Letters in Spatial and Resource Sciences (2023) 16:21
https://doi.org/10.1007/s12076-023-00344-w

ORIGINAL PAPER

# Utility function and location in the Hotelling game

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Received: 18 November 2022 / Accepted: 14 April 2023 / Published online: 28 April 2023


**Abstract**
It is demonstrated herein that a slight expansion of the utility function in the duopolistic Hotelling game enables any symmetric location pair with respect to the center to be in equilibrium, which implies that any level of locational differentiation between the minimum Hotelling (Econ. J.39:41–57, 1929) and maximum D'Aspremont (Econom. 47:1145–1150, 1979) is obtained in one model. The location equilibrium is monotone with respect to the introduced parameter ($k$), while the equilibrium price and profits are not monotone (they are U-shaped). That is, the nearer the two firms are located, the higher their prices are set (with an upper limit) when $k$ is sufficiently large. This counterintuitive phenomenon is interpreted as an example of strategic complementarity that is inherent in the Hotelling games.

**Keywords** Hotelling · Utility function · Location equilibrium · Price term

**JEL Classification** D43 · L13 · R12

## 1 Introduction

In his seminal paper (Hotelling 1929) developed a spatial competition model and showed the tendency of central agglomeration (principle of minimum differentiation). D'Aspremont et al. (1979) found that this principle is incomplete since the price equilibrium does not exist when the firms are near. As a solution for this problem,[1] they assumed a quadratic transportation cost function in distance, instead of the linear function described by Hotelling (1929). More specifically, they assumed that the utility function of consumers is given by $U = r - p - td^2$; where $r$ is a constant, $p$ is the mill price, $d$ is the distance to the firm, and $t$ is the transportation rate.

[1] See, for example, Caplin and Nalebuff (1991) for a comprehensive analysis of the existence of equilibrium.

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D’Aspremont et al. (1979) analyzed a two-stage game in which the firms first decide their locations and then choose the prices. As a result, they observed the opposite tendency, i.e., the principle of maximum differentiation, where firms locate as far apart as possible. Since the D’Aspremont et al. (1979) study, an important theme in the literature is the question of what locational differentiation occurs. For example, Egli (2007) assumed two types of consumers whose transportation costs are linear and quadratic, and he showed that various differentiation patterns emerge depending on the proportions of them. Dragone and Lambertini (2020) assumed convex production costs to restore the existence of equilibrium under a linear transportation cost function, and they also observed the principle of minimum differentiation. Other extended studies focused on the role of the transportation-cost term in the utility function. To the best of our knowledge, there have been no general studies that revised the price term. The objective of the paper is to show what happens in the Hotelling game if the price term is changed into another functional form.

Specifically, we revise the utility function of D’Aspremont et al. (1979) as $U = r - p^k - td^2 \quad (k > 0)$. The greater $k$, the steeper the slope of the utility function is in the domain of $p > 1$. In contrast, when the price is sufficiently small and less than 1, an increase in $k$ makes the slope gradual. The parameter $k$ can be interpreted as a measure of how strong the convexity with respect to the price is for consumers’ utility.[2] We should be careful in choosing the value of $k$. Suppose that $k$ is sufficiently large. Then, the higher the price, the larger the decrease in utility by 1 dollar increase in price. For example, the decrease in utility from $p = 1000$ to $p = 1001$ (0.1 percent increase in price) is much larger than that from $p = 1$ to $p = 2$ (100 percent increase in price), which seems strange. Therefore, although $k$ is arbitrary in the model, we should choose an adequate value of $k$ in reality. For better readability, we begin with the case of $k = 2$ to quickly demonstrate the features of our model, and other cases will be analyzed later.

The remainder of the paper is organized as follows. Section 2 provides our model. In Sect. 3 we obtain the price and location equilibrium under $k = 2$. Section 4 deals with the general cases of $k > 0$. Section 5 provides our conclusions.

## 2 Model

We consider a spatial duopolistic competition à la Hotelling in a linear space $L = [0, 1]$ with quadratic transport costs as in D’Aspremont et al. (1979). Let $i \in \{1, 2\}$ denote the index of firms, and let $x_i \in L$ be the location of firm $i$. Without a loss of generality, we assume $0 \le x_1 \le x_2 \le 1$. The firms provide a homogeneous product with zero marginal cost. We consider a two-stage game: In the first stage the

[2] One may think that $t$ is the same role in the first place. However, $t$ does not affect the location equilibrium in D’Aspremont et al. (1979), and we will see that $k$ matters for the equilibrium. Further, regarding price sensitivity, Puu (2002) dealt with an elastic demand in the Hotelling game and also showed the various location equilibrium values. In their model, the demand function is linear: $f(z) = \alpha - \beta z$, where $z = p + td$ is the linear full cost.

firms simultaneously determine their locations, and in the second stage they simultaneously determine their prices. We adopt subgame perfection as an equilibrium concept, and we only deal with the equilibrium in pure strategies.

There are consumers whose mass is normalized to one. They are evenly distributed over $L$, and inelastically they buy one unit of the good. Each consumer's utility at $z$ is given by Eq. (1)

$$u_z(p_i, x_i) = r - p_i^k - t(z - x_i)^2 \tag{1}$$

if a consumer buys from firm $i$, where $r > 0$ is the reservation utility, $p_i$ is the mill price of firm $i$, and $k > 0$ is the main parameter. As mentioned in the Introduction, let $k = 2$ for the moment. The value of $r$ is assumed to be high enough for the utility to be positive, which ensures that each consumer always buys from either firm.

This utility generates the demand for each firm. When $x_1 = x_2$, all consumers buy from the firm that posts a lower price. If their prices are equal, each firm's demand is assumed to be one-half. When $x_1 \neq x_2$, solving $u_z(p_1, x_1) = u_z(p_2, x_2)$ with respect to $z$, we have:

$$z = \frac{p_2^2 - p_1^2}{2t(x_2 - x_1)} + \frac{x_1 + x_2}{2} \tag{2}$$

as the border between the market areas of both firms if $0 \le z \le 1$. Let the solutions of $z = 1$ and $z = 0$ be $p_1^+$ and $p_1^{++}$, respectively. Then, we have

$$p_1^+ = \{p_2^2 + t(x_2^2 - x_1^2 - 2(x_2 - x_1))\}^{1/2} < \{p_2^2 + t(x_2^2 - x_1^2)\}^{1/2} = p_1^{++}.$$

From these calculations the demand for firm 1 is given by Eq. (3)

$$D_1 = \begin{cases} z, & \text{if} \quad p_1^+ \le p_1 \le p_1^{++}; \\ 0, & \text{if} \quad p_1 > p_1^{++}; \\ 1, & \text{if} \quad 0 \le p_1 < p_1^+. \end{cases} \tag{3}$$

Applying the same computation for firm 2, we get the demand for firm 2 defined as $D_2 = 1 - D_1$. Finally, the profits for firm $i$ are defined as:

$$\pi_i = p_i D_i. \tag{4}$$

We find that $\pi_i$ is continuous and quasi-concave with respect to $p_i$ over the admissive domain, $[0, p_i^{++}]$, which implies that a price equilibrium exists in pure strategies for any location pair.

## 3 Equilibrium under $k = 2$

Applying backward induction, we analyze the the second-stage game problem. First, we find that any agglomerated location $x_1 = x_2$ leads to severe price competition, which results in zero profits due to the zero price at equilibrium. The firms can, however, earn positive profits by being away from the rival and setting a positive price. Thus, $x_1 \neq x_2$ holds in equilibrium.

Let us focus on firm 1. When $p_1 < p_1^+$ ($D_1 = 1$), an increase in price can enhance the profits. When $p_1 > p_1^{++}$ ($D_1 = 0$), the firm can earn positive profits by setting its price under $p_1^{++}$. Therefore, the price equilibria would satisfy $p_1^+ \leq p_1 \leq p_1^{++}$. A similar analysis can be applied to firm 2. As a result, it is sufficient that we seek equilibria in the case in which both firms have positive demand. Then, the fist-order conditions $\partial\pi_i/\partial p_i = 0$ yield:

$$p_1 = \frac{1}{2}\sqrt{t(x_2 - x_1)(x_1 + x_2 + 1)}, \quad p_2 = \frac{1}{2}\sqrt{t(x_2 - x_1)(3 - x_1 - x_2)}, \tag{5}$$

which is the unique price equilibrium.[3] We next consider the first-stage given the above price equilibrium. Substituting Eq. (5) into Eq. (4), we have:

$$\pi_1 = \frac{1}{8}\sqrt{t(x_2 - x_1)(x_1 + x_2 + 1)^3}, \quad \pi_2 = \frac{1}{8}\sqrt{t(x_2 - x_1)(3 - x_1 - x_2)^3}, \tag{6}$$

Straightforward calculations yield the best-response functions for firm 1 and firm 2, respectively:

$$x_1(x_2) = \begin{cases} (2x_2 - 1)/4, & \text{if} \quad 1/2 \leq x_2 \leq 1; \\ 0, & \text{if} \quad 0 \leq x_2 < 1/2. \end{cases} \tag{7}$$

$$x_2(x_1) = \begin{cases} (2x_1 + 3)/4, & \text{if} \quad 0 \leq x_1 \leq 1/2; \\ 1, & \text{if} \quad 1/2 < x_1 \leq 1. \end{cases} \tag{8}$$

By solving these two equations simultaneously, we obtain the unique location equilibrium as $(x_1, x_2) = (1/6, 5/6)$. Substituting it into Eqs. (5) and (6), we have the following proposition.

**Proposition 1** *Assume that the utility function of consumers in the Hotelling game be* (1) *with* $k = 2$. *Then, the unique Nash location equilibrium is* $(x_1, x_2) = (1/6, 5/6)$. *The correspondent prices and the profits are* $p_1 = p_2 = \sqrt{3t}/3$ *and* $\pi_1 = \pi_2 = \sqrt{3t}/6$, *respectively.*

Unlike the maximum differentiation given by D'Aspremont et al. (1979), the firms choose in-between differentiation. Locational differentiation softens price competition in the Hotelling game. In our revised model, the consumers are more price-sensitive; therefore, the firms should set lower prices than in the case of D'Aspremont et al. (1979) ($k = 1$). This implies that our revision weakens the positive effects of locational differentiation, and the firms prefer to expand their markets by approaching the market center.

[3] We readily find that the second-order conditions are satisfied for the equilibrium.

## 4 General cases

Let us deal with the general case with $k > 0$. The utility function is:

$$u_z(p_i, x_i) = r - p_i^k - t(z - x_i)^2. \quad (9)$$

Lambertini (1993) provided more differentiated equilibrium $(x_1, x_2) = (-1/4, 5/4)$ assuming that the firms can be located anywhere in $(-\infty, \infty)$ under $k = 1$. In other words, his model allows outside-city location ($x_i > 1$) in the model of D'Aspremont et al. (1979). To compare our result with his study, let the firms be located in $(-\infty, \infty)$ in our model too, with other things being equal. In order to abbreviate the analysis, we assume the positive demand for each firm. The demand function is then rewritten as:

$$D_1 = \frac{p_2^k - p_1^k}{2t(x_2 - x_1)} + \frac{x_1 + x_2}{2}, \quad D_2 = 1 - D_1. \quad (10)$$

The profit function is defined similarly as in Eq. (4). In the second stage the first-order conditions $\partial \pi_i / \partial p_i = 0$ yield the price equilibrium[4]:

$$p_1 = \left\{ \frac{t(x_2 - x_1)\big(2 + k(x_1 + x_2)\big)}{k(k+2)} \right\}^{1/k}, \quad p_2 = \left\{ \frac{t(x_2 - x_1)\big(2 + k(2 - x_1 - x_2)\big)}{k(k+2)} \right\}^{1/k}. \quad (11)$$

Note that if $k = 2$, the equations degenerate into the price equilibrium given by (5) in Sect. 3 above.

We proceed to the first stage. Substituting Eq. (11) into the profit function, we have:

$$\pi_1 = \frac{p_1\big(2 + k(x_1 + x_2)\big)}{2(k+2)}, \quad \pi_2 = \frac{p_2\big(2 + k(2 - x_1 - x_2)\big)}{2(k+2)}, \quad (12)$$

where $p_1$ and $p_2$ are given by Eq. (11). If $x_1 + x_2 = 1$ (the locations are symmetric with respect to the center), $\pi_i = p_i/2$ holds. Solving the first-order conditions $\partial \pi_i / \partial x_i = 0$, we obtain the symmetric location equilibrium $x_i^*$ as a function of $k$ as follows:

**Proposition 2** *The location equilibrium is symmetric and given by*:

$$x_1^*(k) = \frac{k^2 - 2}{2k(k+1)}, \quad x_2^*(k) = 1 - x_1^*(k). \quad (13)$$

[4] We can easily confirm that the second-order conditions are satisfied ($\partial^2 \pi_i / \partial p_i^2 < 0$).

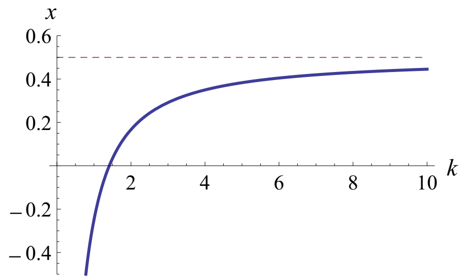



**Fig. 1** (The location equilibrium)

Figure 1 shows the equilibrium. The variable $x_1^*$ is independent from $t$, and it monotonically increases with respect to $k$ and is convergent to 1/2 since $dx_1^*/dk = (k^2 + 4k + 2)/(2k^2(k+1)^2) > 0$ and $\lim_{k\to\infty} x_1^* = 1/2$. In other words, the greater the value of $k$, the nearer to the center the firms are located, which implies that the principle of minimum differentiation *almost* holds if $k$ is large. Note that the firms never choose the same location because a price-cutting war leads to zero profits (non-existence of equilibrium), and the principle of minimum differentiation does not hold literally.

In an opposite direction, we consider a lower $k$ value. The fact of $x_1^*(1) = -1/4$ reproduces the result reported by Lambertini (1993). Further, we find that $x_1^* \leq 0$ when $k \leq \sqrt{2}$. If we reject the firms' outside location ($x_i \notin [0, 1]$), then the firms are to be located at the edges ($x_1^* = 0$ and $x_2^* = 1$) since $\partial\pi_1/\partial x_1 < 0$ and $\partial\pi_2/\partial x_2 > 0$ hold when $x_i \in [0, 1]$.

**Proposition 3** (Principle of Maximum Differentiation) *Assume that the firms must be located in* $L = [0, 1]$ *and* $k \leq \sqrt{2}$. *The principle of maximum differentiation then holds* ($x_1^* = 0$ *and* $x_2^* = 1$).

Without the above-mentioned restriction, since $\lim_{k\to 0} x_1^* = -\infty$, we find that the smaller the $k$, the more differentiated the locations from each other the firms will choose.

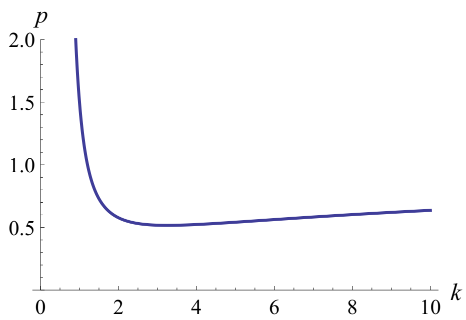
We next analyze the properties of the price and the profits in equilibrium. Remember that the profits are half of the price ($\pi_i^* = p_i^*/2$) because $x_1^* + x_2^* = 1$. Substituting Eq. (13) into Eq. (11), we have:

$$p_1^*(k, t) = p_2^*(k, t) = \left(\frac{t(k+2)}{k^2(k+1)}\right)^{1/k}. \tag{14}$$

We first confirm $\partial p_i^*/\partial t > 0$. Differentiating this price with respect to $k$, we find

$$\frac{\partial p_i^*}{\partial k} \lesseqgtr 0 \Longleftrightarrow k \lesseqgtr \tilde{k}(t), \tag{15}$$

**Fig. 2** (The price curve when $t = 1$)




where $\tilde{k}(t)$ is a unique solution[5] of:

$$\ln t = -\frac{2k^2 + 7k + 4}{k^2 + 3k + 2} - \ln\left(\frac{k+2}{k^2(k+1)}\right) \tag{16}$$

with $\lim_{k\to 0} p_i^* = \infty$ and $\lim_{k\to\infty} p_i^* = 1$. Summarizing the above, we have the following.

**Proposition 4** (Properties of price and profit) *The equilibrium price is always increasing in t, while it is first decreasing in k when $k \in (0, \tilde{k})$ and is increasing in k when $k \in (\tilde{k}, \infty)$. The equilibrium profits are half of the equilibrium price.*

Figure 2 shows the price curves with respect to $k$ when $t = 1$. Solving Eq. (16) with $t = 1$ yields $\tilde{k} \approx 3.25$. Therefore, the curve has an upward-sloping part.

From Proposition 2, the firms approach each other toward the center when $k$ gets large; it thus seems that the competition between the firms becomes fiercer. It might therefore be expected that $p_i^*$ is decreasing in $k$. However, Proposition 4 shows that this intuition is true only when $k$ is relatively low. Recall that when the price is sufficiently low, $k$ can be an inverse measure of price-sensitiveness. One explanation is that when $k$ is large and the price is low, an increase in $k$ does not necessarily imply that consumers are more price-sensitive. Therefore, the competition is relaxed and the price is gradually increasing.

Let us focus on the price term in the utility function. In contrast to the price itself, $(p^*)^k$ is monotonically decreasing in $k$. Figure 3 shows the curve when $t = 1$. Note that the demand (market share) directly depends on the $k$-th power of the price, rather than the price. The price is adjusted in accordance with the optimal $(p^*)^k$. When $k$ is large, a slight reduction in price causes a considerable reduction in $(p^*)^k$. Therefore, firms only need a small reduction in price when $k$ is large, even if the firms approach each other and competition gets intense. This is another explanation of why the price curve is upward-sloping for a larger $k$.

[5] See Appendix for the details.

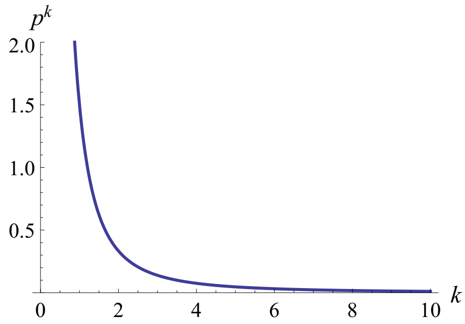



**Fig. 3** (The price term in the utility function: $(p^*)^k$ with $t = 1$

Economically, *strategic complements* are a key for these properties of equilibrium values. That is, the firm chooses a low (high) price when its rival posts a low (high) price in the presence of strategic complements in economic analyses. In our model, when $k$ is large and the rival firm sets a relatively high price, the rival is not so competitive; hence, enhancing the price does not decrease the demand very much. The high-price equilibrium is thus reasonable. U-shaped price (profits) curves are an interesting feature in our model.

## 5 Conclusion

We have shown that a slight change in the price term of the utility function in the Hotelling game enables us to obtain any level of locational differentiation in equilibrium. We can reproduce either the minimum differentiation (Hotelling 1929) or the maximum differentiation (D'Aspremont et al. 1979) in one model by properly choosing the parameter. Further, the locational change is monotone with respect to the parameter, while the price and the profits are U-shaped.

The well-known feature in the spatial competition of Hotelling (1929), i.e., "softening competition by locational differentiation," is not so straightforward, but it is specific in the case of $k = 1$. Our model contributes to a better understanding the Hotelling game.

## Appendix

## On Proposition 2

We have $\partial\pi_1/\partial x_1 = g_0 \cdot g_1$ and $\partial\pi_2/\partial x_2 = g_0 \cdot g_2$, where

$$g_0 = \left\{2k(k+2)(x_2 - x_1)\right\}^{-1}\{t(x_2 - x_1)((2 - x_1 - x_2) + 2)\}^{1/k}k^{-1/k}(k+2)^{-1/k} \neq 0$$

$$g_1 = k^2(x_2 - x_1) + 2kx_1 + 2$$

$$g_2 = k^2(x_2 - x_1) + 2k(1 - x_2) + 2.$$

Therefore, the first-order conditions $\partial\pi_i/\partial x_i = 0$ degenerate into $g_1 = 0$ and $g_2 = 0$. Solving the two equations yield (13) in the proposition. Evaluating the second derivative $\partial^2\pi_i/\partial x_i^2$ at Eq. (13), we have:

$$\left.\frac{\partial^2\pi_i}{\partial x_i^2}\right|_{(x_1,x_2)=(x_1^*,x_2^*)} = -\frac{t}{2k}\left(\frac{t(k+2)}{k^2(k+1)}\right)^{-1+1/k} < 0,$$

which implies that the second-order conditions are satisfied.

## On $\tilde{k}$

Let $f(k)$ be the right-hand side of Eq. (16). We can easily have $df(k)/dk > 0$ with $\lim_{k\to 0} f(k) = -\infty$ and $\lim_{k\to\infty} f(k) = \infty$. This implies that there exists a unique solution for Eq. (16) for any $t \in (0,\infty)$, and we also find $d\tilde{k}/dt = (2k^4 + 14k^3 + 29k^2 + 24k + 8)/k(k^2 + 3k + 2)^2 > 0$.

**Acknowledgements** The author is grateful for the helpful comments of two anonymous referees, which improved the quality of this publication. All remaining errors are my own.

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