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# Complete classification for simple root cyclic codes over the local ring $\mathbb{Z}_4[v]/\langle v^2+2v\rangle$

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**Abstract**
Let $R=\mathbb{Z}_4[v]/\langle v^2+2v\rangle$. Then $R$ is a local non-principal ideal ring of 16 elements. First, we give the structure of every cyclic code of odd length $n$ over $R$ and obtain a complete classification for these codes. Then we determine the cardinality, the type and its dual code for each of these cyclic codes. Moreover, we determine all self-dual cyclic codes of odd length $n$ over $R$ and provide a clear formula to count the number of these self-dual cyclic codes. Finally, we list some optimal 2-quasi-cyclic self-dual linear codes of length 30 over $\mathbb{Z}_4$ and obtain 4-quasi-cyclic and formally self-dual binary linear [60, 30, 12] codes derived from cyclic codes of length 15 over $\mathbb{Z}_4[v]/\langle v^2+2v\rangle$.

**Keywords** Cyclic code · Dual code · Self-dual code · Galois ring · Local ring

**Mathematics Subject Classification (2010)** 94B15 · 94B05 · 11T71

## 1 Introduction

The catalyst for the study of codes over rings was the discovery of the connection between linear codes over $\mathbb{Z}_4$ and the Kerdock and Preparata codes, which are non-linear binary codes [2, 3]. Soon after this discovery, codes over many different rings were studied. This led to many new discoveries and concreted the study of codes over rings as an important part of the coding theory discipline. Since $\mathbb{Z}_4$ is a chain ring, it was natural to expand the

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theory to focus on alphabets that are finite commutative chain rings (see [1, 4–6, 11, 12, 16, 17, 19] and [20], for examples) and other type of rings [7, 18].

In 1999, Wood in [23] showed that for certain reasons finite Frobenius rings are the most general class of rings that should be used for alphabets of codes. Then self-dual codes over commutative Frobenius rings were investigated in Dougherty et al. [13]. Especially, codes over an extension rings of $\mathbb{Z}_4$ were studied in Yildiz et al. [25] and [26] where many good $\mathbb{Z}_4$-codes were obtained as images. The ring in the mentioned works was described as $\mathbb{Z}_4[u]/\langle u^2\rangle = \mathbb{Z}_4 + u\mathbb{Z}_4$ ($u^2 = 0$) which is a local non-principal ring. Shi et al. in [18] studied $(1+2u)$-constacyclic codes of odd length $n$ over the ring $\mathbb{Z}_4[u]/\langle u^2-1\rangle = \mathbb{Z}_4+u\mathbb{Z}_4$ ($u^2 = 1$) which is another extension ring of $\mathbb{Z}_4$. Properties of these codes and their $\mathbb{Z}_4$ images were investigated.

Let $A$ be an arbitrary finite commutative ring with identity $1 \neq 0$, and $A^\times$ be the multiplicative group of invertible elements (units) in $A$. For any $a \in A$, we denote by $\langle a\rangle_A$, or $\langle a\rangle$ for simplicity, the ideal of $A$ generated by $a$, i.e. $\langle a\rangle_A = aA$. For any ideal $I$ of $A$, we will identify the element $a + I$ of the residue class ring $A/I$ with $a$ (mod $I$) in this paper.

For any positive integer $N$, let $A^N = \{(a_0, a_1, \ldots, a_{N-1}) \mid a_i \in A,\ i = 0, 1, \ldots, N-1\}$ which is an $A$-module with componentwise addition and scalar multiplication by elements of $A$. Then an $A$-submodule $\mathcal{C}$ of $A^N$ is called a *linear code* of length $N$ over $A$. For any vectors $a = (a_0, a_1, \ldots, a_{N-1})$, $b = (b_0, b_1, \ldots, b_{N-1}) \in A^N$. The usual *Euclidian inner product* of $a$ and $b$ is defined by $[a, b] = \sum_{j=0}^{N-1} a_j b_j \in A$. Then $[-, -]$ is a symmetric and non-degenerate bilinear form on the $A$-module $A^N$. Let $\mathcal{C}$ be a linear code of length $N$ over $A$. The *dual code* of $\mathcal{C}$ is defined by $\mathcal{C}^\perp = \{a \in A^N \mid [a, b] = 0,\ \forall b \in \mathcal{C}\}$, and $\mathcal{C}$ is said to be *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$. A linear code $\mathcal{C}$ of length $N$ over $A$ is said to be *cyclic* if $(a_{N-1}, a_0, a_1, \ldots, a_{N-2}) \in \mathcal{C}$ for all $(a_0, a_1, \ldots, a_{N-1}) \in \mathcal{C}$. Let $A$ be a local ring with residue class field $F$. Then cyclic codes of length $N$ over $R$ are called *simple root cyclic codes* if $\gcd(N, \mathrm{char}(F)) = 1$.

In this paper, every vector $c = (c_0, c_1, \ldots, c_{N-1}) \in A^N$ is viewed as the polynomial $c(x) = \sum_{j=0}^{N-1} c_j x^j$. Then every cyclic code $\mathcal{C}$ is viewed as an ideal in the polynomial residue ring $A[x]/\langle x^N - 1\rangle$.

In this paper, we adopt the following notation:

$\diamond$ $\mathbb{Z}_2 = \{0, 1\}$ in which the arithmetic is done modulo 2. Then $\mathbb{Z}_2$ is a binary finite field.
$\diamond$ $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ in which the arithmetic is done modulo 4. Then $\mathbb{Z}_4$ is a finite chain ring with the maximal ideal $2\mathbb{Z}_4 = \{0, 2\}$.
$\diamond$ $R = \mathbb{Z}_4[v]/\langle v^2 + 2v\rangle = \{a + bv \mid a, b \in \mathbb{Z}_4\} = \mathbb{Z}_4 + v\mathbb{Z}_4$ ($v^2 = 2v$) in which the operations are defined by:

$$\alpha + \beta = (a + c) + v(b + d) \text{ and } \alpha\beta = ac + (ad + bc + 2bd)v,$$

for any $\alpha = a + bv, \beta = c + dv \in \mathbb{Z}_4 + v\mathbb{Z}_4$ with $a, b, c, d \in \mathbb{Z}_4$. Then $R$ is a local Frobenius non-chain ring of 16 elements.

In 2015, linear codes over $\mathbb{Z}_4 + v\mathbb{Z}_4$ ($v^2 = 2v$) were studied in [15]. In the paper, a duality preserving Gray map was given and used to present MacWilliams identities and self-dual codes. Some extremal Type II $\mathbb{Z}_4$-codes were provided as images of codes over this ring. Recently, we gave a complete classification for negacyclic codes of length $2n$ over $\mathbb{Z}_4 + v\mathbb{Z}_4$, where $n$ is odd, and obtained some good and new self-dual $\mathbb{Z}_4$-codes which are are Gray images of self-dual negacyclic codes over $\mathbb{Z}_4 + v\mathbb{Z}_4$ ([8]).

Now, we follow this line to continue studying cyclic codes of length $n$ over $R = \mathbb{Z}_4 + v\mathbb{Z}_4$, where $n$ is a positive odd integer and $n \geq 3$.

The rest of the paper is organized as follows. In Section 2, we give an explicit representation for every cyclic code of length $n$ over $R$ by determining their generator sets as ideals in the ring $R[x]/\langle x^n-1\rangle$. In Section 3, we determine the dual code for each code, present explicitly all distinct self-dual cyclic codes of length $n$ over $R$ and give a clear formula to count the number of all these self-dual cyclic codes. In Section 4, we present all 583443 cyclic codes of length 15 over $R$ and list all 315 self-dual codes among them. As applications, we obtain 162 good self-dual quasi-cyclic codes of index 2 and length 30 with minimum Lee weight 10 and 12 over $\mathbb{Z}_4$. From these $\mathbb{Z}_4$-codes, we derive 70 quasi-cyclic type II binary formal self-dual [60, 30, 12] codes of index 4.

## 2 Cyclic codes over $\mathbb{Z}_4+v\mathbb{Z}_4$ of odd length

In this section, we consider cyclic codes of odd length $n$ over the ring $R=\mathbb{Z}_4[v]/\langle v^2+2v\rangle=\mathbb{Z}_4+v\mathbb{Z}_4$ ($v^2=2v$), i.e., ideals of the ring

$$R[x]/\langle x^n-1\rangle=\left\{\sum_{i=0}^{n-1}r_ix^i \mid r_i\in R,\ i=0,1,\ldots,n-1\right\}$$

in which the arithmetic is done modulo $x^n-1$. From now on, we denote

- $\mathcal{A}=\mathbb{Z}_4[x]/\langle x^n-1\rangle=\left\{\sum_{i=0}^{n-1}a_ix^i \mid a_i\in\mathbb{Z}_4,\ i=0,1,\ldots,n-1\right\}$
  in which the arithmetic is done modulo $x^n-1$.
  Then $\mathcal{A}$ is a finite commutative ring containing $\mathbb{Z}_4$ as its subring.
- $\mathcal{A}+v\mathcal{A}=\mathcal{A}[v]/\langle v^2+2v\rangle=\{\alpha+\beta v \mid \alpha,\beta\in\mathcal{A}\}$ ($v^2=2v$)
  and the operations are defined by: for any $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathcal{A}$, we have

$$(\alpha_1+\beta_1v)+(\alpha_2+\beta_2v)=(\alpha_1+\alpha_2)+v(\beta_1+\beta_2);$$
$$(\alpha_1+\beta_1v)(\alpha_2+\beta_2v)=\alpha_1\alpha_2+v(\alpha_1\beta_2+\beta_1\alpha_2+2\beta_1\beta_2).$$

Then $\mathcal{A}+v\mathcal{A}$ is a finite commutative local ring containing $\mathcal{A}$ as its subring.

Let $\alpha,\beta\in\mathcal{A}$. Then $\alpha$ and $\beta$ can be uniquely expressed as $\alpha=\sum_{i=0}^{n-1}a_ix^i$ and $\beta=\sum_{i=0}^{n-1}b_ix^i$ respectively, where $a_i,b_i\in\mathbb{Z}_4$ for all $i=0,1,\ldots,n-1$. Now, define a map $\Theta:\mathcal{A}+v\mathcal{A}\to R[x]/\langle x^n-1\rangle$ by

$$\Theta(\alpha+\beta v)=\sum_{i=0}^{n-1}\xi_ix^i,\ \text{where } \xi_i=a_i+b_iv\in R,\ i=0,1,\ldots,n-1.$$

Then one can easily verify the following conclusion.

**Lemma 1** *The map $\Theta$ defined above is an isomorphism of rings from $\mathcal{A}+v\mathcal{A}$ onto $R[x]/\langle x^n-1\rangle$.*

In the following, we will identify $\mathcal{A}+v\mathcal{A}$ with $R[x]/\langle x^n-1\rangle$ under the ring isomorphism $\Theta$. Therefore, in order to determine all cyclic codes of length $n$ over $R$, we only need to determine all ideals of the ring $\mathcal{A}+v\mathcal{A}$. To do this, we investigate the structure of the ring $\mathcal{A}$ first.

In this paper, we will regard $\mathbb{Z}_2$ as a subset of the ring $\mathbb{Z}_4$ although $\mathbb{Z}_2$ is not a subring of $\mathbb{Z}_4$. Then every element $a$ of $\mathbb{Z}_4$ has a unique 2-adic expansion: $a=a_0+2a_1,\ a_0,a_1\in\mathbb{Z}_2$. Define $\overline{a}=a_0=a$ (mod 2) for all $a\in\mathbb{Z}_4$. This map $^-$ is a surjective homomorphism of rings from $\mathbb{Z}_4$ onto $\mathbb{Z}_2$, which can be extended into a surjective homomorphism of rings from $\mathbb{Z}_4[x]$ onto $\mathbb{Z}_2[x]$ in the natural way:

$$\overline{f}(x) = \overline{f(x)} = \sum_k \overline{f}_k x^k, \ \forall f(x) = \sum_k f_k x^k \in \mathbb{Z}_4[x] \text{ where } a_k \in \mathbb{Z}_4.$$

Let $g(x)$ be a monic polynomial in $\mathbb{Z}_4[x]$ of positive integer. Then $g(x)$ is said to be *basic irreducible* if $\overline{g}(x)$ is an irreducible polynomial in $\mathbb{Z}_2[x]$.

As $n$ is odd, by [22, Theorem 13.8] there are pairwise coprime monic basic irreducible polynomials $f_0(x) = x - 1, f_1(x), \ldots, f_r(x) \in \mathbb{Z}_4[x]$ such that

$$x^n - 1 = f_0(x) f_1(x) \ldots f_r(x), \tag{1}$$

where $\overline{f}_j(x)$ is irreducible in $\mathbb{Z}_2[x]$ and $\deg(f_j(x)) = m_j$ for all $j = 0, 1, \ldots, r$. Especially, $f_0(x) = x - 1$ with degree $m_0 = 1$.

For each integer $j$, $0 \le j \le r$, denote $F_j(x) = \frac{x^n-1}{f_j(x)} \in \mathbb{Z}_4[x]$. Since $\gcd(\overline{F}_j(x), \overline{f}_j(x)) = 1$, we see that $F_j(x)$ and $f_j(x)$ are coprime in $\mathbb{Z}_4[x]$ (cf. [22, Lemma 13.5]). Hence there are polynomials $c_j(x), d_j(x) \in \mathbb{Z}_4[x]$ such that

$$c_j(x) F_j(x) + d_j(x) f_j(x) = 1. \tag{2}$$

In this paper, we adopt the following notation:

- Let $e_j(x) \in \mathcal{A}$ satisfying
$$e_j(x) \equiv c_j(x) F_j(x) = 1 - d_j(x) f_j(x) \ (\text{mod } x^n - 1). \tag{3}$$
- Denote $\mathcal{K}_j = \mathbb{Z}_4[x]/\langle f_j(x)\rangle = \left\{\sum_{i=0}^{m_j-1} a_i x^i \mid a_i \in \mathbb{Z}_4,\ i = 0, 1, \ldots, m_j - 1\right\}$ in which the arithmetic is done modulo $f_j(x)$. Then $\mathcal{K}_j$ is a Galois ring of characteristic 4 and cardinality $4^{m_j}$ (cf. [22, Theorem 14.1]).
- Let $\mathcal{K}_j + v\mathcal{K}_j = \mathcal{K}_j[v]/\langle v^2 + 2v\rangle = \{\alpha + \beta v \mid \alpha, \beta \in \mathcal{K}_j\}$ $(v^2 = 2v)$ and the operations are defined by: for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{K}_j$, we have
$$(\alpha_1 + \beta_1 v) + (\alpha_2 + \beta_2 v) = (\alpha_1 + \alpha_2) + v(\beta_1 + \beta_2);$$
$$(\alpha_1 + \beta_1 v)(\alpha_2 + \beta_2 v) = \alpha_1\alpha_2 + v(\alpha_1\beta_2 + \beta_1\alpha_2 + 2\beta_1\beta_2).$$
Then $\mathcal{K}_j + v\mathcal{K}_j$ is a finite commutative local ring containing $\mathcal{K}_j$ as its subring.

By [21, Theorem 2.7] and [7, Lemma 3.2], one can easily deduce the following conclusions.

**Lemma 2** *Using the notation above, we have the following conclusions:*

(i) $e_0(x) + e_1(x) + \ldots + e_r(x) = 1$, $e_j(x)^2 = e_j(x)$ *and* $e_j(x)e_l(x) = 0$ *in the ring* $\mathcal{A}$ *for all* $0 \le j \ne l \le r$.

(ii) *For each integer* $j$, $0 \le j \le r$, $\mathcal{A}e_j(x)$ *is a subring of* $\mathcal{A}$ *with* $e_j(x)$ *as its multiplicative identity. Define*
$$\varphi_j(a(x)) = a(x)e_j(x)\ (mod\ x^n - 1),\ \forall a(x) \in \mathcal{K}_j.$$
*Then* $\varphi_j$ *is a ring isomorphism from* $\mathcal{K}_j$ *onto* $\mathcal{A}e_j(x)$ *with inverse* $\varphi_j^{-1}$*:*
$$\varphi_j^{-1}(c(x)) = c(x)\ (mod\ f_j(x)),\ \forall c(x) \in \mathcal{A}e_j(x).$$

(iii) *For any $a_j(x) \in \mathcal{K}_j$ and $j = 0, 1, \ldots, r$, define*

$$\varphi(a_0(x), a_1(x), \ldots, a_r(x)) = \sum_{j=0}^{r} \varphi_j(a_j(x)) = \sum_{j=0}^{r} e_j(x)a_j(x) \ (mod\ x^n - 1).$$

*Then $\varphi$ is a ring isomorphism from $\mathcal{K}_0 \times \mathcal{K}_1 \times \ldots \times \mathcal{K}_r$ onto $\mathcal{A}$.*

Now, we give structural properties for cyclic codes of length $n$ over $R$.

**Lemma 3** *Let $\mathcal{C} \subseteq \mathcal{A} + v\mathcal{A}$. Then $\mathcal{C}$ is a cyclic code of length n over $R$ if and only if for each integer $j$, $0 \le j \le r$, there is a unique ideal $C_j$ of the ring $\mathcal{K}_j + v\mathcal{K}_j$ ($v^2 = 2v$) such that*

$$\mathcal{C} = e_0(x)C_0 \oplus e_1(x)C_1 \oplus \ldots \oplus e_r(x)C_r \ (mod\ x^n - 1)$$

*where*

$$e_j(x)C_j = \{e_j(x)\alpha + ve_j(x)\beta \mid \alpha + \beta v \in C_j,\ \alpha, \beta \in \mathcal{K}_j\} \subseteq \mathcal{A} + v\mathcal{A}$$

*for all $j = 0, 1, \ldots, r$. Then the number of codewords in $\mathcal{C}$ equals $\prod_{j=0}^{r} |C_j|$.*

*Proof* For any $\xi_j = \alpha_j + \beta_j v \in \mathcal{K}_j + v\mathcal{K}_j$ with $\alpha_j, \beta_j \in \mathcal{K}_j$ for all $j = 0, 1, \ldots, r$, we define

$$\begin{aligned}\Phi(\xi) &= \sum_{j=0}^{r} e_j(x)\xi_j = \sum_{j=0}^{r} e_j(x)\left(\alpha_j + v\beta_j\right) = \sum_{j=0}^{r} e_j(x)\alpha_j + v\sum_{j=0}^{r} e_j(x)\beta_j \\ &= \varphi(\alpha_0, \alpha_1, \ldots, \alpha_r) + v\varphi(\beta_0, \beta_1, \ldots, \beta_r).\end{aligned}$$

Then $\Phi$ is an isomorphism of rings from the direct product ring $(\mathcal{K}_0 + v\mathcal{K}_0) \times (\mathcal{K}_1 + v\mathcal{K}_1) \times \ldots \times (\mathcal{K}_r + v\mathcal{K}_r)$ onto $\mathcal{A} + v\mathcal{A}$. This conclusion can be proved by Lemma 2 similarly as that for [8, Theorem 4.2(i)]. Here, we omit it.

From the properties of ring isomorphisms and direct product rings, we conclude that $\mathcal{C}$ is a cyclic code of length $n$ over $R$, i.e. $\mathcal{C}$ is an ideal of $\mathcal{A} + v\mathcal{A}$, if and only if for each integer $j$, $0 \le j \le r$, there is a unique ideal $C_j$ of the ring $\mathcal{K}_j + v\mathcal{K}_j$ such that

$$\begin{aligned}\mathcal{C} &= \Phi(C_0 \times C_1 \times \ldots \times C_r) = \{\Phi(\xi_0, \xi_1, \ldots, \xi_r) \mid \xi_j \in C_j,\ j = 0, 1, \ldots, r\} \\ &= \left\{\sum_{j=0}^{r} e_j(x)\xi_j \mid \xi_j \in C_j,\ j = 0, 1, \ldots, r\right\} = \sum_{j=0}^{r} e_j(x)\{\xi_j \mid \xi_j \in C_j\}.\end{aligned}$$

Hence $\mathcal{C} = \bigoplus_{j=0}^{r} e_j(x)C_j$ (mod $x^n - 1$) and $|\mathcal{C}| = |C_0 \times C_1 \times \ldots \times C_r| = \prod_{j=0}^{r} |C_j|$. □

By Lemma 3, it is sufficient to determine the ideals of the ring $\mathcal{K}_j + v\mathcal{K}_j$ for all $j$, in order to determine cyclic codes of length $n$ over $R$.

As $\mathcal{K}_j$ is a subring of $\mathcal{K}_j + v\mathcal{K}_j$, we can regard $\mathcal{K}_j + v\mathcal{K}_j$ as a $\mathcal{K}_j$-module. Precisely, $\mathcal{K}_j + v\mathcal{K}_j$ is a free $\mathcal{K}_j$-module with basis $\{1, v\}$. Let $\mathcal{K}_j^2 = \{(\alpha, \beta) \mid \alpha, \beta \in \mathcal{K}_j\}$. Then $\mathcal{K}_j^2$ is a free $\mathcal{K}_j$-module of rank 2 with the componentwise addition and scalar multiplication. Now, define

$$\sigma : \mathcal{K}_j^2 \to \mathcal{K}_j + v\mathcal{K}_j \text{ via } (\alpha, \beta) \mapsto \alpha + \beta v\ (\forall \alpha, \beta \in \mathcal{K}_j).$$

Then it is obvious that $\sigma$ is an isomorphism of $\mathcal{K}_j$-modules from $\mathcal{K}_j^2$ onto $\mathcal{K}_j + v\mathcal{K}_j$. Moreover, we have the following key conclusion.

**Lemma 4** *(cf.* [10, Lemma 3.4]*) Let* $0 \le j \le r$*. Then* $C_j$ *is an ideal of the ring* $\mathcal{K}_j + v\mathcal{K}_j$ *if and only if there is a unique* $\mathcal{K}_j$*-submodule* $S_j$ *of* $\mathcal{K}_j^2$ *satisfying the following condition*

$$(0, \alpha + 2\beta) \in S_j, \ \forall(\alpha, \beta) \in S_j \tag{4}$$

*such that* $\sigma(S_j) = C_j$*.*

Therefore, in order to determine all ideals of $\mathcal{K}_j + v\mathcal{K}_j$ ($0 \le j \le r$), it is sufficient to determine all $\mathcal{K}_j$-submodule of $\mathcal{K}_j^2$ satisfying Condition (4). To do this, we sketch some basic theory of linear codes over Galois rings.

In the rest of the paper, for each integer $j$, $0 \le j \le r$, we denote

- $T_j = \{\sum_{j=0}^{m-1} a_j x^j \mid a_j \in \mathbb{Z}_2\} \subseteq \mathcal{K}_j$;
- $F_j = \mathbb{Z}_2[x]/\langle \overline{f}_j(x)\rangle = \{\sum_{j=0}^{m-1} a_j x^j \mid a_j \in \mathbb{Z}_2\}$ in which the arithmetic is done modulo $\overline{f}_j(x)$ in the polynomial ring $\mathbb{Z}_2[x]$.

Then $F_j$ is an extension field of the binary field $\mathbb{Z}_2$ of $2^{m_j}$ elements, and $T_j$ is a subset of $\mathcal{K}_j$ since we regard $\mathbb{Z}_2$ as a subset of $\mathbb{Z}_4$.

To reduce the number of symbols, in this paper we will identify $T_j$ with $F_j$. Whether $T_j$ is a subset of the Galois ring $\mathcal{K}_j$ or a finite field itself, the reader can easily determine what it means according to the context. In this sense, each element $\alpha$ of $\mathcal{K}_j$ has a unique 2-adic expansion:

$$\alpha = t_0(x) + 2t_1(x), \text{ where } t_0(x), t_1(x) \in T_j.$$

Then $\alpha \in \mathcal{K}_j^{\times}$ if and only if $t_0(x) \neq 0$. Hence $|\mathcal{K}_j^{\times}| = (2^{m_j} - 1)2^{m_j}$.

Let $0 \le j \le r$ and $L \ge 2$ be a positive integer. Assume $S$ is a linear code of length $L$ over the Galois ring $\mathcal{K}_j$. By [16, Definition 3.1], a matrix $G$ is called a *generator matrix* for $S$ if the rows of $G$ span $S$ and none of them can be written as an $\mathcal{K}_j$-linear combination of the other rows of $G$. Moreover, a generator matrix $G$ is said to be *in standard form* if there is a suitable permutation matrix $U$ of size $L \times L$ such that

$$G = \begin{pmatrix} I_{k_0} & M_{0,1} & M_{0,2} \\ 0 & 2I_{k_1} & 2M_{1,2} \end{pmatrix} U$$

where the columns are grouped into blocks of column sizes $k_0, k_1, k$ with $k_i \ge 0$, $k = L - (k_0 + k_1)$, $M_{0,2}$ is a matrix over $\mathcal{K}_j$, $M_{0,1}$ and $M_{1,2}$ are matrices over $T_j$. Of course, if $k_i = 0$, the matrices $2^i I_{k_i}$ and $2^i M_{i,t}$ ($i = 0, 1$) are suppressed in $G$. In this case, the following map

$$(\xi, \eta) \mapsto (\xi, \eta)G = \left(\xi(I_{k_0}, M_{0,1}, M_{0,2}) + \eta(0, 2I_{k_1}, 2M_{1,2})\right) U$$

(for any $\xi = (\alpha_1, \ldots, \alpha_{k_0}) \in \mathcal{K}_j^{k_0}$ and $\eta = (b_1, \ldots, b_{k_1}) \in F_j^{k_1}$) is an isomorphism of groups from $(\mathcal{K}_j^{k_0} \times F_j^{k_1}, +)$ onto $(S, +)$. Hence $S$ is an abelian group of type $4^{k_0 m_j} 2^{k_1 m_j}$ and contain $2^{(2k_0 + k_1)m_j}$ codewords.

All distinct nontrivial linear codes of length 2 over finite chain rings had been determined (by [9, Example 2.5], for example). Moreover, we have

**Lemma 5** *All distinct linear code* $S_j$ *of length 2 over the Galois ring* $\mathcal{K}_j = \mathbb{Z}_4[x]/\langle f_j(x)\rangle$ *satisfying Condition* (4) *in Lemma 4 are given by the following table, where G is a generator matrix of* $S_j$*:*

| *Case* | $G$ | *type of* $S_j$ | $\|S_j\|$ |
|---|---|---|---|
| *(i)* | $I_2$ | $4^{2m_j}2^0$ | $2^{4m_j}$ |
| | $2I_2$ | $4^0 2^{2m_j}$ | $2^{2m_j}$ |
| | *0* | $4^0 2^0$ | 1 |
| *(ii)* | $(2w_j(x), 1)$, *where* $w_j(x) \in T_j$ *arbitrary* | $4^{m_j}2^0$ | $2^{2m_j}$ |
| *(iii)* | $(0, 2)$ | $4^0 2^{m_j}$ | $2^{m_j}$ |
| *(iv)* | $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ | $4^{m_j}2^{m_j}$ | $2^{3m_j}$ |

*Then the number of all codes listed above is equal to* $2^{m_j} + 5$.

*Proof* By $2^2 = 0$ and [9, Example 2.5] we know that the number of linear codes over the Galois ring $\mathcal{K}_j$ of length 2 is equal to $\sum_{i=0}^{2}(2i+1)|T_j|^{2-i} = \sum_{i=0}^{2}(2i+1)2^{(2-i)m_j} = 4^{m_j} + 3 \cdot 2^{m_j} + 5$. Precisely, every nontrivial linear code $S$ over $\mathcal{K}_j$ of length 2 has one and only one of the following matrices $G$ as its generator matrix:

**1.** $G = (1, a(x)), \forall a(x) \in \mathcal{K}_j$. **2.** $G = (2, 2b(x)), \forall b(x) \in T_j$.
**3.** $G = (2w(x), 1), \forall w(x) \in T_j$. **4.** $G = (0, 2)$. **5.** $G = 2I_2$.
**6.** $G = \begin{pmatrix} 1 & c(x) \\ 0 & 2 \end{pmatrix}, \forall c(x) \in T_j$. **7.** $G = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$.

Then by ordinary careful calculations (cf. [10, Appendix]), we obtain the conclusions. □

As the end of this section, for any integer $j$, $0 \le j \le r$, we determine the ideals of $\mathcal{K}_j + v\mathcal{K}_j$ and their annihilating ideals. For any ideal $C$ of $\mathcal{K}_j + v\mathcal{K}_j$, its *annihilating ideal* is defined by $\text{Ann}(C) = \{\beta \in \mathcal{K}_j + v\mathcal{K}_j \mid \alpha\beta = 0, \ \forall \alpha \in C\}$.

**Theorem 1** *Let* $0 \le j \le r$. *Then all distinct ideals* $C_j$ *and their annihilating ideals of the ring* $\mathcal{K}_j + v\mathcal{K}_j$ $(v^2 = 2v)$ *are given by the following table.*

| $N$ | $C_j$ | *Type of* $C_j$ | $\|C_j\|$ | *Ann*$(C_j)$ |
|---|---|---|---|---|
| *1* | $\langle 1 \rangle$ | $4^{2m_j}2^0$ | $2^{4m_j}$ | $\langle 0 \rangle$ |
| *1* | $\langle 2 \rangle$ | $4^0 2^{2m_j}$ | $2^{2m_j}$ | $\langle 2 \rangle$ |
| *1* | $\langle 0 \rangle$ | $4^0 2^0$ | *1* | $\langle 1 \rangle$ |
| $2^{m_j}$ | $\langle 2w_j(x) + v \rangle$ | $4^{m_j}2^0$ | $2^{2m_j}$ | $\langle 2(1 + w_j(x)) + v \rangle$ |
| *1* | $\langle 2v \rangle$ | $4^0 2^{m_j}$ | $2^{m_j}$ | $\langle 2, v \rangle$ |
| *1* | $\langle 2, v \rangle$ | $4^{m_j}2^{m_j}$ | $2^{3m_j}$ | $\langle 2v \rangle$ |

*where* $w_j(x) \in T_j$ *arbitrary and* $N$ *is the number of ideals in the same row.*

*Therefore, the number of ideals in* $\mathcal{K}_j + v\mathcal{K}_j$ *is* $2^{m_j} + 5$.

*Proof* It follows from Lemma 4, Lemma 5 and the definition for the $\mathcal{K}_j$-module isomorphism $\sigma$ from $\mathcal{K}_j^2$ onto $\mathcal{K}_j + v\mathcal{K}_j$, immediately. Here, we omit these details. □

As stated above, by Lemma 3 and Theorem 1 we conclude that the number of cyclic codes over $R$ of odd length $n$ is equal to $\prod_{j=0}^{r}(2^{m_j} + 5)$. Precisely, all these cyclic codes can be listed easily by Lemma 3 and Theorem 1.

Using the notation of Lemma 3, $\mathcal{C} = \bigoplus_{j=0}^{r} e_j(x)C_j$ is called the *canonical form decomposition* of the cyclic code $\mathcal{C}$ with length $n$ over $R$.

## 3 Self-dual cyclic codes over $\mathbb{Z}_4 + v\mathbb{Z}_4$ of odd length

In this section, we consider how to determine all distinct self-dual cyclic codes over $R = \mathbb{Z}_4 + v\mathbb{Z}_4$ of odd length $n$ precisely. To do this, we need to give the dual code for each cyclic code over $R$ of length $n$ first.

As $x^n = 1$ in the ring $\mathcal{A} = \mathbb{Z}_4[x]/\langle x^n - 1\rangle$, we have $x^{-1} = x^{n-1}$. For any $\alpha(x) = \sum_{i=0}^{n-1} \alpha_i x^i \in R[x]/\langle x^n - 1\rangle$, where $\alpha_i = a_i + vb_i \in R$ with $a_i, b_i \in \mathbb{Z}_4$ for all $0 \le i \le n-1$, we have that $\alpha(x) = a(x) + vb(x)$ where $a(x), b(x) \in \mathcal{A}$ given by $a(x) = \sum_{i=0}^{n-1} a_i x^i$ and $b(x) = \sum_{i=0}^{n-1} b_i x^i$. Now, we define

$$\tau(\alpha(x)) = \alpha(x^{-1}) = \alpha_0 + \sum_{i=1}^{n-1} \alpha_i x^{n-i} = a(x^{-1}) + vb(x^{-1}),$$

where $a(x^{-1}) = a_0 + \sum_{i=1}^{n-1} a_i x^{n-i}$, $b(x^{-1}) = b_0 + \sum_{i=1}^{n-1} b_i x^{n-i} \in \mathcal{A}$. It can be verified easily that the map $\tau$ is a ring automorphism of $R[x]/\langle x^n - 1\rangle$. Moreover, the following conclusion is well known.

**Lemma 6** *For any cyclic code $\mathcal{C}$ of length $n$ over $R$, its dual code is given by $\mathcal{C}^{\perp} = \tau(Ann(\mathcal{C})) = \{a(x^{-1}) \mid a(x) \in Ann(\mathcal{C}\}$, where*

$$Ann(\mathcal{C}) = \{\eta \in R[x]/\langle x^n - 1\rangle \mid \xi\eta = 0,\ \forall \xi \in \mathcal{C}\}$$

*is the annihilating ideal of $\mathcal{C}$ in $R[x]/\langle x^n - 1\rangle$.*

Since $R[x]/\langle x^n - 1\rangle = \mathcal{A} + v\mathcal{A}$ ($v^2 = 2v$) by Lemma 1, we see that the restriction of $\tau$ on its subring $\mathcal{A}$ is a ring automorphism of $\mathcal{A}$. We still denote this ring automorphism by $\tau$. Then $\tau(a(x)) = a(x^{-1})$ for all $a(x) \in \mathcal{A}$.

For any polynomial $f(x) = \sum_{i=0}^{m} a_i x^i \in \mathbb{Z}_4[x]$ of degree $m \ge 0$. The *reciprocal polynomial* of $f(x)$ is defined by $\widetilde{f}(x) = x^m f(\frac{1}{x})$, and $f(x)$ is said to be *self-reciprocal* if $\widetilde{f}(x) = \delta f(x)$ for some $\delta \in \mathbb{Z}_4^{\times} = \{1, -1\}$. By (1) in Section 2, we have $x^n - 1 = f_0(x) f_1(x) \ldots f_r(x)$. This implies

$$x^n - 1 = (-1)\widetilde{f_0}(x)\widetilde{f_1}(x)\ldots\widetilde{f_r}(x).$$

Since $f_0(x) = x - 1, f_1(x), \ldots, f_r(x)$ are pairwise coprime monic basic irreducible polynomials in $\mathbb{Z}_4[x]$, $\widetilde{f_0}(x), \widetilde{f_1}(x), \ldots, \widetilde{f_r}(x)$ are pairwise coprime basic irreducible polynomials in $\mathbb{Z}_4[x]$ as well. Hence for any integer $j$, $0 \le j \le r$, there is a unique integer $j'$, $0 \le j' \le r$, such that

$$\widetilde{f_j}(x) = \delta_j f_{j'}(x) \text{ for some } \delta_j \in \mathbb{Z}_4^{\times}.$$

Especially, we have $0' = 0$ since $\widetilde{f_0}(x) = 1 - x = (-1) f_0(x)$. Then by

$$x^n = 1 \text{ and } x^{m_j} f_j(x^{-1}) = \widetilde{f_j}(x) \text{ in } \mathcal{A},$$

from the definition for $e_j(x)$ (see (2) and (3)), we deduce that $e_j(x^{-1}) = e_{j'}(x)$.

We still use $\tau$ to denote the map $j \mapsto j'$. Then we have

$$\widetilde{f_j}(x) = \delta_j f_{\tau(j)}(x) \text{ and } \tau(e_j(x)) = e_j(x^{-1}) = e_{\tau(j)}(x). \tag{5}$$

Whether $\tau$ denotes the ring automorphism of $\mathcal{A}$ or this map is determined by the context. The next lemma shows the compatibility of the two uses of $\tau$.

**Lemma 7** *Using the notation above, we have the following conclusions.*

(i) *$\tau$ is a permutation on the set $\{0, 1, \ldots, r\}$ satisfying $\tau^{-1} = \tau$.*

(ii) *After a rearrangement of* $f_1(x), \ldots, f_r(x)$*, there exists a unique pair* $(\lambda, \rho)$ *of nonnegative integers such that*

- $\lambda + 2\rho = r$*;*
- $\tau(j) = j$*, for all* $0 \le j \le \lambda$*;*
- $\tau(\lambda + l) = \lambda + l + \rho$ *and* $\tau(\lambda + l + \rho) = \lambda + l$*, for all* $1 \le l \le \rho$*.*

(iii) *For any integer* $j$*,* $0 \le j \le r$*, the map* $\tau_j$ *defined by*

$$\tau_j(a(x) + vb(x)) = a(x^{-1}) + vb(x^{-1}) \ (mod\ f_{\tau(j)}(x)), \forall a(x), b(x) \in \mathcal{K}_j$$

*is an isomorphism of rings from* $\mathcal{K}_j + v\mathcal{K}_j$ *onto* $\mathcal{K}_{\tau(j)} + v\mathcal{K}_{\tau(j)}$*, where* $x^{-1} = x^{n-1}$ $(mod\ f_{\tau(j)}(x))$*.*

*Proof* (i) It follows from the definition of the map $\tau$ and that $f_{\tau(\tau(j))}(x) = \delta_{\tau(j)}^{-1}\widetilde{f}_{\tau(j)}(x) = \delta_{\tau(j)}^{-1}\delta_j^{-1}\widetilde{\widetilde{f}}_j(x) = \delta_{\tau(j)}^{-1}\delta_j^{-1}f_j(x) = f_j(x)$ by (5).

(ii) It follows from (i) and the properties of permutations on a finite set.

(iii) The map $\tau_j$ is well-defined and makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{K}_j + v\mathcal{K}_j & \stackrel{\tau_j}{\longrightarrow} & \mathcal{K}_{\tau(j)} + v\mathcal{K}_{\tau(j)} \\ \varphi_j \downarrow & & \downarrow \varphi_j \\ (\mathcal{A} + v\mathcal{A})e_j(x) & \stackrel{\tau|_{(\mathcal{A}+v\mathcal{A})e_j(x)}}{\longrightarrow} & (\mathcal{A} + v\mathcal{A})e_{\tau(j)}(x) \end{array},$$

where $\tau|_{(\mathcal{A}+v\mathcal{A})e_j(x)}$ is the restriction of the ring automorphism $\tau$ to the subring $(\mathcal{A} + v\mathcal{A})e_j(x)$ of $\mathcal{A} + v\mathcal{A}$. From this and by Lemma 2(ii), we deduce that $\tau_j$ is an isomorphism of rings from $\mathcal{K}_j + v\mathcal{K}_j$ onto $\mathcal{K}_{\tau(j)} + v\mathcal{K}_{\tau(j)}$. □

Now, using the notation of Lemma 3 and Theorem 1, we give the dual code for any cyclic code of length $n$ over $R$.

**Theorem 2** *Let* $\mathcal{C}$ *be a cyclic code of length* $n$ *over* $R$ *with the canonical form decomposition*

$$\mathcal{C} = \bigoplus_{0 \le j \le r} e_j(x)C_j,$$

*where* $C_j$ *is an ideal of* $\mathcal{K}_j + v\mathcal{K}_j$ *listed by Theorem 1 for all* $j$*. Then*

$$\mathcal{C}^{\perp} = \bigoplus_{0 \le j \le r} e_{\tau(j)}(x)D_{\tau(j)} \ (mod\ x^n - 1),$$

*where* $D_{\tau(j)} = \tau_j(Ann(C_j))$*,* $0 \le j \le r$*, is an ideal of* $\mathcal{K}_{\tau(j)} + v\mathcal{K}_{\tau(j)}$ *determined by the following table:*

| *N* | $C_j$ *(mod* $f_j(x)$*)* | $D_{\tau(j)}$ *(mod* $f_{\tau(j)}(x)$*)* |
|---|---|---|
| *3* | $\langle 2^k \rangle$ $(k = 0, 1, 2)$ | $\langle 2^{2-k} \rangle$ |
| $2^{m_j}$ | $\langle 2w_j(x) + v \rangle$ $(w_j(x) \in T_j)$ | $\langle 2(1 + w_j(x^{-1})) + v \rangle$ |
| *1* | $\langle 2v \rangle$ | $\langle 2, v \rangle$ |
| *1* | $\langle 2, v \rangle$ | $\langle 2v \rangle$ |

*where* $N$ *is the number of the pair* $(C_j, D_{\tau(j)})$ *of ideals in the same row.*

*Proof* Let $\alpha,\beta\in R[x]/\langle x^n-1\rangle=\mathcal{A}+v\mathcal{A}$. By Lemma 3 and its proof, we have that

$$\alpha=\sum_{j=0}^{r}e_j(x)\xi_j,\ \beta=\sum_{j=0}^{r}e_j(x)\eta_j,\ \text{where } \xi_j,\eta_j\in\mathcal{K}_j+v\mathcal{K}_j.$$

Since $e_j(x)^2=e_j(x)$ and $e_j(x)e_l(x)=0$ for all $0\leq j\neq l\leq r$ by Lemma 2(i), it follows that $\alpha\beta=\sum_{j=0}^{r}e_j(x)(\xi_j\eta_j)$. This implies that

$$\alpha\beta=0 \text{ in } R[x]/\langle x^n-1\rangle \iff \xi_j\eta_j=0 \text{ in } \mathcal{K}_j+v\mathcal{K}_j,\ \forall j=0,1,\ldots,r.$$

From this we deduce $\mathrm{Ann}(\mathcal{C})=\bigoplus_{j=0}^{r}e_j(x)\mathrm{Ann}(C_j)$ (mod $x^n-1$), where $\mathrm{Ann}(C_j)$ is the annihilating ideal of $C_j$ in $\mathcal{K}_j+v\mathcal{K}_j$ for any $0\leq j\leq r$. As $\tau$ is a ring isomorphism, we have

$$\mathcal{C}^{\perp}=\tau(\mathrm{Ann}(\mathcal{C}))=\sum_{j=0}^{r}\tau(e_j(x)\mathrm{Ann}(C_j)).$$

For any integer $j$, $0\leq j\leq r$, by (5) and the definition of $\tau$ we have $\tau(e_j(x))=e_j(x^{-1})=e_{\tau(j)}(x)$ and

$$\tau(e_j(x)c_j(x))=e_j(x^{-1})c_j(x^{-1})=e_{\tau(j)}(x)\tau_j(c_j(x)),$$

where $\tau_j(c_j(x))=c_j(x^{-1})=c(x^{n-1})$ (mod $f_{\tau(j)}(x)$) by Lemma 7(iii), for any $c_j(x)\in\mathrm{Ann}(C_j)$. From these, we deduce that

$$\mathcal{C}^{\perp}=\bigoplus_{j=0}^{r}e_{\tau(j)}(x)\tau_j(\mathrm{Ann}(C_j)).$$

Denote $D_{\tau(j)}=\tau_j(\mathrm{Ann}(C_j))$ where $0\leq j\leq r$. Since $\tau_j$ is a ring isomorphism from $\mathcal{K}_j+v\mathcal{K}_j$ onto $\mathcal{K}_{\tau(j)}+v\mathcal{K}_{\tau(j)}$ by Lemma 7(iii), we see that $D_{\tau(j)}$ is an ideal of the ring $\mathcal{K}_{\tau(j)}+v\mathcal{K}_{\tau(j)}$.

Let $C_j=\langle\langle 2w_j(x)+v\rangle\rangle$ where $w_j(x)\in T_j$. By Theorem 1 we have $\mathrm{Ann}(C_j)=\langle 2(1+w_j(x))+v\rangle$. Then by the definition of $\tau_j$, it follows that

$$D_{\tau(j)}=\tau_j(\mathrm{Ann}(C_j))=\langle\tau_j(2(1+w_j(x))+v)\rangle=\langle 2(1+w_j(x^{-1}))+v\rangle$$

(mod $f_{\tau(j)}(x)$). The expressions for $D_{\tau(j)}$ in other cases can be calculated easily. We omit these here. □

For any integer $j$, $1\leq j\leq\lambda$, as $f_j(x)$ is self-reciprocal in $\mathbb{Z}_4[x]$ by Lemma 7(ii) and (5), we see that $\overline{f}_j(x)$ is self-reciprocal in $\mathbb{Z}_2[x]$ and hence its degree $m_j$ must be even. Then it is well known that

$$x^{-1}=x^{2^{\frac{m_j}{2}}} \text{ in the field } F_j=\mathbb{Z}_2[x]/\langle\overline{f}_j(x)\rangle. \tag{6}$$

In the rest of this paper, we adopt the following notation:

- $\mathcal{H}_j=\left\{\xi\in F_j\mid \xi^{2^{\frac{m_j}{2}}}=\xi\right\}$. Then $\mathcal{H}_j$ is a subfield of $F_j$ with $2^{\frac{m_j}{2}}$ elements.
- Let $\mathrm{Tr}_j$ be the trace function from $F_j$ onto its subfield $\mathcal{H}_j$ defined by:

$$\mathrm{Tr}_j(\xi)=\xi+\xi^{2^{\frac{m_j}{2}}},\ \forall\xi\in F_j.$$

Then by [22, Corollary 7.17], we have that $|\mathrm{Tr}_j^{-1}(1)|=2^{\frac{m_i}{2}}$ where

$$\mathrm{Tr}_j^{-1}(1)=\{\xi\in F_j\mid \mathrm{Tr}_j(\xi)=1\}=\{\xi\in F_j\mid \xi+\xi^{2^{\frac{m_j}{2}}}=1\}. \tag{7}$$

Now is the time to list all self-dual cyclic codes over $R$ of length $n$.

**Theorem 3** *Using the notation above, all distinct self-dual cyclic codes of length n over $R = \mathbb{Z}_4 + v\mathbb{Z}_4$ $(v^2 = 2v)$ are given by:*

$$\mathcal{C} = \bigoplus_{0 \le j \le \lambda+2\rho} e_j(x) C_j (\mathrm{mod} x^n - 1),$$

*where $C_j$ is an ideal of $\mathcal{K}_j + v\mathcal{K}_j$ listed as follows.*

(i) *When $j = 0$, there is 1 ideal: $C_0 = \langle 2 \rangle$.*

(ii) *When $1 \le j \le \lambda$, there are $1 + 2^{\frac{m_j}{2}}$ ideals:*

$$C_j = \langle 2 \rangle, \text{ and } C_j = \langle 2w_j(x) + v \rangle \text{ where } w_j(x) \in Tr_j^{-1}(1) \text{ arbitrary.}$$

(iii) *When $\lambda + 1 \le j \le \lambda + \rho$, there are $2^{m_j} + 5$ pairs $(C_j, C_{j+\rho})$ of ideals listed by the following table:*

| *N* | $C_j$ *(mod $f_j(x)$)* | $\lvert C_j \rvert$ | $C_{j+\rho}$ *(mod $f_{j+\rho}(x)$)* |
|---|---|---|---|
| *3* | $\langle 2^k \rangle$, $0 \le k \le 2$ | $2^{(4-2k)m_j}$ | $\langle 2^{2-k} \rangle$ |
| $2^{m_j}$ | $\langle 2w_j(x) + v \rangle$, $w_j(x) \in T_j$ | $2^{2m_j}$ | $\langle 2(1 + w_j(x^{-1})) + v \rangle$ |
| *1* | $\langle 2v \rangle$ | $2^{m_j}$ | $\langle 2, v \rangle$ |
| *1* | $\langle 2, v \rangle$ | $2^{3m_j}$ | $\langle 2v \rangle$ |

*where N is the number of pairs in the same row.*

*Therefore, the number of self-dual cyclic codes of length n over R is*

$$L_n = \prod_{1 \le j \le \lambda} (1 + 2^{\frac{m_j}{2}}) \prod_{\lambda+1 \le j \le \lambda+\rho} (2^{m_j} + 5).$$

*Proof* By Lemma 7(ii) and Theorem 2, we have

$$\mathcal{C}^{\perp} = \left( \bigoplus_{j=0}^{\lambda} e_j(x) D_j \right) \oplus \left( \bigoplus_{j=\lambda+1}^{\lambda+\rho} e_{j+\rho} D_{j+\rho} \right) \oplus \left( \bigoplus_{j=\lambda+\rho+1}^{\lambda+2\rho} e_{j-\rho} D_{j-\rho} \right),$$

where $D_{j+\rho} = \tau_j(C_j)$ for all $j = \lambda + 1, \ldots, \lambda + \rho$. Hence the cyclic code $\mathcal{C}$ is self-dual, i.e., $\mathcal{C} = \mathcal{C}^{\perp}$, if and only if the following two conditions are satisfied:

(†) $C_j = D_j = \tau_j(\mathrm{Ann}(C_j))$, for all $0 \le j \le \lambda$;
(‡) $C_j$ is an arbitrary ideal of $\mathcal{K}_j + v\mathcal{K}_j$ and $C_{j+\rho} = D_{j+\rho} = \tau_{j+\rho}(\mathrm{Ann}(C_j))$, for all $j = \lambda + 1, \ldots, \lambda + \rho$.

Therefore, the class of self-dual cyclic codes over $R$ of length $n$ is the same as the class of cyclic codes over $R$ of length $n$: $\mathcal{C} = \bigoplus_{j=0}^{\lambda+2\rho} e_j(x) C_j$, where $C_j$ is an ideal of $\mathcal{K}_j + v\mathcal{K}_j$ satisfying Conditions (†) and (‡) for all $j$.

Every pair $(C_j, C_{j+\rho})$ of ideals can be determined by the table in Theorem 2, for all $j = \lambda + 1, \ldots, \lambda + \rho$. Then in order to determine codes in the latter class, we only need to consider ideals of $\mathcal{K}_j + v\mathcal{K}_j$ satisfying Conditions (†) for all $j = 0, 1, \ldots, \lambda$.

Now, let $0 \le j \le \lambda$ and $C_j$ is an ideal of $\mathcal{K}_j + v\mathcal{K}_j$ listed by Theorem 2. Then $C_j$ satisfies condition (†) if and only if $C_j$ is given by one of the following two cases:

(†-1) $C_j = \langle 2 \rangle$.
(†-2) $C_j = \langle 2w_j(x) + v \rangle$,
where $w_j(x) \in T_j$ satisfying $w_j(x) \equiv 1 + w_j(x^{-1})$ (mod $f_j(x)$, mod 2).

Since $w_j(x)$ is a polynomial in $\mathbb{Z}_2[x]$, the condition is equivalent to $w_j(x) + w_j(x^{-1}) \equiv 1$ (mod $\overline{f}_j(x)$), i.e.,

$$w_j(x) + w_j(x^{-1}) = 1 \text{ in } F_j = \mathbb{Z}_2[x]/\langle \overline{f}_j(x) \rangle. \tag{8}$$

Now, we have the following two subcases:

(‡-2-i) When $j = 0$, we have $\overline{f}_0(x) = x - 1$ and $F_0 = \mathbb{Z}_2$. In this case, there is no element $w_j \in \mathbb{Z}_2$ satisfying Condition (8).
(‡-2-ii) Let $1 \le j \le \lambda$. By (6) and $a^2 = a$ for all $a \in \mathbb{Z}_2$, we obtain

$$w_j(x^{-1}) = w_j(x^{2^{\frac{m_j}{2}}}) = (w_j(x))^{2^{\frac{m_j}{2}}}, \ \forall w_j(x) \in F_j.$$

Hence Condition (8) is equivalent to $w_j(x) \in \mathrm{Tr}_j^{-1}(1)$ by (7).

By Lemma 7(ii) and Theorem 2, we conclude that the class of cyclic codes over $R$ of length $n$: $\mathcal{C} = \bigoplus_{j=0}^{\lambda+2\rho} e_j(x)C_j$, where $C_j$ is an ideal of $\mathcal{K}_j + v\mathcal{K}_j$ satisfying Conditions (†) and (‡) for all $j$, is exactly the same as the class of cyclic codes over $R$ of length $n$ listed by the three cases (i)–(iii) of this theorem.

As stated above, we proved the theorem. □

As the end of this section, we list the number $L_n$ of self-dual cyclic codes of length $n$ over $\mathbb{Z}_4 + v\mathbb{Z}_4$ ($v^2 = 2v$), where $n$ is odd and $3 \le n \le 49$, by the following table.

| $n$ | $L_n$ | $\mathcal{M}_n$ | $n$ | $L_n$ | $\mathcal{M}_n$ |
|---|---|---|---|---|---|
| 3 | 3 | (2; ∅) | 27 | 13851 | (2, 6, 18; ∅) |
| 5 | 5 | (4; ∅) | 29 | 16385 | (28; ∅) |
| 7 | 13 | (∅; 3) | 31 | 50653 | (∅; 5, 5, 5) |
| 9 | 27 | (2, 6; ∅) | 33 | 107811 | (2, 10, 10, 10; ∅) |
| 11 | 33 | (10; ∅) | 35 | 266565 | (4; 3, 12) |
| 13 | 65 | (12; ∅) | 37 | 262145 | (36; ∅) |
| 15 | 315 | (2, 4; 4) | 39 | 799695 | (2, 12; 12) |
| 17 | 289 | (8, 8; ∅) | 41 | 1050625 | (20, 20; ∅) |
| 19 | 513 | (18; ∅) | 43 | 2146689 | (14, 14, 14; ∅) |
| 21 | 2691 | (2; 3, 6) | 45 | 11626335 | (2, 4, 6; 4, 12) |
| 23 | 2053 | (∅; 11) | 47 | 8388613 | (∅; 23) |
| 25 | 5125 | (4, 20; ∅) | 49 | 27263041 | (∅; 3, 21) |

where $\mathcal{M}_n = (m_1, \ldots, m_\lambda; m_{\lambda+1}, \ldots, m_{\lambda+\rho})$ corresponding to the degrees of monic basic irreducible divisors $f_1(x), \ldots, f_\lambda(x); f_{\lambda+1}(x), \ldots, f_{\lambda+\rho}(x)$ of $x^n - 1$ in $\mathbb{Z}_4[x]$ (see (1) in Section 1). It is obvious that

$$1 + m_1 + \ldots + m_\lambda + 2(m_{\lambda+1} + \ldots + m_{\lambda+\rho}) = n.$$

## 4 Quasi-cyclic code over $\mathbb{Z}_4$ derived from cyclic codes over $\mathbb{Z}_4 + v\mathbb{Z}_4$

In this section, we consider self-dual 2-quasi-cyclic codes of length $2n$ over $\mathbb{Z}_4$ derived from self-dual cyclic codes of length $n$ over the ring $R = \mathbb{Z}_4 + v\mathbb{Z}_4$. As in [15, Section 3], we define $\varrho : R \to \mathbb{Z}_4^2$ by

$$\varrho(\alpha) = (a+b, b),\ \forall \alpha = a + bv \in R \text{ where } a, b \in \mathbb{Z}_4$$

and let $\theta : R^n \to \mathbb{Z}_4^{2n}$ be such that $\theta(\alpha_1, \ldots, \alpha_n) = (\varrho(\alpha_1), \ldots, \varrho(\alpha_n))$, for all $\alpha_1, \ldots, \alpha_n \in R$. Let $w_L$ denote the Lee weight on $\mathbb{Z}_4$ defined by:

$$w_L(0) = 0,\ w_L(1) = w_L(3) = 1 \text{ and } w_L(2) = 2.$$

We extend $w_L$ in a natural way: for $a + bv \in R$ with $a, b \in \mathbb{Z}_4$, define

$$w_L(a + bv) = w_L(a + b) + w_L(b).$$

With this distance and Gray map definition, the following conclusions have been verified by Martínez-Moro et al. [15].

**Proposition 1** *([15, Theorem 3.1]) Let $\mathcal{C}$ be a linear code of length $n$ and minimum Lee distance $d$ over $R$. Then $\theta(\mathcal{C})$ is a linear code of length $2n$ over $\mathbb{Z}_4$, $|\theta(\mathcal{C})| = |\mathcal{C}|$ and is of minimum Lee distance $d$.*

**Proposition 2** *([15, Proposition 3.3]) Let $\mathcal{C}$ be a linear code of length $n$ over $R$. Then $\theta(\mathcal{C}^{\perp}) = \theta(\mathcal{C})^{\perp}$. In particular, if $\mathcal{C}$ is self-dual, then $\theta(\mathcal{C})$ is a self-dual code of length $2n$ over $\mathbb{Z}_4$ and has the same Lee weight distribution.*

Moreover, we have the following properties for cyclic codes over $R$.

**Proposition 3** *Let $\mathcal{C}$ be a cyclic codes of length $n$ over $R$. Then $\theta(\mathcal{C})$ is a 2-quasi-cyclic code of length $2n$ over $\mathbb{Z}_4$.*

*Proof* Let $\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathcal{C}$, where $\alpha_i = a_i + b_i v$ with $a_i, b_i \in \mathbb{Z}_4$ for all $i = 0, 1, \ldots, n-1$. Then $\theta(\underline{\alpha}) = (a_0+b_0, b_0, a_1+b_1, b_1, \ldots, a_{n-1}+b_{n-1}, b_{n-1}) \in \theta(\mathcal{C})$. Since $\mathcal{C}$ is cyclic, we have $(\alpha_{n-1}, \alpha_0, \alpha_1, \ldots, \alpha_{n-2}) \in \mathcal{C}$. This implies $(a_{n-1} + b_{n-1}, b_{n-1}, a_0 + b_0, b_0, a_1 + b_1, b_1, \ldots, a_{n-2} + b_{n-2}, b_{n-2}) \in \theta(\mathcal{C})$. Hence $\theta(\mathcal{C})$ is a 2-quasi-cyclic code of length $2n$ over $\mathbb{Z}_4$. □

As an application, we consider cyclic codes of length 15 over $R$. In this case, $x^{15} - 1 = f_0(x)f_1(x)f_2(x)f_3(x)f_4(x)$ where $f_0(x) = x - 1$, $f_1(x) = x^2 + x + 1$, $f_2(x) = x^4 + x^3 + x^2 + x + 1$, $f_3(x) = x^4 + 2x^2 + 3x + 1$ and $f_4(x) = \widetilde{f_3}(x)$ are monic basic irreducible polynomials in $\mathbb{Z}_4[x]$. Hence $m_1 = 2$ and $m_2 = m_3 = m_4 = 4$. Hence the number of cyclic codes of length 15 over $\mathbb{Z}_4 + v\mathbb{Z}_4$ is

$$(2^1 + 5) \cdot (2^2 + 5) \cdot (2^4 + 5)^3 = 583443.$$

First, for each integer $j$, $0 \le j \le 4$, we denote $F_j(x) = \frac{x^{15}-1}{f_j(x)}$,

- ⋄ $\mathcal{K}_j = \mathbb{Z}_4[x]/\langle f_j(x)\rangle = \left\{\sum_{i=0}^{m_j-1} a_i x^i \mid a_0, a_1, \ldots, a_{m_j-1} \in \mathbb{Z}_4\right\}$.
- ⋄ $T_j = \left\{\sum_{i=0}^{m_j-1} b_i x^i \mid b_0, b_1, \ldots, b_{m_j-1} \in \mathbb{Z}_2\right\} \subset \mathcal{K}_j$.

Then we find polynomials $c_j(x), d_j(x) \in \mathbb{Z}_4[x]$ satisfying $c_j(x)F_j(x)+d_j(x)f_j(x) = 1$ and set $e_j(x) \in \mathbb{Z}_4[x]/\langle x^{15}-1\rangle$ such that $e_j(x) \equiv c_j(x)F_j(x) \pmod{x^{15}-1}$. Precisely, we have

$$\begin{aligned}
e_0(x) &= 3x^{14}+3x^{13}+3x^{12}+3x^{11}+3x^{10}+3x^9+3x^8+3x^7+3x^6+3x^5+3x^4\\
&\quad +3x^3+3x^2+3x+3,\\
e_1(x) &= x^{14}+x^{13}+2x^{12}+x^{11}+x^{10}+2x^9+x^8+x^7+2x^6+x^5+x^4+2x^3+x^+x+2,\\
e_2(x) &= x^{14}+x^{13}+x^{12}+x^{11}+x^9+x^8+x^7+x^6+x^4+x^3+x^2+x,\\
e_3(x) &= x^{12}+2x^{10}+x^9+3x^8+x^6+2x^5+3x^4+x^3+3x^2+3x,\\
e_4(x) &= 3x^{14}+3x^{13}+x^{12}+3x^{11}+2x^{10}+x^9+3x^7+x^6+2x^5+x^3.
\end{aligned}$$

**(I)** By Lemma 3 and Theorem 1, all distinct 583443 cyclic codes of length 15 over $R$ are given by: $\mathcal{C} = \bigoplus_{j=0}^{4} e_j(x)C_j \pmod{x^{15}-1}$, where

- $C_0$ is one of the following 7 ideals of $\mathbb{Z}_4 + v\mathbb{Z}_4$:

  $C_0 = \langle 2^k\rangle$ with $|C_0| = 2^{4-2k}$, where $0 \le k \le 2$;
  $C_0 = \langle 2a+v\rangle$ with $|C_0| = 4$, where $a \in \mathbb{Z}_2 = \{0, 1\}$ arbitrary;
  $C_0 = \langle 2v\rangle$ with $|C_0| = 2$;
  $C_0 = \langle 2, v\rangle$ with $|C_0| = 8$.

- $C_1$ is one of the following 9 ideals of $\mathcal{K}_1 + v\mathcal{K}_1$:

  $C_1 = \langle 2^k\rangle$ with $|C_1| = 4^{4-2k}$, where $0 \le k \le 2$;
  $C_1 = \langle 2(a_0+a_1x)+v\rangle$ with $|C_1| = 16$, where $a_0, a_1 \in \mathbb{Z}_2$ arbitrary;
  $C_1 = \langle 2v\rangle$ with $|C_1| = 4$;
  $C_1 = \langle 2, v\rangle$ with $|C_1| = 64$.

- $C_j$ is one of the following 21 ideals of $\mathcal{K}_j + v\mathcal{K}_j$ for all $j = 2, 3, 4$:

  $C_j = \langle 2^k\rangle$ with $|C_j| = 16^{4-2k}$, where $0 \le k \le 2$;
  $C_j = \langle 2(a_0+a_1x+a_2x^2+a_3x^3)+v\rangle$ with $|C_j| = 16^2$,
  where $a_0, a_1, a_2, a_3 \in \mathbb{Z}_2$ arbitrary;
  $C_j = \langle 2v\rangle$ with $|C_j| = 16$;
  $C_j = \langle 2, v\rangle$ with $|C_j| = 16^3$.

**(II)** We have $r = 4$, $\lambda = 2$ and $\rho = 1$. For $j = 1, 2$, set

$\diamond$ $F_j = \mathbb{Z}_2[x]/\langle \overline{f}_j(x)\rangle = \{\sum_{i=0}^{m_j-1} a_i x^i \mid a_0, a_1, \ldots, a_{m_j-1} \in \mathbb{Z}_2\}$
in which the arithmetic is done modulo $\overline{f}_j(x)$;
$\diamond$ $\mathrm{Tr}_j^{-1}(1) = \{a(x) \in F_j \mid a(x) + a(x^{-1}) \equiv 1 \pmod{\overline{f}_j(x)}\}$.

Then by $x^{-1} \equiv x^{14} \pmod{\overline{f}_j(x)}$ for $j = 1, 2$, we have the following

$\mathrm{Tr}_1^{-1}(1) = \{a+bx \in \mathbb{Z}_2[x]/\langle \overline{f}_1(x)\rangle \mid (a+bx)+(a+bx^{14}) \equiv 1 \pmod{\overline{f}_1(x)}\} = \{a+bx \mid a+bx+a+b(1+x)+1 = 0,\ a, b \in \mathbb{Z}_2\} = \{x, 1+x\}$.

$\mathrm{Tr}_2^{-1}(1) = \{a+bx+cx^2+dx^3 \in \mathbb{Z}_2[x]/\langle \overline{f}_2(x)\rangle \mid (a+bx+cx^2+dx^3)+(a+bx^{14}+cx^{13}+dx^{12}) \equiv 1 \pmod{\overline{f}_2(x)}\} = \{a+bx+cx^2+dx^3 \mid bx+cx^2+dx^3+b(1+x+x^2+x^3)+cx^3+dx^2+1 = 0,\ a, b, c, d \in \mathbb{Z}_2\} = \{a+x+cx^2+(1+c)x^3 \mid a, c \in \mathbb{Z}_2\}$.

By Theorem 3, all 315 self-dual cyclic codes of length 15 over $R$ are given by: $\mathcal{C} = \bigoplus_{j=0}^{4} e_j(x)C_j \pmod{x^{15}-1}$, where

- $C_0 = \langle 2\rangle = 2(\mathbb{Z}_4 + v\mathbb{Z}_4)$.

- $C_1 = \langle 2\rangle$, $C_1 = \langle 2w_1(x) + v\rangle$ where $w_1(x) \in \mathrm{Tr}_1^{-1}(1)$.
- $C_2 = \langle 2\rangle$, $C_2 = \langle 2w_2(x) + v\rangle$ where $w_2(x) \in \mathrm{Tr}_2^{-1}(1)$.
- $(C_3, C_4)$ is given by the following table:

| N | $C_3$ (mod $f_3(x)$) | $\|C_3\|$ | $C_4$ (mod $f_4(x)$) |
|---|---|---|---|
| 3 | $\langle 2^k\rangle, 0 \le k \le 2$ | $2^{4(4-2k)}$ | $\langle 2^{2-k}\rangle$ |
| 16 | $\langle 2w_3(x) + v\rangle$ | $2^8$ | $\langle 2(1 + w_3(x^{-1})) + v\rangle$ |
| 1 | $\langle 2v\rangle$ | $2^4$ | $\langle 2, v\rangle$ |
| 1 | $\langle 2, v\rangle$ | $2^{12}$ | $\langle 2v\rangle$ |

in which $N$ is the number of pairs $(C_3, C_4)$ in the same row, and

$w_3(x) = a + bx + cx^2 + dx^3$;

$w_3(x^{-1}) = a + d + (d + c)x + (c + b)x^2 + bx^3 \equiv w_3(x^{14}) \pmod{f_4(x), 2}$,

for $a, b, c, d \in \mathbb{Z}_2$ arbitrary.

By Proposition 3, we obtain 315 2-quasi-cyclic self-dual codes $\theta(\mathcal{C})$ of length 30 over $\mathbb{Z}_4$. Among these codes, there are 70 codes with minimum Lee weight 12. These 70 2-quasi-cyclic self-dual codes over $\mathbb{Z}_4$ are given by the following table, with $C_1$, $C_2$, $C_3$, $C_4$ and the type of each $\mathbb{Z}_4$-code $\theta(\mathcal{C})$.

| $C_1$ | $C_2$ | $C_3$ (mod $f_3(x)$) | $C_4$ (mod $f_4(x)$) | Type |
|---|---|---|---|---|
| $\langle v + 2x\rangle$ | $\langle 2x^3 + 2x + v\rangle$ | $\langle 2\rangle$ | $\langle 2\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x + 2\rangle$ | $\langle 2x^3 + 2x + v\rangle$ | $\langle 2\rangle$ | $\langle 2\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x\rangle$ | $\langle 2x^2 + 2x + v\rangle$ | $\langle 2\rangle$ | $\langle 2\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x + 2\rangle$ | $\langle 2x^2 + 2x + v\rangle$ | $\langle 2\rangle$ | $\langle 2\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x\rangle$ | $\langle 2x^3 + 2x + v + 2\rangle$ | $\langle 2\rangle$ | $\langle 2\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x + 2\rangle$ | $\langle 2x^3 + 2x + v + 2\rangle$ | $\langle 2\rangle$ | $\langle 2\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x\rangle$ | $\langle 2x^2 + 2x + v + 2\rangle$ | $\langle 2\rangle$ | $\langle 2\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x + 2\rangle$ | $\langle 2x^2 + 2x + v + 2\rangle$ | $\langle 2\rangle$ | $\langle 2\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x\rangle$ | $\langle 2\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x + 2\rangle$ | $\langle 2\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x\rangle$ | $\langle 2\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x + 2\rangle$ | $\langle 2\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{18}4^6$ |
| $\langle v + 2x\rangle$ | $\langle 2x^3 + 2x + v\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2x^2 + 2x + v\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2x^3 + 2x + v + 2\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2x^2 + 2x + v + 2\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2\rangle$ | $\langle 2x^3 + 2x^2 + v\rangle$ | $\langle 2x^2 + v\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2\rangle$ | $\langle 2x^3 + 2x + v\rangle$ | $\langle 2x^3 + 2x^2 + 2x + v\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2\rangle$ | $\langle 2x^3 + 2x^2 + 2x + v\rangle$ | $\langle 2x^3 + v\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2\rangle$ | $\langle 2x^3 + v + 2\rangle$ | $\langle v + 2x + 2\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x + 2\rangle$ | $\langle 2\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x\rangle$ | $\langle 2\rangle$ | $\langle 0\rangle$ | $\langle 1\rangle$ | $2^{10}4^{10}$ |
| $\langle v + 2x + 2\rangle$ | $\langle 2\rangle$ | $\langle 0\rangle$ | $\langle 1\rangle$ | $2^{10}4^{10}$ |
| $\langle 2\rangle$ | $\langle 2x^3 + 2x + v\rangle$ | $\langle 2x^3 + v + 2\rangle$ | $\langle v + 2x + 2\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^3 + 2x + v\rangle$ | $\langle 2x^3 + 2x^2 + 2x + v + 2\rangle$ | $\langle 2x^3 + v + 2\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^3 + 2x + v\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^2 + 2x + v\rangle$ | $\langle 2x^3 + 2x + v\rangle$ | $\langle 2x^3 + 2x^2 + 2x + v\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^2 + 2x + v\rangle$ | $\langle 2x^3 + 2x^2 + v + 2\rangle$ | $\langle 2x^2 + v + 2\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^2 + 2x + v\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^6 4^{12}$ |

| | | | | |
|---|---|---|---|---|
| $\langle 2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle 0\rangle$ | $\langle 1\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2x^3+v\rangle$ | $\langle v+2x\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2x^3+2x^2+2x+v\rangle$ | $\langle 2x^3+v\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 2x^3+2x^2+v\rangle$ | $\langle 2x^2+v\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2x^3+2x^2+2x+v+2\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^6 4^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 0\rangle$ | $\langle 1\rangle$ | $2^6 4^{12}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle v\rangle$ | $\langle v+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2x^3+2x^2+v\rangle$ | $\langle 2x^2+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2x^3+2x^2+v\rangle$ | $\langle 2x^2+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle v+2\rangle$ | $\langle v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2x^2+v+2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2x^3+2x^2+2x+v+2\rangle$ | $\langle 2x^3+v+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle v\rangle$ | $\langle v+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle v+2x\rangle$ | $\langle 2x^3+2x^2+v+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle v+2x\rangle$ | $\langle 2x^3+2x^2+v+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2x^3+2x^2+2x+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle v+2\rangle$ | $\langle v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle 2x^3+v+2\rangle$ | $\langle v+2x+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle 2x^3+v+2\rangle$ | $\langle v+2x+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle v\rangle$ | $\langle v+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2x^3+v\rangle$ | $\langle v+2x\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2x^2+v\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2x^3+2x^2+2x+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2x^3+2x^2+2x+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle v+2\rangle$ | $\langle v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle v\rangle$ | $\langle v+2\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 2x^3+2x^2+v\rangle$ | $\langle 2x^2+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 2x^3+2x^2+2x+v\rangle$ | $\langle 2x^3+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 2x^3+2x^2+2x+v\rangle$ | $\langle 2x^3+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle v+2\rangle$ | $\langle v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle v+2x+2\rangle$ | $\langle 2x^3+2x^2+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle v+2x+2\rangle$ | $\langle 2x^3+2x^2+v\rangle$ | $2^2 4^{14}$ |
| $\langle v+2x\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $2^2 4^{14}$ |

There are 92 2-quasi-cyclic self-dual codes $\theta(\mathcal{C})$ of length 30 with minimum Lee weight 10 over $\mathbb{Z}_4$ derived from the 315 self-dual codes $\mathcal{C}$ of length 15 over $\mathbb{Z}_4+v\mathbb{Z}_4$. These 92 codes are given by the following table, with $C_1$, $C_2$, $C_3$, $C_4$ and the type of each $\mathbb{Z}_4$-code $\theta(\mathcal{C})$.

| $C_1$ | $C_2$ | $C_3$ (mod $f_3(x)$) | $C_4$ (mod $f_4(x)$) | Type |
|---|---|---|---|---|
| $\langle 2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{14} 4^8$ |
| $\langle 2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{14} 4^8$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{14} 4^8$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{14} 4^8$ |

| | | | | |
|---|---|---|---|---|
| $\langle 2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{14}4^8$ |
| $\langle 2\rangle$ | $\langle 2x^3+2x+v+2\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{14}4^8$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{14}4^8$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{14}4^8$ |
| $\langle v+2x\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{10}4^{10}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{10}4^{10}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2, v\rangle$ | $\langle 2v\rangle$ | $2^{10}4^{10}$ |
| $\langle v+2x+2\rangle$ | $\langle 2x^2+2x+v\rangle$ | $\langle 2v\rangle$ | $\langle 2, v\rangle$ | $2^{10}4^{10}$ |
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| $\langle 2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle v\rangle$ | $\langle v+2\rangle$ | $2^64^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^3+2x+v\rangle$ | $\langle 2x^3+2x^2+v\rangle$ | $\langle 2x^2+v\rangle$ | $2^64^{12}$ |
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| $\langle 2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle v\rangle$ | $\langle v+2\rangle$ | $2^64^{12}$ |
| $\langle 2\rangle$ | $\langle 2x^2+2x+v+2\rangle$ | $\langle 2x^3+v\rangle$ | $\langle v+2x\rangle$ | $2^64^{12}$ |
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Let $\phi$ be the Gray map from $\mathbb{Z}_4^{30}$ onto $\mathbb{F}_2^{60}$ extended by $0\to 00, 1\to 01, 2\to 11, 3\to 10$ in the natural way. Then $\phi$ is a distance and orthogonality preserving bijection from ($\mathbb{Z}_4^{30}$, Lee distance) onto ($\mathbb{F}_2^{60}$, Hamming distance). From the 70 2-quasi-cyclic self-dual codes with minimal Lee weight 12 and 92 2-quasi-cyclic self-dual codes with minimum Lee weight 10 over $\mathbb{Z}_4$ above and by the Gray map $\phi$, we derive 70 4-quasi-cyclic type II binary formally self-dual [60, 30, 12] codes and 92 4-quasi-cyclic type II binary formally self-dual [60, 30, 10] codes. It is well known that binary self-dual [60, 30, 12] codes are extremal (cf. [14] and [24]).

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