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# The *t*-wise intersection and trellis of relative four-weight codes

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## Abstract
Based on the applications of codes with few weights, we define the so-called relative four-weight codes and present a method for constructing such codes by using the finite projective geometry method. Also, the $t$-wise intersection and the trellis of relative four-weight codes are determined.

**Keywords** Relative four-weight code · Projective space · Support · $t$-wise intersecting · Trellis

**Mathematics Subject Classification (2010)** 94B05

## 1 Introduction

A linear code $C$ of length $n$ is defined as a subspace of $\mathbf{GF}(q)^n$, where $\mathbf{GF}(q)$ is a finite field with $q$ elements, and the code $C$ is called binary when $q = 2$. The $t$-wise intersecting codes and their wide applications were first introduced in [13], and then further studied by Cohen et al. [2, 3] and Encheva et al. [5]. The $t$-wise intersecting codes are a generalization of intersecting ones, which correspond to $t = 2$ and satisfy that any two non-zero codewords have intersecting supports. Finding the judgment criteria and a constructing method for $t$-wise intersecting codes is meaningful research work.

The importance of another concept, the *trellis* of a linear code [6, 15, 17], is in that it can be used to estimate the complexity of the Viterbi decoding algorithm.

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A linear code with few weights [4] is useful in authentication codes, secret sharing schemes, and association schemes apart from its applications in consumer electronics, communication and data storage systems. Many recent papers are dedicated to constructing linear codes with few weights [7, 8, 16] by using the defining set and the technique of exponential sums.

The finite geometry method was first introduced in [1] and [14], and it has been effectively generalized at present to study codes with respect to the rank-metric in [12]. By using the finite geometry method, Liu and Wu [10] provided a technique of constructing codes with few weights, namely, the so-called relative two-weight and three-weight codes. Besides the applications already mentioned for codes with few weights, Liu and Wu further showed that relative two-weight and three-weight codes can be applied to the wire-tap channel of type II with the coset coding scheme. The geometry structures of relative two-weight and three-weight codes were given in [10]. By using these geometry structures, the $t$-wise intersection of relative three-weight codes was calculated in [9], and the trellis of relative two-weight codes was estimated in [11].

Based on the results of relative two-weight and three-weight codes, one can find that these codes have good geometric structures, and by using their geometric structures, one can easily construct these codes and determine some important parameters, say, *generalized Hamming weights*. In fact, by using the geometric structures and the combination techniques, the determination of the general support weight distribution of subcodes of relative two-weight and three-weight codes is also possible. All the present results and observations strongly motivate us to generalize relative two-weight and three-weight codes to codes with four weights, that is, relative four-weight codes. The paper will exactly aim at this goal, and we will define relative four-weight codes and then determine their geometric structures. Also, we will calculate the $t$-wise intersection and estimate the trellis of relative four-weight codes by using the geometric structures we have obtained.

The rest of the paper is organized as follows: Section 2 is devoted to basic definitions and results. The geometric construction of relative four-weight codes is presented in Section 3. The $t$-wise intersection of relative four-weight codes is determined in Section 4. The trellis is estimated in Section 5 and in the Appendix we will exhibit some key lemmas which are used to obtain the $t$-wise intersection of relative four-weight codes.

## 2 Definitions and foundations

The support of a codeword c, denoted by $\chi(c)$, is defined as the set of its non-zero coordinate positions. It is obvious that the Hamming weight of c, denoted by $w(c)$, is $w(c) = |\chi(c)|$. The intersection of the two codewords $c$ and $c'$ is denoted by $\chi(c) \cap \chi(c')$. The $t$-wise intersection of a linear $q$-ary code $C$ is defined as $t = \min\left\{| \cap_{k=1}^{t} \chi(c_k)| \mid c_1, c_2, \cdots, c_t \text{ are any } t \text{ linearly independent codewords}\right\}$ and $C$ is $t$-wise intersecting if $t > 0$. Let $C_1$ be a $k_1$-dimensional subcode of $C$ and $C_2$ be a $k_2$-dimensional subcode of $C$ satisfying $C_1 \subseteq C_2 \subseteq C$, then $C$ is called a *relative three-weight code* concerning $C_1$ and $C_2$, provided that $C_1 \setminus \{0\}$, $C_2 \setminus C_1$ and $C \setminus C_2$ are all constant-weight codes.

**Definition 2.1** Let $C$ be an $[n, k]$ code, and $C_1 \subseteq C_2 \subseteq C_3$ be a $k_1$, $k_2$ and $k_3$-dimensional subcode of $C$, respectively. Then, $C$ is called a relative four-weight code with respect to $C_1$, $C_2$ and $C_3$ and is denoted by $C(w_1, w_2, w_3, w_4)$, if $C_1 \setminus \{0\}$, $C_2 \setminus C_1$, $C_3 \setminus C_2$ and $C \setminus C_3$ are all constant-weight codes with weights $w_1$, $w_2$, $w_3$ and $w_4$, respectively.

*Remark 2.2* In Definition 2.1, if the constant-weight codes $C_3 \setminus C_2$ and $C \setminus C_3$ have the same weight, that is, $w_3 = w_4$, then a relative four-weight code reduces to a relative three-weight one defined in [10]. A relative four-weight code is thus a generalization of a relative three-weight one.

A subcode $D$ of $C$ is called a relative $(r, r_1, r_2, r_3)$ subcode of $C$ with $r_1 \leq r_2 \leq r_3 \leq r$ if $D$ satisfies $\dim D = r$, $\dim(D \cap C_1) = r_1$, $\dim(D \cap C_2) = r_2$ and $\dim(D \cap C_3) = r_3$. Let $<c_1, c_2, \cdots, c_t>$ be the subcode of $C$ generated by $c_1, c_2, \cdots, c_t$. Define $t_i^{\max} = \max\{\dim(<c_1, c_2, \cdots, c_t> \cap C_i)$ such that $c_1, c_2, \cdots, c_t$ are any $t$ linearly independent codewords in $C\}$, $i = 1, 2, 3$. Assume that, $G$ to be a generator matrix of a $k$-dimensional $q$-ary linear code. The columns of $G$ as points of the $(k-1)$- dimensional projective space $\mathbf{PG}(k-1, q)$ such a view point induces a map $m(\cdot)$ from $\mathbf{PG}(k-1, q)$ to the set of non-negative integers: $m : \mathbf{PG}(k-1, q) \to \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \cdots\}$ and for any $p \in \mathbf{PG}(k-1, q)$, $m(p)$ is the number of occurrence of $p$ as a column of $G$. The value of $p$ is denoted by $m(p)$ and the $m(\cdot)$ is called a value assignment or value function. Define the value of $S \subseteq \mathbf{PG}(k-1, q)$ by $m(S) = \Sigma_{p \in S} m(p)$. In addition, let $L \subset \{1, 2, \cdots, k\}$ and $p = \{t_1, t_2, \cdots, t_k\} \in \mathbf{PG}(k-1, q)$, then define $P_L(p) = (v_1, v_2, \cdots, v_k)$, where

$$v_i = \begin{cases} t_i & \text{if } i \in \text{L} \\ 0 & \text{otherwise.} \end{cases}$$

For a subset $W \subset \mathbf{PG}(k-1, q)$, define $P_L(W) = \{P_L(p) \mid p \in W\}$. Clearly, $P_L(W)$ is a vector space, since $W$ is a projective subspace of $\mathbf{PG}(k-1, q)$. For a projective subspaces $V$ of $\mathbf{PG}(k-1, q)$ and integers $l = 1, 2, \cdots, k-1$, we define $V^l = \{p \in V \mid p = 0, 0, \cdots, 0, p_{l+1}, \cdots, p_k\}$. That is, $V^l$ is the set of points of $V$ which are all 0 in the first $l$ co-ordinates. Clearly $V^l$ may be an empty set. If $V^l \neq \varnothing$, then it is a projective subspaces of $V$. Let $L_1 = \{1, 2, \cdots, k_1\}$, $L_2 = \{k_1+1, k_1+2, \cdots, k_2\}$, $L_3 = \{k_2+1, k_2+2, \cdots, k_3\}$. For a non-negative integers $\xi, \eta, \gamma, \delta$, we denote by $P^{\xi}_{\eta\gamma\delta}$, a projective subspace of $V$ of $\mathbf{PG}(k-1, q)$ satisfying $\dim P_{L_1}(V) = \xi - 1$, $\dim P_{L_2}(V^{k_1}) = \eta - 1$, $\dim P_{L_3}(V^{k_2}) = \gamma - 1$ and $\dim P_L(V^{k_3}) = \delta - 1$. Hence, a projective subspace of dimension 0 is a set consisting of a single point and the empty set is viewed as a projective space of dimension $-1$. Therefore, $\dim(P^{\xi}_{\eta\gamma\delta}) = \xi + \eta + \gamma + \delta - 1$.

Let

$$\begin{aligned} P^1_{000} &= \{p \in \mathbf{PG}(k-1, q) \mid P_{L_1}(p) \neq 0\} \\ P^0_{100} &= \{p \in \mathbf{PG}(k-1, q) \mid P_{L_1}(p) = 0, P_{L_2}(p) \neq 0 \quad \& \quad P_{L_3}(p) = 0\} \\ P^0_{010} &= \{p \in \mathbf{PG}(k-1, q) \mid P_{L_3}(p) \neq 0\} \\ P^0_{001} &= \{p \in \mathbf{PG}(k-1, q) \mid P_{L_i}(p) = 0 \text{ for } i = 1, 2, 3\}. \end{aligned}$$

We will show that the values $m(P^1_{000})$, $m(P^0_{100})$, $m(P^0_{010})$ and $m(P^0_{001})$ play an important role in the characterization and construction of the relative four-weight codes.

**Lemma 2.3** *Let $C_1$ be a $k_1$-dimensional subcode of $C$, $C_2$ be a $k_2$-dimensional subcode and $C_3$ be a $k_3$-dimensional subcode satisfying $C_1 \subset C_2 \subset C_3 \subset C$. There is a one-one correspondence between the non-zero codewords $c_1 \in C_1$, $c_2 \in C_2 \setminus C_1$, $c_3 \in C_3 \setminus C_2$ and $c \in C \setminus C_3$ and the subspaces $P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}$, $P^{k_1}_{(k_2-k_1-1)(k_3-k_2)(k-k_3)}$, $P^{k_1}_{(k_2-k_1)(k_3-k_2-1)(k-k_3)}$ and $P^{k_1}_{(k_2-k_1)(k_3-k_2)(k-k_3-1)}$, respectively. The one-one correspondence satisfies that if $c_1$, $c_2$, $c_3$ and $c$ correspond to $P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}$,*

$P^{k_1}_{(k_2-k_1-1)(k_3-k_2)(k-k_3)}$, $P^{k_1}_{(k_2-k_1)(k_3-k_2-1)(k-k_3)}$ *and* $P^{k_1}_{(k_2-k_1)(k_3-k_2)(k-k_3-1)}$, *respectively, then*

$$\begin{aligned} m(\mathbf{PG}(k-1,q)) &= n \\ n-w(c_1) &= m(P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}) \\ n-w(c_2) &= m(P^{k_1}_{(k_2-k_1-1)(k_3-k_2)(k-k_3)}) \\ n-w(c_3) &= m(P^{k_1}_{(k_2-k_1)(k_3-k_2-1)(k-k_3)}) \text{ and} \\ n-w(c) &= m(P^{k_1}_{(k_2-k_1)(k_3-k_2)(k-k_3-1)}). \end{aligned}$$

*Proof* First, we start to prove fourth equation. Let $c_3 \in C_3 \setminus C_2$, then the code $c_3$ is given by $c_3 = (x_1, \cdots, x_{k_1}, x_{k_1+1}, \cdots, x_{k_2}, x_{k_2+1}, \cdots, x_{k_3}, 0, \cdots, 0)G$, where $G$ is a generator matrix of $C$. Assume that the first $k_1$ rows of $G$ generate the subcode $C_1$, the next $(k_2 - k_1)$ rows of $C_1$ and the first $k_1$ rows of $G$ together generate the subcode $C_2$, the next $(k_3 - k_2)$ rows of $G$ and the first $k_2$ rows of $G$ together generate the subcode $C_3$. Since $c_3 \in (C_3 \setminus C_2)$ there exists some $j$ satisfying $k_2 + 1 \leq j \leq k_3$ such that $x_j \neq 0$. Consider the space $U$ of $\mathbf{GF}(q)^k$ which is orthogonal to the vector $(x_1, x_2, \cdots, x_{k_1}, x_{k_1+1}, \cdots, x_{k_2}, x_{k_2+1}, \cdots, x_{k_3}, 0, \cdots, 0)$, Then, $\dim P_{L_1}(U) = k_1 - 1$, $\dim P_{L_2}(U^{k_2}) = k_2 - k_1 - 1$, $\dim P_{L_3}(U^{k_2}) = k_3 - k_2 - 2$ and $\dim P(U^{k_3}) = k - k_3 - 1$. This implies, $U$ is exactly $P^{k_1}_{(k_2-k_1)(k_3-k_2-1)(k-k_3)}$ corresponding to the codeword $c_3$. Therefore, $n - w(c_3) = m(P^{k_1}_{(k_2-k_1)(k_3-k_2-1)(k-k_3)})$. Now, the first equation has to be proved, since the columns of the generator matrix as the points of the projective space $\mathbf{PG}(k-1, q)$ is clear. The proof of the other equation is similar to the above one, hence the proof can be skipped. □

## 3 Construction of relative four-weight codes

In this section, we will present the geometric construction of a relative four-weight code and determine the parameters of a relative four-weight code. Though relative four-weight codes have one weight more than relative three-weight codes, the relations of the projective subspaces become much more complicated than that of relative three-weight codes. The way to get the geometric structure of relative four-weight codes is by induction and by fully using the relations of different projective subspaces.

### 3.1 The determination of the geometric structure

**Theorem 3.1** *Consider $C_1 \subset C_2 \subset C_3 \subset C$ and $C$ be $n$ length linear code and let $C_1$, $C_2$ and $C_3$ be generated by the first $k_1$, $k_2$ and $k_3$ rows of $G$, respectively. Then, $C$ is a relative four-weight code with respect to $C_1$, $C_2$ and $C_3$ if and only if their following value functions are true*

$$\begin{aligned} &m(P^1_{000}) \textit{ is a constant for all points } P^1_{000}, \\ &m(P^0_{100}) \textit{ is a constant for all points } P^0_{100}, \\ &m(P^0_{010}) \textit{ is a constant for all points } P^0_{010} \textit{ and} \\ &m(P^0_{001}) \textit{ is a constant for all points } P^0_{001} \end{aligned}$$

*Proof* Assume that the value function $m(\cdot)$ has same values on $P^1_{000}$, $P^0_{100}$, $P^0_{010}$ and $P^0_{001}$. This implies that the subspaces $P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}$ will have the same value. Since $P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}$ contains the same number of points from the set of points $P^1_{000}$, $P^0_{100}$ $P^0_{010}$ and $P^0_{001}$, respectively. It follows from Lemma 2.3 that all the non-zero codewords of $C_1$ have the same weight. Similarly, we know that all the non-zero codewords of $C_2 \setminus C_1$ have the same weight and all the non-zero codewords of $C_3 \setminus C_2$ and $C \setminus C_3$ have the same weight. Therefore, $C$ is a relative four-weight code with respect to $C_1$, $C_2$ and $C_3$.

Conversely, we assume that $C$ is a relative four-weight code. In order to show that the value function has the same values on the points $P^1_{000}$, $P^0_{100}$, $P^0_{010}$ and $P^0_{001}$, respectively. We will prove the following general result

$$m(P^{\xi}_{\eta\gamma\delta}) = \text{ constant for any fixed } (\xi, \eta, \gamma, \delta). \tag{3.1}$$

The (3.1) is true. If we denote $\xi + \eta + \gamma + \delta = k - j$ for any $j \in \{0, 1, \dots, k-1\}$, the equation in (3.1) will be true. We can prove the theorem by induction on $j$.

For $j = 0$, we have $\xi + \eta + \gamma + \delta = k$ and $P^{\xi}_{\eta\gamma\delta} = \mathbf{PG}(k-1, q)$, so $m(P^{\xi}_{\eta\gamma\delta}) = m(\mathbf{PG}(k-1, q)) = n$.

For $j = 1$, we have $\xi + \eta + \gamma + \delta = k - 1$ and $P^{\xi}_{\eta\gamma\delta}$ is equal to one of the four kinds of subspaces $P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}$, $P^{k_1}_{(k_2-k_1-1)(k_3-k_2)(k-k_3)}$, $P^{k_1}_{(k_2-k_1)(k_3-k_2-1)(k-k_3)}$ and $P^{k_1}_{(k_2-k_1)(k_3-k_2)(k-k_3-1)}$. It follows from Lemma 2.3, $m(P^{\xi}_{\eta\gamma\delta})$ is a constant.

Now, we assume (3.1) is true for $j < j_0$, that is, it is true for any fixed four quatral $(\xi, \eta, \gamma, \delta)$ satisfying $\xi + \eta + \gamma + \delta > k - j_0$. We will show (3.1) is true for $j = j_0$ in the following. For any $P^{\xi}_{\eta\gamma\delta}$ satisfying $\xi + \eta + \gamma + \delta = k - j_0$, there exists a $P^{\xi'}_{\eta'\gamma'\delta'}$ satisfying $\xi' + \eta' + \gamma' + \delta' = k - (j_0 - 3)$ such that $P^{\xi}_{\eta\gamma\delta} \subset P^{\xi'}_{\eta'\gamma'\delta'}$. We may distinguish the parameter into the following cases.

**(Case 1)** 1f $\xi' = \xi + 3$, then $\eta' = \eta$, $\gamma' = \gamma$ and $\delta' = \delta$, since
$m(P^{\xi'}_{\eta'\gamma'\delta'}) = (q+2)m(P^{\xi+1}_{\eta\gamma\delta}) - qm(P^{\xi}_{\eta\gamma\delta})$,
$m(P^{\xi}_{\eta\gamma\delta}) = (\frac{q+2}{q})m(P^{\xi+1}_{\eta\gamma\delta}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$. Thus, $m(P^{\xi}_{\eta\gamma\delta})$ is constant, by the inductive hypothesis.

**(Case 2)** If $\xi' = \xi + 2$, $\eta' = \eta + 1$, then $\delta' = \delta$ and $\gamma' = \gamma$, since
$m(P^{\xi'}_{\eta'\gamma'\delta'}) = qm(P^{\xi+2}_{\eta\gamma\delta}) + m(P^{\xi}_{(\eta+1)\gamma\delta})) - qm(P^{\xi}_{\eta\gamma\delta})$,
$m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi+2}_{\eta\gamma\delta}) + \frac{1}{q}m(P^{\xi}_{(\eta+1)\gamma\delta}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$ which is constant, by the inductive hypothesis.

**(Case 3)** If $\delta' = \delta + 2$, $\eta' = \eta + 1$, then $\xi' = \xi$ and $\gamma' = \gamma$, following the procedure in Case 2, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi}_{\eta\gamma(\delta+2)}) + \frac{1}{q}m(P^{\xi}_{(\eta+1)\gamma\delta}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 4)** If $\gamma' = \gamma + 2$, $\eta' = \eta + 1$, then $\xi' = \xi$ and $\delta' = \delta$, following the procedure in Case 2, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi}_{\eta(\gamma+2)\delta} + \frac{1}{q}m(P^{\xi}_{(\eta+1)\gamma\delta}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 5)** If $\xi' = \xi + 2$, $\gamma' = \gamma + 1$, then $\eta' = \eta$ and $\delta' = \delta$, following the procedure in Case 2, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi+2}_{\eta\gamma\delta}) + \frac{1}{q}m(P^{\xi}_{(\eta+1)\gamma\delta}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 6)** If $\gamma' = \gamma + 2$, $\delta' = \delta + 1$, then $\xi' = \xi$ and $\eta' = \eta$, following the procedure in Case 2, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi}_{\eta(\gamma+2)\delta}) + \frac{1}{q}m(P^{\xi}_{\eta\gamma(\delta+1)}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 7)** If $\xi' = \xi + 2$ $\delta' = \delta + 1$, then $\gamma' = \gamma$ and $\eta' = \eta$, following the procedure in Case 2, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi+2}_{\eta\gamma\delta}) + \frac{1}{q}m(P^{\xi}_{\eta\gamma(\delta+1)}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 8)** If $\xi' = \xi + 1$, $\eta' = \eta + 1$ and $\delta' = \delta + 1$, then $\gamma' = \gamma$. Following the procedure in Case 2, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi+1}_{\eta\gamma\delta}) + \frac{1}{q}m(P^{\xi}_{\eta\gamma\delta}) + m(P^{\xi}_{\eta\gamma(\delta+1)}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 9)** If $\xi' = \xi + 1$, $\eta' = \eta + 1$ and $\gamma' = \gamma + 1$, then $\delta' = \delta$. Following the procedure in Case 2, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi+1}_{\eta\gamma\delta}) - \frac{1}{q}m(P^{\xi}_{(\eta+1)\gamma\delta}) + m(P^{\xi}_{\eta\gamma(\delta+1)}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 10)** If $\xi' = \xi + 1$, $\gamma' = \gamma + 1$ and $\delta' = \delta + 1$ , then $\eta' = \eta$. Following the procedure in Case 2, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi+1}_{\eta\gamma\delta}) - \frac{1}{q}m(P^{\xi}_{\eta(\gamma+1)\delta}) + m(P^{\xi}_{\eta\gamma(\delta+1)}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 11)** If $\xi' = \xi$, $\eta' = \eta + 1$, $\gamma' = \gamma + 1$ and $\delta' = \delta + 1$, following the procedure in Case 2, then we obtain $m(P^{\xi}_{\eta\gamma\delta}) = m(P^{\xi}_{(\eta+1)\gamma\delta}) - \frac{1}{q}m(P^{\xi}_{\eta(\gamma+1)\delta}) + m(P^{\xi}_{\eta\gamma(\delta+1)}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 12)** If $\xi' = \xi$, $\eta' = \eta$, $\gamma' = \gamma$, then $\delta' = \delta + 3$. Following the procedure in Case 1, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = (\frac{q+2}{q})m(P^{\xi}_{\eta\gamma(\delta+1)}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 13)** If $\xi' = \xi$, $\eta' = \eta$, $\delta' = \delta$, then $\gamma' = \gamma + 3$. Following the procedure in Case 1, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = (\frac{q+2}{q})m(P^{\xi}_{\eta(\gamma+1)\delta}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

**(Case 14)** If $\xi' = \xi$, $\gamma' = \gamma$, $\delta' = \delta$, then $\eta' = \eta + 3$. Following the procedure in Case 1, we obtain $m(P^{\xi}_{\eta\gamma\delta}) = (\frac{q+2}{q})m(P^{\xi}_{(\eta+1)\gamma\delta}) - \frac{1}{q}m(P^{\xi'}_{\eta'\gamma'\delta'})$, which is constant.

Hence, this is true for $j = j_0$. The theorem is proved by the induction hypothesis.

□

*Remark 3.2* Theorem 3.1 may be viewed as an effective generalization of the main result in [10], and it plays a key role in constructing a relative four-weight code and the calculation of the $t$-wise intersection and the trellis in later sections.

### 3.2 The parameters of a relative four-weight code

From Theorem 3.1, we can construct a generator matrix $G$ for a relative four-weight code as follows: choose the appropriate $k$-dimensional column vectors over $\mathbf{GF}(q)$(or equivalently, points of $\mathbf{PG}(k-1, q)$) and use them as the columns of $G$, such that $m(P^1_{000})$, $m(P^0_{100})$, $m(P^0_{010})$ and $m(P^0_{001})$ are all constant, respectively. Then, the code $C$ by $G$ is a generator matrix for relative four-weight code. Let $k_1$, $k_2$ and $k_3$ be any positive integers such that $k_1 < k_2 < k_3 \leq k$. Taking $L_1 = \{1, 2, \cdots, k_1\}$, $L_2 = \{k_1+1, \cdots, k_2\}$ and $L_3 = \{k_2+1, \cdots, k_3\}$. We know that, there are exactly $\frac{q^k - q^{k-k_1}}{q-1}$ points $P^1_{000}$, $\frac{q^{k-k_1} - q^{k-k_2}}{q-1}$ points $P^0_{100}$, $\frac{q^{k-k_2} - q^{k-k_3}}{q-1}$ points $P^0_{010}$ and $\frac{q^{k-k_3} - 1}{q-1}$ points $P^0_{001}$ and $G$ is a generator matrix with $n$ columns, the length of $C$ is given by $n$, where

$$n = \frac{q^k - q^{k-k_1}}{q-1}m(P^1_{000}) + \frac{q^{k-k_1} - q^{k-k_2}}{q-1}m(P^0_{100}) + \frac{q^{k-k_2} - q^{k-k_3}}{q-1}m(P^0_{010}) + \frac{q^{k-k_3} - 1}{q-1}m(P^0_{001}).$$

From Lemma 2.3, we get

$$w(c_1) = n - m(P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}).$$

It is clear that the projective subspace $P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}$ contains $\frac{q^{k-1}-q^{k-k_1}}{q-1}$ points $P^1_{000}$, $\frac{q^{k-k_1}-q^{k-k_2}}{q-1}$ points $P^0_{100}$, $\frac{q^{k-k_2}-q^{k-k_3}}{q-1}$ points $P^0_{010}$ and $\frac{q^{k-k_3}-1}{q-1}$ points $P^0_{001}$. Thus,

$$\begin{aligned} m(P^{k_1-1}_{(k_2-k_1)(k_3-k_2)(k-k_3)}) &= \frac{q^{k-1}-q^{k-k_1}}{q-1} m(P^1_{000}) + \frac{q^{k-k_1}-q^{k-k_2}}{q-1} m(P^0_{100}) \\ &\quad + \frac{q^{k-k_2}-q^{k-k_3}}{q-1} m(P^0_{010}) + \frac{q^{k-k_3}-1}{q-1} m(P^0_{001}). \end{aligned}$$

From the above two equations, we have

$$\begin{aligned} w_1 &= w(c_1) \\ &= q^{k-1} m(P^1_{000}), \text{ for all } c_1 \in C_1. \end{aligned}$$

Similarly, apply the above method, and we arrive

$$\begin{aligned} w_2 &= (q^{k-1} - q^{k-k_1-1}) m(P^1_{000})) + (q^{k-k_1-1}) m(P^0_{100}), \\ w_3 &= (q^{k-1} - q^{k-k_1-1}) m(P^1_{000}) + (q^{k-k_1-1} - q^{k-k_2-1}) m(P^0_{100}) \\ &\quad + (q^{k-k_2-1}) m(P^0_{010}), \end{aligned}$$

and

$$\begin{aligned} w_4 &= (q^{k-1} - q^{k-k_1-1}) m(P^1_{000}) + (q^{k-k_1-1} - q^{k-k_2-1}) m(P^0_{100}) + (q^{k-k_2-1} - q^{k-k_3-1}) m(P^0_{010}) \\ &\quad + (q^{k-k_3-1}) m(P^0_{001}). \end{aligned}$$

**Lemma 3.3** *Let $C(w_1, w_2, w_3, w_4)$ be a relative four-weight code with respect to a $k_1$-dimensional subcode $C_1$, $k_2$-dimensional subcode $C_2$ and $k_3$-dimensional subcode $C_3$, and let $G$ and $m(\cdot)$ be generator matrix and value function, respectively. Then, $m(\cdot)$ satiesfies*

$$m(p) = \begin{cases} \frac{w_1}{q^{k-1}}, & \text{for } p \in S_1, \\ \frac{q^{k_1} w_2 - (q^{k_1}-1) w_1}{q^{k-1}}, & \text{for } p \in S_2, \\ \frac{q^{k_2} w_3 - (q^{k_1}-1) w_1 - (q^{k_2}-q^{k_1}) w_2}{q^{k-1}}, & \text{for } p \in S_3, \\ \frac{q^{k_3} w_4 - (q^{k_1}-1) w_1 - (q^{k_2}-q^{k_1}) w_2 - (q^{k_3}-q^{k_2}) w_3}{q^{k-1}}, & \text{for } p \in S_4, \end{cases} \tag{3.2}$$

*where $S_i \subset \mathbf{PG}(k-1, q)$ for $1 \le i \le 4$ and $S_1 = \{p \mid P_{L_1} \neq 0\}$, $S_2 = \{p \mid P_{L_1} = 0,\ P_{L_2} \neq 0\ \&\ P_{L_3} = 0\}$, $S_3 = \{p \mid P_{L_3} \neq 0\}$ and $S_4 = \{p \mid P_{L_1} = P_{L_2} = P_{L_3} = 0\}$, where $L_1 = \{1, 2, \cdots, k_1\}$, $L_2 = \{k_1+1, \cdots, k_2\}$ and $L_3 = \{k_2+1, \cdots, k_3\}$.*

*Proof* From the above geometric construction, we have

$$w_1 = w(c_1) = q^{k-1} m(P^1_{000}), \text{ for all } c_1 \in C_1.$$

Then, it will be

$$m(P^1_{000}) = \frac{w_1}{q^{k-1}},$$

again, we have

$$w_2 = (q^{k-1} - q^{k-k_1-1})m(P^1_{000}) + (q^{k-k_1-1})m(P^0_{100}).$$

Substituting the value of $m(P^1_{000})$ into the above equation, we get

$$m(P^0_{100}) = \frac{q^{k_1}w_2 - w_1(q^{k_1} - 1)}{q^{k-1}}.$$

Similarly,

$$w_3 = (q^{k-1} - q^{k-k_1-1})m(P^1_{000}) + (q^{k-k_1-1} - q^{k-k_2-1})m(P^0_{100}) + (q^{k-k_2-1})m(P^0_{010}).$$

Substituting the values of $m(P^1_{000})$ and $m(P^0_{010})$ into the above equation, and after simplification, we arrive

$$m(P^0_{010}) = \frac{q^{k_2}w_3 - (q^{k_1} - 1)w_1 - (q^{k_2} - q^{k_1})w_2}{q^{k-1}}.$$

Again,

$$\begin{aligned} w_4 = {} & (q^{k-1} - q^{k-k_1-1})m(P^1_{000}) + (q^{k-k_1-1} - q^{k-k_2-1}) \\ & m(P^0_{100}) + (q^{k-k_2-1} - q^{k-k_3-1})m(P^0_{010}) + (q^{k-k_3-1})m(P^0_{001}), \end{aligned}$$

and substituting the above all value functions, finally we get

$$m(P^0_{001}) = \frac{q^{k_3}w_4 - (q^{k_1} - 1)w_1 - (q^{k_2} - q^{k_1})w_2 - (q^{k_3} - q^{k_2})w_3}{q^{k-1}}.$$

□

*Example 3.4* Consider a value function

$$m(p) = \begin{cases} 1 \text{ if } p \in S_1, \\ 3 \text{ if } p \in S_2, \\ 4 \text{ if } p \in S_3, \\ 6 \text{ if } p \in S_4, \end{cases}$$

for $q = 2$ and let $k = 5$, $k_1 = 2$, $k_2 = 3$, $k_3 = 4$. Then, we have $w_1 = 16$, $w_2 = 24$, $w_3 = 26$ and $w_4 = 28$, where $L_1 = \{1, 2\}$, $L_2 = \{3\}$ and $L_3 = \{4\}$.

By the above procedure, we generate the generator matrix $G$ as follows.

$$\left[\begin{matrix} 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0 \\ 0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&0&0 \\ 1&1&1&0&0&1&1&0&1&1&1&1&1&1&1&1&0&1&0&1&1&1&0&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&0&1&1&0&1&0&1&1&1&1&1&1&1&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0 \\ 1&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1 \\ 0&1&1&0&0&0&0&0&0&0&1&1&0&1&1&1&0 \\ 0&0&0&1&1&1&1&1&1&1&1&1&1&1&0&0&0 \end{matrix}\right].$$

Note that each of the 24 points $P^1_{000}$ appears in $G$ once; each of the 4 points $P^0_{100}$ appears three times; each of the 2 points $P^0_{010}$ appears in $G$ four times and there is only one point $P^0_{001}$ which appears in $G$ for six times.

*Remark 3.5* Based on the above geometric construction and by borrowing the method in [10], one can show that relative four-weight codes will behave similarly as relative three-weight codes in that they are optimal in certain cases in the wire-tap channel II. Further, since relative four-weight codes have only four-weights and the weight distribution is clear also based on the geometric structure, they can also be applied to secret sharing schemes based on linear codes [4], and we omit these detailed arguments.

## 4 The intersection of relative four-weight codes

The $t$-wise intersection of a linear code is in general difficult to calculate. By using the geometric structure, the $t$-wise intersection of binary relative three-weight codes is obtained in [9]. Since we have gotten the geometric structure of a relative four-weight code, we also expect to calculate the $t$-wise intersection of a relative four-weight code. However, a relative four-weight code has more complicated structure than a relative three-weight code, which leads to the complexity of $t$ linearly independent codewords. It is thus tedious to get the $t$-wise intersection of a relative four-weight code, and we will have to classify the analysis into many cases, and in each case we will generalize the method in [9] by developing the skill of the matrix operation. The novelty of our work is in that in each case we will construct an invertible matrix which is a product of invertible matrices and expanding the generator matrix, which leads to the $t$-wise intersection of binary relative four-weight codes.

**Lemma 4.1** [9] *The $t$-wise intersection of a linear constant-weight code $w$ is equal to $(\frac{q-1}{q})^{t-1}w$.*

As in the case of relative three-weight codes, it is a key to construct the generator matrix of linearly independent codewords. By organize the $t$ linearly independent codewords $c_1, c_2, c_3, \cdots, c_t$ into a matrix form $T_{t\times n}$, we can get the $t$-wise intersection of four-weight codes as below.

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_t \end{bmatrix} = X_{t\times k}G = (X_{t\times k_1}, X_{t\times (k-k_1)}) \begin{bmatrix} G_{k_1\times n} \\ G_{(k-k_1)\times n} \end{bmatrix}$$

$$= (X_{t\times k_2}, X_{t\times (k-k_2)}) \begin{bmatrix} G_{k_2\times n} \\ G_{(k-k_2)\times n} \end{bmatrix}$$

$$= (X_{t\times k_3}, X_{t\times (k-k_3)}) \begin{bmatrix} G_{k_3\times n} \\ G_{(k-k_3)\times n} \end{bmatrix}.$$

Note that $rank(X_{t\times k}) = t$, and that the block matrices $G_{k_1\times n}$, $G_{k_2\times n}$ and $G_{k_3\times n}(k_1 < k_2 < k_3)$ are generator matrices of $C_1$, $C_2$ and $C_3$, respectively.

**Lemma 4.2** *Let $C$ be a relative four-weight code and it has the subcode $D = <c_1, c_2, \cdots, c_t>$ is a relative $(t, t_1, t_2, t_3)$ code, then $rank(X_{t\times(k-k_1)}) = t - t_1$, $rank(X_{t\times(k-k_2)}) = t - t_2$ and $rank(X_{t\times(k-k_3)}) = t - t_3$.*

*Proof* Since $< c_1, c_2, \cdots, c_t >$ is a relative $(t, t_1, t_2, t_3)$ subcode, there is an invertible matrix $Y_{t\times t}$ such that

$$\begin{aligned}
Y_{t\times t}X_{t\times k} &= (Y_{t\times t}X_{t\times k_1}, Y_{t\times t}X_{t\times(k-k_1)})\\
&= (Y_{t\times t}X_{t\times k_2}, Y_{t\times t}X_{t\times(k-k_2)})\\
&= (Y_{t\times t}X_{t\times k_3}, Y_{t\times t}X_{t\times(k-k_3)})\\
&= \begin{bmatrix} X'_{t_1\times k_1} & 0_{t_1\times(k-k_1)} \\ X'_{(t-t_1)\times k_1} & X'_{(t-t_1)\times(k-k_1)} \end{bmatrix}\\
&= \begin{bmatrix} X'_{t_2\times k_2} & 0_{t_1\times(k_2-k_1)} & 0_{t_1\times(k-k_2)} \\ X''_{(t_2-t_1)\times k_1} & X''_{(t_2-t_1)\times(k_2-k_1)} & 0_{(t_2-t_1)\times(k-k_2)} \\ X''_{(t-t_2)\times k_1} & X''_{(t-t_2)\times(k_2-k_1)} & X''_{(t-t_2)\times(k-k_2)} \end{bmatrix}\\
&= \begin{bmatrix} X'_{t_3\times k_3} & 0_{t_1\times(k_2-k_1)} & 0_{t_1\times(k_3-k_2)} & 0_{t_1\times(k-k_3)} \\ X''_{(t_2-t_1)\times k_1} & X''_{(t_2-t_1)\times(k_2-k_1)} & 0_{(t_2-t_1)\times(k_2-k_1)} & 0_{(t_2-t_1)\times(k-k_3)} \\ X''_{(t_3-t_2)\times k_1} & X''_{(t_3-t_2)\times(k_2-k_1)} & X''_{(t_3-t_2)\times(k_3-k_2)} & 0_{(t_3-t_2)\times(k-k_3)} \\ X'''_{(t-t_3)\times k_1} & X'''_{(t-t_3)\times(k_2-k_1)} & X'''_{(t-t_3)\times(k_3-k_2)} & X'''_{(t-t_3)\times(k-k_3)} \end{bmatrix},
\end{aligned}$$

with $rank(X'_{t_1\times k_1}) = t_1$, $rank(X'_{(t-t_1)\times(k-k_1)}) = t - t_1$, $rank(X''_{(t-t_2)\times(k-k_2)}) = t - t_2$ and $rank(X'''_{(t-t_3)\times(k-k_3)}) = t - t_3$. Therefore, we have

$$\begin{aligned}
rank(X_{t\times(k-k_1)}) &= rank(Y_{t\times t})X_{t\times(k-k_1)}\\
&= rank(X'_{(t-t_1)\times(k-k_1)})\\
&= t - t_1,\\
rank(X_{t\times(k-k_2)}) &= rank(Y_{t\times t})X_{t\times(k-k_2)}\\
&= rank(X''_{(t-t_2)\times(k-k_2)})\\
&= t - t_2,\\
rank(X_{t\times(k-k_3)}) &= rank(Y_{t\times t})X_{t\times(k-k_3)}\\
&= rank(X'''_{(t-t_3)\times(k-k_3)})\\
&= t - t_3.
\end{aligned}$$

□

The $t$-wise intersecting of the relative four-weight code $C(w_1, w_2, w_3, w_4)$ is closely related to the size of $w_1$, $w_2$, $w_3$ and $w_4$. Denote $m_j = m(p_j)$ for $j = 1, 2, 3, 4$, where

$p_j \in S_j$ for every $j$. The above notation is same as in Lemma 3.3 and from (3.2), we get the following

$$\begin{aligned} w_1 &= m_1 q^{k-1}, \\ w_1 - w_2 &= q^{k-k_1-1}(m_1 - m_2), \\ w_2 - w_3 &= q^{k-k_2-1}(m_2 - m_3), \\ w_3 - w_4 &= q^{k-k_3-1}(m_3 - m_4). \end{aligned}$$

We move towards the calculation of the $t$-wise intersection of relative four-weight codes. It is based on the relation among $w_1$, $w_2$, $w_3$ and $w_4$. According to these relations, we altogether get twenty four cases, in which cases 1 and 2 are completely different from the rest of the cases. So that, they required a separate calculation analysis. The remaining cases can be grouped into six major classes. The first major class cases (3, 4, 5, 6 and 7) with first key lemma, the second major class cases (8, 9 and 10) with second key lemma, the third major class cases (11, 12, 13, 14 and 15) with third key lemma, the fourth major class cases (16, 17 and 18) with fourth key lemma, the fifth major class cases (19, 20 and 21) with fifth key lemma and the sixth major class cases (22, 23 and 24) with sixth key lemma followed by the other cases, can be proved respectively. To avoid the tedious procedure, all the key lemmas are presented in the Appendix to make the work easier.

Now, we classify the cases of the calculation of the $t$-wise intersection as follows.

**Theorem 4.3** *The $t$-wise intersection of relative four-weight codes $C(w_1, w_2, w_3, w_4)$ is equal to*

1. $(\frac{1}{2})^{t-1} w_1, \quad w_4 > w_3 > w_2 > w_1.$

2. $$\begin{cases} (\frac{1}{2})^{t-1} w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1 - w_2) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2 - w_3) \\ -(\frac{1}{2})^{t-t_3^{\max}-1}(w_3 - w_4), & \begin{cases} t_3^{\max} < t \\ w_1 > w_2 > w_3 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1} w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1 - w_2) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2 - w_3) \\ -(w_3 - w_4), & \begin{cases} t_3^{\max} = t, \ t_2^{\max} < t \\ w_1 > w_2 > w_3 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1} w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1 - w_2) - (w_2 - w_3) \\ -(w_3 - w_4), & \begin{cases} t_3^{\max} = t_2^{\max} = t, \ t_1^{\max} < t \\ w_1 > w_2 > w_3 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1} w_1 - (w_1 - w_4), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_1 > w_2 > w_3 > w_4. \end{cases} \end{cases}$$

3. $$\begin{cases} \min\{(\frac{1}{2})^{t-1} w_1 - (\frac{1}{2})^{t-t_3^{\max}-1}(w_4 - w_3); (\frac{1}{2})^{t-1} w_1\}, & \begin{cases} t_3^{\max} < t \\ w_2 > w_1 > w_4 > w_3 \end{cases} \\ \min\{(\frac{1}{2})^{t-1} w_1 - (w_4 - w_3); (\frac{1}{2})^{t-1} w_1\}, & \begin{cases} t_3^{\max} = t \\ w_2 > w_1 > w_4 > w_3 \end{cases} \end{cases}$$

4. $$\begin{cases} \min\{(\frac{1}{2})^{t-1} w_1 - (\frac{1}{2})^{t-t_3^{\max}-1}(w_4 - w_3); (\frac{1}{2})^{t-1} w_1\}, & \begin{cases} t_3^{\max} < t \\ w_2 > w_4 > w_1 > w_3 \end{cases} \\ \min\{(\frac{1}{2})^{t-1} w_1 - (w_4 - w_3); (\frac{1}{2})^{t-1} w_1\}, & \begin{cases} t_3^{\max} = t \\ w_2 > w_4 > w_1 > w_3 \end{cases} \end{cases}$$

$$5.\quad \begin{cases} \min\{(\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3);\, (\frac{1}{2})^{t-1}w_1\}, & \begin{cases} t_3^{\max} < t \\ w_2 > w_4 > w_3 > w_1 \end{cases} \\ \min\{(\frac{1}{2})^{t-1}w_1 - (w_4-w_3);\, (\frac{1}{2})^{t-1}w_1\}, & \begin{cases} t_3^{\max} = t \\ w_2 > w_4 > w_3 > w_1 \end{cases} \end{cases}$$

$$6.\quad \begin{cases} \min\{(\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3);\, (\frac{1}{2})^{t-1}w_1\}, & \begin{cases} t_3^{\max} < t \\ w_4 > w_2 > w_3 > w_1 \end{cases} \\ \min\{(\frac{1}{2})^{t-1}w_1 - (w_4-w_3);\, (\frac{1}{2})^{t-1}w_1\}, & \begin{cases} t_3^{\max} = t \\ w_4 > w_2 > w_3 > w_1 \end{cases} \end{cases}$$

$$7.\quad \begin{cases} \min\{(\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3);\, (\frac{1}{2})^{t-1}w_1\}, & \begin{cases} t_3^{\max} < t \\ w_4 > w_2 > w_1 > w_3 \end{cases} \\ \min\{(\frac{1}{2})^{t-1}w_1 - (w_4-w_3);\, (\frac{1}{2})^{t-1}w_1\}, & \begin{cases} t_3^{\max} = t \\ w_4 > w_2 > w_1 > w_3 \end{cases} \end{cases}$$

$$8.\quad \begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (\frac{1}{2})^{t-t_2-1}(w_3-w_2), & \text{if } t_2^{\max} < t, t_1^{\max} < t \text{ and } w_3 > w_4 > w_2 > w_1 \\ (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (w_3-w_2), & \text{if } t_2^{\max} = t, t_1^{\max} < t \text{ and } w_3 > w_4 > w_2 > w_1 \\ (\frac{1}{2})^{t-1}w_1 + (w_2-w_1) - (w_3-w_2), & \text{if } t_2^{\max} = t_1^{\max} = t \text{ and } w_3 > w_4 > w_2 > w_1. \end{cases}$$

$$9.\quad \begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (\frac{1}{2})^{t-t_2-1}(w_3-w_2), & \text{if } t_2^{\max} < t, t_1^{\max} < t \text{ and } w_3 > w_2 > w_1 > w_4 \\ (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (w_3-w_2), & \text{if } t_2^{\max} = t, t_1^{\max} < t \text{ and } w_3 > w_2 > w_1 > w_4 \\ (\frac{1}{2})^{t-1}w_1 + (w_2-w_1) - (w_3-w_2), & \text{if } t_2^{\max} = t_1^{\max} = t \text{ and } w_3 > w_2 > w_1 > w_4. \end{cases}$$

$$10.\quad \begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (\frac{1}{2})^{t-t_2-1}(w_3-w_2), & \text{if } t_2^{\max} < t, t_1^{\max} < t \text{ and } w_3 > w_2 > w_4 > w_1 \\ (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (w_3-w_2), & \text{if } t_2^{\max} = t, t_1^{\max} < t \text{ and } w_3 > w_2 > w_4 > w_1 \\ (\frac{1}{2})^{t-1}w_1 + (w_2-w_1) - (w_3-w_2), & \text{if } t_2^{\max} = t_1^{\max} = t \text{ and } w_3 > w_2 > w_4 > w_1. \end{cases}$$

$$11.\quad \begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) & \\ +(\frac{1}{2})^{t-t_3^{\max}-1}(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} < t \\ w_1 > w_3 > w_2 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) & \\ +(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} = t,\ t_2^{\max} < t \\ w_1 > w_3 > w_2 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_3-w_2) & \\ +(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} = t_2^{\max} = t,\ t_1^{\max} < t \\ w_1 > w_3 > w_2 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_3-w_4) - (w_1-w_3), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_1 > w_3 > w_2 > w_4. \end{cases} \end{cases}$$

$$12.\quad \begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) & \\ +(\frac{1}{2})^{t-t_3^{\max}-1}(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} < t \\ w_1 > w_3 > w_4 > w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) & \\ +(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} = t,\ t_2^{\max} < t \\ w_1 > w_3 > w_4 > w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_3-w_2) & \\ +(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} = t_2^{\max} = t,\ t_1^{\max} < t \\ w_1 > w_3 > w_4 > w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_3-w_4) - (w_1-w_3), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_1 > w_3 > w_4 > w_2. \end{cases} \end{cases}$$

$$13.\quad \begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) & \\ +(\frac{1}{2})^{t-t_3^{\max}-1}(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} < t \\ w_3 > w_1 > w_2 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) & \\ +(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} = t_2^{\max} < t \\ w_3 > w_1 > w_2 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_3-w_2) & \\ +(w_3-w_4) - (\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max} = t,\ t_2^{\max} = t,\ t_1^{\max} < t \\ w_3 > w_1 > w_2 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_3-w_4) - (w_1-w_3), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_3 > w_1 > w_2 > w_4. \end{cases} \end{cases}$$

14. $$\begin{cases} (\frac{1}{2})^{t-1}w_1+(\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ +(\frac{1}{2})^{t-t_3^{\max}-1}(w_3-w_4)-(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max}<t \\ w_3>w_1>w_4>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ +(w_3-w_4)-(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max}=t,\ t_2^{\max}<t \\ w_3>w_1>w_4>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(w_3-w_2) \\ +(w_3-w_4)-(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max}=t_2^{\max}=t,\ t_1^{\max}<t \\ w_3>w_1>w_4>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(w_3-w_4)-(w_1-w_3), & \begin{cases} t_3^{\max}=t_2^{\max}=t_1^{\max}=t \\ w_3>w_1>w_4>w_2. \end{cases} \end{cases}$$

15. $$\begin{cases} (\frac{1}{2})^{t-1}w_1+(\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ +(\frac{1}{2})^{t-t_3^{\max}-1}(w_3-w_4)-(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max}<t \\ w_3>w_4>w_1>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ +(w_3-w_4)-(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max}=t,\ t_2^{\max}<t \\ w_3>w_4>w_1>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(w_3-w_2) \\ +(w_3-w_4)-(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2), & \begin{cases} t_3^{\max}=t_2^{\max}=t,\ t_1^{\max}<t \\ w_3>w_4>w_1>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(w_3-w_4)-(w_1-w_3), & \begin{cases} t_3^{\max}=t_2^{\max}=t_1^{\max}=t \\ w_3>w_4>w_1>w_2. \end{cases} \end{cases}$$

16. $$\begin{cases} (\frac{1}{2})^{t-1}w_1+(\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2)-(\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3), & \begin{cases} t_3^{\max}<t \\ w_1>w_4>w_3>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2)-(w_4-w_3), & \begin{cases} t_3^{\max}=t,\ t_2^{\max}<t \\ w_1>w_4>w_3>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2)-(w_4-w_3), & \begin{cases} t_3^{\max}=t_2^{\max}=t,\ t_1^{\max}<t \\ w_1>w_4>w_3>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2)-(w_4-w_3), & \begin{cases} t_3^{\max}=t_2^{\max}=t_1^{\max}=t \\ w_1>w_4>w_3>w_2. \end{cases} \end{cases}$$

17. $$\begin{cases} (\frac{1}{2})^{t-1}w_1+(\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2)-(\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3), & \begin{cases} t_3^{\max}<t \\ w_4>w_1>w_3>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2)-(w_4-w_3), & \begin{cases} t_3^{\max}=t,\ t_2^{\max}<t \\ w_4>w_1>w_3>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2)-(w_4-w_3), & \begin{cases} t_3^{\max}=t_2^{\max}=t,\ t_1^{\max}<t \\ w_4>w_1>w_3>w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1+(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2)-(w_4-w_3), & \begin{cases} t_3^{\max}=t_2^{\max}=t_1^{\max}=t \\ w_4>w_1>w_3>w_2. \end{cases} \end{cases}$$

18. $$\begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3), & \begin{cases} t_3^{\max} < t \\ w_1 > w_4 > w_3 > w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_2^{\max}-1}(w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (w_4-w_3), & \begin{cases} t_3^{\max} = t,\ t_2^{\max} < t \\ w_1 > w_4 > w_3 > w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (w_4-w_3), & \begin{cases} t_3^{\max} = t_2^{\max} = t,\ t_1^{\max} < t \\ w_1 > w_4 > w_3 > w_2 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_3-w_2) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (w_4-w_3), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_1 > w_4 > w_3 > w_2. \end{cases} \end{cases}$$

19. $$\begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2-w_3), & \begin{cases} t_3^{\max} < t \\ w_1 > w_4 > w_2 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2-w_3), & \begin{cases} t_3^{\max} = t,\ t_2^{\max} < t \\ w_1 > w_4 > w_2 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (w_2-w_3), & \begin{cases} t_3^{\max} = t_2^{\max} = t,\ t_1^{\max} < t \\ w_1 > w_4 > w_2 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_1), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_1 > w_4 > w_2 > w_3. \end{cases} \end{cases}$$

20. $$\begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2-w_3), & \begin{cases} t_3^{\max} < t \\ w_1 > w_2 > w_4 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2-w_3), & \begin{cases} t_3^{\max} = t,\ t_2^{\max} < t \\ w_1 > w_2 > w_4 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (w_2-w_3), & \begin{cases} t_3^{\max} = t_2^{\max} = t,\ t_1^{\max} < t \\ w_1 > w_2 > w_4 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_1), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_1 > w_2 > w_4 > w_3. \end{cases} \end{cases}$$

21. $$\begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_3^{\max}-1}(w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2-w_3), & \begin{cases} t_3^{\max} < t \\ w_4 > w_1 > w_2 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (\frac{1}{2})^{t-t_2-1}(w_2-w_3), & \begin{cases} t_3^{\max} = t,\ t_2^{\max} < t \\ w_4 > w_1 > w_2 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_3) \\ -(\frac{1}{2})^{t-t_1^{\max}-1}(w_1-w_2) - (w_2-w_3), & \begin{cases} t_3^{\max} = t_2^{\max} = t,\ t_1^{\max} < t \\ w_4 > w_1 > w_2 > w_3 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 + (w_4-w_1), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_4 > w_1 > w_2 > w_3. \end{cases} \end{cases}$$

22. $$\begin{cases} (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2 - w_3) \\ -(\frac{1}{2})^{t-t_3^{\max}-1}(w_3 - w_4), & \begin{cases} t_3^{\max} < t \\ w_2 > w_1 > w_3 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2 - w_3) \\ -(w_3 - w_4), & \begin{cases} t_3^{\max} = t, \ t_2^{\max} < t \\ w_2 > w_1 > w_3 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (w_2 - w_4), & \begin{cases} t_3^{\max} = t_2^{\max} = t, \ t_1^{\max} < t \\ w_2 > w_1 > w_3 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (w_2 - w_1) - (w_2 - w_4), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_2 > w_1 > w_3 > w_4. \end{cases} \end{cases}$$

23. $$\begin{cases} (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2 - w_3) \\ -(\frac{1}{2})^{t-t_3^{\max}-1}(w_3 - w_4), & \begin{cases} t_3^{\max} < t \\ w_2 > w_3 > w_1 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2 - w_3) \\ -(w_3 - w_4), & \begin{cases} t_3^{\max} = t, \ t_2^{\max} < t \\ w_2 > w_3 > w_1 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (w_2 - w_4), & \begin{cases} t_3^{\max} = t_2^{\max} = t, \ t_1^{\max} < t \\ w_2 > w_3 > w_1 > w_4 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (w_2 - w_1) - (w_2 - w_4), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_2 > w_3 > w_1 > w_4. \end{cases} \end{cases}$$

24. $$\begin{cases} (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2 - w_3) \\ -(\frac{1}{2})^{t-t_3^{\max}-1}(w_3 - w_4), & \begin{cases} t_3^{\max} < t \\ w_2 > w_3 > w_4 > w_1 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (\frac{1}{2})^{t-t_2^{\max}-1}(w_2 - w_3) \\ -(w_3 - w_4), & \begin{cases} t_3^{\max} = t, \ t_2^{\max} < t \\ w_2 > w_3 > w_4 > w_1 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(w_2 - w_1) - (w_2 - w_4), & \begin{cases} t_3^{\max} = t_2^{\max} = t, \ t_1^{\max} < t \\ w_2 > w_3 > w_4 > w_1 \end{cases} \\ (\frac{1}{2})^{t-1}w_1 - (w_2 - w_1) - (w_2 - w_4), & \begin{cases} t_3^{\max} = t_2^{\max} = t_1^{\max} = t \\ w_2 > w_3 > w_4 > w_1. \end{cases} \end{cases}$$

*Proof* Now, we are ready to prove the theorem by considering first two cases independently and the remaining cases can be proved by major classes as explained above.

**Case 1**: Since $w_4 > w_3 > w_2 > w_1$, we prove that $m_4 > m_3 > m_2 > m_1$ holds, then its generator matrix $G$ of the code $C$ can be written as $G = (G_1, G_2, G_3, G_4)$, where $G_1$ consists of all points in $\mathbf{PG}(k-1, 2)$ with each point repeating $m_1$ times and all the points in $S_2 \cup S_3 \cup S_4$ constitute the columns of $G_2$ with each point repeating $m_2 - m_1$ times. Columns of the generator matrix $G_3$ consist of all points of $S_3$ and each point repeats $m_3 - m_2$ times and columns of the generator matrix $G_4$ include of all points of $S_4$ and each point repeats $m_4 - m_3$ times. Then, $G_1$ generates a $k$-dimensional constant-weight code $C'$ with weight $m_1 2^{k-1}$ and length $l_1 = m_1(2^k - 1)$, $G_2$ generates a $(k - k_1)$-dimensional weight code $C''$ with weight $(m_2 - m_1)2^{k-k_1-1}$ and length $l_2 = (m_2 - m_1)(2^{k-k_1} - 1)$, $G_3$ generates a $(k - k_2)$-dimensional weight code $C'''$ with weight $(m_3 - m_2)2^{k-k_2-1}$ and length $l_3 = (m_3 - m_2)(2^{k-k_2} - 1)$ and $G_4$ generates a $(k - k_3)$-dimensional weight code $C''''$ with weight $(m_4 - m_3)2^{k-k_3-1}$ and length $l_4 = (m_4 - m_3)(2^{k-k_3} - 1)$. Let $c_1, \cdots, c_t$ be any $t$ linearly independent codewords in $C$ such that

$\begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} = X_{t\times k}G$, $\begin{pmatrix} c'_1 \\ \vdots \\ c'_t \end{pmatrix} = X_{t\times k}G_1$, $\begin{pmatrix} c''_1 \\ \vdots \\ c''_t \end{pmatrix} = X_{t\times k}G_2$, $\begin{pmatrix} c'''_1 \\ \vdots \\ c'''_t \end{pmatrix} = X_{t\times k}G_3$ and $\begin{pmatrix} c''''_1 \\ \vdots \\ c''''_t \end{pmatrix} = X_{t\times k}G_4$. It can be concluded that each above codeword $c_i (i = 1, 2, \cdots, t)$ can be divided into four sectors. That is, $c_i = (c'_i, c''_i, c'''_i, c''''_i)$ with $c'_i \in C'$, $c''_i \in C''$, $c'''_i \in C'''$ and $c''''_i \in C''''$. Obviously, the codewords $c'_1, \cdots, c'_t$ are linearly independent. Moreover based on Lemma 4.2, the rank of the codewords $c''_1, \cdots, c''_t$, $c'''_1, \cdots, c'''_t$ and $c''''_1, \cdots, c''''_t$ are $(t - t_1)$, $(t - t_2)$ and $(t - t_3)$, respectively. From Lemma 4.1, we conclude that $inter_1 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$, $0 \leq inter_2 \leq (\frac{1}{2})^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1}$, $0 \leq inter_3 \leq (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$ and $0 \leq inter_4 \leq (\frac{1}{2})^{t-t_3-1}(m_4 - m_3)2^{k-k_3-1}$. Therefore, $inter = inter_1 + inter_2 + inter_3 + inter_4$. Thus, $inter = (\frac{1}{2})^{t-1} m_1 2^{k-1}$ is reachable, whenever $c''_1 = 0$, $c'''_1 = 0$ and $c''''_1 = 0$ are equivalent to $c_1 \in C_1$, $c_1 \in C_2$ and $c_1 \in C_3$, respectively. Since $dim(C_1) \geq 1$, we can select an arbitrary non-zero codeword $c_1$ from $C_1$ and expand it to $t$ linearly independent codewords $c_1, \cdots, c_t$ in $C$. Therefore, the $t$-wise intersection of binary relative four-weight code is $(\frac{1}{2})^{t-1} m_1 2^{k-1} = (\frac{1}{2})^{t-1} w_1$.

**Case 2**: If $w_1 > w_2 > w_3 > w_4$ then $m_1 > m_2 > m_3 > m_4$, similar to the analysis in Case 1, these matrices $G_1$, $G_2$, $G_3$ and $G_4$ can be introduced. $G_4 = (G, G_1, G_2, G_3)$ and $G_1$ includes all the points in $S_2 \cup S_3 \cup S_4$, which constitute the columns of $G_1$ with each point repeating $m_1 - m_2$ times and the columns of $G_2$ consist of all points of $S_3$ with each point repeating $m_2 - m_3$ times. Columns of the generator matrix $G_3$ consist of all points of $S_4$ and each point repeats $m_3 - m_4$ times and columns of the generator matrix $G_4$ consist of all points in $\mathbf{PG}(k - 1, 2)$ with each point repeating $m_1$ times. Then, $G_1$ generates a $(k - k_1)$-dimensional constant-weight code $C'$ with weight $(m_1 - m_2)2^{k-k_1-1}$ and length $l_1 = (m_1 - m_2)(2^{k-k_1} - 1)$, $G_2$ generates a $(k - k_2)$-dimensional weight code $C''$ with weight $(m_2 - m_3)2^{k-k_2-1}$ and length $l_2 = (m_2 - m_3)(2^{k-k_2} - 1)$, $G_3$ generates a $(k - k_3)$-dimensional weight code $C'''$ with weight $(m_3 - m_4)2^{k-k_3-1}$ and length $l_3 = (m_3 - m_4)(2^{k-k_3} - 1)$ and $G_4$ generates a $k$-dimensional weight code $C''''$ with weight $m_1 2^{k-1}$ and length $l_4 = m_1(2^k - 1)$. Let $c_i$ be an arbitrary codeword of the $t$ linearly independent codewords $c_1, \cdots, c_t \in C$ with the matrix form $\begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} = X_{t\times k}G$. Then, we have $c''''_i = (c_i, c'_i, c''_i, c'''_i)$ for any $i \in \{1, 2, \cdots, t\}$ with $c'_i \in C'$, $c''_i \in C''$, $c'''_i \in C'''$ and $c''''_i \in C''''$. Besides $rank(c''''_1, \cdots, c''''_t) = t$, whereas $rank(c'_1, \cdots, c'_t) = t - t_1$, $rank(c''_1, \cdots, c''_t) = t - t_2$ and $rank(c'''_1, \cdots, c'''_t) = t - t_3$ by Lemma 4.2. Furthermore, $inter = inter_4 - inter_1 - inter_2 - inter_3$ with $0 \leq inter_1 \leq (\frac{1}{2})^{t-t_1-1}(m_1 - m_2)2^{k-k_1-1}$, $0 \leq inter_2 \leq (\frac{1}{2})^{t-t_2-1}(m_2 - m_3)2^{k-k_2-1}$, $0 \leq inter_3 \leq (\frac{1}{2})^{t-t_3-1}(m_3 - m_4)2^{k-k_3-1}$ and $inter_4 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$, by Lemma 4.1.

Next, we state that

$$inter = \left(\frac{1}{2}\right)^{t-1} m_1 2^{k-1} - \left(\frac{1}{2}\right)^{t-t_1-1} (m_1 - m_2) 2^{k-k_1-1} - \left(\frac{1}{2}\right)^{t-t_2-1} (m_2 - m_3) 2^{k-k_2-1} - \left(\frac{1}{2}\right)^{t-t_3-1} (m_3 - m_4) 2^{k-k_3-1}$$

can reachable when $< c_1, c_2, \cdots, c_t >$ is a relative $(t, t_1, t_2, t_3)$ subcode $(t_1 \leq t_2 < t_3 < t)$. Let $c_1, \cdots, c_t$ be arbitrary $t$ linearly independent codewords and $< c_1, \cdots, c_t >$ is a relative $(t, t_1, t_2, t_3)$ subcode of $C$. According to the proof of Lemma 2.3, there exists an invertible matrix $Y_{t\times t}$, $Z_{t\times t}$ and $W_{t\times t}$ such that

$$W_{t\times t}Z_{t\times t}Y_{t\times t}X_{t\times k} = \begin{bmatrix} X''''_{t_1\times k_3} & X''''_{t_1\times(k_2-k_1)} & X''''_{t_1\times(k_3-k_2)} & X''''_{t_1\times(k-k_3)} \\ X''''_{(t_2-t_1)\times k_1} & X''''_{(t_2-t_1)\times(k_2-k_1)} & X''''_{(t_2-t_1)\times(k_3-k_2)} & X''''_{(t_2-t_1)\times(k-k_3)} \\ X''''_{(t_3-t_2)\times k_1} & X''''_{(t_3-t_2)\times(k_2-k_1)} & X''''_{(t_3-t_2)\times(k_3-k_2)} & X''''_{(t_3-t_2)\times(k-k_3)} \\ X'''_{(t-t_3)\times k_1} & X'''_{(t-t_3)\times(k_2-k_1)} & X'''_{(t-t_3)\times(k_3-k_2)} & X'''_{(t-t_3)\times(k-k_3)} \end{bmatrix},$$

with each row of $X''''_{t_1\times(k-k_3)}$, $X''''_{(t_2-t_1)\times(k-k_3)}$ and $X''''_{(t_3-t_2)\times(k-k_3)}$ is the same as the last row of $X'''_{(t-t_3)\times(k-k_3)}$. Similarly with each row of $X''''_{t_1\times(k_3-k_2)}$ and $X''''_{(t_2-t_1)\times(k_3-k_2)}$ is the same as the last row of $X''''_{(t-t_3)\times(k_3-k_2)}$. Denote $c_1, \cdots, c_t$ the rows of matrix $W_{t\times t}Z_{t\times t}Y_{t\times t}X_{t\times k}G$. Then, we conclude that these $t$ linearly independent codewords have the intersection

$$\begin{aligned} inter &= inter_4 - inter_1 - inter_2 - inter_3 \\ &= \left(\frac{1}{2}\right)^{t-1} m_1 2^{k-1} - \left(\frac{1}{2}\right)^{t-t_1-1}(m_1-m_2)2^{k-k_1-1} - \left(\frac{1}{2}\right)^{t-t_2-1}(m_2-m_3)2^{k-k_2-1} \\ &\quad - \left(\frac{1}{2}\right)^{t-t_3-1}(m_3-m_4)2^{k-k_3-1} \\ &= \left(\frac{1}{2}\right)^{t-1} w_1 - \left(\frac{1}{2}\right)^{t-t_1-1}(w_1-w_2) - \left(\frac{1}{2}\right)^{t-t_2-1}(w_2-w_3) - \left(\frac{1}{2}\right)^{t-t_3-1}(w_3-w_4). \end{aligned}$$

Thus, for all parameters $t_1$, $t_2$ and $t_3$, we get the $t$-wise intersection of binary relative four-weight codes in the case $w_1 > w_2 > w_3 > w_4$,

$$\min_{t_1,t_2,t_3} inter = \begin{cases} (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1-1}(w_1-w_2) - (\frac{1}{2})^{t-t_2-1}(w_2-w_3) & \\ -(\frac{1}{2})^{t-t_3-1}(w_3-w_4), & \text{if } t_3^{\max} < t \\ (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1-1}(w_1-w_2) - (\frac{1}{2})^{t-t_2-1}(w_2-w_3) & \\ -(w_3-w_4), & \text{if } t_3^{\max} = t \text{ and } t_2^{\max} < t \\ (\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_1-1}(w_1-w_2) - (w_2-w_3) & \\ -(w_3-w_4), & \text{if } t_3^{\max} = t_2^{\max} = t \text{ and } t_1^{\max} < t \\ (\frac{1}{2})^{t-1}w_1 - (w_1-w_4), & \text{if } t_3^{\max} = t_2^{\max} = t_1^{\max} = t. \end{cases}$$

Hereafter, we have to prove the major classes one by one which consist of all the remaining cases.

**Major class 1:** In this major class, it can be considered that all the five cases (3, 4, 5, 6 and 7) have similar proof. Although the cases seem to be different, the condition will be same that is $m_1 < m_2$, $m_2 > m_3$ and $m_3 < m_4$. Using the first key lemma stated in the Appendix and the procedure adopted in Case 2, we can estimate the intersection of the relative $(t, t_1, t_2, t_3)$ $(t_1 < t_2 \leq t_3 < t)$ subcode,

$$\begin{aligned} inter &= \left(\frac{1}{2}\right)^{t-1} m_1 2^{k-1} + \left(\frac{1}{2}\right)^{t-t_1-1}(m_2-m_1)2^{k-k_1-1} \\ &\quad - \left(\frac{1}{2}\right)^{t-t_2-1}(m_2-m_3)2^{k-k_2-1} - \left(\frac{1}{2}\right)^{t-t_3-1}(m_4-m_3)2^{k-k_3-1}. \end{aligned}$$

According to Lemma 1.1 in the Appendix, for any $t$ linearly independent codewords with property that their generating subspace is relative $(t_1, t_2, t_3)$ subcode of $C$, if the corresponding $inter_4 \neq 0$, we have, $inter = inter_1 + inter_2 - inter_3 - inter_4$, with the $inter_1 = (\frac{1}{2})^{t-1}m_1, inter_2 = (\frac{1}{2})^{t-t_1-1}(m_2-m_1)2^{k-k_1-1}$ and $inter_3 = (\frac{1}{2})^{t-t_2-1}(m_3-m_2)2^{k-k_2-1}$. When $inter_4 = (\frac{1}{2})^{t-t_3-1}(m_4 - m_3)2^{k-k_3-1}$ is reachable, $inter$ will have its minimum value.

For any given $t$ codewords with the aforementioned properties, there exists three invertible matrices $Y_{t\times t}$, $Z_{t\times t}$ and $W_{t\times t}$ such that

$$W_{t\times t}Z_{t\times t}Y_{t\times t}X_{t\times k} = \begin{bmatrix} X''''_{t_1\times k_3} & X''''_{t_1\times(k_2-k_1)} & X''''_{t_1\times(k_3-k_2)} & X''''_{t_1\times(k-k_3)} \\ X''''_{(t_2-t_1)\times k_1} & X''''_{(t_2-t_1)\times(k_2-k_1)} & X''''_{(t_2-t_1)\times(k_3-k_2)} & X''''_{(t_2-t_1)\times(k-k_3)} \\ X''''_{(t_3-t_2)\times k_1} & X''''_{(t_3-t_2)\times(k_2-k_1)} & X''''_{(t_3-t_2)\times(k_3-k_2)} & X''''_{(t_3-t_2)\times(k-k_3)} \\ X'''_{(t-t_3)\times k_1} & X'''_{(t-t_3)\times(k_2-k_1)} & X'''_{(t-t_3)\times(k_3-k_2)} & X'''_{(t-t_3)\times(k-k_3)} \end{bmatrix},$$

where each row of $X''''_{t_1\times(k-k_3)}$, $X''''_{(t_2-t_1)\times(k-k_3)}$ and $X''''_{(t_3-t_2)\times(k-k_3)}$ is equal to the last row of the matrix $X'''_{(t-t_3)\times(k-k_3)}$ and each rows of $X''''_{t_1\times(k_3-k_2)}$, $X''''_{(t_2-t_1)\times(k_3-k_2)}$ and $X''''_{(t_3-t_2)\times(k_3-k_2)}$ are the same as the last row of $X'''_{(t-t_3)\times(k_3-k_2)}$. Thus, taking the rows of matrix $W_{t\times t}Z_{t\times t}Y_{t\times t}X_{t\times k}G$ to be new $t$ linearly independent codewords and still denoting them by $c_1, \cdots, c_t$, we can conclude that intersection, $inter = (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2 - w_1) - (\frac{1}{2})^{t-t_2-1}(w_3 - w_2) - (\frac{1}{2})^{t-t_3-1}(w_4 - w_3)$.

In addition, if $inter_4 = 0$, we will have $inter = inter_1 - inter_2 + inter_3 - inter_4$, with $inter_1 = (\frac{1}{2})^{t-1}m_1$ , $inter_2 = 0$ and $inter_3 = 0$. Thus, the minimum value of $inter$ is $(\frac{1}{2})^{t-1}m_1 2^{k-k_1}$, when $inter_2 = inter_3 = 0$. Next, we state that $inter_2 = inter_3 = 0$ can be reached, since $dim(C_1) \geq 1$, we select $t$ linearly independent codewords $c_1 \in C_1$, and expand to it $t$ linearly independent codewords $c_1, \cdots, c_t$. It can be checked that $inter_2 = inter_3 = 0$. Thus, if $inter_4 = 0$, the minimum intersection of $t$ linearly independent codewords will be $inter = (\frac{1}{2})^{t-1}m_1 2^{k-1}$. Summarizing the above discussion, we conclude that all $t$ linearly independent codewords $(c_1, c_2, \cdots, c_t)$ with $t_1 < t_2 \leq t_3 < t$ subcodes of $C$ will have minimum intersection

$$\begin{aligned} inter &= \min\left\{\left(\frac{1}{2}\right)^{t-1} m_1 2^{k-1} + \left(\frac{1}{2}\right)^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1} - \left(\frac{1}{2}\right)^{t-t_2-1}(m_2 - m_3)2^{k-k_2-1}\right. \\ &\quad \left. - \left(\frac{1}{2}\right)^{t-t_3-1}(m_4 - m_3)2^{k-k_3-1}; \left(\frac{1}{2}\right)^{t-1} m_1 2^{k-1}\right\} \\ &= \left\{\left(\frac{1}{2}\right)^{t-1} w_1 + \left(\frac{1}{2}\right)^{t-t_1-1}(w_2 - w_1) - \left(\frac{1}{2}\right)^{t-t_2-1}(w_2 - w_3)\right. \\ &\quad \left. - \left(\frac{1}{2}\right)^{t-t_3-1}(w_4 - w_3); \left(\frac{1}{2}\right)^{t-1} w_1\right\}. \end{aligned}$$

Therefore, the $t$-wise intersection of binary relative four-weight codes, for $m_1 < m_2$, $m_2 > m_3$ and $m_3 < m_4$, is

$$\min_{t_1,t_2,t_3} inter = \begin{cases} \{(\frac{1}{2})^{t-1}w_1 - (\frac{1}{2})^{t-t_3^{\max}-1}(w_4 - w_3);\ (\frac{1}{2})^{t-1}w_1\}, & \text{if } t_3^{\max} < t \\ \{(\frac{1}{2})^{t-1}w_1 - ((w_4 - w_3);\ (\frac{1}{2})^{t-1}w_1\}, & \text{if } t_3^{\max} = t. \end{cases}$$

**Major class 2**: In this the cases 8, 9 and 10 are considered. Since the condition $m_1 < m_2$, $m_2 < m_3$ and $m_3 > m_4$ is same for all the three cases, we will get similar result as in Case

2. Following the second key lemma in the Appendix, the intersection will be

$$\begin{aligned} inter &= \left(\frac{1}{2}\right)^{t-1} m_1 2^{k-1} + \left(\frac{1}{2}\right)^{t-t_1-1} (m_2-m_1)2^{k-k_1-1} \\ &\quad - \left(\frac{1}{2}\right)^{t-t_2-1} (m_3-m_2)2^{k-k_2-1} - \left(\frac{1}{2}\right)^{t-t_3-1} (m_3-m_4)2^{k-k_3-1}. \end{aligned}$$

Next, it is stated that the equations $inter_4 = 0$, $inter_3 = (\frac{1}{2})^{t-t_2-1}(m_3-m_2)2^{k-k_2-1}$ and $inter_2 = (\frac{1}{2})^{t-t_1-1}(m_2-m_1)2^{k-k_1-1}$ are reachable whenever $< c_1, c_2, \cdots, c_t >$ is a relative $(t, t_1, t_2, t_3)$ $(t_1 < t_2 < t_3 \le t)$ subcode. Any $t$ linearly independent codewords, $< c_1, \cdots, c_t >$ is a relative $(t, t_1, t_2, t_3)$ subcode of $C$. So that we can always find that there are invertible matrices $Y_{t\times t}$, $Z_{t\times t}$ and $W_{t\times t}$, such as,

$$W_{t\times t}Z_{t\times t}Y_{t\times t}X_{t\times k} = \begin{bmatrix} X''''_{t_1\times k_3} & X''''_{t_1\times(k_2-k_1)} & X''''_{t_1\times(k_3-k_2)} & X''''_{t_1\times(k-k_3)} \\ X''''_{(t_2-t_1)\times k_1} & X''''_{(t_2-t_1)\times(k_2-k_1)} & X''''_{(t_2-t_1)\times(k_3-k_2)} & X''''_{(t_2-t_1)\times(k-k_3)} \\ X''''_{(t_3-t_2)\times k_1} & X''''_{(t_3-t_2)\times(k_2-k_1)} & X''''_{(t_3-t_2)\times(k_3-k_2)} & 0_{(t_3-t_2)\times(k-k_3)} \\ X'''_{(t-t_3)\times k_1} & X'''_{(t-t_3)\times(k_2-k_1)} & X'''_{(t-t_3)\times(k_3-k_2)} & X'''_{(t-t_3)\times(k-k_3)} \end{bmatrix},$$

where each row of $X''''_{t_1\times(k-k_3)}$ and $X''''_{(t_2-t_1)\times(k_3-k_2)}$ is equal to the last row of the matrix $X'''_{(t-t_3)\times(k-k_3)}$ and each row of $X''''_{t_1\times(k_3-k_2)}$ and $X''''_{(t_2-t_1)\times(k_3-k_2)}$ is the same as the last row of $X'''_{(t-t_3)\times(k_3-k_2)}$. Then, considering the rows of matrix $W_{t\times t}Z_{t\times t}Y_{t\times t}X_{t\times k}G$ a new $t$ linearly independent codewords but still denoting them by $c_1, \cdots, c_t$, we can infer that $inter_4 = 0$, $inter_1 = (\frac{1}{2})^{t-1}m_1 2^{k-1}$, $inter_2 = (\frac{1}{2})^{t-t_1-1}(m_2-m_1)2^{k-k_1-1}$ and $inter_3 = (\frac{1}{2})^{t-t_2-1}(m_3-m_2)2^{k-k_2-1}$. Hence, all the $t$ linearly independent codewords of which generating subspaces are relative $(t, t_1, t_2, t_3)$ $(t_1 < t_2 < t_3 \le t)$ subcodes, have the minimum intersection $inter = (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (\frac{1}{2})^{t-t_2-1}(w_3-w_2)$. Therefore, the $t$-wise intersection of binary relative four- weight codes in major class 2 is

$$\min_{t_1,t_2,t_3} inter = \begin{cases} (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (\frac{1}{2})^{t-t_2-1}(w_3-w_2), & \text{if } t_2^{\max} < t \text{ and } t_1^{\max} < t \\ (\frac{1}{2})^{t-1}w_1 + (\frac{1}{2})^{t-t_1-1}(w_2-w_1) - (w_3-w_2), & \text{if } t_2^{\max} = t \text{ and } t_1^{\max} < t \\ (\frac{1}{2})^{t-1}w_1 + (w_2-w_1) - (w_3-w_2), & \text{if } t_2^{\max} = t_1^{\max} = t. \end{cases}$$

Similar to this, by using the corresponding key lemmas stated in the Appendix, we can also prove the major classes 3, 4, 5 and 6 in the same method so, we omit the detailed proof. □

*Remark 4.4* While generalizing the above theorem for any $q > 2$, we may also obtain the $t$-wise intersection of any $q$-ary relative four-weight code for cases 1, 2 and major class 2. For major class 1, it is more complicated to generalize the $t$-wise intersection of $q$-ary $(q > 2)$code, since we are not able to arrive a similar result as in Lemma 1.1 (Refer to the Appendix) which can be used to prove that major class.

## 5 The trellis of relative four-weight codes

A trellis of a $k$-dimensional block code $C$ is a directed graph. The set of nodes (or called states) of the graph can be partitioned into subsets $S_0 = \{S_0\}$, $S_1$, ..., $S_{n-1}$ , $S_n = \{S_n\}$. An edge from a state in $S_{i-1}$ terminates at a state in $S_i$, $1 \le i \le n$. For binary codes, each edge is labeled by 0 or 1, such that $C$ is the set of edge label sequences obtained by traversing all paths from $S_0$ to $S_n$. For a linear code, each $S_i$ is a vector space.

Given a trellis of the code, the maximum-likelihood soft-decision decoding is achieved by applying the well-known Viterbi algorithm. The complexity of this decoding is determined by the trellis complexity.

Define

$$s^*(C) = \max_{0\leq i\leq n} dim\{S_i\} \tag{5.1}$$

and

$$s(C) = \min_{\pi} S^*\{\pi(C)\}, \tag{5.2}$$

where the minimization is performed over all permutations $\pi$ acting on the coordinate positions of $C$.

Let $S_{i,0}$ be the state in $S_i$ that corresponds to all zero path from $S_0$. Then, the sets of label sequences associated with the sets of paths from $S_0$ to $S_{i,0}$ and from $S_{i,0}$ to $S_n$ , are called the past subcode $C_p^{(i)}$ and the future subcode $C_f^{(i)}$, respectively.

It is mandatory to recall $\chi(c)$ stands for support of the codeword $c$ in other words, the set of the non-zero co-ordinate positions of $c$, whereas $\{1, \cdots, n\}$ the set of the coordinate positions and $i^- = \{1, ..., i\}$ , $i^+ = \{i+1, ..., n\}$ for $1 \leq i \leq n$ are well known. Using these notations, we may also describe $C_p^{(i)}$ and $C_f^{(i)}$ as follows:

$$\left.\begin{aligned} C_p^{(i)} &= \quad \{c \in C, \chi(c) \subseteq i^-\} \\ C_f^{(i)} &= \quad \{c \in C, \chi(c) \subseteq i^+\} \end{aligned}\right\} \tag{5.3}$$

Note that

$$dim(S_i) = k - dim(C_p^{(i)}) - dim(C_f^{(i)}). \tag{5.4}$$

The following lemma is a relation between trellis complexity and intersection of binary relative four-weight codes.

**Lemma 5.1** *Let $C$ be a binary relative four-weight code. If the intersection of any two non-zero codewords is at least three, then $s(C) \geq k - 2$.*

*Proof* We prove this lemma by contradiction. Suppose $s(C) \leq k - 3$.

Then, according to the definition of (5.1) and (5.2), there exists a permutation $\pi_0$ of a co-ordinate position of the code $C$ such that $s(C) = s^*(\pi_0(C)) \leq k - 3$. Thus, for any co-ordinate position $1 \leq i \leq n$, by using the (5.4), we have

$$dim(\pi_0(C_p)^{(i)}) + dim(\pi_0(C_f)^{(i)}) \geq 3. \tag{5.5}$$

If there exists some $i$ such that $dim(\pi_0(C_p)^{(i)}) > 0$ and $dim(\pi_0(C_f)^{(i)}) > 0$, then there are non-zero codewords $c_1 \in \pi_0(C_p)^{(i)}$, $c_2 \in \pi_0(C_f)^{(i)}$ and, they have no intersection by (5.3). This contradiction shows that $C$ is intersecting. This implies that $dim(\pi_0(C_p)^{(i)}) > 0$ and $dim(\pi_0(C_f)^{(i)}) > 0$ are not possible. Hence, only one of the following can occur.

1. If $dim(\pi_0(C_p)^{(i)}) = 0$, then $dim(\pi_0(C_f)^{(i)}) > 0$.
2. If $dim(\pi_0(C_p)^{(i)}) > 0$, then $dim(\pi_0(C_f)^{(i)}) = 0$.

The above facts yield that $C$ is a mimimum code with a distance $d$. We can find $i_0 > d - 1$ such that $dim(\pi_0(C_p)^{(d-1)}) = 0$ and $dim(\pi_0(C_f)^{(d-1)}) > 0$ or $dim(\pi_0(C_p)^{(n-d+1)}) > 0$ and $dim(\pi_0(C_f)^{(n-d+1)}) = 0$.

We state that $dim(\pi_0(C_p)^{(i_0+1)}) = 1$. Otherwise there exists two linearly independent codewords $c_1$ and $c_2$ such that $c_1, c_2 \in \pi_0(C_p)^{(i_0+1)}$. Thus, we can find non-zero codewords $c_1$ and $c_2$ intersect exactly at the $(i_0 + 1)^{\text{th}}$ co-ordinate position due to $dim(\pi_0(C_p)^{(i_0)}) = 0$.

We can thus find a non-zero element $\alpha \in \mathbf{GF}(q)$ such that $0 \neq c_1+\alpha c_2 \in \pi_0(C_p)^{(i_0)}$, which is a contradiction to $dim(\pi_0(C_p)^{(i_0)}) = 0$. This argument shows that $dim(\pi_0(C_p)^{(i_0+1)}) = 1$.

Finally, we get

$$\begin{aligned} s(C) &= s^*(\pi_0(C)) \\ &\geq k - dim(\pi_0(C_p)^{(i_0+1)} + dim(\pi_0(C_f)^{(i_0+1)}) \\ &= k-2. \end{aligned}$$

Which is a contradiction to $s(C) \leq k-3$. □

## Appendix

**Key Lemmas for Theorem 4.3** In order to prove all the major classes in Theorem 4.3, we introduce the first key lemma which will be used in the cases 3, 4, 5, 6 and 7 in which $m_1 < m_2$ , $m_2 > m_3$ and $m_3 < m_4$ . Then, the generator matrix $G$ of $C$ can be written in the following form $(G, G_3, G_4) = (G_1, G_2)$ in which $G_1$ consists of all points in $\mathbf{PG}(k-1, 2)$ with each point repeating $m_1$ times and all the points in $S_2 \cup S_3 \cup S_4$, constitute the columns of $G_2$ with each point repeating $m_2-m_1$ times. Columns of the generator matrix $G_3$ consist of all points of $S_3$ and each point repeats $m_2-m_3$ times and columns of the generator matrix $G_4$ consist of all points of $S_4$ and each point repeats $m_4-m_3$ times. Then, $G_1$ generates a $k$-dimensional constant- weight code $C'$ with weight $m_1 2^{k-1}$ and length $l_1 = m_1 2^k - 1$, $G_2$ generates a $(k-k_1)$-dimensional weight code $C''$ with weight $(m_2-m_1)2^{k-k_1-1}$ and length $l_2 = (m_2-m_1)2^{k-k_1} - 1$, $G_3$ generates a $(k-k_2)$-dimensional constant-weight code $C'''$ with weight $(m_2-m_3)2^{k-k_2-1}$ and length $l_3 = (m_2-m_3)2^{k-k_2} - 1$ and $G_4$ generates a $(k-k_3)$-dimensional constant-weight code $C''''$ with weight $(m_4-m_3)2^{k-k_3-1}$ and length $l_4 = (m_4-m_3)2^{k-k_3} - 1$. Let $c_1, \cdots, c_t$ be a relative $(t, t_1, t_2, t_3)(t_3 < t_2 < t_1 < t)$ subcode.

Denote $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_t \end{bmatrix} = X_{t\times k}G$, $\begin{bmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_t \end{bmatrix} = X_{t\times k}G_1$, $\begin{bmatrix} c''_1 \\ c''_2 \\ \vdots \\ c''_t \end{bmatrix} = X_{t\times k}G_2$, $\begin{bmatrix} c'''_1 \\ c'''_2 \\ \vdots \\ c'''_t \end{bmatrix} = X_{t\times k}G_3$, $\begin{bmatrix} c''''_1 \\ c''''_2 \\ \vdots \\ c''''_t \end{bmatrix} = X_{t\times k}G_4$.

Then, we have for any $i \in \{1, 2, \cdots, t\}$, $(c_i, c'''_i, c''''_i) = (c'_i, c''_i)$, where $c'_i \in C'$, $c''_i \in C''$, $c'''_i \in C'''$ and $c''''_i \in C''''$. In addition that, $c'_1, \cdots, c'_t$ are linearly independent codewords, whereas $rank(c''_1, \cdots, c''_t) = t - t_1$, $rank(c'''_1, \cdots, c'''_t) = t - t_2$ and $rank(c''''_1, \cdots, c''''_t) = t - t_3$ by Lemma 4.2. For satisfication, $inter$, $inter_1$, $inter_2$, $inter_3$ and $inter_4$ will be represented as follows. $inter = |\bigcap_{i=1}^t \chi(c_i)|$, $inter_1 = |\bigcap_{i=1}^t \chi(c'_i)|$, $inter_2 = |\bigcap_{i=1}^t \chi(c''_i)|$, $inter_3 = |\bigcap_{i=1}^t \chi(c'''_i)|$ and $inter_4 = |\bigcap_{i=1}^t \chi(c''''_i)|$. Based on the Lemma 4.1, we have $inter = inter_1 + inter_2 - inter_3 - inter_4$ with $inter_1 =$

$(\frac{1}{2})^{t-1}m_1 2^{k-1}$, $0 \leq inter_2 \leq (\frac{1}{2})^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1}$, $0 \leq inter_3 \leq (\frac{1}{2})^{t-t_2-1}(m_2 - m_3)2^{k-k_2-1}$ and $0 \leq inter_4 \leq (\frac{1}{2})^{t-t_3-1}(m_4 - m_3)2^{k-k_3-1}$, then we have the following lemma.

**Lemma 1.1** *Assume* $q = 2$ *and* $w_3 > \max\{w_1, w_2, w_4\}$ *and let* $D =< c_1, \cdots, c_t >$ *be a relative* $(t, t_1, t_2, t_3)(t_3 < t_2 < t_1 < t)$ *subcode of* $C$ *with* $inter_4 \neq 0$*, then* $inter_3 \leq (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$.

*Proof* Write $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_t \end{bmatrix} = X_{t\times k}G$, then similar to the proof of the Lemma 4.2, there exists an invertible matrix $Y_{t\times t}$ such that

$$Y_{t\times t}X_{t\times k} = \begin{bmatrix} X'_{t_1\times k_1} & 0_{t_1\times(k_2-k_1)} & 0_{t_1\times(k_3-k_2)} & 0_{t_1\times(k-k_3)} \\ X''_{(t_2-t_1)\times k_1} & X''_{(t_2-t_1)\times(k_2-k_1)} & 0_{(t_2-t_1)\times(k_3-k_2)} & 0_{(t_2-t_1)\times(k-k_3)} \\ X''_{(t_3-t_2)\times k_1} & X''_{(t_3-t_2)\times(k_2-k_1)} & X''_{(t_3-t_2)\times(k_3-k_2)} & 0_{(t_3-t_2)\times(k-k_3)} \\ X'''_{(t-t_3)\times k_1} & X'''_{(t-t_3)\times(k_2-k_1)} & X'''_{(t-t_3)\times(k_3-k_2)} & X'''_{(t-t_3)\times(k-k_3)} \end{bmatrix},$$

with $rank(X'_{t_1\times k_1}) = t_1$, $rank(X''_{(t_2-t_1)\times(k_2-k_1)}) = t_2 - t_1$, $rank(X''_{(t_3-t_2)\times(k_3-k_2)}) = t_3 - t_2$ and $rank(X'''_{(t-t_3)\times(k-k_3)}) = t - t_3$.

Thus,

$$Y_{t\times t}\left(\begin{array}{c|c} c'''_1 & c''''_1 \\ \vdots & \vdots \\ c'''_t & c''''_t \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \overline{c}'''_{t_2+1} & 0 \\ \vdots & \vdots \\ \overline{c}'''_{t_3} & 0 \\ \overline{c}'''_{t_3+1} & \overline{c}''''_{t_3+1} \\ \vdots & \vdots \\ \overline{c}'''_t & \overline{c}''''_t \end{array}\right), \tag{A.1}$$

in which $rank(\overline{c}'''_{t_2+1}, \cdots, \overline{c}'''_t) = t - t_2$ and $rank(\overline{c}''''_{t_3+1}, \cdots, \overline{c}''''_t) = t - t_3$. We have $rank\left(\begin{array}{c|c} c'''_1 & c''''_1 \\ \vdots & \vdots \\ c'''_t & c''''_t \end{array}\right) = t - t_2$. Without loss of generality, let $(c'''_{t_2+1}, \cdots, c'''_t)$ be a maximal linearly independent set of $(c'''_1, \cdots, c'''_t)$. Then, the last $(t - t_2)$ rows of the matrix $\left(\begin{array}{c|c} c'''_1 & c''''_1 \\ \vdots & \vdots \\ c'''_t & c''''_t \end{array}\right)$, which is $\left(\begin{array}{c|c} c'''_{t_2+1} & c''''_{t_2+1} \\ \vdots & \vdots \\ c'''_t & c''''_t \end{array}\right)$, is a maximal linearly independent set of its all rows. So, there exists a matrix $\begin{pmatrix} p_{1\times(t_2+1)} & \cdots & p_{1\times t} \\ \vdots & \ddots & \vdots \\ p_{t_2\times(t_2+1)} & \cdots & p_{t_2\times t} \end{pmatrix}$ such that $\left(\begin{array}{c|c} c'''_1 & c''''_1 \\ \vdots & \vdots \\ c'''_t & c''''_t \end{array}\right) =$

$\begin{pmatrix} p_{1\times(t_2+1)} & \cdots & p_{1\times t} \\ \vdots & \ddots & \vdots \\ p_{t_2\times(t_2+1)} & \cdots & p_{t_2\times t} \end{pmatrix} \left(\begin{array}{c|c} c'''_{t_2+1} & c''''_{t_2+1} \\ \vdots & \vdots \\ c'''_t & c''''_t \end{array}\right)$, that is

$$\begin{pmatrix} c'''_1 \\ \vdots \\ c'''_{t_2} \end{pmatrix} = \begin{pmatrix} p_{1\times(t_2+1)} & \cdots & p_{1\times t} \\ \vdots & \ddots & \vdots \\ p_{t_2\times(t_2+1)} & \cdots & p_{t_2\times t} \end{pmatrix} \begin{pmatrix} c'''_{t_2+1} \\ \vdots \\ c'''_t \end{pmatrix} \tag{A.2}$$

and

$$\begin{pmatrix} c''''_1 \\ \vdots \\ c''''_{t_2} \end{pmatrix} = \begin{pmatrix} p_{1\times(t_2+1)} & \cdots & p_{1\times t} \\ \vdots & \ddots & \vdots \\ p_{t_2\times(t_2+1)} & \cdots & p_{t_2\times t} \end{pmatrix} \begin{pmatrix} c''''_{t_2+1} \\ \vdots \\ c''''_t \end{pmatrix}. \tag{A.3}$$

Based on (A.3) and $rank(c''''_1, \cdots, c''''_{t_1}) = t - t_3$, without loss of generality, we assume $(c''''_{t_3+1}, \cdots, c''''_t)$ to be a maximal linearly independent set of $(c''''_1, \cdots, c''''_t)$. Then, there exists a matrix $\begin{pmatrix} r_{(t_2+1)\times(t_3+1)} & \cdots & r_{(t_2+1)\times t} \\ \vdots & \ddots & \vdots \\ r_{t_3\times(t_3+1)} & \cdots & r_{t_3\times t} \end{pmatrix}$, such that

$$\begin{pmatrix} c''''_{t_2+1} \\ \vdots \\ c''''_{t_3} \end{pmatrix} = \begin{pmatrix} r_{(t_2+1)\times(t_3+1)} & \cdots & r_{(t_2+1)\times t} \\ \vdots & \ddots & \vdots \\ r_{t_3\times(t_3+1)} & \cdots & r_{t_3\times t} \end{pmatrix} \begin{pmatrix} c''''_{t_3+1} \\ \vdots \\ c''''_t \end{pmatrix}. \tag{A.4}$$

Substitute the (A.4) in (A.3), we get

$$\begin{pmatrix} c''''_1 \\ \vdots \\ c''''_{t_2} \end{pmatrix} = \begin{pmatrix} p_{1\times(t_2+1)} & \cdots & p_{1\times t} \\ \vdots & \ddots & \vdots \\ p_{t_2\times(t_2+1)} & \cdots & p_{t_2\times t} \end{pmatrix} \begin{pmatrix} r_{(t_2+1)\times(t_3+1)} & \cdots & r_{(t_2+1)\times t} \\ \vdots & \ddots & \vdots \\ r_{t_3\times(t_3+1)} & \cdots & r_{t_3\times t} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c''''_{t_3+1} \\ \vdots \\ c''''_t \end{pmatrix}. \tag{A.5}$$

Since $inter_4 \neq 0$, there must be a co-ordinate position $j_0 \in \{1, 2, \cdots, l_4\}$ such that $j_0 \in \chi(c''''_i), \forall 1 \leq i \leq t$.

Then, (A.4) implies that $\begin{pmatrix} r_{(t_2+1)\times(t_3+1)} & \cdots & r_{(t_2+1)\times t} \\ \vdots & \ddots & \vdots \\ r_{t_3\times(t_3+1)} & \cdots & r_{t_3\times t} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and (A.5) implies

$$\begin{pmatrix} p_{1\times(t_2+1)} & \cdots & p_{1\times t} \\ \vdots & \ddots & \vdots \\ p_{t_2\times(t_2+1)} & \cdots & p_{t_2\times t} \end{pmatrix} \begin{pmatrix} r_{(t_2+1)\times(t_3+1)} & \cdots & r_{(t_2+1)\times t} \\ \vdots & \ddots & \vdots \\ r_{t_3\times(t_3+1)} & \cdots & r_{t_3\times t} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} p_{1\times(t_2+1)} & \cdots & p_{1\times t} \\ \vdots & \ddots & \vdots \\ p_{t_2\times(t_2+1)} & \cdots & p_{t_2\times t} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{A.6}$$

We set $\cap_{i=t_2+1}^{t}\chi(c_i''') = \{j_1, j_2, \cdots, j_r\}$ and let $\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$ be the matrix which consists of the $\mathrm{j}_1^{\mathrm{th}}$, $\mathrm{j}_2^{\mathrm{th}}$, $\cdots$, $\mathrm{j}_r^{\mathrm{th}}$ columns of the matrix $\begin{pmatrix} c_{t_2+1}''' \\ \vdots \\ c_t''' \end{pmatrix}$.

From (A.2) and (A.6), we have $\begin{pmatrix} p_{1\times(t_2+1)} & \cdots & p_{1\times t} \\ \vdots & \ddots & \vdots \\ p_{t_2\times(t_2+1)} & \cdots & p_{t_2\times t} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$, which gives $\bigcap_{i=1}^{t}\chi(c_i''') = \bigcap_{i=t_2+1}^{t}\chi(c_i''')$. When $inter_3 = |\bigcap_{i=t_2+1}^{t}\chi(c_i''')|$, we will get $inter_3 = (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$.

Since $(c_{t_2+1}'', \cdots, c_t'')$ are $t - t_2$ linearly independent codewords of constant-weight $c'''$ with weight $(m_3 - m_2)2^{k-k_2-1}$, using Lemma 4.1, it will follow that $|\bigcap_{i=t_2+1}^{t}\chi(c_i''')| = (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$. Thus, $inter_3 = (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$. □

We introduce the second key lemma in the cases 8, 9 and 10 in which $m_1 < m_2$, $m_2 < m_3$ and $m_3 > m_4$ holds. Their generator matrix $G$ of the code $C$ can be written as $(G, G_3, G_4) = (G_1, G_2)$ in which $G_1$ consists of all points in $\mathbf{PG}(k-1, 2)$ with each point repeating $m_1$ times and all the points in $S_2 \cup S_3 \cup S_4$ constitute the columns of $G_2$ with each point repeating $m_2 - m_1$ times. Columns of the generator matrix $G_3$ consist of all points of $S_3$ and each point repeats $m_3 - m_2$ times and columns of the generator matrix $G_4$ consist of all points of $S_4$ and each point repeats $m_3 - m_4$ times. Then, $G_1$ generates a $k$-dimensional constant-weight code $C'$ with weight $m_1 2^{k-1}$ and length $l_1 = m_1(2^k - 1)$, $G_2$ generates a $(k - k_1)$-dimensional weight code $C''$ with weight $(m_2 - m_1)2^{k-k_1-1}$ and length $l_2 = (m_2 - m_1)(2^{k-k_1} - 1)$, $G_3$ generates a $(k - k_2)$-dimensional weight code $C'''$ with weight $(m_3 - m_2)2^{k-k_2-1}$ and length $l_3 = (m_3 - m_2)(2^{k-k_2} - 1)$ and $G_4$ generates a $(k - k_3)$-dimensional weight code $C''''$ with weight $(m_3 - m_4)2^{k-k_3-1}$ and length $l_4 = (m_3 - m_4)(2^{k-k_3} - 1)$. Assume that $c_1, \cdots, c_t$ with the matrix form $\begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} = X_{t\times k}G$, are any $t$ linearly independent codewords in $C$. Obviously, for any $i \in \{1, 2, \cdots, t\}$, we have $(c_i, c_i''', c_i'''') = (c_i', c_i'')$, where $c_i' \in C'$, $c_i'' \in C''$, $c_i''' \in C'''$ and $c_i'''' \in C''''$. Additionally $rank(c_1', \cdots, c_t') = t$, based on Lemma 4.2, we have $rank(c_1'', \cdots, c_t'') = t - t_1$, $rank(c_1''', \cdots, c_t''') = t - t_2$ and $rank(c_1'''', \cdots, c_t'''') = t - t_3$. Furthermore, $inter = inter_1 + inter_2 - inter_3 - inter_4$ with $inter_1 = (\frac{1}{2})^{t-1}m_1 2^{k-1}$, $0 \le inter_2 \le (\frac{1}{2})^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1}$, $0 \le inter_3 \le (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$ and $0 \le inter_4 \le (\frac{1}{2})^{t-t_3-1}(m_3 - m_4)2^{k-k_3-1}$ by Lemma 4.1.

We introduce the third key lemma in cases 11, 12, 13, 14 and 15 in which we deduce that $w_1$ is greater than $w_2$, $w_2$ is less than $w_3$ and $w_3$ is greater than $w_4$ yields $m_1 > m_2$, $m_2 < m_3$ and $m_3 > m_4$. Then, the generator matrix $G$ of $C$ can be written in the following form

$(G, G_1) = (G_2, G_3, G_4)$ in which the block matrix $G_1$ consists of all points in $S_2 \cup S_3 \cup S_4$ and constitute the columns of $G_1$ with each point repeating $m_1 - m_2$ times. $G_2$ consists of all points in $\mathbf{PG}(k-1, 2)$ and each point appears $m_1$ times. All points in $S_3$ constitute columns of $G_3$ and each point occurs $m_3 - m_2$ times and all the points in $S_4$ constitute columns of $G_4$ and each point occurs $m_3 - m_4$ times. Thus, $G_1$ generates a $(k - k_1)$-dimensional constant-weight code $C'$ with weight $(m_1 - m_2)2^{k-k_1-1}$ and length $l_1 = (m_1 - m_2)(2^{k-k_1} - 1)$. $G_2$ generates a $k$- dimensional constant-weight code $C''$ with weight $m_1 2^{k-1}$ and length $l_2 = m_1(2^k - 1)$. $G_3$ generates a $(k-k_2)$-dimensional constant-weight code $C'''$ with weight $(m_3 - m_2)2^{k-k_2-1}$ and length $l_3 = (m_3 - m_2)(2^{k-k_2} - 1)$ and $G_4$ generates $(k - k_3)$-dimensional constant-weight $(m_3 - m_4)2^{k-k_3-1}$ and length $l_4 = (m_3 - m_4)(2^{k-k_3} - 1)$. Assume that, $c_1, \cdots, c_t$ are any $t$ linearly independent codewords with the matrix form $\begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} = X_{t\times k} G$, $\begin{pmatrix} c'_1 \\ \vdots \\ c'_t \end{pmatrix} = X_{t\times k} G_1$, $\begin{pmatrix} c''_1 \\ \vdots \\ c''_t \end{pmatrix} = X_{t\times k} G_2$, $\begin{pmatrix} c'''_1 \\ \vdots \\ c'''_t \end{pmatrix} = X_{t\times k} G_3$ and $\begin{pmatrix} c''''_1 \\ \vdots \\ c''''_t \end{pmatrix} = X_{t\times k} G_4$. Obviously, for any $i \in \{1, 2, \cdots, t\}$, we have $(c_i, c'_i) = (c''_i, c'''_i, c''''_i)$, where $c'_i \in C'$, $c''_i \in C''$, $c'''_i \in C'''$ and $c''''_i \in C''''$. Additionally, $rank(c''_1, \cdots, c''_t) = t$. From Lemma 4.2, we have $rank(c'_1, \cdots, c'_t) = (t - t_1)$, $rank(c''_1, \cdots, c''_t) = t - t_2$, $rank(c'''_1, \cdots, c'''_t) = t - t_3$ and $rank(c''''_1, \cdots, c''''_t) = t - t_3$. Therefore, we have $inter = inter_2 + inter_3 + inter_4 - inter_1$ with $inter_2 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$, $0 \le inter_1 \le (\frac{1}{2})^{t-t_1-1}(m_1 - m_2)2^{k-k_1-1}$, $0 \le inter_3 \le (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$ and $0 \le inter_4 \le (\frac{1}{2})^{t-t_3-1}(m_3 - m_4)2^{k-k_3-1}$ by Lemma 4.1.

We apply the fourth key lemma in the cases 16, 17 and 18 in which $m_1 > m_2$, $m_2 < m_3$ and $m_3 < m_4$. Then, the generator matrix $G$ can be written in the following form $(G, G_1, G_4) = (G_2, G_3)$ in which the block matrix $G_1$ includes all the points in $S_2 \cup S_3 \cup S_4$, which constitute the columns of $G_1$ with each point repeating $m_1 - m_2$ times. Columns of the generator matrix $G_2$ consist of all points in $\mathbf{PG}(k - 1, 2)$ and each point appears $m_1$ times and all points in $S_3$ constitute columns of $G_3$ and each point occurs $m_3 - m_2$ times, columns of the generator matrix $G_4$ consist of all points of $S_4$ and each point repeats $m_4 - m_3$ times. Then, $G_1$ generates a $(k - k_1)$-dimensional constant-weight code $C'$ with weight $(m_1 - m_2)2^{k-k_1-1}$ and length $l_1 = (m_1 - m_2)(2^{k-k_1} - 1)$. $G_2$ generates a $k$-dimensional constant-weight code $C''$ with weight $m_1 2^{k-1}$ and length $l_2 = m_1(2^k - 1)$, $G_3$ generates $(k - k_3)$-dimensional constant-weight code $C'''$ with weight $(m_3 - m_2)2^{k-k_2-1}$ and length $l_3 = (m_3 - m_2)(2^{k-k_2} - 1)$ and $G_4$ generates $(k - k_4)$-dimensional constant-weight $(m_4 - m_3)2^{k-k_3-1}$ and length $l_4 = (m_4 - m_3)(2^{k-k_3} - 1)$. Using the same procedure as above, we get the intersection $inter = inter_2 + inter_3 - inter_1 - inter_4$ with $inter_2 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$, $0 \le inter_1 \le (\frac{1}{2})^{t-t_1-1}(m_1 - m_2)2^{k-k_1-1}$, $0 \le inter_3 \le (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$ and $0 \le inter_4 \le (\frac{1}{2})^{t-t_3-1}(m_4 - m_3)2^{k-k_3-1}$.

Next, we apply the fifth key lemma in the cases 19, 20 and 21 in which $m_1 > m_2$, $m_2 > m_3$ and $m_3 < m_4$ . Then, the generator matrix $G$ of $C$ can be written in the following form $(G, G_1, G_2) = (G_3, G_4)$ in which the block matrix $G_1$ consists of all points in $S_2 \cup S_3 \cup S_4$, constitute the columns of $G_1$ with each point repeating $m_1 - m_2$ times. Columns of the generator matrix $G_2$ consist of all points of $S_3$ and each point repeats $m_2 - m_3$ times and columns of the generator matrix $G_4$ consist of all points of $S_4$ and each point repeats $m_2 - m_3$ times. $G_3$ consists of all points in $\mathbf{PG}(k - 1, 2)$ and each point appears $m_1$ times and all the points in $S_4$, constitute columns of $G_4$ and each point occurs $m_3 - m_4$ times. Then, $G_1$ generates a $(k - k_1)$-dimensional constant-weight code $C'$ with weight $(m_1 - m_2)2^{k-k_1-1}$ and length $l_1 = (m_1 - m_2)(2^{k-k_1} - 1)$. $G_2$ generates $(k - k_2)$-dimensional constant-weight code $C''$ with weight $(m_2-m_3)2^{k-k_2-1}$ and length $l_2=(m_2-m_3)(2^{k-k_2}-1)$, $G_3$ generates a $k$-dimensional constant-weight code $C'''$ with weight $m_1 2^{k-1}$ and length $l_3=m_1(2^k-1)$ and $G_4$ generates $(k-k_3)$-dimensional constant-weight $(m_4-m_3)2^{k-k_3-1}$ and length $l_4=(m_4-m_3)(2^{k-k_3}-1)$. Similar to this, using the above procedure, we have $inter = inter_3 + inter_4 - inter_1 - inter_2$ with $inter_3 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$, $0 \le inter_1 \le (\frac{1}{2})^{t-t_1-1}(m_1-m_2)2^{k-k_1-1}$, $0 \le inter_2 \le (\frac{1}{2})^{t-t_2-1}(m_2-m_3)2^{k-k_2-1}$ and $0 \le inter_4 \le (\frac{1}{2})^{t-t_3-1}(m_4-m_3)2^{k-k_3-1}$ by Lemma 4.1.

Again, we use the sixth key lemma in the cases 22, 23 and 24 in which $m_1 < m_2$ , $m_2 > m_3$ and $m_3 > m_4$ . Then, the generator matrix $G$ of $C$ can be written in the following form $(G, G_2, G_3, G_4) = (G_1)$ in which the block matrix $G_1$ consists of all points in $\mathbf{PG}(k-1, 2)$ and each point appears $m_1$ times. Columns of the generator matrix $G_2$ consist of all points of $S_2 \cup S_3 \cup S_4$ and each point repeats $m_2 - m_1$ times. All points in $S_3$ constitute columns of $G_3$ and each point occurs $m_2 - m_3$ times , $G_4$ consists of all points in $S_4$ and each point appears $m_3 - m_4$ times. Thus, $G_1$ generates a $k$-dimensional constant-weight code $C'$ with weight $m_1 2^{k-1}$ and length $l_1 = m_1(2^k-1)$, $G_2$ generates $(k-k_1)$-dimensional constant-weight code $C''$ with weight $(m_2-m_1)2^{k-k_1-1}$ and length $l_2=(m_2-m_1)(2^{k-k_1}-1)$, $G_3$ generates $(k-k_2)$-dimensional constant-weight code $C'''$ with weight $(m_2-m_3)(2^{k-k_2-1})$ and length $l_3=(m_2-m_3)2^{k-k_2}-1$ and $G_4$ generates a $k$-dimensional constant-weight code $C''''$ with weight $(m_3-m_4)2^{k-k_3-1}$ and length $l_4=(m_3-m_4)(2^{k-k_3}-1)$. Likewise, using the same procedure as above, we have $inter = inter_1 - inter_2 - inter_3 - inter_4$ with $inter_1 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$, $0 \le inter_2 \le (\frac{1}{2})^{t-t_1-1}(m_2-m_1)2^{k-k_1-1}$, $0 \le inter_3 \le (\frac{1}{2})^{t-t_2-1}(m_2-m_3)2^{k-k_2-1}$ and $0 \le inter_4 \le (\frac{1}{2})^{t-t_3-1}(m_3-m_4)2^{k-k_3-1}$ by Lemma 4.1.

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