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# Generalized block inserting for constructing new constant dimension codes

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**Abstract**
Constant dimension codes (CDCs) have drawn extensive attention due to their applications in random network coding. A fundamental problem for CDCs is to explore the maximum possible cardinality $A_q(n,d,k)$ of a set of $k$-dimensional subspaces in $\mathbb{F}_q^n$ such that the subspace distance statisfies $\mathrm{dis}(U,V) = 2k - 2\dim(U \cap V) \geq d$ for all pairs of distinct subspaces $U$ and $V$ in this set. In this paper, by means of an appropriate combination of the matrix blocks from rank metric codes and small CDCs, we present three constructions of CDCs based on the generalized block inserting construction by Niu et al. in 2021. According to our constructions, we obtain 28 new lower bounds for CDCs which are better than the previously known lower bounds.

**Keywords** Subspace coding · Constant dimension code · Inserting construction · Rank metric code

**Mathematics Subject Classification 2010** 94B60 · 94B65

## 1 Introduction

Let $\mathbb{F}_q^n$ be an $n$-dimensional vector space over the finite field $\mathbb{F}_q$ with $q$ elements. Given a non-negative integer $k \leq n$, let $\mathcal{G}_q(n,k)$ be the set of all $k$-dimensional subspaces in $\mathbb{F}_q^n$. The cardinality of $\mathcal{G}_q(n,k)$ is equal to the *q-ary Gaussian coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}$. For any two subspaces $U, V \in \mathcal{G}_q(n,k)$, the *subspace distance* between $U$ and $V$ is defined as $\mathrm{dis}(U,V) = \dim(U) + \dim(V) - 2\dim(U \cap V) = 2k - 2\dim(U \cap V)$, where $\dim(\cdot)$ denotes the dimension

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of a vector space over $\mathbb{F}_q$. An $(n,d,k)_q$ *constant dimension code* (CDC) $\mathcal{C}$ is a nonempty set in $\mathcal{G}_q(n,k)$ such that $\mathrm{dis}(U,V) \geq d$ for all pairs of distinct subspaces $U, V \in \mathcal{C}$. In particular, we call $\mathcal{C}$ an $(n,M,d,k)_q$ CDC if $\mathcal{C}$ has $M$ codewords. The maximum cardinality among all $(n,d,k)_q$ CDCs is denoted by $A_q(n,d,k)$.

Due to the work of Kötter and Kschishang [18], where they presented an application of subspace codes for correcting errors and erasures in random network coding, constant dimension codes have been studied extensively. The main question of constant dimension subspace coding asks for the maximum cardinality $A_q(n,d,k)$. However, it is hard to obtain the exact value of $A_q(n,d,k)$. A lot of constant dimension codes were constructed in the literature, readers can refer to [5, 6, 10–17, 22, 26–28]. The homepage [15] lists the latest lower and upper bounds of $A_q(n,d,k)$ for $4 \leq n \leq 19$ and $q \in \{2,3,4,5,7,8,9\}$. Rank metric codes, maximum rank distance codes in particular, are utilized to construct CDCs. The rank metric on the matrix space $\mathbb{F}_q^{m\times n}$ is defined by the rank of matrices. For any two matrices $A, B \in \mathbb{F}_q^{m\times n}$, the *rank distance* between $A$ and $B$ is defined as $d_r(A,B) = \mathrm{rank}(A - B)$. The *minimum rank distance* of a code $\mathcal{M} \subset \mathbb{F}_q^{m\times n}$ is defined as $d_r(\mathcal{M}) = \min\{d_r(A,B) : A \neq B,\ A, B \in \mathcal{M}\}$. If the minimum rank distance of a nonempty subset $\mathcal{M}$ in $\mathbb{F}_q^{m\times n}$ is at least $d$, then $\mathcal{M}$ is called an $(m,n,d)_q$ *rank metric code* (RMC). Delsarte [4] and Gabidulin [8] independently showed that the cardinality of an $(m,n,d)_q$ RMC is upper bounded by $\Delta(m,n,d)_q = q^{\max\{m,n\}\times(\min\{m,n\}-d+1)}$. We call $\mathcal{M}$ a *maximum rank distance* (MRD) *code* if $\mathcal{M}$ is an RMC achieving this bound.

A method for constructing CDCs is the *lifting construction* [18]: for any fixed $(m,n,d)_q$ MRD code $\mathcal{M}$, the corresponding CDC is the set consisting of all subspaces spanned by the $m$ rows of matrix $(I|M)$, where $I$ is the $m \times m$ identity matrix and $M \in \mathcal{M}$. Nevertheless, this method usually does not generate codes with large cardinalities. In [5], by lifting Ferrers diagram rank metric codes, Etzion and Silberstein proposed the multilevel construction which generalizes the lifted MRD codes construction. Gluesing-Luerssen and Troha presented the linkage construction by linking two CDCs in [9] and Chen et al. in [1] further presented the so-called parrallel linkage construction by modifying the linkage construction. Later, He [11] constructed CDCs from two parallel linkage constructions which generalize the result of [1]. In [23], Liu et al. generalized the parallel construction and the multilevel construction by introducing rank-restricted rank metric codes. For other recent constructions of CDCs by the multilevel construction, we can refer to [7, 14, 19, 21, 25]. Cossidenta et al. [2] and Heinlei [16] respectively generalized the parallel linkage construction in different ways. In [20], Lao et al. proposed two block inserting constructions which insert flexibly CDCs constructed by matrix blocks from small CDCs and RMCs into the parallel linkage constructions in [11], see also [21]. Recently, Niu et al. in [24] generalized the block inserting construction by padding more matrices.

In this paper, inspired by the ideas of [20] and [24], we propose three constructions of CDCs with restricted parameters based on the generalized block inserting construction through an appropriate combination of matrix blocks from rank metric codes and small CDCs. Our results improve the previously best known lower bounds for CDCs. We organize the remaining part of this paper as follows. In Section 2, we introduce some essential definitions and results which are useful for our main results. In Section 3, we propose our three improved constructions of constant dimension codes and obtain 28 new lower bounds for CDCs. We conclude this paper in Section 4.

## 2 Preliminaries

In this section, we will recall some basic definitions and known results which are necessary to prove our results. More details are available in [2, 4, 5, 8].

### 2.1 Identifying vector and Delsarte Theorem

Let $U$ be a $k$-dimensional subspace in $\mathbb{F}_q^n$, a *generator matrix* of $U$ is a $k \times n$ matrix whose $k$ rows form a basis of $U$. It is a well-known fact that there exists a unique generator matrix of $U$ in reduced row echelon form (RREF) and denote it by $E(U)$, refer to [5].

**Definition 2.1** Let $U \in \mathcal{G}_q(n,k)$ and $E(U)$ be the generator matrix in reduced row echelon form of $U$. The identifying vector of $U$ is denoted by $i(U)$, which is a binary vector of length $n$ and weight $k$ such that $i(U)$ has exactly $k$ ones in the positions of the pivot columns of $E(U)$.

***Example 2.2*** Suppose $U$ is a 4-dimensional subspace in $\mathbb{F}_2^6$ with the generator matrix

$$E(U) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the identifying vector of $U$ is $i(U) = (110011)$.

The following Lemma 2.3 is crucial, we can obtain a lower bound of the subspace distance related to the Hamming distance.

**Lemma 2.3 (Lemma 2 in 5)** *Let $U, V \in \mathcal{G}_q(n,k)$. Assume $i(U)$ and $i(V)$ are the identifying vectors of $U$ and $V$, respectively. Then $dis(U,V) \geq d_H(i(U),i(V))$, where $d_H$ denotes the Hamming distance.*

*If a rank metric code is a linear subspace over $\mathbb{F}_q$ in the matrix space $\mathbb{F}_q^{m\times n}$, then it is said to be $\mathbb{F}_q$-linear and we abbreviate it by linear in this paper. Linear MRD codes exist for all possible parameters (cf. [4, 8]). Moreover, if $d' > d$, then we can assume that there exists a linear $(m,n,d)_q$ MRD code containing a linear $(m,n,d')_q$ MRD code as a subcode. If any codeword $M$ in an $(m,n,d)_q$ RMC satisfies $rank(M) \leq r$, then we call it a rank-restricted rank metric code (RRMC) and denote it by $(m,n,d;r)_q$. The theory of Delsarte allows to determine the rank distribution of a linear MRD code by its parameters, refer to Theorem 5.6 in [4] or Corollary 26 in [3]. Thus a lower bound for the cardinality of RRMC can be given.*

**Theorem 2.4 (Delsarte Theorem 4)** *Let $m$, $n$, $d$ and $r$ be positive integers such that $m \geq n$ and $d \leq r \leq n$. Assume $M$ is an $(m,n,d)_q$ MRD code. Then the number of codewords with rank $r$ in $M$ is given by*

$$D(m,n,d,r)_q = \begin{bmatrix} n \\ r \end{bmatrix}_q \sum_{i=0}^{r-d} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}_q \left(q^{m(r-i-d+1)} - 1\right).$$

*We can construct an* $(m,n,d;r)_q$ *RRMC from a subset of an* $(m,n,d)_q$ *linear MRD code satisfying the rank restriction* (*cf.* [15, 16]). *Hence, the maximum cardinality of an* $(m,n,d;r)_q$ *RRMC is lower bounded by* $1 + \sum_{i=d}^{r} D(m, n, d;i)_q$, *which is abbreviated by* $\Delta(m,n,d;r)_q$.

### 2.2 Parallel linkage construction and subcodes construction

In this subsection, we first review some essential notation and known results about the parallel linkage construction. We finally introduce the subcode construction in [2] since it is important for constructing the desired rank metric codes in our constructions.

For any set $\mathcal{M} \subset \mathbb{F}_q^{k\times n}$, if 1) for any $M \subset \mathcal{M}$, rank($M$) $= k$, and 2) for any two distinct matrices $M_1$ and $M_2$ in $\mathcal{M}$, $rs(M_1) \neq rs(M_2)$, where $rs(M_1)$ denotes the subspace spanned by the $k$ rows of $M_1$, then $\mathcal{M}$ is called an *SC-representation set*[9]. It is obvious that $\{rs(M) : M \in \mathcal{M}\}$ corresponds to a constant dimension code. The following Lemma 2.5 is the *parallel linkage construction* which generalizes Theorem 4 in [1]. In this paper, $(M_1|M_2)$ denotes a matrix concatenated from $M_1$ and $M_2$.

**Lemma 2.5 (Theorem 2 in** 11) *Let* $m$, $n$, $k$ *and* $d$ *be positive integers such that* $m \geq k$ *and* $n \geq k$. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be two SC-representation sets of* $(m,2d,k)_q$ *and* $(n,2d,k)_q$ *CDCs, respectively. Let* $\mathcal{Q}_1$ *be a* $(k,n,d)_q$ *RMC and* $\mathcal{Q}_2$ *be a* $(k,m,d;k-d)_q$ *RRMC. Define* $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, *where*

$$\begin{aligned} \mathcal{C}_1 &= \{rs(A_1|Q_1) : A_1 \in \mathcal{A}_1, Q_1 \in \mathcal{Q}_1\}, \\ \mathcal{C}_2 &= \{rs(Q_2|A_2) : A_2 \in \mathcal{A}_2, Q_2 \in \mathcal{Q}_2\}. \end{aligned}$$

*Then* $\mathcal{C}$ *is an* $(m+n,2d,k)_q$ *CDC. Particularly,* $A_q(m+n,2d,k) \geq A_q(m,2d,k) \cdot \Delta(k,n,d)_q + A_q(n,2d,k) \cdot \Delta(k,m,d;k-d)_q$.

*The subcode construction occured in* [2], *which can be described as the following Lemma* 2.6. *This construction can be used for the block inserting construction in Section* 3. *We give a complete proof here because* [2] *didn't prove the uniqueness.*

**Lemma 2.6 (Subcode construction in** 2) *Let* $\mathcal{M}$ *be an* $(m,n,d)_q$ *linear MRD containing an* $(m,n,d')_q$ *linear MRD code* $\mathcal{C}$ *as a subcode, where* $d$ *and* $d'$ *are positive integers such that* $d' > d$. *Then there exist* $s$ *subcodes of* $\mathcal{M}$ *fulfilling the following conditions*: (1) $\mathcal{M}_i$ *is an* $(m,n,d')_q$ *MRD*, $1 \leq i \leq s$; (2) *for* $M \in \mathcal{M}_i$, $M' \in \mathcal{M}_j (1 \leq i < j \leq s)$, $M \neq M'$ *and rank*$(M - M') \geq d$. *Here,* $s = \frac{\Delta(m,n,d)_q}{\Delta(m,n,d')_q}$. *Moreover,* $\mathcal{C}$ *is the unique linear MRD code in these* $s$ *subcodes.*

***Proof*** Let $M_j \in \mathcal{M}$ and denote $\mathcal{M}_j = M_j + \mathcal{C}$. Then $\mathcal{M}_j$ is a subcode of $\mathcal{M}$. For any two distinct codewords $C_1, C_2 \in \mathcal{C}$, we have $M_j + C_1 \neq M_j + C_2$, which implies that $\mathcal{M}_j$ is an $(m,n,d')_q$ MRD subcode of $\mathcal{M}$. If $M_i + C_1 = M_j + C_2$, where $M_i, M_j \in \mathcal{M}$ and $C_1, C_2 \in \mathcal{C}$, then $M_i - M_j \in \mathcal{C}$ since $\mathcal{C}$ is a linear MRD code. Thus $\mathcal{M}_j$ can be viewed as a coset of $\mathcal{M}$ and there exist $s = \frac{\Delta(m,n,d)_q}{\Delta(m,n,d')_q}$ distinct $(m,n,d')_q$ MRD codes. Assume $M_j + \mathcal{C}$ is a linear MRD code, then $a(M_j + C) \in M_j + \mathcal{C}$ for any $a \in \mathbb{F}_q$ and any $C \in \mathcal{C}$, which implies $M_j \in \mathcal{C}$ and thus $M_j + \mathcal{C} = \mathcal{C}$. So, we can conclude that $\mathcal{C}$ is the unique linear MRD code in these $s$ subcodes.

## 3 Our constructions of constant dimension codes

In this section, we first introduce the generalized block inserting construction. Later, we describe our three improved constructions based on the generalized block inserting construction.

Suppose $A \in \mathbb{F}_q^{k\times m}$ is a matrix in RREF with $\mathrm{rank}(A) = k$. Define the embedding map $\sigma_A : \mathbb{F}_q^{l\times(m-k)} \to \mathbb{F}_q^{l\times m}$ by inserting $k$ zero columns $\underbrace{(0, 0, \cdots, 0)^T}_{l}$ into $F$ in the positions of the pivot columns of $A$, where $F \in \mathbb{F}_q^{l\times(m-k)}$. For example,

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \text{and } F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

where $A$ has pivot columns 1,3,5. Then

$$\sigma_A(F) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In [20], Lao et al. proposed the *block inserting construction* which adds subspaces with generator matrix concatenated by small matrix blocks from CDCs and RMCs into the CDC in Lemma 2.5 by restricting the rank of matrices. Using the above embedding map $\sigma_A(F)$, Niu et al. generalized the block inserting construction in the following Lemma 3.1.

**Lemma 3.1 (Theorem 3.2 in 24)** *Let $m$, $n$, $k$, $d$, $a_1$, $a_2$, $b_1$, $b_2$, $t_1$ and $t_2$ be positive integers with $m \geq k$, $n \geq k$, $a_1 + a_2 = k$, $b_1 + b_2 \geq d$, $a_1 \leq t_1 \leq m - d$, $a_2 \leq t_2 \leq n - d$, and $a_i \geq d$, $1 \leq b_i \leq d$ for $i = 1,2$. Let $\mathcal{U}_i = \{U_i \in \mathbb{F}_q^{a_i\times t_i} : U_i \text{ in RREF}, \mathrm{rank}(U_i) = a_i\}$ be an SC-representation set of $(t_i,2d,a_i)_q$ CDC for $i = 1, 2$. Let*

$$\mathcal{M}_3 \subset \{(F_3|M_3) : F_3 \in \mathbb{F}_q^{a_1\times(t_2-a_2)}, M_3 \in \mathbb{F}_q^{a_1\times(n-t_2)}\}$$

*and*

$$\mathcal{M}_4 \subset \{(F_4|M_4) : F_4 \in \mathbb{F}_q^{a_2\times(t_1-a_1)}, M_4 \in \mathbb{F}_q^{a_2\times(m-t_1)}\}$$

*be two RRMCs with respective parameters $(a_1,n - a_2,d;a_1 - d)_q$ and $(a_2,m - a_1,d;a_2 - d)_q$. Given integer $s$, for $1 \leq r \leq s$, let $\mathcal{M}_1^r$ be an $(a_1,m - t_1,d)_q$ MRD code and $\mathcal{M}_2^r$ be an $(a_2,n - t_2,d)_q$ MRD code. For $i = 1, 2$, let $M \in \mathcal{M}_i^r$ and $M' \in \mathcal{M}_i^{r'}$ for $1 \leq r < r' \leq s$, assume that $M \neq M'$ and $\mathrm{rank}(M - M') \geq b_i$. Define $\mathcal{C}_3 = \cup_{r=1}^s \mathcal{X}_r$,*

$$\mathcal{X}_r = \left\{ rs\begin{pmatrix} U_1 & M_1 & \sigma_{U_2}(F_3) & M_3 \\ \sigma_{U_1}(F_4) & M_4 & U_2 & M_2 \end{pmatrix} \right\},$$

*where* $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$, $M_i \in \mathcal{M}_i^r$ *for* $i = 1, 2$, *and* $(F_i|M_i) \in \mathcal{M}_i$ *for* $i = 3, 4$. *Then* $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ *is an* $(m+n,2d,k)_q$ *CDC, where* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are defined in Lemma* 2.5. *As a consequence,*

$$\begin{aligned} A_q(m+n,2d,k) \geq\ & A_q(m,2d,k)\cdot\Delta(k,n,d)_q + A_q(n,2d,k)\cdot\Delta(k,m,d;k-d)_q \\ & + s\cdot A_q(t_1,2d,a_1)\cdot A_q(t_2,2d,a_2)\cdot\Delta(a_1,m-t_1,d)_q\cdot\Delta(a_2,n-t_2,d)_q \\ & \cdot\Delta(a_1,n-a_2,d;a_1-d)_q\cdot\Delta(a_2,m-a_1,d;a_2-d)_q. \end{aligned}$$

*Next, we construct new CDCs with larger cardinalities based on Lemma* 3.1. *If the parameters* $a_2$ *and* $t_2$ *satisfy* $n - t_2 \geq a_2$, *then more subspaces can be inserted into the CDC in Lemma* 3.1. *Thus, using the similar block inserting construction via exchanging the positions of the matrix blocks, we propose the following Construction A. In this paper,* $O_{m\times n}$ *denotes the zero matrix with size* $m \times n$.

**Theorem 3.2 (Construction A)** *With the same notation used in Lemma* 3.1. *Additionally,* $n - t_2 \geq a_2$ *and* $k - t_1 \geq 2d$. *Let* $c_1$ *and* $c_2$ *be positive integers with* $c_1 + c_2 \geq d$ *and* $1 \leq c_i \leq d$ *for* $i = 1, 2$. *Let* $\mathcal{E} = \{E \in \mathbb{F}_q^{a_1\times t_1} : E \text{ in RREF}, \text{rank}(E) = a_1\}$ $\left(\text{resp. } \mathcal{H} = \{H \in \mathbb{F}_q^{a_2\times(n-t_2)} : H \text{ in RREF}, \text{rank}(H) = a_2\}\right)$ *be an SC-representation set of* $(t_1,2d,a_1)_q$ *(resp.* $(n-t_2,2d,a_2)_q$*) CDC. Let*

$$\mathcal{M}_{1,2} \subset \{(M_{1,2}|F_{1,2}) : F_{1,2} \in \mathbb{F}_q^{a_1\times(n-t_2-a_2)}, M_{1,2} \in \mathbb{F}_q^{a_1\times t_2}\}$$

*and*

$$\mathcal{M}_{2,1} \subset \{(F_{2,1}|M_{2,1}) : F_{2,1} \in \mathbb{F}_q^{a_2\times(t_1-a_1)}, M_{2,1} \in \mathbb{F}_q^{a_2\times(m-t_1)}\}$$

*be two RRMCs with respective parameters* $(a_1,n-a_2,d;a_1-d)_q$ *and* $(a_2,m-a_1,d;a_2-d)_q$. *Given integer t, for* $1 \leq r \leq t$, *let* $\mathcal{M}_{1,1}^r$ *be an* $(a_1,m-t_1,d)_q$ *MRD code and* $\mathcal{M}_{2,2}^r$ *be an* $(a_2,t_2,d;k-t_1-d)_q$ *RRMC. For* $i = 1, 2$, *let* $M \in \mathcal{M}_{i,i}^r$ *and* $M' \in \mathcal{M}_{i,i}^{r'}$ *for* $1 \leq r < r' \leq t$, *assume that* $M \neq M'$ *and rank*$(M - M') \geq c_i$. *Let* $\mathcal{C}_1$, $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *be the codes defined in Lemma* 3.1. *Define* $\mathcal{C}_4 = \cup_{r=t}^{l}\mathcal{B}_r$,

$$\mathcal{B}_r = \left\{ rs\begin{pmatrix} E & M_{1,1} & M_{1,2} & \sigma_H(F_{1,2}) \\ \sigma_E(F_{2,1}) & M_{2,1} & M_{2,2} & H \end{pmatrix} \right\},$$

*where* $E \in \mathcal{E}$, $H \in \mathcal{H}$, $M_{i,i} \in \mathcal{M}_{i,i}^r$ *for* $i = 1, 2$, $(M_{1,2}|F_{1,2}) \in \mathcal{M}_{1,2}$ *and* $(F_{2,1}|M_{2,1}) \in \mathcal{M}_{2,1}$. *Then* $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ *is an* $(m+n,2d,k)_q$ *CDC.*

***Proof*** Obviously, the subpsaces in $\mathcal{C}_4$ are $k$-dimensional. We first show that $\mathcal{C}_4$ is an $(m+n,2d,k)_q$ CDC. Let

$$W_1 = rs(P_1) = \left\{ rs\begin{pmatrix} E & M_{1,1} & M_{1,2} & \sigma_H(F_{1,2}) \\ \sigma_E(F_{2,1}) & M_{2,1} & M_{2,2} & H \end{pmatrix} \right\} \in \mathcal{B}_r,$$

$$W_1' = rs(P_1') = \left\{ rs\begin{pmatrix} E' & M_{1,1}' & M_{1,2}' & \sigma_{H'}(F_{1,2}') \\ \sigma_{E'}(F_{2,1}') & M_{2,1}' & M_{2,2}' & H' \end{pmatrix} \right\} \in \mathcal{B}_{r'}$$

be two distinct $k$-dimensionl subspaces. Then the dimension of $W_1 \cap W_1'$ is

$$\dim(W_1 \cap W_1') = \dim\left(\{\left(\alpha_1, \alpha_2\right) : \exists\left(\beta_1, \beta_2\right) \text{ s.t. } \left(\alpha_1, \alpha_2\right)P_1 = \left(\beta_1, \beta_2\right)P_1'\}\right),$$

where $\alpha_1$, $\beta_1 \in \mathbb{F}_q^{a_1}$, and $\alpha_2$, $\beta_2 \in \mathbb{F}_q^{a_2}$. We analyze the dimension of $W_1 \cap W_1'$ from three cases.

**Case 1** If $E \neq E'$, then $\dim(W_1 \cap W_1') \leq \dim(E \cap E') + a_2 \leq a_1 - d + a_2 = k - d$.

**Case 2** If $H \neq H'$, then $\dim(W_1 \cap W_1') \leq \dim(H \cap H') + a_1 \leq a_2 - d + a_1 = k - d$.

**Case 3** If $E = E'$ and $H = H'$, then

$$\begin{cases} \alpha_1 E + \alpha_2 \sigma_E(F_{2,1}) = \beta_1 E + \beta_2 \sigma_E(F_{2,1}'), \\ \alpha_1 \sigma_H(F_{1,2}) + \alpha_2 H = \beta_1 \sigma_H(F_{1,2}') + \beta_2 H. \end{cases}$$

By the above equation system, we get

$$\begin{cases} \left(\alpha_1 - \beta_1\right)E = \beta_2 \sigma_E(F_{2,1}') - \alpha_2 \sigma_E(F_{2,1}), \\ \left(\alpha_2 - \beta_2\right)H = \beta_1 \sigma_H(F_{1,2}') - \alpha_1 \sigma_H(F_{1,2}). \end{cases}$$

According to the definitions of $\sigma_E$ and $\sigma_H$, we can induce that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Thus,

$$\dim(W_1 \cap W_1') = \dim\left\{\left(\alpha_1, \alpha_2\right) : \left(\alpha_1, \alpha_2\right)P = 0\right\},$$

where

$$P = \left\{ \begin{pmatrix} O_{a_1 \times t_1} & M_{1,1} - M_{1,1}' & M_{1,2} - M_{1,2}' & \sigma_H(F_{1,2} - F_{1,2}') \\ \sigma_E(F_{2,1} - F_{2,1}') & M_{2,1} - M_{2,1}' & M_{2,2} - M_{2,2}' & O_{a_2 \times (n - t_2)} \end{pmatrix} \right\}.$$

We continue analyzing the dimension of $\dim(W_1 \cap W_1')$ from the following three subscases.

(a) If $(M_{1,2}|F_{1,2}) \neq (M_{1,2}'|F_{1,2}')$ or $(F_{2,1}|M_{2,1}) \neq (F_{2,1}'|M_{2,1}')$, then $\dim(W_1 \cap W_1') = k - \mathrm{rank}(P) \leq k - d$.

(b) If $(M_{1,2}|F_{1,2}) = (M_{1,2}'|F_{1,2}')$, $(F_{2,1}|M_{2,1}) = (F_{2,1}'|M_{2,1}')$ and $r = r'$, then $M_{1,1} \neq M_{1,1}'$ or $M_{2,2} \neq M_{2,2}'$. It follows that $\dim(W_1 \cap W_1') = k - \mathrm{rank}(P) \leq k - d$.

(c) If $(M_{1,2}|F_{1,2}) = (M_{1,2}'|F_{1,2}')$, $(F_{2,1}|M_{2,1}) = (F_{2,1}'|M_{2,1}')$ and $r \neq r'$, then $\mathrm{rank}(M_{1,1} - M_{1,1}') \geq c_1$ and $\mathrm{rank}(M_{2,2} - M_{2,2}') \geq c_2$. It follows that

$$\begin{aligned} \dim(W_1 \cap W_1') &= k - \mathrm{rank}(P) \\ &= k - (\mathrm{rank}(M_{1,1} - M_{1,1}') + \mathrm{rank}(M_{2,2} - M_{2,2}')) \\ &\leq k - (c_1 + c_2) \\ &\leq k - d. \end{aligned}$$

Thus, $\mathrm{dis}(W_1, W_1') \geq 2k - 2\dim(W_1 \cap W_1') \geq 2d$.

Finally, we prove that the distance between $W_1 \in \mathcal{C}_4$ and $W_2 \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is at least $2d$. We can discuss from the following three cases.

(1) If $W_2 \in \mathcal{C}_1$, then there exist $k$ ones in the first $m$ positions of $i(W_2)$. However, $i(W_1)$ has no more than $k-d$ ones in the first $m$ positions since $\mathrm{rank}(E) + \mathrm{rank}(F_{2,1}|M_{2,1}) \le a_1 + a_2 - d = k - d$.

(2) If $W_2 \in \mathcal{C}_2$, then $W_2 = rs(P_2) = rs(Q_2|A_2)$. The dimension of $W_1 \cap W_2$ is

$$\dim(\{(\alpha_1, \alpha_2) : \exists\, (\beta_1, \beta_2) \text{ s.t. } (\alpha_1, \alpha_2)P_1 = (\beta_1, \beta_2)P_2\}),$$

where $\alpha_i, \beta_i \in \mathbb{F}_q^{a_i}$ for $i = 1{,}2$. As $\mathrm{rank}(A_2) = k$ and

$$\begin{aligned}
\mathrm{rank}\begin{pmatrix} M_{1,2} & \sigma_H(F_{1,2}) \\ M_{2,2} & H \end{pmatrix} &\le \mathrm{rank}(H) + \mathrm{rank}(M_{1,2}|\sigma_H(F_{1,2})) \\
&= \mathrm{rank}(H) + \mathrm{rank}(M_{1,2}|F_{1,2}) \\
&\le a_2 + a_1 - d \\
&= k - d,
\end{aligned}$$

we have

$$\dim(W_1 \cap W_2) \le \dim(\{(\alpha_1, \alpha_2) : \exists\, (\beta_1, \beta_2) \text{ s.t. } (\alpha_1, \alpha_2)\begin{pmatrix} M_{1,2} & \sigma_H(F_{1,2}) \\ M_{2,2} & H \end{pmatrix} = (\beta_1, \beta_2)A_2\}) \le k - d.$$

(3) If $W_2 \in \mathcal{C}_3$, then $W_2 = rs(P_3) = \begin{pmatrix} U_1 & M_1 & \sigma_{U_2}(F_3) & M_3 \\ \sigma_{U_1}(F_4) & M_4 & U_2 & M_2 \end{pmatrix}$. Thus, the subspace distance between $W_1$ and $W_2$ is

$$\begin{aligned}
\mathrm{dis}(W_1, W_2) &= 2\mathrm{rank}\begin{pmatrix} U_1 & M_1 & \sigma_{U_2}(F_3) & M_3 \\ \sigma_{U_1}(F_4) & M_4 & U_2 & M_2 \\ E & M_{1,1} & M_{1,2} & \sigma_H(F_{1,2}) \\ \sigma_E(F_{2,1}) & M_{2,1} & M_{2,2} & H \end{pmatrix} - 2k \\
&= 2\mathrm{rank}\begin{pmatrix} U_1 & \sigma_{U_2}(F_3) & M_1 & M_3 \\ \sigma_{U_1}(F_4) & U_2 & M_4 & M_2 \\ E & M_{1,2} & M_{1,1} & \sigma_H(F_{1,2}) \\ \sigma_E(F_{2,1}) & M_{2,2} & M_{2,1} & H \end{pmatrix} - 2k \\
&= \mathrm{dis}(W_1', W_2'),
\end{aligned}$$

where

$$W_1' = rs\begin{pmatrix} E & M_{1,2} & M_{1,1} & \sigma_H(F_{1,2}) \\ \sigma_E(F_{2,1}) & M_{2,2} & M_{2,1} & H \end{pmatrix}$$

and

$$W_2' = rs\begin{pmatrix} U_1 & \sigma_{U_2}(F_3) & M_1 & M_3 \\ \sigma_{U_1}(F_4) & U_2 & M_4 & M_2 \end{pmatrix}.$$

It is obvious that $i(W_2')$ has $k$ ones in the first $t_1 + t_2$ positions. But $i(W_1')$ has $t_1 + \mathrm{rank}(M_{2,2})$ ones in the first $t_1 + t_2$ positions since $a_1 + \mathrm{rank}(F_{2,1}|M_{2,2}) \le a_1 + (t_1 - a_1 + \mathrm{rank}(M_{2,2})) = t_1 + \mathrm{rank}(M_{2,2})$. In consequence,

$$\begin{aligned}\mathrm{dis}(W_1, W_2) &= \mathrm{dis}(W_1', W_2')\\ &\geq d_H(i(W_1'), i(W_2'))\\ &\geq (k - t_1 - \mathrm{rank}(M_{2,2})) + (k - t_1 - \mathrm{rank}(M_{2,2}))\\ &= 2(k - t_1 - \mathrm{rank}(M_{2,2}))\\ &\geq 2d\end{aligned}$$

by Lemma 2.3. In conclusion, $\cup_{i=1}^4 \mathcal{C}_i$ is an $(m+n,2d,k)_q$ CDC. This completes the proof.

***Remark 3.3*** The condition $k - t_1 \geq 2d$ is vital for Construction A. According to the subcode construction in Lemma 2.6, this condition can guarantee the existence of $M_{2,2}^r$. Hence, comapred to Lemma 3.1, Construction A inserts more subspace into $\mathcal{C}_1 \cup \mathcal{C}_2$ by exchanging the positions of matrix blocks and improves the lower bounds of CDCs for some parameters.

From Theorem 3.2, we obtain some new lower bounds for CDCs in the following corollary. Denote the cardinality of $\mathcal{C}$ by $|\mathcal{C}|$. We define $\Lambda(m,n,d;r)_q = \sum_{i=d}^{r} D(m,n,d,i)_q$ if $r \geq d$ and $\Lambda(m,n,d;r)_q = 1$ if $r < d$.

**Corollary 3.4** *Let $m$, $n$, $d$, $k$, $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $t_1$ and $t_2$ be positive integers with $m \geq k$, $n \geq k$, $a_1 + a_2 = k$, $b_1 + b_2 \geq d$, $c_1 + c_2 \geq d$, $m - t_1 \geq d$, $n - t_2 \geq a_2$, $d \leq a_i \leq t_i$, $k - t_1 \geq 2d$, $1 \leq b_i \leq d$ and $1 \leq c_i \leq d$ for $i = 1, 2$. Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ be the codes defined in Thoerem* 3.2. *Then*

$$\begin{aligned}A_q(m+n,2d,k) \geq\ & A_q(m,2d,k)\cdot\Delta(k,n,d)_q + A_q(n,2d,k)\cdot\Delta(k,m,d;k-d)_q\\ &+ s\cdot A_q(t_1,2d,a_1)\cdot A_q(t_2,2d,a_2)\cdot\Delta(a_1,m-t_1,d)_q\cdot\Delta(a_2,n-t_2,d)_q\\ &\cdot\Delta(a_1,n-a_2,d;a_1-d)_q\cdot\Delta(a_2,m-a_1,d;a_2-d)_q + A_q(t_1,2d,a_1)\\ &\cdot A_q(n-t_2,2d,a_2)\cdot\Delta(a_1,n-a_2,d;a_1-d)_q\cdot\Delta(a_2,m-a_1,d;a_2-d)_q\\ &\cdot\Delta(a_1,m-t_1,d)_q\cdot\left(\Delta(a_2,t_2,d;k-t_1-d)_q + (t-1)\cdot\Lambda(a_2,t_2,d;k-t_1-d)_q\right),\end{aligned}$$

*where* $s = \min\left(\frac{\Delta(a_1,m-t_1,b_1)_q}{\Delta(a_1,m-t_1,d)_q}, \frac{\Delta(a_2,n-t_2,b_2)_q}{\Delta(a_2,n-t_2,d)_q}\right)$ *and* $t = \min\left(\frac{\Delta(a_1,m-t_1,c_1)_q}{\Delta(a_1,m-t_1,d)_q}, \frac{\Delta(a_2,t_2,c_2)_q}{\Delta(a_2,t_2,d)_q}\right)$.

***Proof*** By Lemma 2.6, we can construct $M_{1,1}^r$ and $M_{2,2}^r$ satisfying Theorem 3.2. Note that the subcode construction in Lemma 2.6 only provides a unique linear MRD code. Let $M_2$ be the unique linear $(a_2,t_2,d)_q$ MRD code constructed by Lemma 2.6. Without loss of generality, let $M_{2,2}^1 = \{M \in M_2 : \mathrm{rank}(M) \leq k - t_1 - d\}$, and $M_{2,2}^r$ be constructed from a subset of a non-linear MRD code such that the rank of each matrix is at most $k - t_1 - d$ for $2 \leq r \leq t$. Then, $|\mathcal{C}_4| = |\mathcal{E}|\cdot|\mathcal{H}|\cdot|\mathcal{M}_{1,2}|\cdot|\mathcal{M}_{2,1}|\cdot|M_{1,1}^1|\cdot\left(|\mathcal{M}_{2,2}^1| + (t-1)\cdot|\mathcal{M}_{2,2}^2|\right)$. We can see that $|\mathcal{M}_{2,2}^2| = \Lambda(a_2,t_2,d;k-t_1-d)_q$. Therefore, the desired conclusion follows.

***Example 3.5*** We adopt the notation in Corollary 3.4. Take $m = n = k = 8$, $d = 2$, $a_1 = a_2 = 4$, $b_1 = c_1 = 1$, $b_2 = c_2 = 1$, and $t_1 = t_2 = 4$. Then $t = q^4$. Take $M_{1,1}^r$ and $M_{2,2}^r$ in the same way as the proof of Corollary 3.4 for $1 \leq r \leq t$. Then $|\mathcal{M}_{1,1}^1| = \Delta(4,4,2)_q$, $|\mathcal{M}_{2,2}^1| = \Delta(4,4,2;2)_q$ and $|\mathcal{M}_{2,2}^1| = \Lambda(4,4,2;2)_q$. Moreover, $A_q(4,4,4) = 1$. It follows that

$$|\mathcal{C}_4| = \Delta(4,4,2)_q\cdot\left(\Delta(4,4,2;2)_q + (q^4-1)\cdot\Lambda(4,4,2;2)_q\right)\cdot\Delta(4,4,2;2)_q\cdot\Delta(4,4,2;2)_q.$$

For $q=2$, $|\mathcal{C}_4|=9520558391296$. The cardinality of $\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$ is 61680045647822848 in [20] and [24], which is the previously best known lower bound of $A_2(16,4,8)$. Hence, $A_2(16,4,8)\geq|\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3|+|\mathcal{C}_4|=61680045647822848+9520558391296=61689566206214144$.

We continue presenting new construction of CDCs based on Lemma 3.1, which is slightly different from Theorem 3.2. If the parameters $a_1$, $a_2$, $t_1$ and $t_2$ satisfy $m-t_1\geq a_1$ and $n-t_2\geq a_2$, then more subspaces can be inserted into the CDC in Lemma 3.1. Hence, using the similar block inserting construction through an appropriate combination of the matrix blocks from small CDCs and RRMCs, we give the following *Construction B* and improve the lower bounds for CDCs.

**Theorem 3.6 (Construction B)** *With the same notation used in Lemma* 3.1. *Additionally*, $m-t_1\geq a_1$ *and* $n-t_2\geq a_2$. *Let* $c_1'$, $c_2'$, $e_1$ *and* $e_2$ *be positive integers with* $c_1'+c_2'\geq d$, $e_1+e_2\leq k-d$ *and* $1\leq c_i'\leq d$ *for* $i=1,2$. *Let* $\mathcal{V}_1=\{V_1\in\mathbb{F}_q^{a_1\times(m-t_1)}:V_1 \text{ in RREF}, \operatorname{rank}(V_1)=a_1\}$ $\big($resp. $\mathcal{V}_2=\{V_2\in\mathbb{F}_q^{a_2\times(n-t_2)}:V_2 \text{ in RREF}, \operatorname{rank}(V_2)=a_2\}\big)$ *be an SC-representation set of* $(m-t_1,2d,a_1)_q$ *(resp.* $(n-t_2,2d,a_2)_q$*) CDC. Let*

$$\mathcal{N}_3\subset\{(N_3|G_3):G_3\in\mathbb{F}_q^{a_1\times(n-t_2-a_2)},N_3\in\mathbb{F}_q^{a_1\times t_2}\}$$

*and*

$$\mathcal{N}_4\subset\{(N_4|G_4):G_4\in\mathbb{F}_q^{a_2\times(m-t_1-a_1)},N_4\in\mathbb{F}_q^{a_2\times t_1}\}$$

*be two RRMCs with respective parameters* $(a_1,n-a_2,d;a_1-d)_q$ *and* $(a_2,m-a_1,d;a_2-d)_q$. *Given integer* $l$, *for* $1\leq r\leq l$, *let* $\mathcal{N}_1^r$ *be an* $(a_1,t_1,d;e_1)_q$ *RRMC and* $\mathcal{N}_2^r$ *be an* $(a_2,t_2,d;e_2)_q$ *RRMC. For* $i=1,2$, *let* $N\in\mathcal{N}_i^r$ *and* $N'\in\mathcal{N}_i^{r'}$ *for* $1\leq r<r'\leq l$, *assume that* $N\neq N'$ *and* *rank*$(N-N')\geq c_i'$. *Define* $\mathcal{C}_4'=\cup_{r=1}^{l}\mathcal{B}_r'$,

$$\mathcal{B}_r'=\left\{rs\begin{pmatrix}N_1 & V_1 & N_3 & \sigma_{V_2}(G_3)\\ N_4 & \sigma_{V_1}(G_4) & N_2 & V_2\end{pmatrix}\right\},$$

*where* $V_1\in\mathcal{V}_1$, $V_2\in\mathcal{V}_2$, $N_i\in\mathcal{N}_i^r$ *for* $i=1,2$, *and* $(N_i|G_i)\in\mathcal{N}_i$ *for* $i=3,4$. *Then* $\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3\cup\mathcal{C}_4'$ *is an* $(m+n,2d,k)_q$ *CDC, where* $\mathcal{C}_1$, $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *are defined in Lemma* 3.1.

***Proof*** Let $W_4$ be a subspace in $\mathcal{C}_4'$, then

$$W_4=rs(P_4)=rs\begin{pmatrix}N_1 & V_1 & N_3 & \sigma_{V_2}(G_3)\\ N_4 & \sigma_{V_1}(G_4) & N_2 & V_2\end{pmatrix}.$$

We can prove that $\mathcal{C}_4'$ is an $(m+n,2d,k)_q$ CDC from a similar proof as Theorem 3.2, hence we omit the detail of it here. It remains to show that the distance between $W_4\in\mathcal{C}_4'$ and $W_2\in\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$ is at least $2d$. There are the following three cases.

(1) If $W_2\in\mathcal{C}_1$, then there exist $k$ ones in the first $m$ positions of $i(W_2)$. But $i(W_4)$ has no more than $k-d$ ones in the first $m$ positions since $\operatorname{rank}(V_1)+\operatorname{rank}(N_4|\sigma_{V_1}(G_4))=a_1+\operatorname{rank}(N_4|G_4)\leq a_1+a_2-d=k-d$. Thus $\operatorname{dis}(W_4,W_2)\geq d_H(i(W_4),i(W_2))\geq 2d$.

(2) If $W_2\in\mathcal{C}_2$, then $W_2=rs(P_2)=rs(Q_2|A_2)$. The dimension of $W_4\cap W_2$ is $\dim(\{(\alpha_1,\alpha_2):\exists\,(\beta_1,\beta_2) \text{ s.t. } (\alpha_1,\alpha_2)P_4=(\beta_1,\beta_2)P_2\})$, where $\alpha_i$, $\beta_i\in\mathbb{F}_q^{a_i}$ for $i=1,2$. Since $\operatorname{rank}(A_2)=k$ and

$$\operatorname{rank}\begin{pmatrix} N_3 & \sigma_{V_2}(G_3) \\ N_2 & V_2 \end{pmatrix} \begin{array}{l} \le \operatorname{rank}(V_2) + \operatorname{rank}(N_3|\sigma_{V_2}(G_3)) \\ = \operatorname{rank}(V_2) + \operatorname{rank}(N_3|G_3) \\ \le a_2 + a_1 - d \\ = k - d, \end{array}$$

we have

$$\dim(W_4 \cap W_2) \le \dim(\{(\alpha_1, \alpha_2) : \exists\, (\beta_1, \beta_2) \text{ s.t. } (\alpha_1, \alpha_2)\begin{pmatrix} N_3 & \sigma_{V_2}(G_3) \\ N_2 & V_2 \end{pmatrix} = (\beta_1, \beta_2)A_2\}) \le k - d.$$

(3) If $W_2 \in \mathcal{C}_3$, then $W_2 = rs(P_3) = \begin{pmatrix} U_1 & M_1 & \sigma_{U_2}(F_3) & M_3 \\ \sigma_{U_1}(F_4) & M_4 & U_2 & M_2 \end{pmatrix}$. Since $\operatorname{rank}(N_1) \le e_1$, $\operatorname{rank}(N_2) \le e_2$, and $\operatorname{rank}(U_i) = a_i$ for $i = 1, 2$,

$$\begin{aligned} \dim(W_4 \cap W_2) &\le \dim(\{\alpha_1 : \exists\, \beta_1 \text{ s.t. } \alpha_1 U_1 = \beta_1 N_1,\ \alpha_1, \beta_1 \in \mathbb{F}_q^{a_1}\}) \\ &\quad + \dim(\{\alpha_2 : \exists\, \beta_2 \text{ s.t. } \alpha_2 U_2 = \beta_2 N_2,\ \alpha_2, \beta_2 \in \mathbb{F}_q^{a_2}\}) \\ &\le e_1 + e_2 \\ &\le k - d. \end{aligned}$$

By the above discussion, we can conclude that $\operatorname{dis}(W_4, W_2) = \dim(W_4) + \dim(W_2) - 2\ \dim(W_4 \cap W_2) \ge 2k - 2(k - d) = 2d$. Hence, we complete the proof.

***Remark 3.7*** It is easy to see that Construction B is different from Theorem 5 in [20], which we pad more matrix blocks and use subcode construction in Lemma 2.6. Moreover, compared to Lemma 3.1, Construction B inserts more subspaces into $\mathcal{C}_1 \cup \mathcal{C}_2$ for some parameters.

Combining with Lemma 3.1, we have the following result from Theorem 3.6 which improves some lower bounds for CDCs.

**Corollary 3.8** *Let $m, n, d, k, a_1, a_2, b_1, b_2, c_1', c_2', e_1, e_2, t_1$ and $t_2$ be positive integers with $m \ge k$, $n \ge k$, $a_1 + a_2 = k$, $b_1 + b_2 \ge d$, $c_1' + c_2' \ge d$, $e_1 + e_2 \le k - d$, $a_1 + t_1 \le m$, $a_2 + t_2 \le n$, $d \le a_i \le t_i$, $1 \le b_i \le d$ and $1 \le c_i' \le d$ for $i = 1, 2$. Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4'$ be the codes defined in Thoerem* 3.6. *Then*

$$\begin{aligned} A_q(m + n, 2d, k) \ge\ & A_q(m, 2d, k) \cdot \Delta(k, n, d)_q + A_q(n, 2d, k) \cdot \Delta(k, m, d; k - d)_q \\ & + s \cdot A_q(t_1, 2d, a_1) \cdot A_q(t_2, 2d, a_2) \cdot \Delta(a_1, m - t_1, d)_q \cdot \Delta(a_2, n - t_2, d)_q \\ & \cdot \Delta(a_1, n - a_2, d; a_1 - d)_q \cdot \Delta(a_2, m - a_1, d; a_2 - d)_q \\ & + A_q(m - t_1, 2d, a_1) \cdot A_q(n - t_2, 2d, a_2) \cdot \Delta(a_1, n - a_2, d; a_1 - d)_q \\ & \cdot \Delta(a_2, m - a_1, d; a_2 - d)_q \cdot \Big( \Delta(a_1, t_1, d; e_1)_q \cdot \Delta(a_2, t_2, d; e_2)_q \\ & + (l - 1) \cdot \Lambda(a_1, t_1, d; e_1)_q \cdot \Lambda(a_2, t_2, d; e_2)_q \Big). \end{aligned}$$

*Here* $s = \min\left(\frac{\Delta(a_1, m - t_1, b_1)_q}{\Delta(a_1, m - t_1, d)_q}, \frac{\Delta(a_2, n - t_2, b_2)_q}{\Delta(a_2, n - t_2, d)_q}\right)$ *and*

$$l = \min\left(l', \Delta(a_1, t_1, c_1'; e_1)_q - \Delta(a_1, t_1, d; e_1)_q, \Delta(a_2, t_2, c_2'; e_2)_q - \Delta(a_2, t_2, d; e_2)_q\right)$$

*with* $l' = \min\left(\frac{\Delta(a_1, t_1, c_1')_q}{\Delta(a_1, t_1, d)_q}, \frac{\Delta(a_2, t_2, c_2')_q}{\Delta(a_2, t_2, d)_q}\right)$.

***Proof*** By Lemma 2.6, we can construct the desired $\mathcal{M}_1^r$ and $\mathcal{M}_2^r$ MRD codes used to construct $\mathcal{C}_3$ for $1 \le r \le s$. Similarly, for $1 \le r' \le l' = \min\left(\frac{\Delta(a_1,t_1,c_1')_q}{\Delta(a_1,t_1,d)_q}, \frac{\Delta(a_2,t_2,c_2')_q}{\Delta(a_2,t_2,d)_q}\right)$, let $\mathcal{D}_i$ be an $(a_i,t_i,c_i')_q$ MRD code, then we can construct an $(a_i,t_i,d)_q$ MRD code $\mathcal{E}_i^{r'} \subset \mathcal{D}_i$ for $i = 1, 2$. By Lemma 2.6 again, we know that there exists only one linear MRD code in the constructed $(a_i,t_i,= d)_q$ MRD codes for $i = 1, 2$. Without loss of generality, assume $\mathcal{E}_1^1$ (resp. $\mathcal{E}_2^1$) is the unique linear $(a_1,t_1,d)_q$ (resp. $(a_2,t_2,d)_q$) MRD code. Let $\mathcal{N}_i^1 = \{N_i : N_i \in \mathcal{E}_i^{r'}, \mathrm{rank}(N_i) \le e_i\}$ for $i = 1, 2$. Then, $|\mathcal{N}_i^1| = \Delta(a_i,t_i,d;e_i)_q$ for $i = 1, 2$. If $e_1 < d$, let $\mathcal{N}_1^{r'}$ contain only one matrix in $\mathcal{D}_1 \setminus \mathcal{E}_1^1$ with rank at most $e_1$ for $2 \le r'$, otherwise, let $\mathcal{N}_1^{r'} = \{N_1 : N_1 \in \mathcal{E}_1^{r'}, \mathrm{rank}(N_1) \le e_1\}$ for $2 \le r'$. We construct $\mathcal{N}_2^{r'}$ for $2 \le r'$ in the same way. Then $|\mathcal{N}_1^{r'}| = \Lambda(a_1,t_1,d;e_1)_q$ and $|\mathcal{N}_2^{r'}| = \Lambda(a_2,t_2,d;e_2)_q$ for $2 \le r'$. From the construction of $\mathcal{N}_i^{r'}$, we get $l$ $(a_1,t_1,d;e_1)_q$ RRMCs and $l$ $(a_2,t_2,d;e_2)_q$ RRMCs, where

$$l = \min\left(l', \Delta(a_1,t_1,c_1';e_1)_q - \Delta(a_1,t_1,d;e_1)_q, \Delta(a_2,t_2,c_2';e_2)_q - \Delta(a_2,t_2,d;e_2)_q\right).$$

Then from Theorem 3.6, we have $A_q(m+n,2d,k) \ge |\mathcal{C}_1| + |\mathcal{C}_2| + |\mathcal{C}_3| + |\mathcal{C}_4'|$, where $|\mathcal{C}_4'| = \left(|\mathcal{V}_1| \cdot |\mathcal{V}_2| \cdot |\mathcal{N}_3| \cdot |\mathcal{N}_4|\right) \cdot \left(|\mathcal{N}_1^1| \cdot |\mathcal{N}_2^1| + (l-1) \cdot (|\mathcal{N}_1^2| \cdot |\mathcal{N}_2^2|)\right)$. Combining this result with the lower bound of $|\mathcal{C}_1| + |\mathcal{C}_2| + |\mathcal{C}_3|$ in Lemma 3.1, the desired conclusion follows.

***Example 3.9*** We adopt the notation in Corollary 3.8. Take $m = n = k = 8$, $d = 3$, $a_1 = a_2 = 4$, $b_1 = c_1' = 2$, $b_2 = c_2' = 1$, $e_1 = 3$, $e_2 = 2$ and $t_1 = t_2 = 4$. Then $l = q^4$. Take $\mathcal{N}_1^{r'}$ and $\mathcal{N}_2^{r'}$ in the same way as the proof of Corollary 3.8 for $1 \le r' \le l$. Then $|\mathcal{N}_2^{r'}| = 1$ for $2 \le r' \le l$ since $e_2 < d$. Moreover, $A_q(4,6,4) = 1$. Thus, from Corollary 3.8,

$$|\mathcal{C}_4'| = \Delta(4,4,3;1)_q \cdot \Delta(4,4,3;1)_q \cdot \left(\Delta(4,4,3;3)_q \cdot \Delta(4,4,3;2)_q + (q^4-1) \cdot \Lambda(4,4,3;3)_q\right).$$

For $q = 2$, $|\mathcal{C}_4'| = 3601$. Then our construction leads to $A_2(16,6,8) \ge 282927684888529$.

If the parameters $a_2$ and $d$ satisfy $a_2 < 2d$, then $\Delta(a_2, m - a_1, d; a_2 - d) = 1$ by Lemma 2.4. Hence, we can assume that $\mathcal{N}_4$ contains only zero matrix. Based on this assumption, more subspaces can be added into the CDC in Theorem 3.6. Niu et al. further showed that the CDC in Lemma 3.1 can combine with other special subspaces. Hence, for constructing CDCs with larger cardinalities, we quote their result in the following Lemma 3.10 and give the following *Construction C*.

**Lemma 3.10 (Theorem 3.3 in 24)** *With the same notation used in Lemma* 3.1. *In addition, $a_2 < 2d$. Let $\mathcal{D} = \{(D_1|D_2) \in \mathbb{F}_q^{k\times 2(k-d)} : D_1, D_2 \in \mathbb{F}_q^{k\times(k-d)}\}$ be an SC-representation set of a $(2(k-d),2d,k)_q$ CDC. Define $\mathcal{C}_5 = \{rs(O_{k\times(m-k+d)}|D_1|O_{k\times(n-k+d)}|D_2) : (D_1|D_2) \in \mathcal{D}\}$. Assume $x = t_1 - m + k - d \ge 0$ and $y = t_2 - n + k - d \ge 0$. Let $\mathcal{C}_i$ be the code defined in Lemma* 3.1 *for $i = 1, 2, 3$. If $k - x - y \ge d$, then $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_5$ is an $(m+n,2d,k)_q$ CDC.*

**Theorem 3.11 (Construction C)** *With the same notation used in Lemma* 3.1 *and Theorem* 3.6. *In addition, $a_2 < 2d$. Let $\mathcal{C}_4'$ be the code defined in Theorem* 3.6 *and $\mathcal{N}_4 = \{O_{a_2\times(m-a_1)}\}$. Let $\mathcal{C}_5$ be the same as the one in Lemma* 3.10. *Assume $x = t_1 - m + k - d \ge 0$ and $y = t_2 - n + k - d \ge 0$. If $k - x - y \ge d$ and $e_1 + e_2 - x - y \ge d$, then $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4' \cup \mathcal{C}_5$ is an $(m+n,2d,k)_q$ CDC, where $\mathcal{C}_i$ is defined in Lemma* 3.1 *for $1 \le i \le 3$. Consequently, $A_q(m+n,2d,k) \ge |\mathcal{C}_1| + |\mathcal{C}_2| + |\mathcal{C}_3| + |\mathcal{C}_4'| + A_q(2(k-d),2d,k)$.*

***Proof*** From Lemma 3.10, we know that $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_5$ is an $(m+n,2d,k)_q$ CDC. Thus, it remains to show that $\mathcal{C}_4' \cup \mathcal{C}_5$ is also an $(m+n,2d,k)_q$ CDC, i.e., $\mathrm{dis}(\mathcal{C}_4',\mathcal{C}_5) \geq 2d$. Let

$$W_4 = rs\begin{pmatrix} N_1 & v_1 & N_3 & \sigma_{V_2}(G_3) \\ O_{a_2\times t_1} & O_{a_2\times(m-t_1)} & N_2 & V_2 \end{pmatrix} \in \mathcal{C}_4'$$

and $W_5 = rs(O_{k\times(m-k+d)}|D_1|O_{k\times(n-k+d)}|D_2) \in \mathcal{C}_5$ be two $k$-dimensional subspaces with respective identifying vectors $i(W_4)$ and $i(W_5) = (v^m,v^n)$. where $v^m \in \mathbb{F}_2^m$ and $v^n \in \mathbb{F}_2^n$. By Lemma 2.3,

$$\begin{aligned}
\mathrm{dis}(W_4, W_5) &\geq d_H(i(W_4), i(W_5)) \\
&\geq e_1 - x + w_H(v^m) - (a_1 - e_1 + x) + e_2 - y + w_H(v^n) - (a_2 - e_2 + y) \\
&= 2(e_1 + e_2) - 2(x+y) - a_1 - a_2 + w_H(v^m) + w_H(v^n) \\
&= 2(e_1 + e_2 - x - y) \\
&\geq 2d,
\end{aligned}$$

where $w_H(v^m)$ and $w_H(v^n)$ denote the Hamming weights of $v^m$ and $v^n$, respectively. It is easy to check the lower bound of $A_q(m+n,2d,k)$. Thus, we complete the proof.

***Example 3.12*** Take the same values for all parameters used in Example 3.9. Then the maximum cardinality of $\mathcal{C}_5$ is $A_2(10,6,8) = A_2(10,6,2) = 341$ [2]. Combining this result with the lower bound in Example 3.9, we have $A_2(16,6,8) \geq |\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4'| + |\mathcal{C}_5| \geq 282927684888529 + 341 = 282927684888870$, which is better than 282927684887704 in [20].

In the end of this section, we give some new lower bounds for CDCs from Corollary 3.4, Corollary 3.8 and Theorem 3.11 in what follows, where $q \in \{2,3,4,5,7,8,9\}$.

**Corollary 3.13** *With $m = n = 8$, $d = 2$, $k = 8$, $a_1 = a_2 = 4$, $b_1 = b_2 = 1$, $c_1 = c_2 = 1$ and $t_1 = t_2 = 4$, we have*

$$\begin{aligned}
A_q(16,4,8) \geq\ & q^{56} + q^{52} + q^{51} + 2q^{50} + q^{49} + q^{48} - q^{46} - 3q^{45} - 3q^{44} - 3q^{43} + q^{42} + 5q^{41} \\
& + 8q^{40} + 9q^{39} + 10q^{38} + 14q^{37} + 19q^{36} + 21q^{35} + 10q^{34} - 14q^{33} - 49q^{32} \\
& - 74q^{31} - 75q^{30} - 42q^{29} + 11q^{28} + 56q^{27} + 78q^{26} + 66q^{25} + 37q^{24} + 3q^{23} \\
& - 21q^{22} - 28q^{21} - 26q^{20} - 16q^{19} - 5q^{18} + q^{17} + 4q^{16} + 3q^{15} + q^{14}.
\end{aligned}$$

*With $m = 6$, $n = 12$, $d = 2$, $k = 6$, $a_1 = 2$, $a_2 = 4$, $b_1 = b_2 = 1$ $c_1 = c_2 = 1$, $t_1 = 2$ and $t_2 = 8$, we have*

$$\begin{aligned}
A_q(18,4,6) \geq\ & q^{60} + A_q(12,4,6)\big(q^{26} + q^{25} + 2q^{24} + q^{23} + q^{22} - q^{21} - 3q^{20} - 4q^{19} - 3q^{18} \\
& - 2q^{17} + 4q^{15} + 5q^{14} + 5q^{13} + 3q^{12} + q^{11} - q^{10} - 3q^9 - 3q^8 - 2q^7 - q^6\big) \\
& + A_q(8,4,4)\big(q^{28} + q^{27} + 2q^{26} + q^{25} - q^{23} - 2q^{22} - q^{21}\big) + q^{28} + 2q^{27} \\
& + 5q^{26} + 6q^{25} + 7q^{24} + 4q^{23} - 4q^{21} - 7q^{20} - 7q^{19} - 8q^{18} - 7q^{17} - 7q^{16} \\
& - 4q^{15} + 4q^{13} + 7q^{12} + 6q^{11} + 5q^{10} + 2q^9 - q^7 - 2q^6 + q^5.
\end{aligned}$$

*With $m = 6$, $n = 13$, $d = 2$, $k = 6$, $a_1 = 2$, $a_2 = 4$, $t_1 = 2$, $t_2 = 8$, $b_1 = b_2 = 1$, and $c_1 = c_2 = 1$, we have*

**Table 1** New lower bounds of $A_q(n,d,k)$ for $q=2$

| $A_2(n,d,k)$ | New | Old | References |
|---|---|---|---|
| $A_2(16, 4, 8)$ | 61689566206214144 | 61680045647822848 | [20, 24] |
| $A_2(16, 6, 8)$ | 282927684888870 | 282927684887704 | [20] |
| $A_2(18, 4, 6)$ | 1321068381747920544 | 1321068380546107328 | [20, 24] |
| $A_2(19, 4, 6)$ | 42242622286590856096 | 42242622285389042880 | [20, 24] |

$$\begin{aligned}A_q(19,4,6) \geq\ & q^{65} + A_q(13,4,6)\big(q^{26} + q^{25} + 2q^{24} + q^{23} + q^{22} - q^{21} - 3q^{20} - 4q^{19} - 3q^{18}\\ & -2q^{17} + 4q^{15} + 5q^{14} + 5q^{13} + 3q^{12} + q^{11} - q^{10} - 3q^9 - 3q^8 - 2q^7 - q^6\big)\\ & +A_q(8,4,4)\big(q^{31} + q^{30} + 2q^{29} + q^{28} - q^{26} - 2q^{25} - q^{24}\big) + q^{28} + 2q^{27}\\ & +5q^{26} + 6q^{25} + 7q^{24} + 4q^{23} - 4q^{21} - 7q^{20} - 7q^{19} - 8q^{18} - 7q^{17} - 7q^{16}\\ & -4q^{15} + 4q^{13} + 7q^{12} + 6q^{11} + 5q^{10} + 2q^9 - q^7 - 2q^6 - q^5.\end{aligned}$$

*With* $m=n=8$, $d=3$, $k=8$, $a_1=a_2=4$, $t_1=t_2=4$, $b_1=c_1'=2$, $b_2=c_2'=1$, $e_1=3$ *and* $e_2=2$, *we have*

$$\begin{aligned}A_q(16,6,8) \geq\ & q^{48} + q^{39} + q^{38} + 2q^{37} + 3q^{36} + 3q^{35} + 3q^{34} + 2q^{33} - 4q^{31} - 6q^{30} - 10q^{29}\\ & -10q^{28} - 11q^{27} - 7q^{26} - 3q^{25} + 6q^{24} + 12q^{23} + 19q^{22} + 23q^{21} + 25q^{20}\\ & +22q^{19} + 16q^{18} + 9q^{17} - 7q^{15} - 13q^{14} - 15q^{13} - 17q^{12} - 13q^{11} - 11q^{10}\\ & -7q^9 - 4q^8 - 3q^7 - 2q^6 - q^5 - q^4 + 1.\end{aligned}$$

*From Corollary* 3.13, *we can obtain* 28 *new lower bounds for CDCs. In Table* 1, *we list the improved lower bounds for CDCs from our constructions for* $q=2$ *and the previously best known results with the corresponding references. It should be noted that the authors in* [13] *obtained lower bounds of* $A_2(18,4,6)$ *and* $A_2(19,4,6)$, *but Lao et al.* [20] *pointed out that the result in Theorem* 4 *in* [13] *is incorrect.*

## 4 Conclusion

A lot of new lower bounds for CDCs have been given from a variety of constructions. However, it is obvious that there exists still a big gap between the best lower bounds and upper bounds for small parameters $n \leq 19$ and $q \leq 9$ in [2]. In this paper, we present three constructions of constant dimension codes via an appropriate combination of the matrix blocks from rank metric codes and small constant dimension codes. Both *Construction A* and *Construction B* construct CDCs through exchanging the positions of matrix blocks. The *Construction A* provides some new CDCs of distance 4, while the *Construction B* provides some new CDCs of distance 6. The *Construction C* further improves the lower bounds for certain CDCs of distance 6. Our constructions provide 28 improved lower bounds for CDCs.

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