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Two constructions of asymptotically optimal codebooks

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Abstract
Codebooks with low-coherence have wide applications in many fields such as direct spread code division multiple access communications, compressed sensing, signal processing and so on. In this paper, we propose two constructions of complex codebooks from the operations of certain sets. The complex codebooks produced by these constructions are proved to be asymptotically optimal with respect to the Welch bound. In addition, the parameters of the complex codebooks presented in this paper are new and flexible in some cases.

Keywords Codebook · Asymptotic optimality · Welch bound · Hyper Eisenstein sum

Mathematics Subject Classification (2010) 94B15 · 11T71

1 Introduction

An (N, K) codebook $\mathcal{C}$ (also called a signal set) is a set of N unit-norm complex vectors $\mathbf{c}_i$ with length K, where $0 \le i \le N-1$. For all pairs of distinct vectors in $\mathcal{C}$, the maximum cross-correlation amplitude of $\mathcal{C}$ is defined by

$$I_{max}(\mathcal{C}) = \max_{0\le i\ne j\le N-1} |\mathbf{c}_i \mathbf{c}_j^H|,$$

where $\mathbf{c}_j^H$ is the conjugate transpose of the complex vector $\mathbf{c}_j$. In code division multiple access (CDMA) systems, codebooks with small $I_{max}(\mathcal{C})$ are utilized to separate the sig-

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nals of different users. The optimization of various performance indicators such as outage probability, average signal-to-noise ratio and symbol error probability for multiple-antenna transmit beamforming from limited-rate feedback can be achieved by minimizing the maximum cross-correlation amplitude $I_{max}(\mathcal{C})$ of a codebook $\mathcal{C}$ [16, 19]. For a certain length K, it is desirable to design a codebook such that the number of the vectors N is as large as possible and the maximum cross-correlation amplitude $I_{max}(\mathcal{C})$ is as small as possible. However, between the parameters N, K and $I_{max}(\mathcal{C})$ of a codebook $\mathcal{C}$, there exists a tradeoff, known as Welch bound [27].

Lemma 1 [27] *For any (N, K) codebook $\mathcal{C}$ with $N \geq K$,*

$$I_{max}(\mathcal{C}) \geq I_W = \sqrt{\frac{N-K}{(N-1)K}}.$$

Moreover, the equality holds if and only if for all pairs of (i, j) with $i \neq j$, it holds that

$$|\mathbf{c}_i \mathbf{c}_j^H| = \sqrt{\frac{N-K}{(N-1)K}}.$$

We term $\mathcal{C}$ a maximum-Welch-bound-equality (MWBE) codebook [23] if $\mathcal{C}$ achieves the Welch bound. MWBE codebooks have many applications in communications [23], CDMA systems [20], compressed sensing [1], coding theory [6] and quantum computing [22]. MWBE codebooks can also be described by equiangular tight frames [25] or line packing in Grassmannian spaces [24]. With the wide utilization, the constructions of MWBE codebooks have attracted much attention and the known classes of MWBE codebooks are presented as follows:

- (N, N) orthogonal MWBE codebooks for any $N > 1$ [23, 28];
- $(N, N-1)$ MWBE codebooks for $N > 1$ based on discrete Fourier transformation matrices [23, 28] or m-sequences [23];
- (N, K) MWBE codebooks from conference matrices [2, 24], where $N = 2K = 2^{d+1}$ for a positive integer d or $N = 2K = p^d + 1$ for a prime p and a positive integer d;
- (N, K) MWBE codebooks based on (N, K, λ) difference sets in cyclic groups [28] and abelian groups [4, 5];
- (N, K) MWBE codebooks from $(2, k, v)$-Steiner systems [7];
- (N, K) MWBE codebooks depended on graph theory and finite geometries [8–10, 21].

Aside from MWBE codebooks, many researchers concern about asymptotically optimal codebooks $\mathcal{C}$, i.e., the maximum cross-correlation amplitude asymptotically achieves the Welch bound for the sufficiently large length of the vectors. As a generalization of the MWBE codebooks derived from difference sets, several classes of asymptotically optimal codebooks were generated from almost difference sets [11, 15, 30, 31], relative difference sets [32]. In [3, 12, 29], asymptotically optimal codebooks constructed from binary row selection sequences were proposed. In [26], Tan et al. proposed a class of asymptotically optimal codebooks by using Gauss sums. Employing Jacobi sums, Heng et al. [13] presented two classes of asymptotically optimal codebooks. As generalizations of Heng et al.'s work, the authors provided several classes of asymptotically optimal codebooks in [14, 17].

Assume that G_1 and G_2 are sets. Let D_1 and D_2 be subsets of G_1 and G_2, respectively. We define the operation of D_1 and D_2 by

$$D_1 \bigtriangledown D_2 = (D_1 \times \overline{D}_2) \cup (\overline{D}_1 \times D_2),$$

where $\overline{D}_1 = G_1 \setminus D_1$, $\overline{D}_2 = G_2 \setminus D_2$ and $D_1 \times D_2 = \{(a, b) : a \in D_1, b \in D_2\}$. When D_1 and D_2 are difference sets, Hu et al. [11] constructed two classes of asymptotically optimal codebooks. Motivated by their work, we investigate the case that D_1 and D_2 are certain sets and propose two classes of codebooks asymptotically meeting the Welch bound. Notably, the parameters of our codebooks are new and flexible in some cases. For reference, the parameters of known classes of codebooks asymptotically achieving the Welch bound and the new ones are given in Table 1.

This paper is organized as follows. In Section 2, we briefly recall some definitions and notation which will be needed in our discussion. We devote Sections 3 and 4 to our constructions of asymptotically optimal codebooks. Finally, Section 5 concludes this paper.

Table 1 The parameters of codebooks asymptotically meeting the Welch bound

Parameters (N, K)	Constraints	I_{max}	References
$\left(p^n, K = \frac{p-1}{2p}(p^n + p^{n/2}) + 1\right)$	p is an odd prime	$\frac{(p+1)p^{n/2}}{2pK}$	[12]
$\left(q^2, \frac{(q-1)^2}{2}\right)$	q is a power of an odd prime,	$\frac{q+1}{(q-1)^2}$	[30]
$\left(q(q+4), \frac{(q+3)(q+1)}{2}\right)$	q is a prime power,	$\frac{1}{q+1}$	[15]
$\left(q, \frac{q+1}{2}\right)$	q is a prime power,	$\frac{\sqrt{q}+1}{q-1}$	[15]
$\left(p^n - 1, \frac{p^n-1}{2}\right)$	p is an odd prime	$\frac{\sqrt{p^n}+1}{p^n-1}$	[29]
$(q^l + q^{l-1} - 1, q^{l-1})$	q is a prime power, and $l > 2$ is an integer	$\frac{1}{\sqrt{q^{l-1}}}$	[32]
$((q-1)^k + q^{k-1}, q^{k-1})$	$q \geq 4$ is a prime power and $k > 2$	$\frac{\sqrt{q^{k+1}}}{(q-1)^k+(-1)^{k+1}}$	[13]
$((q-1)^k + K, K)$	$K = \frac{(q-1)^k+(-1)^{k+1}}{q}$, $k > 2$ and q is a prime power	$\frac{\sqrt{q^{k-1}}}{K}$	[13]
$((q^s-1)^n + K, K)$	$K = \frac{(q^s-1)^n+(-1)^{n+1}}{q}$, $s > 1, n > 1$, and q is a prime power	$\frac{\sqrt{q^{sn+1}}}{(q^s-1)^n+(-1)^{n+1}}$	[17]
$((q^s-1)^n + q^{sn-1}, q^{sn-1})$	$s > 1, n > 1$, and q is a prime power	$\frac{\sqrt{q^{sn+1}}}{(q^s-1)^n+(-1)^{n+1}}$	[17]
$\left(N_1 N_2, \frac{N_1 N_2 - 1}{2}\right)$	$N_1 \equiv 3 \bmod 4$, $N_2 \equiv 3 \bmod 4$	$\frac{\sqrt{(N_1+1)(N_2+1)}}{N_1 N_2 - 1}$	[11]
$\left(N_1 \cdots N_l, \frac{N_1 \cdots N_l - 1}{2}\right)$	$N_i \equiv 3 \bmod 4$ for any $l > 1$	$\frac{\sqrt{(N_1+1)\cdots(N_l+1)}}{N_1 \cdots N_l - 1}$	[11]
$(2K + 1, K)$	$K = \frac{(2^{s_1}-1)^n(2^{s_2}-1)^n-1}{2}$, $n \geq 1, s_1, s_2 > 1$	$\frac{2^{\frac{s_1 n + s_2 n}{2}}}{(2^{s_1}-1)^n(2^{s_2}-1)^n-1}$	Theorem 1
$(2K + (-1)^{ln}, K)$	$K = \frac{(2^{s_1}-1)^n \cdots (2^{s_l}-(-1)^{ln})^n - 1}{2}$, $n \geq 1, l > 1$, $s_i > 1$ for any $1 \leq i \leq l$	$\frac{2^{(s_1 n + s_2 n + \cdots + s_l n)/2}}{2K}$	Theorem 3

2 Preliminaries

Let $q = p^s$ and $\mathbb{F}_q$ denote the finite field with q elements, where p is a prime number and s is a positive integer. The trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$ is defined by

$$\mathrm{Tr}_q(x) = x + x^p + \cdots + x^{p^{s-1}},$$

where $x \in \mathbb{F}_q$.

Denote by $\mathbb{F}_q^*$ the multiplicative group of $\mathbb{F}_q$. It is well-known that $\mathbb{F}_q^*$ is a cyclic group of order $q-1$. A generator of $\mathbb{F}_q^*$ is said to be a primitive element of the finite field $\mathbb{F}_q$. Assume that α is a primitive element of $\mathbb{F}_q$. For each integer i with $0 \le i \le q-2$, a multiplicative character φ_i of $\mathbb{F}_q$ is defined by

$$\varphi_i(\alpha^j) = \xi_{q-1}^{ij},$$

where $0 \le j \le q-2$ and $\xi_{q-1} = e^{2\pi\sqrt{-1}/(q-1)}$. Each multiplicative character of $\mathbb{F}_q$ can be obtained in this way. If $i = 0$, the multiplicative character φ_0 is called the trivial multiplicative character of $\mathbb{F}_q$. For any multiplicative character φ_i of $\mathbb{F}_q$, its conjugate $\overline{\varphi_i}$ is given by $\overline{\varphi_i}(g) = \overline{\varphi_i(g)} = \varphi_i(g^{-1})$, where $g \in \mathbb{F}_q^*$. All the multiplicative characters of $\mathbb{F}_q$ form a cyclic group $\widehat{\mathbb{F}_q^*}$ under the multiplication of characters defined as follows:

$$\chi\psi(g) = \chi(g)\psi(g),$$

for every $g \in \mathbb{F}_q^*$, where χ, ψ are multiplicative characters of $\mathbb{F}_q$. The group $\widehat{\mathbb{F}_q^*}$ is isomorphic to the multiplicative group $\mathbb{F}_q^*$. The orthogonal relations of $\mathbb{F}_q^*$ and $\widehat{\mathbb{F}_q^*}$ are

$$\sum_{g\in\mathbb{F}_q^*} \varphi(g) = \begin{cases} 0, & \text{if } \varphi \text{ is a nontrivial multiplicative character of } \mathbb{F}_q, \\ q-1, & \text{if } \varphi \text{ is the trivial multiplicative character of } \mathbb{F}_q, \end{cases}$$

and

$$\sum_{\varphi\in\widehat{\mathbb{F}_q^*}} \varphi(g) = \begin{cases} 0, & \text{if } g \neq 1 \in \mathbb{F}_q^*, \\ q-1, & \text{if } g = 1. \end{cases}$$

For each $a \in \mathbb{F}_q$, an additive character of $\mathbb{F}_q$ is defined by the function $\chi_a(x) = \zeta_p^{\mathrm{Tr}_q(ax)}$, where $\zeta_p = e^{2\pi\sqrt{-1}/p}$. If $a = 1$, then $\chi_1(x)$ is the canonical additive character of $\mathbb{F}_q$. Let $\chi_0(x)$ denote the trivial additive character of $\mathbb{F}_q$. The additive character of $\mathbb{F}_q$ has the similar properties as the multiplicative character of $\mathbb{F}_q$. For more details on the character theory over finite fields, we refer the reader to [18, Chapter. 5].

Luo et al. [17] established the hyper Eisenstein sum over $\mathbb{F}_q$. Let $D = \{(x_1, x_2, \cdots, x_n) \in (\mathbb{F}_q^*)^n : \mathrm{Tr}_q(x_1 + x_2 + \cdots + x_n) = 1\}$ and assume that $\varphi_1, \cdots, \varphi_n$ are multiplicative characters of $\mathbb{F}_q$. Then the hyper Eisenstein sum is given by

$$E_q(\varphi_1, \cdots, \varphi_n) = \sum_{(x_1,x_2,\cdots,x_n)\in D} \varphi_1(x_1)\cdots\varphi_n(x_n).$$

The following two lemmas evaluate the absolute value of the hyper Eisenstein sums.

Lemma 2 [17] *For $1 \le k < n$, assume that $\varphi_1, \cdots, \varphi_k$ are nontrivial multiplicative characters of $\mathbb{F}_q$ and $\varphi_{k+1}, \cdots, \varphi_n$ are trivial multiplicative characters of $\mathbb{F}_q$. Then*

$$E_q(\varphi_1, \cdots, \varphi_n) = (-1)^{n-k} E_q(\varphi_1, \cdots, \varphi_k).$$

Lemma 3 [17] *Let $\varphi_1, \cdots, \varphi_n$ be nontrivial multiplicative characters of $\mathbb{F}_q$. Denote by $(\varphi_1 \cdots \varphi_n)^*$ the restriction of $\varphi_1 \cdots \varphi_n$ to $\mathbb{F}_p$. Then*

$$|E_q(\varphi_1, \cdots, \varphi_n)| = \begin{cases} p^{\frac{sn-1}{2}}, & \text{if } (\varphi_1 \cdots \varphi_n)^* \text{ is nontrivial,} \\ p^{\frac{sn-2}{2}}, & \text{if } (\varphi_1 \cdots \varphi_n)^* \text{ is trivial.} \end{cases}$$

If $p = 2$, $(\varphi_1 \cdots \varphi_n)^*$ is always trivial which leads to the following proposition.

Proposition 1 *Let $q = 2^s$, where s is a positive integer. Assume that $\varphi_1, \cdots, \varphi_n$ are nontrivial multiplicative characters of $\mathbb{F}_q$. Then*

$$|E_q(\varphi_1, \cdots, \varphi_n)| = 2^{\frac{sn-2}{2}}.$$

3 A new construction of asymptotically optimal codebooks

In this section, we propose a construction of asymptotically optimal codebooks by the operation $D_1 \bigtriangledown D_2$. We begin with the definitions of two certain sets.

Let $n \geq 1$, $s_1 > 1$ and $s_2 > 1$ be positive integers. Throughout the section, we write $q_1 = 2^{s_1}$ and $q_2 = 2^{s_2}$. Put $D_1 = \{(x_1, x_2, \cdots, x_n) \in (\mathbb{F}_{q_1}^*)^n : \mathrm{Tr}_{q_1}(x_1+x_2+\cdots+x_n) = 1\}$ and $D_2 = \{(x_1, x_2, \cdots, x_n) \in (\mathbb{F}_{q_2}^*)^n : \mathrm{Tr}_{q_2}(x_1 + x_2 + \cdots + x_n) = 1\}$. To determine the cardinalities of D_1 and D_2, we need the following lemma.

Lemma 4 [17] *Let $q = p^s$, where p is a prime and $s > 1$ is a positive integer. Assume that $F = \{(x_1, x_2, \cdots, x_n) \in (\mathbb{F}_q^*)^n : \mathrm{Tr}_q(x_1 + x_2 + \cdots + x_n) = a\}$. Then*

$$\#F = \begin{cases} \frac{(p^s-1)^n+(-1)^{n+1}}{p}, & \text{if } a \in \mathbb{F}_p^*, \\ \frac{(p^s-1)^n+(-1)^n(p-1)}{p}, & \text{if } a = 0. \end{cases}$$

Applying Lemma 4, we deduce that $\#D_1 = \frac{(q_1-1)^n+(-1)^{n+1}}{2}$ and $\#D_2 = \frac{(q_2-1)^n+(-1)^{n+1}}{2}$. Let $D = (D_1 \times \overline{D}_2) \cup (\overline{D}_1 \times D_2)$, where $\overline{D}_1 = (\mathbb{F}_{q_1}^*)^n \setminus D_1$, $\overline{D}_2 = (\mathbb{F}_{q_2}^*)^n \setminus D_2$. It is easy to check that $\#D = \frac{(q_1-1)^n(q_2-1)^n-1}{2}$. For simplicity, we write $K = \frac{(q_1-1)^n(q_2-1)^n-1}{2}$.

Assume that $\varphi_1, \cdots, \varphi_n$ are multiplicative characters of $\mathbb{F}_{q_1}$ and $\lambda_1, \cdots, \lambda_n$ are multiplicative characters of $\mathbb{F}_{q_2}$. Then we define a vector of length K by

$$\mathbf{c}_{U,V} = \frac{1}{\sqrt{K}}(\varphi_1(x_1) \cdots \varphi_n(x_n)\lambda_1(y_1) \cdots \lambda_n(y_n))_{X \times Y \in D},$$

where $U = (\varphi_1, \cdots, \varphi_n)$, $V = (\lambda_1, \cdots, \lambda_n)$, $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$. When U and V run over the multiplicative characters groups $(\widehat{\mathbb{F}}_{q_1}^*)^n$ and $(\widehat{\mathbb{F}}_{q_2}^*)^n$, respectively, we obtain a set $\mathcal{C}$ of $(q_1 - 1)^n(q_2 - 1)^n$ unit-norm complex vectors as follows:

$$\mathcal{C} = \{\mathbf{c}_{U,V} : U \in (\widehat{\mathbb{F}}_{q_1}^*)^n, V \in (\widehat{\mathbb{F}}_{q_2}^*)^n\}. \tag{1}$$

Theorem 1 *Let $s_1 > 1$ and $s_2 > 1$ be two positive integers. Assume that $q_1 = 2^{s_1}$ and $q_2 = 2^{s_2}$. Then the set $\mathcal{C}$ defined by (1) is an $\left(N = (q_1 - 1)^n(q_2 - 1)^n, K = \frac{(q_1-1)^n(q_2-1)^n-1}{2}\right)$ codebook with the maximum cross-correlation amplitude $I_{max}(\mathcal{C}) = \frac{2^{\frac{s_1 n+s_2 n}{2}}}{(2^{s_1}-1)^n(2^{s_2}-1)^n-1}$.*

Proof According to the definition of the codebook $\mathcal{C}$, it can be easily shown that $N = (q_1-1)^n(q_2-1)^n$ and $K = \frac{(q_1-1)^n(q_2-1)^n-1}{2}$. Our task now is to calculate the maximum cross-correlation amplitude. For simplicity, we take $N_1 = (q_1-1)^n$, $N_2 = (q_2-1)^n$, $K_1 = \#D_1 = \frac{(q_1-1)^n+(-1)^{n+1}}{2}$ and $K_2 = \#D_2 = \frac{(q_2-1)^n+(-1)^{n+1}}{2}$. Assume that $U = (\varphi_1, \cdots, \varphi_n)$, $U' = (\varphi'_1, \cdots, \varphi'_n) \in (\widehat{\mathbb{F}}^*_{q_1})^n$ and $V = (\lambda_1, \cdots, \lambda_n)$, $V' = (\lambda'_1, \cdots, \lambda'_n) \in (\widehat{\mathbb{F}}^*_{q_2})^n$. For any two distinct vectors $\mathbf{c}_{U,V}$ and $\mathbf{c}_{U',V'}$ with $(U,V) \neq (U',V')$, we have

$$\left|\mathbf{c}_{U,V}\mathbf{c}^H_{U',V'}\right| = \frac{1}{K}\left|\sum_{(x_1,\cdots x_n)\in D_1}\varphi_1\overline{\varphi'_1}(x_1)\cdots\varphi_n\overline{\varphi'_n}(x_n)\sum_{(y_1,\cdots y_n)\in\overline{D}_2}\lambda_1\overline{\lambda'_1}(y_1)\cdots\lambda_n\overline{\lambda'_n}(y_n)\right.$$
$$\left.+\sum_{(x_1,\cdots x_n)\in\overline{D}_1}\varphi_1\overline{\varphi'_1}(x_1)\cdots\varphi_n\overline{\varphi'_n}(x_n)\sum_{(y_1,\cdots y_n)\in D_2}\lambda_1\overline{\lambda'_1}(y_1)\cdots\lambda_n\overline{\lambda'_n}(y_n)\right|.$$

In the following, we divide the computation of $\left|\mathbf{c}_{U,V}\mathbf{c}^H_{U',V'}\right|$ into three cases.

(1) If $\varphi_1\overline{\varphi'_1}, \cdots, \varphi_n\overline{\varphi'_n}$ are trivial, then $\lambda_1\overline{\lambda'_1}, \cdots, \lambda_n\overline{\lambda'_n}$ are not all trivial. Without loss of generality, we assume that $\lambda_1\overline{\lambda'_1}, \cdots, \lambda_k\overline{\lambda'_k}$ are nontrivial, where $1 \le k \le n$. So we obtain

$$\begin{aligned}\left|\mathbf{c}_{U,V}\mathbf{c}^H_{U',V'}\right| &= \frac{1}{K}\left|K_1\sum_{(y_1,\cdots y_n)\in\overline{D}_2}\lambda_1\overline{\lambda'_1}(y_1)\cdots\lambda_n\overline{\lambda'_n}(y_n)\right. \\ &\quad\left.+(N_1-K_1)\sum_{(y_1,\cdots y_n)\in D_2}\lambda_1\overline{\lambda'_1}(y_1)\cdots\lambda_n\overline{\lambda'_n}(y_n)\right| \\ &= \frac{|N_1-2K_1|}{K}\left|E_{q_2}(\lambda_1\overline{\lambda'_1}, \cdots, \lambda_n\overline{\lambda'_n})\right|,\end{aligned}$$

where the last equality is derived from the definition of the hyper Eisenstein sums over $\mathbb{F}_{q_2}$ and the fact that

$$\begin{aligned}&\sum_{(y_1,\cdots y_n)\in\overline{D}_2}\lambda_1\overline{\lambda'_1}(y_1)\cdots\lambda_n\overline{\lambda'_n}(y_n)+\sum_{(y_1,\cdots y_n)\in D_2}\lambda_1\overline{\lambda'_1}(y_1)\cdots\lambda_n\overline{\lambda'_n}(y_n) \\ &= \sum_{y_1\in\mathbb{F}^*_{q_2}}\lambda_1\overline{\lambda'_1}(y_1)\cdots\sum_{y_n\in\mathbb{F}^*_{q_2}}\lambda_n\overline{\lambda'_n}(y_n) = 0.\end{aligned}$$

It follows from Lemma 2 and Proposition 1 that

$$\left|\mathbf{c}_{U,V}\mathbf{c}^H_{U',V'}\right| = \frac{2^{\frac{s_2k-2}{2}}}{K} \le \frac{2^{\frac{s_2n-2}{2}}}{K}.$$

The equality holds if and only if $\lambda_1\overline{\lambda'_1}, \cdots, \lambda_n\overline{\lambda'_n}$ are nontrivial.

(2) If $\lambda_1\overline{\lambda'_1}, \cdots, \lambda_n\overline{\lambda'_n}$ are trivial, using the same argument as in the first case, we can easily deduce that

$$\left|\mathbf{c}_{U,V}\mathbf{c}^H_{U',V'}\right| = \frac{2^{\frac{s_1k-2}{2}}}{K} \le \frac{2^{\frac{s_1n-2}{2}}}{K}.$$

The equality holds if and only if $\varphi_1\overline{\varphi'_1}, \cdots, \varphi_n\overline{\varphi'_n}$ are nontrivial.

(3) If $\varphi_1\overline{\varphi_1'}, \cdots, \varphi_n\overline{\varphi_n'}$ are not all trivial and $\lambda_1\overline{\lambda_1'}, \cdots, \lambda_n\overline{\lambda_n'}$ are not all trivial, without loss of generality, suppose that $\varphi_1\overline{\varphi_1'}, \cdots, \varphi_l\overline{\varphi_l'}$ are nontrivial and $\lambda_1\overline{\lambda_1'}, \cdots, \lambda_k\overline{\lambda_k'}$ are nontrivial, where $1 \le l, k \le n$. By Lemma 2 and Proposition 1, we have

$$\begin{aligned}\left|\mathbf{c}_{U,V}\mathbf{c}_{U',V'}^H\right| &= \frac{2}{K}\left|E_{q_1}(\varphi_1\overline{\varphi_1'}, \cdots, \varphi_n\overline{\varphi_n'})\right|\left|E_{q_2}(\lambda_1\overline{\lambda_1'}, \cdots, \lambda_n\overline{\lambda_n'})\right| \\ &= \frac{2^{\frac{s_1 l+s_2 k-2}{2}}}{K} \le \frac{2^{\frac{s_1 n+s_2 n-2}{2}}}{K}.\end{aligned}$$

The equality holds if and only if $\varphi_1\overline{\varphi_1'}, \cdots, \varphi_n\overline{\varphi_n'}, \lambda_1\overline{\lambda_1'}, \cdots, \lambda_n\overline{\lambda_n'}$ are nontrivial.

Combining three cases above, we see that the maximum cross-correlation amplitude

$$I_{max}(\mathcal{C}) = \frac{2^{\frac{s_1 n+s_2 n-2}{2}}}{K} = \frac{2^{\frac{s_1 n+s_2 n}{2}}}{(2^{s_1}-1)^n(2^{s_2}-1)^n-1}.$$

□

Theorem 2 *Let the symbols be the same as those in Theorem 1. Then the codebook $\mathcal{C}$ defined by* (1) *asymptotically achieves the Welch bound.*

Proof Since $\mathcal{C}$ has parameters $(N = (q_1-1)^n(q_2-1)^n, K = \frac{(q_1-1)^n(q_2-1)^n-1}{2})$, the corresponding Welch bound is

$$I_W = \sqrt{\frac{N-K}{(N-1)K}} = \sqrt{\frac{K+1}{2K^2}}.$$

Note that the maximum cross-correlation amplitude of $\mathcal{C}$ is $I_{max}(\mathcal{C}) = \frac{2^{\frac{s_1 n+s_2 n-2}{2}}}{K}$. Therefore,

$$\frac{I_{max}(\mathcal{C})}{I_W} = \sqrt{\frac{2^{s_1 n+s_2 n-2}}{K^2}\frac{2K^2}{K+1}} = \sqrt{\frac{2^{s_1 n+s_2 n}}{(2^{s_1}-1)^n(2^{s_2}-1)^n+1}} \to 1,$$

when $s_1 \to +\infty$ and $s_2 \to +\infty$. □

Remark 1 Let $Q_1 \equiv 3 \bmod 4$ and $Q_2 \equiv 3 \bmod 4$. Hu et al. [11, Corollary 1] proposed a $(Q_1Q_2, \frac{Q_1Q_2-1}{2})$ codebook with the maximum cross-correlation amplitude $I_{max} = \frac{\sqrt{(Q_1+1)(Q_2+1)}}{Q_1Q_2-1}$. If $n = 1$, then the codebooks generated by Theorem 1 have the same parameters as the codebooks constructed by Hu. If $n > 1$ is odd, the codebooks generated by Theorem 1 have the same parameters N, K as the codebooks proposed by Hu. However, the maximum cross-correlation amplitude of our construction is greater than the maximum cross-correlation amplitude of Hu's construction. If $n > 1$ is even, the codebooks produced by Theorem 1 have the new parameters due to $(2^{s_1}-1)^n \equiv 1 \bmod 4$ and $(2^{s_2}-1)^n \equiv 1 \bmod 4$. In Table 2, we list some examples of codebooks generated by Theorem 1 for $n = 2$ and some given s_1 and s_2 such that $s = s_1 = s_2$. As can be seen, the codebook $\mathcal{C}$ asymptotically achieves the Welch bound.

Table 2 Parameters of the (N, K) codebook of Theorem 1 for $n = 2$ and some given $s = s_1 = s_2$

s	N	K	$I_{max}(\mathcal{C})$	I_W	$\frac{I_{max}(\mathcal{C})}{I_W}$
2	81	40	0.200000	0.113192	1.76690
3	2401	1200	0.0266667	0.0204209	1.30585
4	50625	25312	0.00505689	0.00444458	1.13777
5	923521	461760	0.00110880	0.00104059	1.06556
6	15752961	7876480	0.000260015	0.000251953	1.03200
7	260144641	130072320	6.29803×10^{-5}	6.20001×10^{-5}	1.01581
8	4228250625	2114125312	1.54996×10^{-5}	1.53787×10^{-5}	1.00786
9	68184176641	34092088320	3.84465×10^{-6}	3.82964×10^{-6}	1.00392
10	1095222947841	547611473920	9.57409×10^{-7}	9.55541×10^{-7}	1.00196
11	17557851463681	8778925731840	2.38885×10^{-7}	2.38651×10^{-7}	1.00098
12	281200199450625	140600099725312	5.96629×10^{-8}	5.96337×10^{-8}	1.00049
13	4501401006735361	2250700503367680	1.49084×10^{-8}	1.49048×10^{-8}	1.00024
14	72040003462430721	36020001731215360	3.72620×10^{-9}	3.72575×10^{-9}	1.00012
15	1152780773560811521	576390386780405760	9.31436×10^{-10}	9.31379×10^{-10}	1.00006

4 A recursive construction

Based on the construction of asymptotically optimal codebooks in the previous section, we study how to recursively construct asymptotically optimal codebooks with large and flexible parameters.

For convenience, we adopt the following notation throughout this section.

- Let $s_i > 1$ be positive integers and $q_i = 2^{s_i}$, where $1 \le i \le l$ and $l > 1$ is an integer.
- For any $1 \le i \le l$, we write $N_i = (q_i - 1)^n$, where $n \ge 1$ is an integer.
- $\mathbb{F}_{q_i}$ is a finite filed with q_i elements, where $1 \le i \le l$.
- $\mathrm{Tr}_{q_i}(\cdot)$ is the trace function from $\mathbb{F}_{q_i}$ to $\mathbb{F}_2$.
- $D_i = \{(x_1, x_2, \cdots, x_n) \in (\mathbb{F}_{q_i}^*)^n : \mathrm{Tr}_{q_i}(x_1 + x_2 + \cdots + x_n) = 1\}$, where $1 \le i \le l$.
- $X_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in (\mathbb{F}_{q_i}^*)^n$ for any $1 \le i \le l$.

In Section 3, we present a construction of asymptotically optimal codebooks from the operation $D_1 \bigtriangledown D_2$. Here, we consider the similar operation of $D_1, D_2, \cdots, D_l$ by a recursive method. Let $(\mathbb{F}_{q_1}^*)^n \times (\mathbb{F}_{q_2}^*)^n \times \cdots \times (\mathbb{F}_{q_l}^*)^n = \{(X_1, X_2, \cdots, X_l) : X_i \in (\mathbb{F}_{q_i}^*)^n, 1 \le i \le l\}$. Now we construct a subset D of $(\mathbb{F}_{q_1}^*)^n \times (\mathbb{F}_{q_2}^*)^n \times \cdots \times (\mathbb{F}_{q_l}^*)^n$ as follows. Assume that $P_1 = D_1$. For any $1 \le i \le l-1$, we put $P_{i+1} = (P_i \times \overline{D}_{i+1}) \cup (\overline{P}_i \times D_{i+1})$, where $\overline{P}_i = (\mathbb{F}_{q_1}^*)^n \times (\mathbb{F}_{q_2}^*)^n \times \cdots \times (\mathbb{F}_{q_i}^*)^n \backslash P_i$. In the end, let $D = P_l$.

Suppose that $\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{in}$ are multiplicative characters of $\mathbb{F}_{q_i}$ and $U_i = (\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{in})$ for any $1 \le i \le l$. For simplicity, denote by $U_i(X_i) = \varphi_{i1}(x_{i1})\varphi_{i2}(x_{i2}) \cdots \varphi_{in}(x_{in})$. Then we define a vector of length $K = \#D$ as

$$\mathbf{c}_{(U_1, \cdots, U_l)} = \frac{1}{\sqrt{K}}(U_1(X_1)U_2(X_2) \cdots U_l(X_l))_{(X_1, X_2, \cdots, X_l) \in D},$$

and define a set $\mathcal{C}$ of $N = N_1 N_2 \cdots N_l$ unit-norm complex vectors by

$$\mathcal{C} = \{\mathbf{c}_{(U_1, \cdots, U_l)} : U_i \in (\widehat{\mathbb{F}_{q_i}^*})^n, 1 \le i \le l\}. \tag{2}$$

Obviously, the set $\mathcal{C}$ is an (N, K) codebook. In order to determine the length K and the maximum cross-correlation amplitude of $\mathcal{C}$, we need the following two lemmas.

Lemma 5 *With the notation as above,* $\#P_i = \frac{N_1N_2\cdots N_i-(-1)^{in}}{2}$ *for any* $1 \le i \le l$. *Furthermore,* $K = \frac{N_1N_2\cdots N_l-(-1)^{ln}}{2}$.

Proof We prove this lemma by mathematical induction. It follows from Lemma 4 that $P_1 = D_1 = \frac{N_1-(-1)^n}{2}$. Assume that $\#P_i = \frac{N_1N_2\cdots N_i-(-1)^{in}}{2}$ for $i > 1$. Then, by the definition of the set P_{i+1}, we have

$$\begin{aligned}\#P_{i+1} &= \#P_i \times (N_{i+1} - \#D_{i+1}) + (N_1N_2\cdots N_i - \#P_i) \times \#D_{i+1}\\ &= \frac{N_1N_2\cdots N_i - (-1)^{in}}{2} \times \frac{N_{i+1} + (-1)^n}{2}\\ &\quad + \frac{N_1N_2\cdots N_i + (-1)^{in}}{2} \times \frac{N_{i+1} - (-1)^n}{2}\\ &= \frac{N_1N_2\cdots N_{i+1} - (-1)^{(i+1)n}}{2}.\end{aligned}$$

Therefore, $\#P_i = \frac{N_1N_2\cdots N_i-(-1)^{in}}{2}$ for any $1 \le i \le l$ and $K = \#P_l = \frac{N_1N_2\cdots N_l-(-1)^{ln}}{2}$. □

Lemma 6 *With the notation as above, for any multiplicative characters* $U_i \in (\widehat{\mathbb{F}_{q_i}^*})^n$, $1 \le i \le l$, *we get*

$$\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| = K,$$

if $U_i = (1, 1, \cdots, 1)$ *for any* $1 \le i \le l$. *Otherwise,*

$$\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| \le 2^{\frac{s_1n+\cdots+s_ln-2}{2}}.$$

The equality holds if and only if $\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{in}$ *are all nontrivial for any* $1 \le i \le l$.

Proof If $U_i = (1, 1, \cdots, 1)$ for any $1 \le i \le l$, it is easy to show that

$$\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| = \#D = K.$$

Otherwise, we verify $\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| \le 2^{\frac{s_1n+\cdots+s_ln-2}{2}}$ by using mathematical induction. According to the proof of Theorem 1, the result is correct for $l = 2$. If $l > 2$, assume that

$$\left|\sum_{(X_1,X_2,\cdots,X_{l-1})\in P_{l-1}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1})\right| \le 2^{\frac{s_1n+\cdots+s_{l-1}n-2}{2}}.$$

We check that

$$\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| \le 2^{\frac{s_1n+\cdots+s_ln-2}{2}},$$

and the proof is divided into the following three cases.

(1) If $U_l = (1, 1, \cdots, 1)$ and there is at least one $U_i \neq (1, 1, \cdots, 1)$, where $1 \le i < l$, then we obtain

$$\begin{aligned}
&\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| \\
&= \left|(N_l - \#D_l) \sum_{(X_1,X_2,\cdots,X_{l-1})\in P_{l-1}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1})\right. \\
&\quad \left. + \#D_l \sum_{(X_1,X_2,\cdots,X_{l-1})\in \overline{P_{l-1}}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1})\right| \\
&= |N_l - 2\#D_l| \left|\sum_{(X_1,X_2,\cdots,X_{l-1})\in P_{l-1}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1})\right| \\
&\le 2^{\frac{s_1n+\cdots+s_{l-1}n-2}{2}},
\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
&\sum_{(X_1,X_2,\cdots,X_{l-1})\in P_{l-1}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1}) \\
&+ \sum_{(X_1,X_2,\cdots,X_{l-1})\in \overline{P_{l-1}}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1}) = 0.
\end{aligned}$$

(2) If $U_i = (1, 1, \cdots, 1)$ for any $1 \le i \le l-1$ and $U_l \neq (1, 1, \cdots, 1)$, it follows from Lemma 2 and Proposition 1 that

$$\begin{aligned}
&\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| \\
&= \left|\#P_{l-1} \sum_{X_l\in \overline{D_l}} U_l(X_l) + (N_1N_2\cdots N_{l-1} - \#P_{l-1}) \sum_{X_l\in D_l} U_l(X_l)\right| \\
&= |N_1N_2\cdots N_{l-1} - 2\#P_{l-1}| \left|\sum_{X_l\in D_l} U_l(X_l)\right| \\
&\le 2^{\frac{s_ln-2}{2}}.
\end{aligned}$$

(3) If $U_l \neq (1, 1, \cdots, 1)$ and there is at least one $U_i \neq (1, 1, \cdots, 1)$, where $1 \leq i < l$, then we obtain

$$\begin{aligned}
&\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| \\
&= \left|\sum_{(X_1,X_2,\cdots,X_{l-1})\in P_{l-1}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1}) \sum_{X_l\in\overline{D_l}} U_l(X_l)\right. \\
&\quad \left. + \sum_{(X_1,X_2,\cdots,X_{l-1})\in\overline{P_{l-1}}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1}) \sum_{X_l\in D_l} U_l(X_l)\right| \\
&= 2\left|\sum_{(X_1,X_2,\cdots,X_{l-1})\in P_{l-1}} U_1(X_1)U_2(X_2)\cdots U_{l-1}(X_{l-1})\right|\left|\sum_{X_l\in D_l} U_l(X_l)\right| \\
&\leq 2^{\frac{s_1n+\cdots+s_ln-2}{2}}.
\end{aligned}$$

Consequently, we infer that

$$\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1(X_1)U_2(X_2)\cdots U_l(X_l)\right| \leq 2^{\frac{s_1n+\cdots+s_ln-2}{2}}.$$

By the method analogous to that used above and Proposition 1, we can show that the equality holds if and only if $\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{in}$ are all nontrivial for any $1 \leq i \leq l$. □

Next, we present our second class of codebooks in the following theorem.

Theorem 3 *Let the symbols be the same as above and $n \geq 1$, $l > 1$ be integers. We write $N = N_1N_2\cdots N_l$ and $K = \frac{N_1N_2\cdots N_l-(-1)^{ln}}{2}$. Then the set $\mathcal{C}$ defined by (2) is an (N, K) codebook and its maximum cross-correlation amplitude is $I_{max}(\mathcal{C}) = \frac{2^{(s_1n+s_2n+\cdots+s_ln)/2}}{N_1N_2\cdots N_l-(-1)^{ln}}$.*

Proof From the definition of the set $\mathcal{C}$ and Lemma 5, it follows that $\mathcal{C}$ has $N_1N_2\cdots N_l$ vectors and the length of each vector is $\frac{N_1N_2\cdots N_l-(-1)^{ln}}{2}$. For any two distinct vectors $\mathbf{c}_{(U_1,\cdots,U_l)}$ and $\mathbf{c}_{(V_1,\cdots,V_l)}$ of $\mathcal{C}$, where $U_i = (\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{in})$ and $V_i = (\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in})$ for any $1 \leq i \leq l$, we get

$$\left|\mathbf{c}_{(U_1,\cdots,U_l)}\mathbf{c}^H_{(V_1,\cdots,V_l)}\right| = \frac{1}{K}\left|\sum_{(X_1,X_2,\cdots,X_l)\in D} U_1V_1^{-1}(X_1)U_2V_2^{-1}(X_2)\cdots U_lV_l^{-1}(X_l)\right|,$$

where $U_iV_i^{-1}(X_i) = \varphi_{i1}\lambda_{i1}^{-1}(x_{i1})\varphi_{i2}\lambda_{i2}^{-1}(x_{i2})\cdots\varphi_{in}\lambda_{in}^{-1}(x_{in})$. Due to $(U_1, \cdots, U_l) \neq (V_1, \cdots, V_l)$, it follows from Lemma 6 that

$$\left|\mathbf{c}_{(U_1,\cdots,U_l)}\mathbf{c}^H_{(V_1,\cdots,V_l)}\right| \leq \frac{2^{\frac{s_1n+\cdots+s_ln-2}{2}}}{K} = \frac{2^{(s_1n+s_2n+\cdots+s_ln)/2}}{N_1N_2\cdots N_l-(-1)^{ln}}.$$

The equality holds if and only if $\varphi_{i1}\lambda_{i1}^{-1}, \varphi_{i2}\lambda_{i2}^{-1}, \cdots, \varphi_{in}\lambda_{in}^{-1}$ are all nontrivial for any $1 \leq i \leq l$. □

Theorem 4 *Let the symbols be the same as those in Theorem 3. Then the codebook $\mathcal{C}$ defined by* (2) *is asymptotically optimal with respect to the Welch bound.*

Proof Observe that $\mathcal{C}$ is an $\left(N = N_1N_2\cdots N_l, K = \frac{N_1N_2\cdots N_l-(-1)^{ln}}{2}\right)$ codebook. The corresponding Welch bound of $\mathcal{C}$ is

$$I_W = \sqrt{\frac{K+(-1)^{ln}}{2K^2+((-1)^{ln}-1)K}}.$$

Due to $I_{max}(\mathcal{C}) = \frac{2^{(s_1n+s_2n+\cdots+s_ln)/2}}{N_1N_2\cdots N_l-(-1)^{ln}}$, it follows that

$$\frac{I_{max}(\mathcal{C})}{I_W} = \sqrt{\frac{2^{s_1n+\cdots s_ln}\left(1+\frac{(-1)^{ln}-1}{(2^{s_1}-1)^n\cdots(2^{s_l}-1)^n}\right)}{(2^{s_1}-1)^n\cdots(2^{s_l}-1)^n+(-1)^{ln}}} \to 1,$$

if $s_i \to 1$ for any $1 \le i \le l$. □

Remark 2 Let $l > 1$ be an integer and $Q_i \equiv 3 \bmod 4$, where $1 \le i \le l$. The codebooks generated by [11, Theorem 2] has parameters $N = Q_1Q_2\cdots Q_l$, $K = \frac{Q_1Q_2\cdots Q_l-1}{2}$ and the maximum cross-correlation amplitude $I_{max} = \frac{\sqrt{(Q_1+1)(Q_2+1)\cdots(Q_l+1)}}{Q_1Q_2\cdots Q_l-1}$. If $n = 1$, then the codebooks in Theorem 3 has the same parameters as the codebooks constructed by [11, Theorem 2]. For any $n > 1$ such that n is odd, the codebooks produced by Theorem 3 has the same parameters N, K as the codebooks proposed by [11, Theorem 2]. While the maximum cross-correlation amplitude of our construction is greater than the maximum cross-correlation amplitude of codebooks generated by [11, Theorem 2]. If $n > 1$ is even, the codebooks produced by Theorem 3 has the new parameters due to $(2^{s_i}-1)^n \equiv 1 \bmod 4$ for any $1 \le i \le l$. In Table 3, we provide some explicit examples of the codebook $\mathcal{C}$ in Theorem 3 for $n = 2$, $l = 3$ and some given $s = s_1 = s_2 = s_3$. It is indicated that the codebook $\mathcal{C}$ produced by Theorem 3 is asymptotically optimal with respect to the Welch bound.

Table 3 Parameters of the (N, K) codebook of Theorem 3 for $n = 2, l = 3$ and some given $s = s_1 = s_2 = s_3$

s	N	K	$I_{max}(\mathcal{C})$	I_W	$\frac{I_{max}(\mathcal{C})}{I_W}$
2	729	364	0.0879121	0.0371134	2.36875
3	117649	58824	0.00435197	0.00291549	1.49270
4	11390625	5695312	0.000359594	0.000296297	1.21363
5	887503681	443751840	3.69215×10^{-5}	3.35672×10^{-5}	1.09993
6	62523502209	31261751104	4.19273×10^{-6}	3.99924×10^{-6}	1.04838
7	4195872914689	2097936457344	4.99813×10^{-7}	4.88190×10^{-7}	1.02381
8	274941996890625	137470998445312	6.10210×10^{-8}	6.03086×10^{-8}	1.01181
9	17804320388674561	8902160194337280	7.53849×10^{-9}	7.49441×10^{-9}	1.00588
10	1146182576381093889	573091288190546944	9.36799×10^{-10}	9.34056×10^{-10}	1.00294

5 Concluding remarks

In this paper, we investigated the operation of certain sets and its recursive construction. Based on the operations of sets, we proposed two constructions of codebooks and determined the maximum cross-correlation amplitude of codebooks generated by these two constructions. We verified that the codebooks generated by these two constructions are asymptotically optimal with respect to the Welch bound. Although our constructions are very similar to the constructions proposed in [11], the codebooks generated by our constructions have new parameters in some cases.

The technique of compression while sampling, usually referred to as Compressed Sensing, has been the center of attention for a decade. In Compressed Sensing, an (N, K) codebook can be viewed as a $K \times N$ compressed sensing matrix. As an application, we can employ our codebooks to construct compressed sensing matrices with low coherence.

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