(\gamma)$, by (17), it follows that the number of this kind of γ is at most 3. When $I_1 \notin Iv(\gamma)$, by (26), we have that the number of this kind of γ is at most $3n - 6$. Hence, we have that

$$|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}| \leq \binom{3}{2}(n-2) + 3 = 3n - 3. \tag{27}$$

If $|j_1 - i_1| = 2$, then I_1 may be an element of $Iv(\gamma)$. For convenience, let $I_1 = (i_1, j_1) = (i, i+2)$. Consider $\{(i, i+1), (i+1, i+2)\} \subset Iv(\beta)$. When $I_1 \in Iv(\gamma)$, by (18) and (19), the number of this kind of γ is at most $2(n-2) + 2$. Moreover, if $(i, i+1), (i+1, i+2) \in Iv(\gamma)$, then $I_1 = (i, i+2) \in Iv(\gamma)$. Thus, when $I_1 \notin Iv(\gamma)$, by Lemma 12, the number of this kind of γ is at most $2(n-2)$. Hence, when $|j_1 - i_1| = 2$ and $\{(i, i+1), (i+1, i+2)\} \subset Iv(\beta)$, we have that

$$|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}| \leq 4(n-2) + 2 = 4n - 6. \tag{28}$$

Consider $\{(i, i+1), (i+1, i+2)\} \not\subset Iv(\beta)$. When $I_1 \in Iv(\gamma)$, by (18) and (19), the number of this kind of γ is at most $(n-2) + 4$. When $I_1 \notin Iv(\gamma)$, by Lemma 12, the number of this kind of γ is at most $3(n-2)$. Hence, when $|j_1 - i_1| = 2$ and $\{(i, i+1), (i+1, i+2)\} \not\subset Iv(\beta)$, we have that

$$|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}| \leq 4(n-2) + 4 = 4n - 4. \tag{29}$$

If $|j_1 - i_1| = 1$, then I_1, I_2, I_3, I_4 may be an element of $Iv(\gamma)$. By (20) and (21), the number of this kind of γ is at most $\binom{4}{2}(n-3)$. Hence, when $|j_1 - i_1| = 1$, we have that

$$|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}| \leq \binom{4}{2}(n-3) = 6n - 18. \tag{30}$$

Similarly, when $|i_1 - j_1| \geq 2$, we have that $|\{\gamma | w_K(\gamma) = 2, d_K(\gamma,\beta) \leq 3\}| \leq 5$, $|\{\gamma | w_K(\gamma) = 1, d_K(\gamma,\beta) \leq 3\}| \leq 3$ and $|\{\gamma | w_K(\gamma) = 0, d_K(\gamma,\beta) \leq 3\}| = 0$. When $|i_1 - j_1| = 1$, we obtain that $|\{\gamma | w_K(\gamma) = 2, d_K(\gamma,\beta) \leq 3\}| \leq 6$, $|\{\gamma | w_K(\gamma) = 1, d_K(\gamma,\beta) \leq 3\}| \leq 4$ and $|\{\gamma | w_K(\gamma) = 0, d_K(\gamma,\beta) \leq 3\}| = 0$. By (26)-(29) and the above discussion, when $|i_1 - j_1| \geq 2$, it follows that

$$\sum_{i=0}^{3} |\{\gamma | w_K(\gamma) = i, d_K(\gamma,\beta) \leq 3\}| \leq 4n - 4 + 8 = 4n + 4, \tag{31}$$

for $n \geq 6$. By (30) and the above discussion, when $|i_1 - j_1| = 1$, it follows that

$$\sum_{i=0}^{3} |\{\gamma | w_K(\gamma) = i, d_K(\gamma, \beta) \le 3\}| \le 6n - 18 + 10 = 6n - 8. \tag{32}$$

Therefore, for $n \geq 6$, by (31) and (32), we have that

$$I(n, 4, 3) = \sum_{i=0}^{3} |\{\gamma | w_K(\gamma) = i, d_K(\gamma, \beta) \le 3\}| \le 6n - 8.$$

□

Lemma 15 For $n \geq 5$, we have

$$\begin{aligned} I(n, d, 3) &\le 20 \quad \text{if } d = 5 \text{ or } 6, \\ I(n, d, 3) &= 0 \quad \text{if } d \ge 7. \end{aligned} \tag{33}$$

Proof When $d = 5$, let $\beta \in \mathcal{S}_n$ be a permutation such that $w_K(\beta) = 5$ and $I(n, 5, 3) = |B_K^n(\epsilon_n, 3) \cap B_K^n(\beta, 3)|$. For convenience, let $Iv(\beta) = \{I_1, I_2, I_3, I_4, I_5\}$. Next, we estimate the value of $|\{\gamma | w_K(\gamma) = i, d_K(\gamma, \beta) \le 3\}|$ for each $0 \le i \le 3$.

Let $w_K(\gamma) = i$ and $d_K(\gamma,\beta) \le 3$. By Lemma 8, then we have $d_K(\gamma,\beta) = |Iv(\gamma)| + |Iv(\beta)| - 2|(Iv(\beta) \cap Iv(\gamma))|$. Hence, we obtain that $d_K(\gamma,\beta) = 5 + i - 2|(Iv(\beta) \cap Iv(\gamma))|$ and $0 \le |(Iv(\beta) \cap Iv(\gamma))| \le i$ for any $0 \le i \le 3$. Since $d_K(\gamma,\beta) \le 3$, then we have $i = 2$ or 3, and $Iv(\gamma) \subset Iv(\beta)$. When $i = 2$, the number of γ of $w_K(\gamma) = 2$ is at most $\binom{5}{2}$. When $i = 3$, we also have that the number of γ of $w_K(\gamma) = 3$ is at most $\binom{5}{3}$. So, we get

$$I(n, 5, 3) \le \binom{5}{2} + \binom{5}{3} = 20.$$

Similarly, we can obtain that $I(n,6,3) \le 20$ and $I(n,d,3) = 0$ for all $d \ge 7$. □

By Lemmas 13-15, we can summarize up some properties of $N(n,d,3)$ for $3 \le d$ as follows.

Proposition 1 Let n,d be integers. Then we have

$$I(n, d, 3) \le \begin{cases} 6n - 7 & \text{if } d = 3 \text{ and } n \ge 6, \\ 6n - 8 & \text{if } d = 4 \text{ and } n \ge 6, \\ 20 & \text{if } d = 5 \text{ or } 6 \text{ and } n \ge 5, \end{cases}$$

and $I(n,d,3) = 0$ for all $d \ge 7$.

By the above discussion, we can obtain the following theorem. Moreover, the upper bounds on $I(4,3,3)$,$I(4,4,3)$,$I(5,3,3)$,$I(5,4,3)$ will be discussed in the A.

Theorem 2 For all $n \ge 3$, we have

$$N(n, 1, 3) = N(n, 2, 3) = n^2 - n. \tag{34}$$

Proof By (14), Lemma 3, and Proposition 1, we can obtain that $N(n,1,3) = N(n,2,3) = n^2 - n$ for any $n \ge 6$. When $3 \le n \le 5$, we have $N(3,1,3) = N(3,2,3) = 6$,$N(4,1,3) = N(4,2,3)$

$=12$,$N(5,1,3)$ = $N(5,2,3)$ $=20$. Specifically, more details on the proof of $N(n,d,3)$ for all $d=1,2$ and $3 \leq n \leq 5$ can be found in the A. Therefore, for all $n \geq 3$, we have $N(n,1,3) = N(n,2,3) = n^2 - n$. □

4 Conclusions

In this paper, we studied the reconstruction problem for permutations on n elements from their erroneous patterns which are distorted by at most three Kendall τ-errors. Specially, it is shown that $n^2 - n + 1$ erroneous patterns are required in order to reconstruct an unknown permutation from some permutation code of minimum Kendall τ-distance 2 or an arbitrary unknown permutation for any $n \geq 3$. That is, we proved that $N(n,1,3) = N(n,2,3) = n^2 - n$ for any $n \geq 3$.

Appendix

In this appendix, we will discuss the size of $N(n,d, 3)$ for all $3 \leq n \leq 5$ and $d=1, 2$ as follows. Assume that there exists some permutation $\beta \in \mathcal{S}_n$ such that $I(n,d,3) = |B_K^n(\beta,3) \cap B_K^n(\epsilon_n,3)|$ and $w_K(\beta) = d$ for any $3 \leq n \leq 5$ and $d=3, 4$.

When $n=3$, we can easily obtain $I(3, 1, 3) = I(3, 2, 3) = I(3, 3, 3) = 6$. Hence we have $N(3, 1, 3) = N(3, 2, 3) = 6$.

When $n=4$, we have $I(4, 1, 3) = I(4, 2, 3) = 12$. By Lemma 4, we have $S_K^4(0) = 1, S_K^4(1) = 3, S_K^4(2) = 5$, and $S_K^4(3) = 6$. If $(i,i+1) \in Iv(\beta)$ for each $i \in [3]$, then $\beta = [4, 3, 2, 1]$ and $w_K(\beta) = 6$. Since $S_K^4(1) = 3$, then $|\{\gamma | w_K(\gamma) = 1, d_K(\gamma,\beta) \leq 3\}| \leq 2$ for $w_K(\beta) = 3$ or 4. First, we estimate the value of $I(4, 4, 3)$ for $w_K(\beta) = 4$. Suppose $Iv(e_i) = I_1$ such that $I_1 \notin Iv(\beta)$ for some $i \in [3]$. Then, there exists a permutation α such that $Iv(\alpha) = \{I_1,I_2\}$. Hence, $|Iv(\beta) \cap Iv(\alpha)| \leq 1$ and $d_K(\beta,\alpha) \geq 4$. Since $S_K^4(2) = 5$, then we have $|\{\gamma | w_K(\gamma) = 2, d_K(\gamma,\beta) \leq 3\}| \leq 4$ for $w_K(\beta) = 4$. Therefore, we have

$$I(4,4,3) = \sum_{i=1}^{3} |\{\gamma | w_K(\gamma) = i, d_K(\gamma,\beta) \leq 3\}| \leq 2 + 4 + 6 = 12.$$

Second, we estimate the value of $I(4, 3, 3)$ for $w_K(\beta) = 3$. When $w_K(\gamma) = 3$, we have that $|\{\gamma | w_K(\gamma) = 3, d_K(\gamma,\beta) \leq 3\}| \leq 4$. Therefore, we have

$$I(4,3,3) = \sum_{i=0}^{3} |\{\gamma | w_K(\gamma) = i, d_K(\gamma,\beta) \leq 3\}| \leq 1 + 2 + 5 + 4 = 12.$$

So, we can obtain that $N(4, 1, 3) = N(4, 2, 3) = 12$.

When $n=5$, we have $I(5, 1, 3) = I(5, 2, 3) = 20$. By Lemma 4, we have $S_K^5(0) = 1, S_K^5(1) = 4, S_K^5(2) = 9$, and $S_K^5(3) = 15$. First, we estimate the value of $I(5, 4, 3)$ for $w_K(\beta) = 4$. The inversions of all the elements of $S_K^5(\epsilon_5, 1)$ are (1, 2), (2, 3), (3, 4), (4, 5). Suppose $(i_0,j_0) \in Iv(\beta)$ is an inversion with the maximum value of $|i_0 - j_0|$. It is easily verified that $j_0 - i_0 = 2$, 3, or 4. When $j_0 - i_0 = 4$, then $(1, 5) \in Iv(\beta)$. Since $w_K(\beta) = 4$ and $(1, 5) \in Iv(\beta)$, it follows that $\beta = [5, 1, 2, 3, 4]$ or $[2, 3, 4, 5, 1]$. Without loss of generality, let $\beta = [5, 1, 2, 3, 4]$. Thus, $Iv(\beta) = \{(1, 5), (2, 5), (3, 5), (4, 5)\}$. let $\gamma \in \{\gamma | w_K(\gamma) = 1, d_K(\gamma,\beta) \leq 3\}$. Since $w_K(\beta) = 4$, $w_K(\gamma) = 1$, and $d_K(\gamma,\beta) \leq 3$, by Lemma 8, then $Iv(\gamma) \subset Iv(\beta)$. Hence, $|\{\gamma | w_K(\gamma) = 1, d_K(\gamma,\beta) \leq 3\}| = 1$. Similarly, let $\gamma \in \{\gamma | w_K(\gamma) = 2, d_K(\gamma,\beta) \leq 3\}$, then $Iv(\gamma) \subset$

$Iv(\beta)$ and $|\{\gamma|w_K(\gamma)=2, d_K(\gamma,\beta)\leq 3\}|=1$. Since $S_K^5(3)=15$, then $|\{\gamma|w_K(\gamma)=2, d_K(\gamma,\beta)\leq 3\}|\leq 15$. So, we have

$$\sum_{i=1}^{3}|\{\gamma|w_K(\gamma)=i, d_K(\gamma,\beta)\leq 3\}|\leq 1+1+15=17.$$

When $j_0-i_0=3$, then (1, 4) or (2, 5) $\in Iv(\beta)$. Without loss of generality, let (1, 4) $\in Iv(\beta)$. Since $w_K(\beta)=4$ and (1, 4) $\in Iv(\beta)$, it follows that {(1, 4), (2, 4), (3, 4)} or {(1, 4), (1, 2), (1, 3)}$\subset Iv(\beta)$. Consider {(1, 4), (2, 4),(3, 4)}$\subset Iv(\beta)$, we easily have that $|\{\gamma|w_K(\gamma)=1, d_K(\gamma,\beta)\leq 3\}|\leq 2$ and $|\{\gamma|w_K(\gamma)=2, d_K(\gamma,\beta)\leq 3\}|\leq 3$. So, we have

$$\sum_{i=1}^{3}|\{\gamma|w_K(\gamma)=i, d_K(\gamma,\beta)\leq 3\}|\leq 2+3+15=20.$$

When $j_0-i_0=2$, then (1, 3), (2, 4), or (3, 5) $\in Iv(\beta)$. If $Iv(\beta)$ has at least two elements of {(1, 3), (2, 4), (3, 5)}, then (1, 3), (2, 4) $\in Iv(\beta)$ or (2, 4), (3, 5) $\in Iv(\beta)$. Without loss of generality, consider (1, 3), (2, 4) $\in Iv(\beta)$, it follows that $|\{\gamma|w_K(\gamma)=1, d_K(\gamma,\beta)\leq 3\}|\leq 2$ and $|\{\gamma|w_K(\gamma)=2, d_K(\gamma,\beta)\leq 3\}|\leq 3$. So, we have

$$\sum_{i=1}^{3}|\{\gamma|w_K(\gamma)=i, d_K(\gamma,\beta)\leq 3\}|\leq 2+3+15=20.$$

If $Iv(\beta)$ has only an element of (1, 3), (2, 4), (3, 5), it is easily verified that $\beta=[3,2,1,5,4]$ or $[2,1,5,4,3]$. Without loss of generality, let $\beta=[3,2,1,5,4]$. Then we have that $|\{\gamma|w_K(\gamma)=1, d_K(\gamma,\beta)\leq 3\}|=3$ and $|\{\gamma|w_K(\gamma)=2, d_K(\gamma,\beta)\leq 3\}|=4$. Obviously, when $\gamma=[4,1,2,3,5]$ *or* $[1,5,2,3,4]$, we have $d_K(\beta,\gamma)\geq 4$. Since $S_K^5(3)=15$, then $|\{\gamma|w_K(\gamma)=2, d_K(\gamma,\beta)\leq 3\}|\leq 13$. So, we have

$$\sum_{i=1}^{3}|\{\gamma|w_K(\gamma)=i, d_K(\gamma,\beta)\leq 3\}|\leq 3+4+13=20.$$

By the above discussion, we have that

$$I(5,4,3)=\sum_{i=1}^{3}|\{\gamma|w_K(\gamma)=i, d_K(\gamma,\beta)\leq 3\}|\leq 20.$$

Second, we estimate the size of $I(5,3,3)$ and $w_K(\beta)=3$. If $Iv(\beta)$ contains all the inversions of any three distinct elements of $S_K^5(\epsilon_5,1)$, then $|Iv(\beta)|\geq 4$. Hence, $|\{\gamma|w_K(\gamma)=1, d_K(\gamma,\beta)\leq 3\}|\leq 2$. Since $S_K^5(2)=9$, then $|\{\gamma|w_K(\gamma)=2, d_K(\gamma,\beta)\leq 3\}|\leq 9$. Therefore, by (23), we have

$$I(5,3,3)=\sum_{i=0}^{3}|\{\gamma|w_K(\gamma)=i, d_K(\gamma,\beta)\leq 3\}|\leq 1+2+9+7=19.$$

So, we can get $N(5,1,3)=N(5,2,3)=20$.

Acknowledgements The authors would like to express their sincere gratefulness to the editor and the two anonymous reviewers for their valuable suggestions and comments which have greatly improved this paper. This work is supported in part by the National Natural Science Foundation of China (Grant No. 12001134) and the National Natural Science Foundation of China - Join Fund of Basic Research of General Technology (Grant U1836111).

References

1. Haeupler, B., Mitzenmacher, M.: Repeated deletion channels. In: Proc. IEEE Inf. Theory Workshop, pp. 152–156 (2014)
2. Holenstein, T., Mitzenmacher, M., Panigrahy, R., Wieder, U.: Trace reconstruction with constant deletion probability and related results. In: Proc. of the Nineteenth Annual ACM-SIAM Symp. on Discrete Algorithms, pp. 389–398 (2008)
3. Jiang, A., Schwartz, M., Bruck, J.: Correcting charge-constrained errors in the rank-modulation scheme. IEEE Trans. Inf. Theory **56**, 2112–2120 (2010)
4. Konstantinova, E.: Reconstruction of permutations distorted by single reversal errors. Discrete Applied Math. **155**, 2426–2434 (2007)
5. Konstantinova, E.: On reconstruction of signed permutations distorted by reversal errors. Discret. Math. **308**, 974–984 (2008)
6. Konstantinova, E., Levenshtein V., Siemons J.: Reconstruction of permutations distorted by single transposition errors. math/0702191v1 (2007)
7. Levenshtein, V.: Efficient reconstruction of sequences. IEEE Trans. on Inf. Theory **47**, 2–22 (2001)
8. Levenshtein, V.: Efficient reconstruction of sequences from their subsequences or supersequences. Journal of Combin. Theory, Ser. A **93**, 310–332 (2001)
9. Levenshtein, V., Konstantinova, E., Konstantinov, E., Molodtsov, S.: Reconstruction of a graph from 2-vicinities of its vertices. Discret. Appl. Math. **156**, 1399–1406 (2008)
10. Levenshtein, V., Siemons, J.: Error graphs and the reconstruction of elements in groups. Journal of Combin. Theory, Ser. A **116**, 795–815 (2009)
11. Sala, F., Gabrys, R., Schoeny, C., Dolecek, L.: Exact reconstruction from insertions in synchronization codes. IEEE Trans. Inf. Theory **63**, 2428–2445 (2017)
12. Yaakobi, E., Schwartz, M., Langberg, M., Bruck, J.: Sequence reconstruction for Grassmann graphs and permutations. In: Proc. Int. Symp. Inf. Theory, pp. 874–878 (2013)
13. Yaakobi, E., Bruck, J.: On the uncertainty of information retrieval in associative memories. In: Proc. Int. Symp. Inf. Theory, pp. 106–110 (2012)
14. Wang, X., Wang, Y., Yin, W., Fu, F.-W.: Nonexistence of perfect permutation codes under the Kendall τ-metric. Des. Codes Cryptogr. **89**, 2511–2531 (2021)
15. Deza, M, Huang, H.: Metrics on permutations, a survey. J. Comb. Inf. Sys. Sci. **23**, 173–185 (1988)
16. Wang, X., Zhang, Y.W., Yang, Y.T., Ge, G.N.: New bounds of permutation codes under Hamming metric and Kendall's τ-metric. Des. Codes Cryptogr. **85**, 533–545 (2017)