



Critical speeding up as an early warning signal of stochastic regime shifts

Mathew Titus^{1,2} · James Watson¹

Received: 8 August 2019 / Accepted: 3 February 2020 / Published online: 26 February 2020
© Springer Nature B.V. 2020

Abstract

The use of critical slowing down as an early warning indicator for regime switching in observations from noisy dynamical systems and models has been widely studied and implemented in recent years. Some systems, however, have been shown to avoid critical slowing down prior to a transition between equilibria (Ditlevsen and Johnsen, *Geophysical Research Letters*, 37(19), 2010; Hastings and Wysham, *Ecol Lett* 13(4):464–472, 2010). Possible explanations include a non-smooth potential driving the dynamic (Hastings and Wysham, *Ecol Lett* 13(4):464–472, 2010) or large perturbations driving the system out of the initial basin of attraction (Boettiger and Batt 2020). In this paper, we discuss a phenomenon analogous to critical slowing down, where a slow parameter change leads to a high likelihood of a regime shift and creates signature warning signs in the statistics of the process's sample paths. This effect, which we dub “critical speeding up,” is demonstrated using a simple population model exhibiting an Allee effect. In short, if a basin of attraction is compressed under a parameter change then the potential well steepens, leading to a drop in the time series' variance and autocorrelation; precisely the opposite warning signs exhibited by critical slowing down. The fact that either falling or rising variance / autocorrelation can indicate imminent state change should underline the need for reliable modeling of any empirical system where one desires to forecast regime change.

Keywords Early warning signals · Regime change · Critical slowing down · Critical speeding up · Complex systems · Stability

Introduction

When studying time series data for dynamical systems which exhibit critical transitions—that is, sudden changes in equilibrium behavior—a widely used early warning signal for an oncoming transition is critical slowing down (Dakos et al. 2008, 2012; Scheffer et al. 2009). This signature for the system being at risk of a large transition is based on the theory of stochastic dynamical systems (Hastings and Wysham 2010; Drake 2013; Kuehn 2013) and has been observed in a variety of empirical tests, both in nature and in the laboratory (e.g., Carpenter and Brock 2006; Scheffer et al. 2012; van Belzen et al. 2017; Wen et al. 2018). At its core, critical slowing down (CSD)

assumes that the stochastic process X experiences a smooth potential V_t which is varying slowly in time, $dX_t = -\nabla V_t(X_t)dt + \xi_t$ where ξ is some random process. If the potential nears a bifurcation point, such as a pitchfork or fold bifurcation, then the shape of V around its equilibrium necessarily flattens out as its minimum (the stable point) becomes a degenerate critical point. The lessening or loss of local curvature, responsible for the mean-reverting property within the basin of attraction, means that excursions of X away from its stable point grow both in extent and length of time. The concomitant increase in variance and autocorrelation of the sample path of X when approaching a bifurcation is what we refer to as critical slowing down.

The catalogued examples where critical slowing down is observed are often carefully controlled laboratory settings (Kramer and Ross 1985) or models of natural systems subject to relatively small perturbations (Carpenter and Brock 2006; Dakos and Bascompte 2014; Lade and Gross 2012), but many empirical observations of dynamical systems fail to exhibit critical slowing down prior to making a change of regime (Ditlevsen and Johnsen 2010; Hastings

✉ Mathew Titus
mtitus@ceoas.oregonstate.edu

¹ College of Earth, Ocean and Atmospheric Sciences, Oregon State University, Corvallis, OR, USA

² The Prediction Lab, Corvallis, OR, USA

and Wysham 2010; Jessica et al. 2017). That critical state transitions appear to commonly occur in nature away from a bifurcation in the underlying governing dynamic speaks to the value in devising early warning indicators for other “high-risk” situations where a regime shift may be becoming increasingly likely. This lack of critical slowing down prior to transition has been discussed in the literature recently, with explanations including large exogenous perturbations (Boettiger and Hastings 2013; Boettiger and Batt 2020) or the potential V lacking smoothness (Hastings and Wysham 2010); these translate, respectively, to a sudden change in the governing equation (a fundamental change in the statistics of ξ) or a departure from the classical setting of modeling the dynamics with a system of smooth partial differential equations perturbed by a noise source ξ .

In the present paper, we study an alternative culprit which retains both smoothness of the potential and the (quasi-)stationarity of the system. We assume the usual setting for critical slowing down: a dynamical system governed by a slowly changing potential V_t , and a fixed noise term, $\xi_t = aB_t$ for some $a > 0$ and B a brownian motion. If V experiences a bifurcation, the degeneracy of ∇V_t will create a slowing effect, increasing both variance and autocorrelation. However, we consider instead potentials where the long-term evolution of V narrows the basin of attraction that the process is initiated within, avoiding bifurcations. We illustrate this in Fig. 1: given a potential well with fixed height but shrinking width leads to a rattling effect, wherein the stronger restoring force of the gradient punishes excursions away from the stable point (minimum of V), shortening their extent. This manifests itself in decreased variance and decreased autocorrelation of the sample path of X . We show that these signs indicate that a critical transition is becoming more likely, for though the excursions are smaller, they are occurring more often in a narrow well. This gives us an early warning of a stochastic transition, rather than a transition facilitated by bifurcation. Note that this is not at odds with the conclusions of Boettiger and Hastings (2013), for the transition is not purely due to noise.

We further show that if the narrowing of the potential is uniform (a linear rescaling of the spatial axis) then the effect on the exit time distribution is equivalent to rescaling the time variable. That is, in terms of the risk of escaping the basin of attraction within the next T units of time, narrowing the potential is the same as speeding up the process’s evolution, and from this standpoint it is clear that such a transformation increases the chance of regime switching. This result is expounded in Section 1.1 below, and is the motivation for our term “critical speeding up” (CSU), as the time until a critical transition is shortened.

The result is particularly surprising in light of well-known estimates for particles’ rates of escape from potential

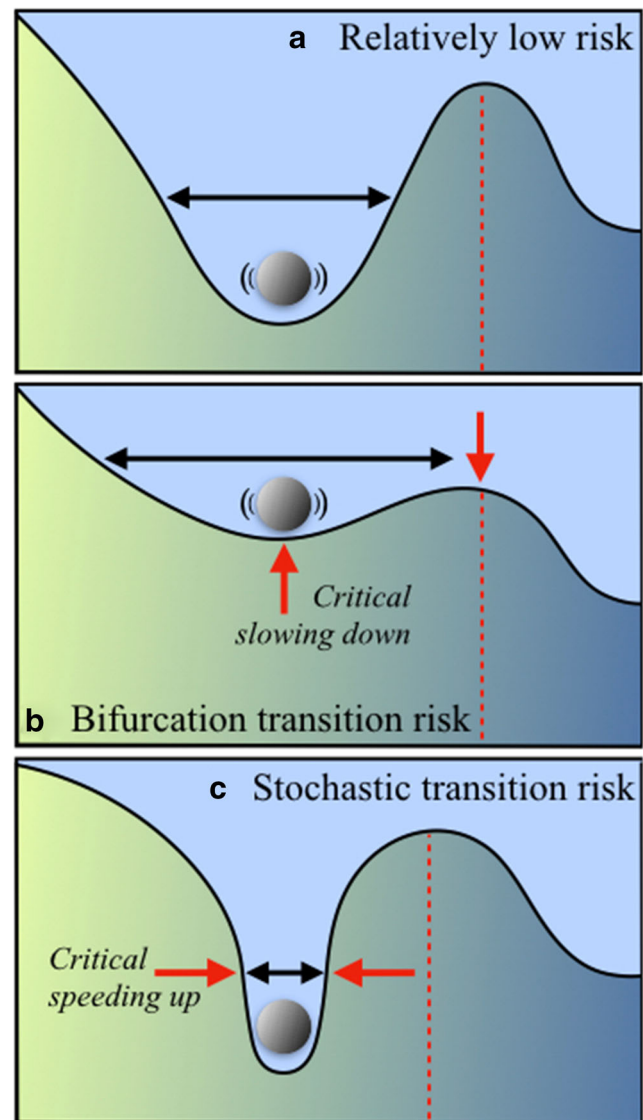


Fig. 1 **a** The standard well potential diagram, where the ball identifies the state of the system, and the valleys of the landscape identify (multiple) basins of attraction. Here, the red line identifies the tipping point or separatrix delineating the basins of attraction. **b** When undergoing a bifurcation, a slowly changing parameter causes a shallowing of a given basin of attraction. This leads to increasing variance in measured variables and critical slowing down. **c** When a parameter change leads to a narrowing of the occupied basin of attraction, the variance in measured variables diminishes and there is critical speeding up. In some cases, the separatrix is moved left. In any case, well narrowing leads to an increased chance of stochastic regime shift

wells, e.g., Kramer’s law or the Arrhenius equation. These estimates, based on large deviations theory, do not apply in the present setting as we are not operating in the small noise regime typically considered when discussing (meta)stability, i.e., Freidlin-Wentzell theory. See, for example, Berglund (2011). This is expanded on in the “Discussion” section below.

In the next section, we give a simple mathematical description of both critical slowing down and critical speeding up. As an example of critical speeding up occurring in a simple ecological model, in the following section, we introduce a population model for a species under limited environment size and an Allee effect. We assume that for this species the rate of procreation is proportional to the population densities of both male and female members; then, under a shrinking habitat, the system begins to exhibit critical speeding up effects as risk of extinction grows (see Fig. 3). In the final section, we discuss the broader implications of this effect: even if an ecological model accurately reflects reality and slow parameter changes alter the time series’ statistics, there is no direct mapping from a statistical signature (decreased autocorrelation, say) to a change in risk of transition.

Mathematical background: critical slowing down and critical speeding up

Suppose for simplicity that we are dealing with a one-dimensional process $X = \{X_t : t \geq 0\}$ where X_t may be a population size, or a concentration of a chemical, or a fraction of people who subscribe to a given belief, at time t . The systems we are interested in allow X to be modeled by a stochastic differential equation (SDE) as it generally obeys some dynamical law, but is subject to exogenous perturbations or uncertainty in measurement. We write this equation as

$$dX_t = b(X_t, t)dt + a(X_t, t)dB_t. \tag{1}$$

Here, $b \in C^\infty$ is the (smooth) drift function, describing the deterministic differential equation X would obey if $a \equiv 0$; the noise is modeled by the product of the coefficient $a \in C^\infty$ and $B = \{B_t : t \geq 0\}$, a one-dimensional brownian motion. Since we are working in one-dimension, a potential function exists

$$V(x, t) = - \int_0^x b(x, t)dx \tag{2}$$

so that Eq. 1 becomes $dX_t = -\nabla V(X_t, t)dt + a(X_t, t)dB_t$.

If the process is initiated in a basin of attraction which, over time, is bounded on the left (right) by $x_l(t)$ (respectively, $x_r(t)$), i.e., $X_0 = x \in (x_l(0), x_r(0))$, we write $T(x)$ for the exit time of the process from that basin:

$$T(x_0) = \inf_{t \geq 0} \{X_t \notin (x_l(t), x_r(t)) | X_0 = x\}. \tag{3}$$

Define as well the cumulative distribution of the exit time,

$$p_t(x_0) := P(T(x_0) < t). \tag{4}$$

In the sequel, we will use the potential

$$V_{\beta, \gamma}(x) = \gamma\beta \left(x - \sqrt{\gamma/3\beta^2}\right) - \beta^3 \left(x - \sqrt{\gamma/3\beta^2}\right)^3 \tag{5}$$

as a testbed for both phenomena. (This is just an affine transformation composed with the simpler function, $V(x) = \gamma x - x^3$, the simplest polynomial with a “basin” which can be escaped.) Note that the origin is an equilibrium point, $\nabla V_{\beta, \gamma}(0) = 0$, for all $\gamma, \beta \geq 0$. See Fig. 2 for illustrations of characteristic potential wells, their sample paths (taking $a = 2$), and observed distributions of $\log(T(x_0))$ where x_0 is the unique metastable state of the system.

Equivalence of well narrowing and time change

Suppose the process $X = \{X_t\}$ obeys the SDE $dX_t = -\nabla V(X_t)dt + adB_t$. If the spatial scale is compressed by a factor $k > 1$ so that the potential defining the dynamic becomes $\hat{V}(x) := V(kx)$, then (leaving the diffusion term unchanged) we have a second process, $\hat{X} = \{\hat{X}_t\}$, which evolves under the influence of the narrowed potential \hat{V} , and so experiences at point x a drift $-\nabla \hat{V}(x) = -k\nabla V(kx)$:

$$d\hat{X}_t = -k\nabla V(k\hat{X}_t) + adB_t. \tag{6}$$

Let us define $Y_t := k\hat{X}_t/k^2$. Then from Eq. 6 and the identity $\sqrt{c}B_t \sim B_{ct}$, we have

$$\begin{aligned} kd\hat{X}_t &= -k^2\nabla V(k\hat{X}_t)dt + akdB_t, \\ dY_{k^2t} &= -\nabla V(Y_{k^2t})d(k^2t) + adB_{k^2t}, \end{aligned}$$

and so taking $s := k^2t$ we find $Y = \{Y_s : s \geq 0\}$ obeys $dY_s = -\nabla V(Y_s)ds + adB_s$, the same equation as X .

Since $Y = \{Y_s\}$ obeys the same dynamic as $X = \{X_t\}$, the probability of Y exiting the basin (x_l, x_r) by some time $s = k^2t < k^2\tau$ (when initiated from x) is equal to $p_{k^2\tau}(x)$. Define $\hat{T}(x) = \inf_{t \geq 0} \{\hat{X}_t \notin (x_l/k, x_r/k)\} = \inf_{s \geq 0} \{Y_s \notin (x_l, x_r)\}$ and $\hat{p}_\tau(x) = P(\hat{T}(x) < \tau)$. Then, finally we have the following relation between exit time distributions:

$$\hat{p}_\tau(x) = p_{k^2\tau}(x) \tag{7}$$

So we see that contracting space by a factor $k > 1$ leads to the same exit time statistics (i.e., critical transition rates) as contracting time by a factor of k^2 . Effectively, a uniform narrowing of V can be interpreted as speeding up the evolution of X , increasing the rate of extreme events.

We note that the above argument also applies for $0 < k < 1$, in which case the potential widens, and has the same statistics as a copy of the process which evolves a factor of k^2 slower, reducing the frequency of critical transitions.

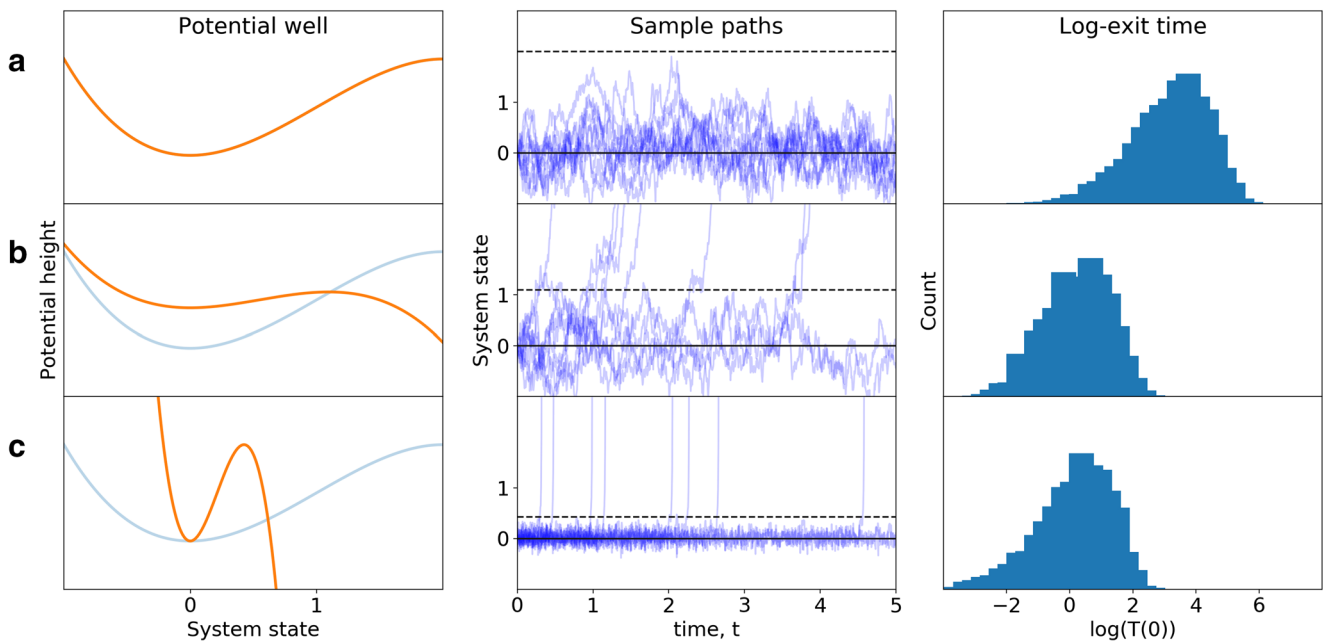


Fig. 2 All simulated paths take the noise coefficient a in Eq. 1 equal to 2. Panels in row **a** correspond to sample paths with parameters $(\gamma, \beta) = (3, 1)$. Row **b** takes $(\gamma, \beta) = (0.9, 1)$, placing the system closer to the bifurcation at $\gamma = 0$. Row **c** has $(\gamma, \beta) = (3, 4.7)$, narrowing the potential well. The leftmost panels plot the potential V defining the dynamics of X_t ; note that the unique stable point has been translated to the origin, $x = 0$. The light blue curves display the potential

from row **a** for comparison. The center panels display 10 sample paths generated from the process X governed by the corresponding potential V ; the stable (solid line) and unstable (dashed line) equilibria are plotted in black. On the right, we plot the exit time distributions for the sample paths. Note that the bottom two panels both experience shorter exit times than the top panel, while their sample paths have either higher or lower variance and autocorrelation

Critical slowing down versus speeding up

In this subsection, we discuss the fundamentals of critical slowing down; it is not intended to be an exhaustive treatment and readers should turn to the literature for full details and generality (i.e., Dakos et al. 2008; Drake 2013; Kuehn 2013). As before, we assume that the process of interest is one-dimensional and described by the SDE $dX_t = -\nabla V_t(X_t)dt + adB_t$, where V is smooth, a is a positive constant, and B is a brownian motion. As we are not interested in deriving the explicit estimates of critical slowing down using normal forms and slow-fast systems theory (Berglund and Gentz 2006; Kuehn 2013), but rather the qualitative features, we approximate the potentials with the lowest order term in their Taylor expansions. This simplifies the dynamics to those of the Ornstein-Uhlenbeck processes or brownian motions with drift. We also assume that the process originates within a basin of attraction, $X_0 \in (x_l(0), x_r(0)) \subset \mathbb{R}$, and that the (smooth) functions $x_l(t), x_r(t)$ are defined so that the basin of attraction is given by $(x_l(t), x_r(t))$ for all future times t that it exists.

Near the unique minimum of V_t within the basin $(x_l(t), x_r(t))$, the second-order approximation of V_t can be used in place of the true potential to define a process whose dynamics are close to those of X while it remains near equilibrium. Since we are interested in the system (5), we

consider the modified equation

$$dX_t = -2\beta^2\sqrt{3\gamma}X_tdt + adB_t \quad (8)$$

which simply follows from the Taylor expansion of V about the stable point $x_0 = 0$.

As mentioned above, this process is an Ornstein-Uhlenbeck (OU) process, whose characteristics are well-known. The definitions and formulas below can be found in Borodin (2017). Recall that the covariance function of two random variables X and Y is

$$\text{Cov}(X, Y) := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \quad (9)$$

and the variance of X , when defined, can be written $\text{Var}(X) := \text{Cov}(X, X)$. Then the s -autocorrelation function at time t is defined by

$$C_s(t) := \text{Cov}(X_t, X_{t+s}). \quad (10)$$

In truth, the above is the autocovariance function, which is more commonly used in the applied probability literature than the autocorrelation function. We refer to this as the autocorrelation in the sequel as the two functions are proportional to one another in all of our examples and autocorrelation is far and away the more commonly used term in the study of early warning signals and time series analysis.

The autocorrelation of the Ornstein-Uhlenbeck process $dX_t = -bX_t dt + adB_t$ is given by the formula

$$C_s(t) = \frac{a^2}{2b} \left(e^{-sb} - e^{-(2t+s)b} \right). \tag{11}$$

From this, it is immediate that the variance is given by

$$\text{Var}(X_t) := \mathbb{E} \left[(X_t - \mathbb{E}[X_t])^2 \right] = \frac{a^2}{2b} \left(1 - e^{-2bt} \right), \tag{12}$$

and so we have for Eq. 8 that the autocorrelation with $s = 1$ is given by

$$\text{Cov}(X_t, X_{t+1}) = \frac{a^2}{4\beta^2\sqrt{3}\gamma} e^{-2\beta^2\sqrt{3}\gamma} \left(1 - \exp \left(-4\beta^2\sqrt{3}\gamma t \right) \right). \tag{13}$$

Now, if we allow γ to change in time so that it slowly approaches zero from above, a bifurcation occurs at $\gamma = 0$. Let us define $\gamma(t) := t^{-2}$; then, γ will approach 0 slowly enough that critical slowing down is observable. Then, the OU process has

$$bx = \nabla V_{\beta,t^{-2}}(x) = (2\beta^2\sqrt{3}/t)x,$$

so inputting $b = 2\beta^2\sqrt{3}/t$ in Eq. 12, we see that the variance grows linearly in time, $\text{Var}(X_t) = mt$ for a constant m depending on β and a . A similar result holds for the autocorrelation. Compare rows (A) and (B) in Fig. 2.

On the other hand, if we do not approach the bifurcation point, fixing $\gamma > 0$ in time, and instead take β to be increasing in time, say $\beta(t) = t$ and restrict to $t \geq 1$, then we have

$$\begin{aligned} \nabla V_{t,\gamma}(x) &= 2\sqrt{3}\gamma t^2 x \\ &= t \left(2\sqrt{3}\gamma t x \right) \\ &= t \left[(\nabla V_{1,\gamma})(tx) \right] \\ &= \nabla \left[V_{1,\gamma}(tx) \right] \end{aligned}$$

demonstrating the spatial contraction of the well. We may again apply (11) and (12) to recover the variance and autocorrelation. One sees that for this process $b = 2\sqrt{3}\gamma t^2$, leading to

$$\text{Var}(X_t) = \frac{a^2}{4\sqrt{3}\gamma t^2} \left(1 - e^{-4\sqrt{3}\gamma t^3} \right) = O(t^{-2}),$$

and the 1-step autocorrelation decays like e^{-Ct^2} (compare rows (A) and (C) in Fig. 2). While we have restricted our attention to a rather specific example, the decay of autocorrelation and variance indicating an increasing risk of transitions, which we call critical speeding up, is characteristic of contracting potentials.

Observing critical speeding up in a population model

Here, we introduce a population model that exhibits critical speeding up before a stochastic state transition (population collapse). We assume a strong Allee effect, proportional to the territory size; this could model a species which has displaced its competitors, benefitting from a cooperative advantage against another species. We also assume that the reproduction rate is a function of population density, rather than total population. These features create an attracting region for the fixed point at zero, so that after passing below a critical threshold the species will become extinct (discounting a restoring perturbation).

This deterministic model is described by the following equation

$$\frac{dx}{dt} = \frac{r}{\beta^2} x \left(\frac{x}{\beta A} - 1 \right) \left(1 - \frac{x}{\beta C} \right), \quad \beta \in [0, 1], \tag{14}$$

where all parameters and the population size x are assumed to be nonnegative.

We interpret C as the maximum carrying capacity of the species, say if they control 100% of their potential territory, while βC is the carrying capacity when a fraction β is controlled. The product βA represents the strength of the Allee effect; suppose for example that the population only controls one-quarter of the contested territory ($\beta = 0.25$) rather than half ($\beta = 0.5$), then the Allee effect is weakened as their competitors have control of a greater swath of the environment, reducing the degree to which the individuals are able to benefit from cooperation. Finally, r is chosen so that a population of size x reproduces at a rate rx when distributed over the entire territory. When restricted to a fraction β of the total land, the population densities of males and females both increase by a factor of $1/\beta$, so their rate of “collision” is increased by a factor of β^{-2} . Hence, the reproductive rate is given by $\frac{r}{\beta^2}x$.

As a final addition, we assume the model originates at its positive equilibrium ($X_0 = \beta C$) and add the noise term adB to model exogenous perturbations (change in population through fatal accidents, immigration from another distant population, twins, etc.), and write $X = \{X_t : t \geq 0\}$ for the population process:

$$dX_t = \frac{r}{\beta^2} X_t \left(\frac{X_t}{\beta A} - 1 \right) \left(1 - \frac{X_t}{\beta C} \right) dt + adB_t. \tag{15}$$

Notice that the drift term can be written $-\nabla V$ with V a fourth-order polynomial in x/β . Thus, as β decreases V will be compressed by a factor of $1/\beta$ and we expect to observe CSU.

We study this model’s vulnerability to extinction events under the increasing stress of an encroaching competing

species, which decreases the available territory, driving β toward zero:

$$\beta(t) = \frac{4}{1 + 0.01t}. \quad (16)$$

In Fig. 3, we see the effect of slowly decreasing β over 10,000 independent trials; the lower panel shows the distribution of the time of collapse, which we define as the first time the system's population X_t drops below $\beta(t)A$, as this separates the two dynamical regimes (extinction and survival). For all of these trials, we fix $a = 0.22$, $C = 2.5$, $A = 1.5$, and $r = 1$. Also pictured in Fig. 3 are the variance and autocorrelation functions (calculated over a rolling window of length 10), after averaging over all sample paths which have yet to exit the basin of attraction. The hallmarks of critical speeding up are evident in the time series; as β drops, the equilibrium points $(C\beta(t), A\beta(t))$, and 0) gather toward the origin, and both the variance and autocorrelation of the series are reduced. Despite the variance and autocorrelation dropping by over 50%, none of the 10,000 trials survived past 989 units of time (surviving sample path not pictured).

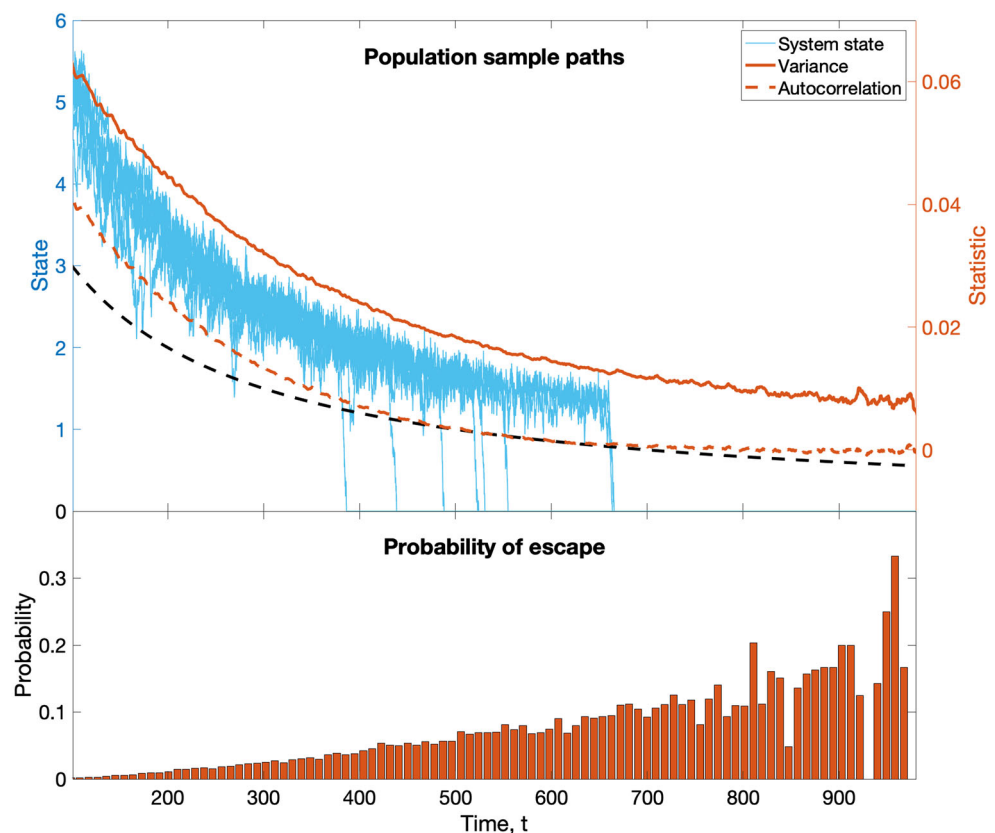
As another verification of the CSU theory, we also run 2000 repeated simulations of X_t starting from $X_0 = \beta C$ for β equal to a fixed value between 0.2 and 1.2 (ten

equally distributed values were used, making for 20,000 independent trials). Statistics of the results are plotted in Figure 4. It is shown that the time for the population X_t to exit $(\beta A, \infty)$, the basin of attraction about βC , decreases with decreasing β as expected from the previous analysis, i.e., the risk of a critical transition grows as the potential narrows. However, the calculation of the covariance and autocorrelation (see Fig. 4) show that the system becomes more brittle as β drops toward 0.2, and critical speeding up gives the precursor signal for population collapse. Note that all values are plotted on a semi-log scale.

Discussion

Using stochastic differential systems theory, we have shown that noisy dynamical systems can not only fail to exhibit critical slowing down prior to a regime shift, they may actually exhibit a *speeding up* of their dynamic, with a decrease in both variance and autocorrelation. To a practitioner observing a system with a narrowing potential, a naive application of the theory of critical slowing down would suggest the system is stabilizing, despite the impending stochastically driven critical transition. We used an example model of population growth to demonstrate this

Fig. 3 The top panel displays in blue eight of the 10,000 calculated sample paths of $\{X_t, 100 \leq t \leq 1000\}$, as β decreases from 2 to roughly $1/2$. The mean variance of the surviving sample paths, calculated over a rolling window of length 10 is shown in orange; the mean autocorrelation of lag 1 for the surviving paths within the same window is plotted in dashed orange. NB: On the far right of the figure the low number of surviving samples leads to a greater variance in these averages, as the law of large numbers fails to hold. The dashed black line denotes $A\beta(t)$, the unstable equilibrium point dividing the extinction regime from the survival regime. Below we plot the fraction of sample paths escaping into the basin of attraction of $x = 0$ (extinction regime) during a given interval of time; this clearly shows the increasing risk of system collapse



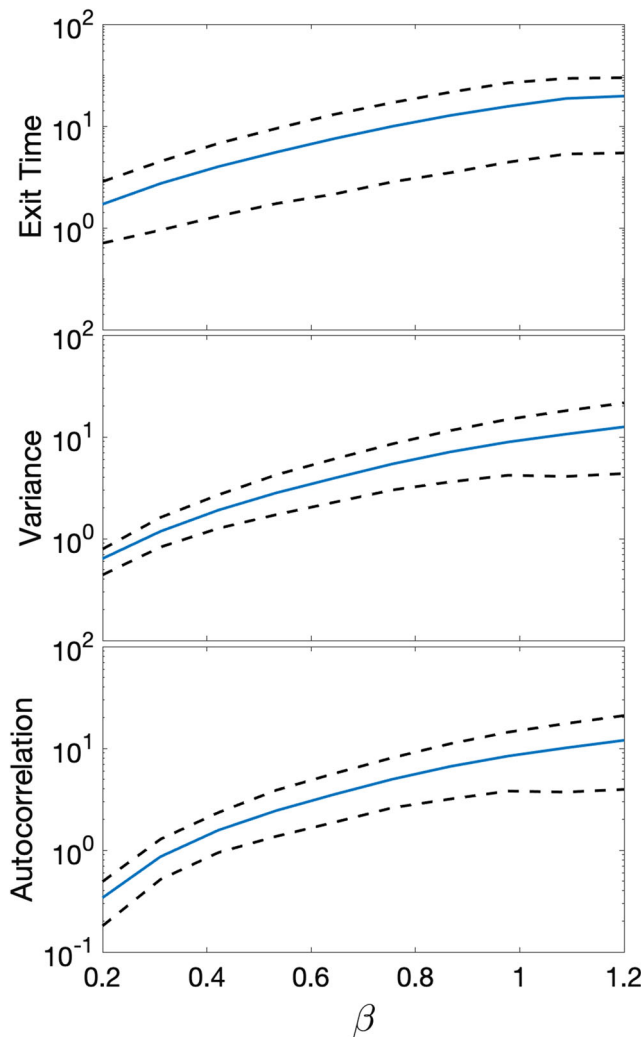


Fig. 4 Here, we give semi-log plots of the mean, 5th percentile, and 95th percentile for three statistics defining critical speeding up. Top: exit time of X_t from the basin of attraction $(A\beta, \infty)$. Middle: variance of X_t over the 10 time units preceding its exit from the basin. Bottom: 1-step autocorrelation of X_t over the 10 time units preceding its exit from the basin. One clearly observes critical speeding up in the marked similarity of these panels

in practice, showing that a population experiencing territory loss and an Allee effect exhibits critical speeding up prior to collapse. We hope that this study underscores the necessity of having high-fidelity descriptions of the system dynamics before applying an early warning indicator such as critical slowing down or critical speeding up, a sentiment expressed also by others (Hastings and Wysham 2010; Boettiger and Hastings 2012).

It is worth emphasizing the limits of CSU. If a system’s potential well deepens while narrowing, the precursor signal of critical speeding up will be a false positive. However, this is analogous to observing a false positive under the critical slowing down paradigm due to a potential well broadening without becoming more shallow (variance and

autocorrelation increase, but the excursion size necessary to leave the basin grows, leaving the chance of escape low). Both signals offer an early warning under the appropriate conditions, but neither is a panacea for predicting regime change.

We contrast our results with the classical approach of measuring engineering resilience, i.e., linearizing the system about the equilibrium and observing the distance between the eigenvalue with greatest real part, i.e., the dominant eigenvalue, and the set $\mathcal{Z} = \{z \in \mathbb{C} | \text{Re}(z) > 0\}$ of complex numbers with positive real part. The larger the distance, the steeper the potential well at the equilibrium point, and the greater the (local) restoring force of the potential. This leads to shorter return times and lower variance. This distance is often used as a measure of the system’s stability. One can check that by shrinking the spatial scale by a factor of k one multiplies the dominant eigenvalue by k as well. Thus, as the potential well narrows ($k > 1$), the eigenvalue is pushed away from \mathcal{Z} , which according to the engineering resilience paradigm, indicates greater stability. Of course, as mentioned in the “Introduction” section, the usual connections between potential well shape, eigenvalues of the linearized dynamics, and probability of escape rely on the amplitude of the driving noise process adB_t being small. Many classical results on rates of escape hold only in the limit of small noise ($a \rightarrow 0^+$). The present article addresses what results from this small noise condition being violated.

There is an expansive literature on the resilience of ecological and engineering systems, e.g., Peterson et al. (1998), Holling (1996), and Walker et al. (2004), and their differences. Borrowing the language of Holling in Walker et al. (2004), resilience depends on the system’s latitude, resistance, precariousness, and panarchy. The effects of CSD can be thought of qualitatively as a loss of resistance; small perturbations are not damped out, and large excursions become more common. Critical speeding up, on the other hand, features increased precariousness and resistance, and possibly decreased latitude. This gives a partial answer to when one should expect critical speeding up to occur in nature. Systems which are designed or managed in a way that increases resistance, but only with a concomitant increase in precariousness, are candidates for unwanted stochastic transitions.

As a second example of time series statistics belying the approach of a transition, one can imagine the model system (5) evolving such that both $\gamma \rightarrow 0^+$ and $\beta \rightarrow \infty$. This would create both critical slowing down and critical speeding up effects in the time series. As these signals interfere with one another, simply observing the autocorrelation and variance of the time series may fail to display either a slowing down or a speeding up effect in the time series, yet the system would quickly become

unstable. This is another example of abrupt transitions occurring without a precursor slowing down and may be responsible for some of the examples in the literature of critical transitions without critical slowing down (such as the regime switching in Ditlevsen and Johnsen 2010; Hastings and Wysham 2010). This should stress that the use of critical slowing down as an early warning indicator may face more obstacles than previously imagined. This strengthens the findings of the metastudy (Litzow and Hunsicker 2016), which suggested that CSD may require nonlinear dynamics to be observable in nature; here, we see that nonlinearity is certainly an insufficient condition for CSD to provide an early indicator of transition.

In spite of the preceding discussion, there are several reasons that we expect to find systems in nature that exhibit critical speeding up. First, the model above describes a population governed by three simple rules, so it would be no surprise to find such a population within the earth's ecology. Further, suppose a sequence of failures must occur for a complex system to experience catastrophe, such as in a collection of power grids servicing a region. In this example, we consider the system state to be described by the percentage of demand being met. If the grid is initiated at normal operating conditions, we have $X_0 = 1$, and this is a stable point: when a light is turned on or a generator is shut down, it will cause a fluctuation, but the system soon recovers to a stable operating position. The system is considered to have failed if $X_t = 0$, i.e., a blackout occurs. If two power subsystems become connected, their probability of failing may be lowered due to the subgrids' pooled resilience (resistance has increased). However, if a failure *does* occur in one subsystem, it will cascade to the other, eventually effecting a larger portion of the system. Thus, the number of failures required to reach a blackout is lowered as well (precariousness has increased). In this case, the probability of an excursion of the state from stability ($X_0 = 1$) to instability ($X_t = X_{T(1)}$) may decrease, but the number of events necessary has been reduced. This sort of thresholding between stability and instability is not uncommon in man-made systems (see, for example, Reason (1995)) and leads to the qualitative ingredients for CSU discussed in the second paragraph of this section.

As a third example, there are many examples of ecological models that exhibit two equilibrium states—e.g., algae-dominated or coral-dominated reef systems, barren desert vs. forest systems, and so on. General models can be fit to a variety of different systems, for example, in May (1977), there is a simple model of a constant density herbivore population feeding on a vegetative resource of biomass V . The biomass dynamic is given by $\dot{V} = G(V) - Hc(V)$ where G is the growth rate function, c is the per capita feeding rate, and H is the density of herbivores present. Simple choices for the functions c and G lead to a

dual-well potential for a range of H values (see the figure on page 472 of May (1977)). By considering a sequence of herbivore-resource pairs or by observing a system respond to evolutionary or seasonal pressures, one expects to see the features of the potential well shift, and of course a narrowing of the potential is a feasible result (specifically, by lowering the carrying capacity of the vegetation and decreasing $c(V)$ at an appropriate rate), which we would expect to exhibit CSU. As the dual-well model is very common, one can similarly conjecture other scenarios where signs of CSU may be observed in empirical systems.

In sum, when studying a complex system with a variety of possible stable states, the notion of critical speeding up may be useful in determining the likelihood of the system occupying a given equilibrium. Despite stabilizing drift that may be quite strong, if the basin of attraction about the stable point is prone to narrowing, the system may not inhabit the basin for long periods of time. This principle may be useful in studying the time-dynamics and evolution of protein folding, food webs, and ecosystems, coupled social-ecological systems or entirely social systems for example (Zwanzig 1997; Dakos et al. 2008; Guttal and Jayaprakash 2008; Wang et al. 2012; Lade et al. 2013; Ma et al. 2017; Nekovee et al. 2007).

Acknowledgments The authors would like to acknowledge support from the DARPA YFA project N66001-17-1-4038 and to thank George Hagstrom for helpful conversations that led to the population model above. The authors are also happy to recognize the contributions of the anonymous reviewers in catching our mistakes and making helpful suggestions improving the presentation of the paper.

References

- Berglund N (2011) Kramers' law: validity derivations and generalisations
- Berglund N, Gentz B (2006) Noise-induced phenomena in slow-fast dynamical systems: a sample-paths approach. Springer Science & Business Media, Berlin
- Boettiger C, Hastings A (2012) Quantifying limits to detection of early warning for critical transitions. *Journal of the Royal Society Interface*, page rsif20120125
- Boettiger C, Hastings A (2013) No early warning signals for stochastic transitions: insights from large deviation theory. *Proc R Soc B: Biol Sci* 280(1766):20131372
- Boettiger C, Batt R (2020) Bifurcation or state tipping: assessing transition type in a model trophic cascade. *J Math Biol* 80 (143–155). <https://doi.org/10.1007/s00285-019-01358-z>
- Borodin AN (2017) Diffusion processes. In: Stochastic processes. Cham, Birkhäuser
- Carpenter SR, Brock WA (2006) Rising variance: a leading indicator of ecological transition. *Ecol Lett* 9(3):311–318
- Dakos V, Bascompte J (2014) Critical slowing down as early warning for the onset of collapse in mutualistic communities. *Proc Natl Acad Sci* 111(49):17546–17551
- Dakos V, Carpenter SR, Brock WA, Ellison AM, Guttal V, Ives AR, Kéfi S, Livina V, Seekell DA, van Nes EH, Scheffer M (2012) Methods for detecting early warnings of critical transitions in

- time series illustrated using simulated ecological data. *PLOS ONE* 07(7):1–20
- Dakos V, Scheffer M, van Nes EH, Brovkin V, Petoukhov V, Held H (2008) Slowing down as an early warning signal for abrupt climate change. *Proc Natl Acad Sci* 105(38):14308–14312
- Ditlevsen PD, Johnsen SJ (2010) Tipping points: early warning and wishful thinking. *Geophysical Research Letters*, 37(19)
- Drake JM (2013) Early warning signals of stochastic switching. *Proc R Soc B: Biol Sci* 280(1766):20130686
- Guttal V, Jayaprakash C (2008) Changing skewness: an early warning signal of regime shifts in ecosystems. *Ecol Lett* 11(5):450–460
- Hastings A, Wysham DB (2010) Regime shifts in ecological systems can occur with no warning. *Ecol Lett* 13(4):464–472
- Holling CS (1996) Engineering resilience versus ecological resilience. *Engineering within Ecological Constraints* 31(1996):32
- Jessica C, Rozek R, Camp J, Reed JM (2017) No evidence of critical slowing down in two endangered Hawaiian honeycreepers. *PLOS ONE* 11(11):1–18
- Kramer J, Ross J (1985) Stabilization of unstable states, relaxation, and critical slowing down in a bistable system. *J Chem Phys* 83(12):6234–6241
- Kuehn C (2013) A mathematical framework for critical transitions: normal forms, variance and applications. *J Nonlinear Sci* 23(3):457–510
- Lade SJ, Gross T (2012) Early warning signals for critical transitions: a generalized modeling approach. *PLoS Computational Biology* 8(2):e1002360
- Lade SJ, Tavoni A, Levin SA, Schlüter M (2013) Regime shifts in a social-ecological system. *Theoretical Ecology* 6(3):359–372
- Litzow MA, Hunsicker ME (2016) Early warning signals, nonlinearity, and signs of hysteresis in real ecosystems. *Ecosphere* 7(12):e01614
- Ma Y, Zhang Q, Wang L, Kang T (2017) Dissipative control of a three-species food chain stochastic system with a hidden Markov chain. *Adv Diff Eq* 2017(1):102
- May RM (1977) Thresholds and breakpoints in ecosystems with a multiplicity of stable states. *Nature* 269(5628):471
- Nekovee M, Moreno Y, Bianconi G, Marsili M (2007) Theory of rumour spreading in complex social networks. *Physica A: Statistical Mechanics and its Applications* 374(1):457–470
- Peterson G, Allen CR, Holling CS (1998) Ecological resilience, biodiversity, and scale. *Ecosystems* 1(1):6–18
- Reason J (1995) A systems approach to organizational error. *Ergonomics* 38(8):1708–1721
- Scheffer M, Bascompte J, Brock WA, Brovkin V, Carpenter SR, Dakos V, Held H, Van Nes EH, Rietkerk M, Sugihara G (2009) Early-warning signals for critical transitions. *Nature* 461(7260):53
- Scheffer M, Carpenter SR, Lenton TM, Bascompte J, Brock W, Dakos V, Van de Koppel J, Van de Leemput IA, Levin SA, Van Nes EH, et al. (2012) Anticipating critical transitions. *Science* 338(6105):344–348
- van Belzen Jim, van de Koppel J, Kirwan MatthewL., van der Wal D, Herman PMJ, Dakos V, Kéfi S., Scheffer M, Guntenspergen GR, Bouma TJ (2017) Vegetation recovery in tidal marshes reveals critical slowing down under increased inundation. *Nat Commun* 8:15811 EP –, 06
- Walker B, Holling CS, Carpenter SR, Kinzig A (2004) Resilience, adaptability and transformability in social-ecological systems. *Ecol. Soc*, 9
- Wang R, Dearing JA, Langdon PG, Zhang E, Yang X, Dakos V, Scheffer M (2012) Flickering gives early warning signals of a critical transition to a eutrophic lake state. *Nature* 492(7429):419
- Wen H., Ciamarra MP, Cheong SA (2018) How one might miss early warning signals of critical transitions in time series data: a systematic study of two major currency pairs. *PLoS one* 13(3):e0191439
- Zwanzig R (1997) Two-state models of protein folding kinetics. *Proceedings of the National Academy of Sciences of the United States of America* 01(1):148–150