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ORIGINAL PAPER

Mixed duopoly in a Hotelling framework with cubic transportation costs

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Abstract
We examine a two-stage location-price model of a mixed duopoly where a private profit-maximizing firm competes with a public welfare-maximizing firm in a Hotelling-type framework. A noteworthy result in this model is that, with quadratic transportation costs, which has become the usual assumption in the literature, the socially optimal locations are obtained in equilibrium. We show here that under the alternative assumption of cubic transportation costs this result no longer holds: equilibrium locations are socially suboptimal. We find that just as in the case of linear transportation costs, previously studied in the literature, for some locations there is price equilibrium in the second stage of the game and for other locations there is not. But, in contrast with such a case, there is a location pair for which there is price equilibrium in the second stage of the game and neither firm has an incentive to marginally change its location. We also find that, in contrast with the case of quadratic transportation costs, this location pair is not socially optimal.

Keywords Mixed duopoly · Hotelling line · Socially optimal locations · Product differentiation

JEl Classification L13 · L32 · L33 · H44

1 Introduction

There are markets in which public firms coexist in competition with private firms. For example, in the banking sector Caixa Geral de Depositos in Portugal, Banco Nacional in Costa Rica, BancoEstado in Chile and Banco de la República Oriental in Uruguay, all government-owned banks, are major players in their respective

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countries.[1] Similar examples can be found in sectors like telecommunications, energy, health, postal services or education. In consonance with the importance of this kind of markets, which have come to be known as mixed markets, a growing literature[2] has developed studying their different facets, among them their spatial dimension. A noteworthy result[3] is that competition in a mixed duopoly between a profit-maximizing private firm and a welfare-maximizing public firm[4] yields the socially optimal outcome in the standard location-price Hotelling model. This result is obtained under the assumption that transportations costs are quadratic, and contrasts with the socially suboptimal equilibrium locations in a purely private duopoly.[5]

Here we study if the result that a mixed duopoly yields the socially optimal locations continues to hold when we abandon the assumption of quadratic transportation costs-which has become the standard assumption in Hotelling-type models of mixed markets-by examining the alternative assumption of cubic transportation costs. We find that under this extension the result no longer holds: firms' locations are socially suboptimal.

The most noticeable examples of steeply increasing transportations costs, such as cubic transportation costs, may be those where the Hotelling model is interpreted as referring to consumers that have heterogeneous tastes over some characteristic of a product, such as the sweetness of cereal.[6] In this interpretation, the location of a consumer refers to his or her most preferred specification of this characteristic (the most preferred degree of sweetness) and the transportation costs refer to the disutility from consuming a product with a different specification. This disutility may steeply increase as a product characteristic gets increasingly different from the consumer's preferred choice, which is the equivalent of a store being located at a longer distance from a consumer. Similarly, in the interpretation of actual physical locations, the transportation costs may be also steeply increasing if they include not only the monetary cost of travelling but also the disutility from the time spent in the travel and from the uncertainty in its length.[7] One can easily envisage situations where as the distance, and thus the time spent in the journey and the uncertainty in

[1] Information on these banks can be found at www.cgd.pt, www.bncr.fi.cr, www.corporativo.bancoestado.cl and https://www.the-brow.com/ for the cases of Portugal, Costa Rica, Chile and Uruguay, respectively. Barros and Modesto (1999) provide a detailed analysis of the Portugal banking system in the 90 s.

[2] This literature has expanded to study a variety of issues, such as partial privatization (Matsumura 1998; Jain and Pal 2012), mergers (Bárcena-Ruiz and Garzón 2003; Artz et al. 2009; Méndez-Naya 2008) and capacity choice (Nishimori and Ogawa 2004; Lu and Poddar 2005), to name but a few.

[3] See Cremer et al. (1991) and Matsumura and Matsushima (2004).

[4] Modelling the public firm as maximizing social welfare is typical in the mixed oligopoly literature. For an alternative approach see Bennett and La Manna (2012) and Zhang and Zhong (2015).

[5] D'Aspremont et al. (1979) show that in a private duopoly with quadratic transportation costs firms locate too far away from each other from a social point of view, at the edges of the line.

[6] Hotelling (1929) original paper refers to this interpretation with the example of the sweetness of cider. For an exposition of the Hotelling model and alternative interpretations of it see also Tirole (1988) and Waterson (1994).

[7] The literature of transportation costs takes into account not only the average travel time, but also its variability. See for example Lam and Small (2001), Small et al. (2005) and Asensio and Matas (2008).

its duration, increases, the disutility steeply increases.[8] More importantly, since there is no certainty in the exact functional form of transportation costs, it is of relevance to examine if and how different assumptions on them change the outcomes that we obtain. Cubic transportation costs provide us with a natural alternative assumption to perform such an investigation.

To explain our results let us mention that Lu (2006) finds that there is no equilibrium in the two-stage location-price model of a mixed duopoly under linear transportation costs, while Cremer et al. (1991) show that an equilibrium does exist and yields the socially optimal locations under the alternative assumption of quadratic transportation costs. Here we show that under cubic transportation costs, just as in Lu (2006), for some locations there is price equilibrium in the second stage of the game and for other locations there is not. But, in contrast to Lu, there is a location pair for which there is price equilibrium in the second stage of the game and neither firm has an incentive to marginally change its location and, moreover, in contrast to Cremer et al. (1991), this location pair is not socially optimal.

By comparing the incentives that firms face when choosing their locations under linear, quadratic and cubic transportation costs, our analysis also sheds light on why the result that a mixed duopoly yields the socially optimal outcome is not robust. To minimize transportation costs, the public firm always chooses its location such that it is half as far away from the closest edge of the Hotelling line as it is from the location of the private firm, irrespective of the convexity of these costs. In contrast, the location choice of the private firm involves a trade-off between getting closer to the public firm to increase its market share given fixed prices, and getting away from the public firm to increase prices,[9] and the convexity of the transportation costs affects the relative strength of these two forces. As the degree of convexity changes, the resolution of the trade-off also changes. This results in the private firm wanting to get as close as possible to its competitor under linear transportation costs,[10] locating at just the right distance from it—from a social point of view-under quadratic transportation costs, and moving too far away from it, to the extent that it locates at one edge of the line, under cubic transportation costs.

The robustness of the result that a mixed duopoly yields the socially optimal outcome has been studied with respect to other assumptions of the model. Matsumura and Matsushima (2004) show that it continues to hold when the private and public firms have different marginal costs, and Matsumura and Matsushima (2003) show that it also holds if there is a sequential choice of locations with the public firm acting as the leader. On the other hand, Kitahara and Matsumura (2013) show that the result no longer holds when the assumption of inelastic demand is dropped, and Benassi et al. (2017) show that it also fails to hold when the assumption of a uniform consumers' distribution is replaced by other distributions.

[8] An extreme example would be the choice of an obstetric unit or birth center.

[9] This trade-off also appears in the private duopoly, as explained in Tirole (1988), and has been previously mentioned in the location choice of the private firm in a mixed duopoly in, for instance, Benassi et al. (2017).

[10] Which in turns leads to the non-existence of equilibrium.

Yet, none of these papers examines the role of transportation costs, which is the focus of our paper.

2 The model

We use a standard location-price model of a mixed duopoly as studied by Cremer et al. (1991). Consumers are uniformly distributed with unit density on the interval [0,1] and inelastically demand one unit of a product. Two firms produce this product with zero marginal cost: firm 0, a welfare-maximizing public firm located at x_0, and firm 1, a profit-maximizing private firm located at x_1. We assume that $0 \leq x_0 < x_1 \leq 1$. The firms compete in the following two-stage game: in the first stage, they simultaneously choose their locations.[11] In the second stage, they simultaneously choose prices. We also assume, contrary to the standard assumption of quadratic transportation costs, that a consumer located at y incurs a transportation cost of $|y - x_0|^3$ from buying firm 0's product and $|y - x_1|^3$ from buying firm 1's product.

3 Results

Let p_i be firm i's price and q_i firm i's demand, i = 0, 1. A consumer located at y incurs a total cost (price plus transportation cost) of $p_0 + |y - x_0|^3$ to buy from firm 0 and of $p_1 + |y - x_1|^3$ to buy from firm 1. Let y^* be the solution of the equation[12]:

$$p_0 + |y^* - x_0|^3 = p_1 + |y^* - x_1|^3 \tag{1}$$

If $0 \leq y^* \leq 1$, y^* is the location of the consumer indifferent between buying from either firm. Moreover, since the consumers located to the left of y^* prefer to buy from firm 0, while those located to the right of y^* prefer to buy from firm 1, y^* is also firm 0's demand. On the other hand, if $y^* < 0\,(y^* > 1)$ then all consumers prefer to buy from firm 1 (firm 0). We therefore have:

$$q_0 = \begin{cases} 0 & \textit{if } y^* \leq 0 \\ y^* & \textit{if } 0 \leq y^* \leq 1 \\ 1 & \textit{if } 1 \leq y^* \end{cases}$$

[11] We assume for simplicity that firms choose different locations. Just as in Lu (2006) there would be infinite price equilibria in the second stage if they chose the same location. As we explain below, we restrict our attention to a subset of first-stage locations for which there is a unique price equilibrium in the second stage.

[12] To see that Eq. (1) has a solution and it is unique, notice that the continuous function $d(y) = p_0 + |y - x_0|^3 - p_1 - |y - x_1|^3$: (1) approaches $-\infty$ as y approaches $-\infty$, (2) approaches $+\infty$ as y approaches $+\infty$, and (3) is strictly increasing.

It is easy to check that q_0 satisfies[13]:

$$q_0 = \begin{cases} = 0 & \text{if } p_1 - p_0 \le -\left(x_1^3 - x_0^3\right) \\ \in \left[0, x_0\right] & \text{if } -\left(x_1^3 - x_0^3\right) \le p_1 - p_0 \le -\left(x_1 - x_0\right)^3 \\ \in \left[x_0, x_1\right] & \text{if } -\left(x_1 - x_0\right)^3 \le p_1 - p_0 \le \left(x_1 - x_0\right)^3 \\ \in \left[x_1, 1\right] & \text{if } \left(x_1 - x_0\right)^3 \le p_1 - p_0 \le \left(1 - x_0\right)^3 - \left(1 - x_1\right)^3 \\ = 1 & \text{if } \left(1 - x_0\right)^3 - \left(1 - x_1\right)^3 \le p_1 - p_0 \end{cases} \tag{2}$$

while q_1 is given by

$$q_1 = 1 - q_0 \tag{3}$$

Since demand is completely inelastic, maximization of social surplus is equivalent to minimization of transportation costs, which are given by:

$$TC = \int_0^{q_0} \left|y - x_0\right|^3 dy + \int_{q_0}^{1} \left|y - x_1\right|^3 dy \tag{4}$$

and can be written as:

$$TC = \begin{cases} \frac{\left\{x_0^4 - \left(x_0 - q_0\right)^4 + \left(x_1 - q_0\right)^4 + \left(1 - x_1\right)^4\right\}}{4} & \text{if } 0 \le q_0 \le x_0 \\ \frac{\left\{x_0^4 + \left(q_0 - x_0\right)^4 + \left(q_0 - x_1\right)^4 + \left(1 - x_1\right)^4\right\}}{4} & \text{if } x_0 \le q_0 \le x_1 \\ \frac{\left\{x_0^4 + \left(q_0 - x_0\right)^4 - \left(q_0 - x_1\right)^4 + \left(1 - x_1\right)^4\right\}}{4} & \text{if } x_1 \le q_0 \le 1 \end{cases} \tag{5}$$

Firm 0 chooses p_0 to minimize TC. It follows from (1) and (5) that to solve this problem it sets

$$p_0 = p_1 \tag{6}$$

To see this notice first that, from (5), transportation costs are minimized when $q_0 = \frac{x_0 + x_1}{2}$ because TC is strictly decreasing (increasing) in q_0 when $0 \le q_0 \le x_0$ ($x_1 \le q_0 \le 1$), $\frac{dTC}{dq_0} = 0$ when $q_0 = \frac{x_0 + x_1}{2}$ and $\frac{d^2TC}{dq_0^2} > 0$ in the interval $x_0 \le q_0 \le x_1$. Notice then that, from (1), firm 0 must set $p_0 = p_1$ to achieve $q_0 = \frac{x_0 + x_1}{2}$.

Firm 1 chooses p_1 to maximize Π_1, where

$$\Pi_1 = p_1\left(1 - q_0\right) \tag{7}$$

Suppose first that firm 1 chooses a price p_1 in the interval $-\left(x_1 - x_0\right)^3 \le p_1 - p_0 \le \left(x_1 - x_0\right)^3$, which corresponds to $x_0 \le q_0 \le x_1$. The first-order condition to maximize profits in the interior of this interval is given by:

[13] We present in the appendix q_0 written as an explicit function of p_0 and p_1.

$$\frac{\partial \Pi_1}{\partial p_1} = -p_1 \frac{\partial q_0}{\partial p_1} + 1 - q_0 = 0$$

It is convenient to obtain $\frac{\partial q_0}{\partial p_1}$ directly from the condition (1), which takes the form

$$p_0 + \left(q_0 - x_0\right)^3 = p_1 + \left(x_1 - q_0\right)^3$$

when $x_0 \leq q_0 \leq x_1$. This condition implies that

$$\frac{\partial q_0}{\partial p_1} = \frac{1}{3\left(q_0 - x_0\right)^2 + 3\left(x_1 - q_0\right)^2}$$

and therefore firm 1's first-order condition is:

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{-p_1}{3\left(q_0 - x_0\right)^2 + 3\left(x_1 - q_0\right)^2} + 1 - q_0 = 0 \tag{8}$$

Let us solve for p_0 and p_1 in (6) and (8). Condition $p_0 = p_1$ from (6) implies $q_0 = \frac{x_0 + x_1}{2}$ which in turn implies, using (8), that:

$$p_0 = p_1 = \frac{3\left(2 - x_0 - x_1\right)\left(x_1 - x_0\right)^2}{4} \tag{9}$$

Suppose instead that firm 1 chooses a price p_1 such that $p_1 - p_0 \leq -\left(x_1 - x_0\right)^3$. We will then have $p_1 < p_0$ and firm 0 will not minimize TC (which is done by choosing $p_0 = p_1$). Similarly, if firm 1 chooses a price p_1 such that $\left(x_1 - x_0\right)^3 \leq p_1 - p_0$ we will have $p_1 > p_0$ and, again, firm 0 will not minimize TC.

Therefore, if there is an equilibrium in the second stage of the game, it is given by the prices in Eq. (9). To know if these prices are indeed an equilibrium, we need to make sure that neither firm can improve its objective function by deviating to another price.

As argued above, given any p_1 [and therefore also firm 1's price in Eq. (9)], firm 0 minimizes transportation costs by setting $p_0 = p_1$. Thus, firm 0 cannot lower transportation costs by deviating to another price. However, given p_0 as in Eq. (9), firm 1's response, according to condition (8), only tells us that p_1 satisfies a necessary condition to maximize Π_1 in the interior of the range $p_0 - \left(x_1 - x_0\right)^3 \leq p_1 \leq p_0 + (x_1 - x_0)^3$. Therefore, we have to examine if firm 1 can increase its profits by deviating to another price either in this range or outside of it. It turns out that for some locations (x_0, x_1), firm 1 is able to increase its profits by deviating from the price in Eq. (9). For these locations, there is no equilibrium in the second stage of the game. Technically, the reason for the non-existence of equilibrium is the lack of quasi-concavity of firm 1's profit function. For these locations, given p_0 as in Eq. (9) although firm 1's profit function has one local maximum at $p_1 = p_0$, it has another one that yields higher profits at another value of p_1.

Proposition 1 *For every x_0, there exists $x_1^* \in \left(\frac{3x_0+2}{5}, \frac{\left(\sqrt{13}-2\right)x_0+5-\sqrt{13}}{3}\right)$ such that there exists a Nash equilibrium in prices in the second stage of the game if $x_1 > x_1^*$ while there does not exist equilibrium if $x_1 < x_1^*$. Whenever it exists, the equilibrium is given by Eq.* (9).

Proposition 1 tells us that there exists price equilibrium in the second stage of the game if firms are sufficiently far apart, but not otherwise. It tells us that, for each x_0, there is a cutoff point x_1^* such that x_1 needs to be further away from x_0 than this point to have equilibrium.

It follows from Proposition 1 that the restriction $x_0 \leq 0.34$, $x_1 \geq 0.66$, which amounts to banning firms from locating at distance smaller than 0.16 from the city center, is a sufficient condition for a price equilibrium to exist in the second stage of the game. We impose such assumption on the locations of firms before proceeding with the analysis of the first stage. This restriction may be justified by zoning regulations [see for example Lai and Tsai (2004), Chen and Lai (2008) or Matsumura and Matsushima (2012)]. A particular type of zoning regulation bans some businesses from locating around the city center for several reasons. One of them is pollution, which tends to be particularly intense in this area[14] and thus calls for the prohibition of polluting businesses. Another reason is traffic congestion, for which businesses that by their nature generate a lot of traffic may also be banned around the city center.[15]

Let us consider the first stage of the game. Replacing the equilibrium prices p_0 and p_1 in (9), and the implied firm 0's demand q_0, into the transportation costs in (5) and firm 1's profits in (7), we obtain both transportation costs and firm 1's profits as a function of the first-stage locations x_0 and x_1 as follows:

$$TC = \frac{x_0^4 + 2\left(\frac{x_1-x_0}{2}\right)^4 + \left(1-x_1\right)^4}{4} \tag{10}$$

$$\Pi_1 = \frac{3\left(x_1-x_0\right)^2\left(2-x_1-x_0\right)^2}{8} \tag{11}$$

In the first stage of the game, firm 0 chooses x_0 to minimize TC as given in (10) and firm 1 chooses x_1 to maximize Π_1 as given in (11). Minimization of TC with respect to x_0 leads to the following first-order condition:

14 As evidenced by the closely related fact that some cities have severely limited or completely ban most vehicles from the city center, putting special attention in the most polluting vehicles.

15 Also, for historic reasons, the city center in many places has an outstanding cultural value, which makes certain businesses incompatible with the preservation of its character.

$$\frac{\partial TC}{\partial x_0} = x_0^3 - \left(\frac{x_1 - x_0}{2}\right)^3 = 0 \tag{12}$$

The second-order condition is:

$$\frac{\partial^2 TC}{\partial x_0^2} = 3x_0^2 + \frac{3}{2}\left(\frac{x_1 - x_0}{2}\right)^2 > 0 \tag{13}$$

Maximization of Π_1 with respect to x_1 leads to the following first-order condition:

$$\frac{\partial \Pi_1}{\partial x_1} = \left(\frac{3}{2}\right)\left(2 - x_0 - x_1\right)\left(x_1 - x_0\right)\left(1 - x_1\right) = 0 \tag{14}$$

It follows from (14) that, since $\frac{\partial \Pi_1}{\partial x_1} > 0$ for all $x_0 < x_1 < 1$ and $\frac{\partial \Pi_1}{\partial x_1} = 0$ for $x_1 = 1$, firm 1's optimal choice is $x_1 = 1$. It follows from (12) and (13) that firm 0's optimal choice is $x_0 = \frac{x_1}{3}$. Simultaneous optimization of (10) and (11) therefore yields:

$$x_0 = \frac{1}{3}, \quad x_1 = 1 \tag{15}$$

which is a location pair outside the restricted zone and, thus, for which there is equilibrium in the second stage. This is the unique subgame-perfect equilibrium, or location equilibrium, of the whole game.

The problem of non-existence of price equilibrium for some locations in the second stage of two-stage location-price models also arises in private oligopolies. We follow the approach of this literature (Economides 1984, 1986, 1989)[16] and define the direction in which $\frac{\partial \Pi_1}{\partial x_1}$ is positive as the 'relocation tendency' of firm 1. Similarly, we define the direction in which $\frac{\partial TC}{\partial x_0}$ is negative as the 'relocation tendency' of firm 0.[17]

We have shown that with cubic transportation costs, $\frac{\partial \Pi_1}{\partial x_1} > 0$ for all $x_0 < x_1 < 1$ and thus firm 1's relocation tendency is toward the right edge, away from firm 0, for all interior points, and it is zero at the edge of the line: $\frac{\partial \Pi_1}{\partial x_1} = 0$ for $x_1 = 1$. When $x_1 = 1$, the relocation tendency of the public firm is towards the private firm when $x_0 < \frac{1}{3}\left(\frac{\partial TC}{\partial x_0} < 0\right)$, away from the private firm when $x_0 > \frac{1}{3}\left(\frac{\partial TC}{\partial x_0} > 0\right)$ and it is zero when $x_0 = \frac{1}{3}$. Therefore, when $(x_0, x_1) = (1/3, 1)$ the relocation tendency is zero for

[16] See also Hinloopen and Van Marrewijk (1999) and Posada and Straume (2004).

[17] The literature on private oligopolies uses a slightly different definition for equilibrium in the location-price game. It considers the whole set E of first-stage locations for which there is unique price equilibrium in the second stage, instead of considering only a subset of E as we do here. It then considers the zero-relocation locus (in our case is $\frac{\partial \Pi_1}{\partial x_1} = \frac{\partial TC}{\partial x_0} = 0$). The intersection of this locus with E defines an equilibrium (Economides 1986). It is easy to see that our results do not change if we use this alternative definition. Notice that with linear transportation costs this alternative approach underscores that whenever firms choose locations in E, they have a tendency to move away from this zone and into the zone where there is no equilibrium.

both firms. At this location pair, neither firm improves its objective function by marginally changing its location.

We can compare the results of linear, quadratic and cubic transportation costs as follows. The reaction function of the public firm is the same with all three transportation costs. This firm always minimizes transportation costs by setting $x_0 = \frac{x_1}{3}$. For location pairs satisfying this condition, the public firm will have no incentives to relocate.

The private firm, in contrast, will behave differently as we change the convexity of the transportation costs. There are two forces that drive firm 1's behavior. First, there is an incentive for firm 1 to move closer to firm 0 because with fixed prices, this movement increases its demand. Second, there is an incentive for firm 1 to move away from firm 0 to increase second-stage prices. An increase in the convexity of the cost function increases the strength of the second effect relative to the first one. With linear transportation costs, the trade-off of the two effects results in a relocation tendency for firm 1 towards firm 0's location, which results in turn in the absence of equilibrium. If we increase the convexity of transportation costs and consider quadratic transportation costs, firm 1 has a relocation tendency towards firm 0 if it is far from this firm, but away from firm 0's location if it is close to it. The relocation tendency vanishes exactly at the social optimum. If we further increase the convexity of transportation costs and consider cubic transportation costs, firm 1 has a relocation tendency away from firm 0. Since the existence zone of second-stage price equilibrium is precisely the zone of locations where firms are not too close, the tendency of firm 1 to move away from firm 0 keeps firms in the existence zone and leads to an equilibrium with firm 1 at the right edge and firm 0 one-third of the line away from the other edge.

Importantly, for the socially optimal locations $(1/4, 3/4)$ there is also equilibrium in the second stage of the game, but the private firm exhibits a tendency to move away from the public firm: $\frac{\partial \Pi_1}{\partial x_1} > 0$. Given the public firm's location $x_0 = 1/4$, the private firm increases its profits by marginally changing its location from $x_1 = 3/4$ to the right.

Proposition 2 $(x_0, x_1) = (1/3, 1)$ *is the unique location equilibrium of the location-price game with cubic transportation costs. In this equilibrium locations are different from the socially optimal locations* $(x_0, x_1) = (1/4, 3/4)$.

4 Conclusion

We have studied a standard Hotelling location-price model of a mixed duopoly with cubic transportation costs. We have shown that, in contrast with the case of linear transportation costs, there exists a location pair for which there exists a unique price equilibrium in the second stage and neither firm has incentives to marginally relocate. We have also shown that, in contrast with the case of quadratic transportation costs, this location pair is not socially optimal.

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Appendix

Proof of Proposition 1

Let $p^* = \frac{3(2-x_0-x_1)(x_1-x_0)^2}{4}$ be the candidate equilibrium common price in Eq. (9) and $\Pi_1^* = \frac{3(2-x_0-x_1)^2(x_1-x_0)^2}{8}$ be the candidate equilibrium firm 1's profits as given in Eq. (11). We now examine if given $p_0 = p^*$, firm 1 can increase its profits above Π_1^* by choosing a price $p_1 \neq p^*$.

(1) if firm 1 chooses a price in the interval $p_0 - \left(x_1^3 - x_0^3\right) \leq p_1 \leq p_0 - \left(x_1 - x_0\right)^3$, then $q_0 \in \left[0, x_0\right]$ and we have

$$p_0 + \left(x_0 - q_0\right)^3 = p_1 + \left(x_1 - q_0\right)^3 \tag{16}$$

from where

$$\frac{\partial q_0}{\partial p_1} = \frac{1}{3\left(x_1 - q_0\right)^2 - 3\left(x_0 - q_0\right)^2}$$

and therefore:

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{-p_1}{3\left(x_1 - q_0\right)^2 - 3\left(x_0 - q_0\right)^2} + 1 - q_0 \tag{17}$$

Replacing p_1 from (16) into (17) and using $p_0 = p^*$ we obtain

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{36q_0^2 - 24\left(x_0 + x_1 + 1\right)q_0 + 7x_1^2 + 4x_0x_1 + 6x_1 + x_0^2 + 18x_0}{12\left(-2q_0 + x_0 + x_1\right)} \tag{18}$$

The denominator in the RHS of (18) is positive because $q_0 \leq x_0 < x_1$. The numerator is also positive because (a) it is decreasing in q_0 (its derivative with respect to q_0 is equal to $72q_0 - 24\left(x_0 + x_1 + 1\right) \leq 72x_0 - 24\left(x_0 + x_1 + 1\right) = -24\left(x_1 - x_0\right) - 24\left(1 - x_0\right) < 0$) and (b) it is positive when $q_0 = x_0$ (it is then equal to $\left(x_1 - x_0\right)\left(-13x_0 + 7x_1 + 6\right) > 0$).

It follows that Π_1 is increasing over the whole range $p_0 - \left(x_1^3 - x_0^3\right) \leq p_1 \leq p_0 - \left(x_1 - x_0\right)^3$. Let Π_1^a be firm 1's profits when $p_1 = p_0 - \left(x_1 - x_0\right)^3$-which implies $q_0 = x_0$ - and $p_0 = p^*$. Then

$$\Pi_1^a = \left(\frac{3\left(2 - x_1 - x_0\right)\left(x_1 - x_0\right)^2}{4} - \left(x_1 - x_0\right)^3\right)\left(1 - x_0\right)$$

If we substract from these profits the candidate equilibrium profits Π_1^* we obtain, after some simplifications,

$$\Pi_1^a - \Pi_1^* = \frac{\left(x_1 - x_0\right)^3\left(-3x_1 + 5x_0 - 2\right)}{8} < 0$$

which implies that firm 1 will not deviate to such a price, and will thus neither deviate to any price p_1 with $p_0 - \left(x_1^3 - x_0^3\right) \le p_1 \le p_0 - \left(x_1 - x_0\right)^3$.

(2) If firm 1 chooses a price in the interval $p_0 - \left(x_1 - x_0\right)^3 \le p_1 \le p_0 + \left(x_1 - x_0\right)^3$, then $q_0 \in \left[x_0, x_1\right]$ and we will have:

$$p_0 + \left(q_0 - x_0\right)^3 = p_1 + \left(x_1 - q_0\right)^3 \tag{19}$$

and thus

$$\frac{\partial q_0}{\partial p_1} = \frac{1}{3\left(q_0 - x_0\right)^2 + 3\left(x_1 - q_0\right)^2}$$

Therefore:

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{-p_1}{3\left(q_0 - x_0\right)^2 + 3\left(x_1 - q_0\right)^2} + 1 - q_0 \tag{20}$$

Replacing p_1 from (19) into (20) and using $p_0 = p^*$ we obtain, after some simplifications:

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{\left[2q_0 - x_0 - x_1\right]\left[16q_0^2 - \left(10\left(x_0 + x_1\right) + 12\right)q_0 + \left(7x_1 - 10x_0 + 6\right)x_1 + 7x_0^2 + 6x_0\right]}{-12\left(q_0 - x_0\right)^2 - 12\left(x_1 - q_0\right)^2}$$

Therefore, $\frac{\partial \Pi_1}{\partial p_1} = 0$ if $q_0 = \frac{x_0 + x_1}{2}$, which implies $p_1 = p_0 = p^*$. Notice also that, from (20):

$$\frac{\partial^2 \Pi_1}{\partial p_1^2} = \frac{-\left(3\left(q_0 - x_0\right)^2 + 3\left(x_1 - q_0\right)^2\right) + 6p_1\left(\left(q_0 - x_0\right) - \left(x_1 - q_0\right)\right)\frac{\partial q_0}{\partial p_1}}{\left(3\left(q_0 - x_0\right)^2 + 3\left(x_1 - q_0\right)^2\right)^2} - \frac{\partial q_0}{\partial p_1}$$

and that this second derivative evaluated at $p_1 = p_0 = p^*$ (and thus at $q_0 = \frac{x_0 + x_1}{2}$) is negative:

$$\frac{\partial^2 \Pi_1}{\partial p_1^2} = \frac{-4}{3\left(x_1 - x_0\right)^2} < 0$$

Therefore, $p_1 = p^*$ is a local maximum, and it is also the only value where $\frac{\partial \Pi_1}{\partial p_1}$ vanishes (in the segment $q_0 \in \left[x_0, x_1\right]$) unless the following equation has roots in $q_0 \in \left[x_0, x_1\right]$:

$$16q_0^2 - \left(10\left(x_0 + x_1\right) + 12\right)q_0 + \left(7x_1 - 10x_0 + 6\right)x_1 + 7x_0^2 + 6x_0 = 0 \tag{21}$$

Equation (21) is a second degree equation in q_0 with discriminant equal to

$$12\left(-29x_1^2 + \left(70x_0 - 12\right)x_1 - 29x_0^2 - 12x_0 + 12\right) \tag{22}$$

If the discriminant in (22) is negative $\frac{\partial \Pi_1}{\partial p_1}$ will only vanish at $q_0 = \frac{x_0+x_1}{2}$ (which corresponds to $p_1 = p^*$) in the segment $q_0 \in \left[x_0, x_1\right]$. If this discriminat is positive, then Eq. (21) will have two roots:

$$q_0^H = \frac{5x_0 + 5x_1 + 6 + \sqrt{3}\sqrt{-29x_1^2 + \left(70x_0 - 12\right)x_1 - 29x_0^2 - 12x_0 + 12}}{16} \tag{23}$$

$$q_0^L = \frac{5x_0 + 5x_1 + 6 - \sqrt{3}\sqrt{-29x_1^2 + \left(70x_0 - 12\right)x_1 - 29x_0^2 - 12x_0 + 12}}{16} \tag{24}$$

But, $q_0^H > x_1$, since the positiveness of the discriminant in (22) implies that (a) $q_0^H > \frac{5x_0+5x_1+6}{11}$, and (b) $x_1 < \frac{5x_0+6}{11}$, because this discriminat is strictly decreasing in x_1 and it is negative when $x_1 = \frac{5x_0+6}{11}$. These two facts imply $q_0^H - x_1 \geq \frac{5x_0+5x_1+6}{11} - x_1 = \frac{5x_0-11x_1+6}{11} > 0$. Thus, q_0^L is the only possible root additional to $q_0 = \frac{(x_0+x_1)}{2}$ in the range $q_0 \in \left[x_0, x_1\right]$, which cannot therefore be a local maximum (since $p_1 = p^*$ is a local maximum and $\frac{\partial \Pi_1}{\partial p_1}$ only vanishes at p^* and p^L (associated to q_0^L) then Π_1 is strictly decreasing in $p_1 \in \left(p^*, p^L\right)$ if $p^* < p^L$ and, similarly, it is strictly increasing in $p_1 \in \left(p^L, p^*\right)$ if $p^L < p^*$).

Since we proved above that firm 1 will not deviate to a price p_1 such that $q_0 = x_0$, the only price that remains to be considered in this interval is $p_1 = p_0 + \left(x_1 - x_0\right)^3$ which corresponds to $q_0 = x_1$. If $p_0 = p^*$ and firm 1 chooses such a price, firm 1's profits will be equal to

$$\Pi_1^b = \left(\frac{3\left(2 - x_0 - x_1\right)\left(x_1 - x_0\right)^2}{4} + \left(x_1 - x_0\right)^3\right)\left(1 - x_1\right)$$

Subtracting the candidate equilibrium profits Π_1^* from Π_1^b we obtain:

$$gb\left(x_0, x_1\right) = \Pi_1^b - \Pi_1^* = \left(\frac{3\left(2 - x_0 - x_1\right)\left(x_1 - x_0\right)^2}{4} + \left(x_1 - x_0\right)^3\right)\left(1 - x_1\right) - \frac{3\left(2 - x_0 - x_1\right)^2\left(x_1 - x_0\right)^2}{8}$$

with

$$\frac{\partial gb(x_0,x_1)}{\partial x_1}=\frac{-(x_1-x_0)^2(10x_1-7x_0-3)}{4}$$

For any given x_0,$gb(x_0,x_0)=0$, $gb(x_0,x_1)$ reaches a maximum at $x_1=\frac{(7x_0+3)}{10}$ with $gb\left(x_0,\frac{(7x_0+3)}{10}\right)=\frac{27(1-x_0)^4}{16{,}000}>0$, and $gb(x_0,1)=\frac{-3(1-x_0)^4}{8}<0$. Since $gb(x_0,x_1)$ is strictly increasing in x_1 for $x_0<x_1<\frac{(7x_0+3)}{10}$, strictly decreasing in x_1 for $\frac{(7x_0+3)}{10}<x_1<1$, and $gb\left(x_0,\frac{(3x_0+2)}{5}\right)=0$, it follows that, when $\frac{3x_0+2}{5}\leq x_1\leq 1$, $gb(x_0,x_1)\leq 0$, and firm 1 does not find this deviation profitable, while for $x_1<\frac{3x_0+2}{5}$, $gb(x_0,x_1)>0$, firm 1 does deviate from the candidate equilibrium price and there does not exist an equilibrium in the second stage of the game.

(3) if firm 1 chooses a price in the interval $p_0+(x_1-x_0)^3\leq p_1\leq p_0+(1-x_0)^3-(1-x_1)^3$, then $q_0\in[x_1,1]$ and we have

$$p_0+(q_0-x_0)^3=p_1+(q_0-x_1)^3 \tag{25}$$

from where

$$\frac{\partial q_0}{\partial p_1}=\frac{1}{3(q_0-x_0)^2-3(q_0-x_1)^2}$$

and therefore:

$$\frac{\partial \Pi_1}{\partial p_1}=\frac{-p_1}{3(q_0-x_0)^2-3(q_0-x_1)^2}+1-q_0 \tag{26}$$

Replacing p_1 from (25) into (26) and using $p_0=p^*$ we obtain

$$\frac{\partial \Pi_1}{\partial p_1}=\frac{-36q_0^2+24(x_0+x_1+1)q_0-x_1^2-4x_0x_1-18x_1-7x_0^2-6x_0}{12(2q_0-x_0-x_1)}$$

The denominator of $\frac{\partial \Pi_1}{\partial p_1}$ is positive because $x_0<x_1\leq q_0$. $\frac{\partial \Pi_1}{\partial p_1}$ will vanish when the numerator does so, which yields a quadratic equation in q_0 with discriminant equal to

$$144\left(3x_1^2+(4x_0-10)x_1-3x_0^2+2x_0+4\right) \tag{27}$$

The discriminant in (27) is strictly decreasing in x_1 and it is equal to zero when $x_1=x_1^R$, with

$$x_1^R=\frac{\left(\sqrt{13}-2\right)x_0+5-\sqrt{13}}{3}\in\left(\frac{3x_0+2}{5},1\right)$$

Therefore,

(a) when $x_1 > x_1^R$, the discriminant in (27) is negative and $\frac{\partial \Pi_1}{\partial p_1}$ never vanishes (it is negative). It then suffices to consider the price deviation associated to $q_0 = x_1$ (which yields profits Π_1^b). Since $x_1^R > \frac{3x_0+2}{5}$, we have that $x_1 > \frac{3x_0+2}{5}$ and thus $gb(x_0, x_1) < 0$, $\Pi_1^b < \Pi_1^*$ and there is equilibrium.

(b) when $x_1 \le x_1^R$ the discriminant in (27) is positive and $\frac{\partial \Pi_1}{\partial p_1} = 0$ has the following roots:

$$q_0^c = \frac{2x_1 + 2x_0 + 2 + \sqrt{3x_1^2 + (4x_0 - 10)x_1 - 3x_0^2 + 2x_0 + 4}}{6}$$

$$q_0^d = \frac{2x_1 + 2x_0 + 2 - \sqrt{3x_1^2 + (4x_0 - 10)x_1 - 3x_0^2 + 2x_0 + 4}}{6}$$

Now, from (26)

$$\frac{\partial^2 \Pi_1}{\partial p_1^2} = \frac{-3(q_0 - x_0)^2 + 3(q_0 - x_1)^2 + p_1\left(6(q_0 - x_0) - 6(q_0 - x_1)\right)\frac{\partial q_0}{\partial p_1}}{\left(3(q_0 - x_0)^2 - 3(q_0 - x_1)^2\right)^2} - \frac{\partial q_0}{\partial p_1} \tag{28}$$

Also, from (26), $\frac{\partial \Pi_1}{\partial p_1} = 0$ implies that

$$p_1 = (1 - q_0)\left(3(q_0 - x_0)^2 - 3(q_0 - x_1)^2\right) \tag{29}$$

Replacing p_1 from (29) and q_0 with q_0^c in (28) we get:

$$\frac{\partial^2 \Pi_1}{\partial p_1^2} = \frac{-3\sqrt{3x_1^2 + (4x_0 - 10)x_1 - 3x_0^2 + 2x_0 + 4}}{(x_1 - x_0)\left(\sqrt{3x_1^2 + (4x_0 - 10)x_1 - 3x_0^2 + 2x_0 + 4} - x_0 - x_1 + 2\right)^2}$$

Similarly, when $q_0 = q_0^d$ we get:

$$\frac{\partial^2 \Pi_1}{\partial p_1^2} = \frac{3\sqrt{3x_1^2 + (4x_0 - 10)x_1 - 3x_0^2 + 2x_0 + 4}}{(x_1 - x_0)\left(\sqrt{3x_1^2 + (4x_0 - 10)x_1 - 3x_0^2 + 2x_0 + 4} + x_0 + x_1 - 2\right)^2}$$

Therefore Π_1 reaches a local maximum at q_0^c and a local minimum at q_0^d. Let Π_1^c be firm 1's profits when p_1 is such that $q_0 = q_0^c$,

$$\Pi_1^c = \left(\frac{3(2 - x_0 - x_1)(x_1 - x_0)^2}{4} + (q_0^c - x_0)^3 - (q_0^c - x_1)^3\right)(1 - q_0^c)$$

and let $gc(x_0, x_1) = \Pi_1^c - \Pi_1^*$. $gc(x_0, x_1)$ is the increase in profits when firm 1 chooses p_1 such that $q_0 = q_0^c$ instead of the candidate equilibrium price p^*.

Since we know that for $x_1 < \frac{3x_0+2}{5}$ there is no equilibrium (because $\Pi_1^b > \Pi_1^*$), while for $x_1 > x_1^R$ there is equilibrium, we will focus our attention on the behaviour of $gc(x_0, x_1)$ in the interval $x_1 \in \left[\frac{3x_0+2}{5}, x_1^R\right]$.

We have that $gc\left(x_0, \frac{3x_0+2}{5}\right) > 0$, $gc(x_0, x_1^R) < 0$,and we will now prove that $\frac{\partial gc(x_0,x_1)}{\partial x_1} < 0$ for $x_1 \in \left[\frac{3x_0+2}{5}, x_1^R\right]$. This implies that there exists $x_1^* \in \left(\frac{3x_0+2}{5}, x_1^R\right)$ such that $\Pi_1^c = \Pi_1^*$ if $x_1 = x_1^*$ and $\Pi_1^c > \Pi_1^*$ $\left(\Pi_1^c < \Pi_1^*\right)$ if $x_1 < x_1^*$ $\left(x_1 > x_1^*\right)$. This in turn implies that there does not exist (there exists) equilibrium if $x_1 < x_1^*$ $\left(x_1 > x_1^*\right)$.

We have:

$$\frac{\partial gc(x_0, x_1)}{\partial x_1} = \frac{f(x_0, x_1)\sqrt{3x_1^2 + (4x_0 - 10)x_1 - 3x_0^2 + 2x_0 + 4} + h(x_0, x_1)}{36\sqrt{3x_1^2 + (4x_0 - 10)x_1 - 3x_0^2 + 2x_0 + 4}}$$

with

$$f(x_0, x_1) = -\left(34x_1^3 + (-12x_0 - 90)x_1^2 + (-18x_0^2 + 60x_0 + 60)x_1 + 4x_0^3 + 6x_0^2 - 36x_0 - 8\right)$$

and

$$\begin{aligned} h(x_0, x_1) = {} & 36x_1^4 + (51x_0 - 195)x_1^3 + (-59x_0^2 - 35x_0 + 310)x_1^2 \\ & + (-39x_0^3 + 235x_0^2 - 200x_0 - 140)x_1 + 27x_0^4 - 69x_0^3 \\ & - 14x_0^2 + 76x_0 + 16 \end{aligned}$$

Now, $f(x_0, x_1) < 0$ because, as can be easily checked, $\frac{\partial^2 f}{\partial x_1^2} > 0$, $\frac{\partial f}{\partial x_1} < 0$ when $x_1 = \frac{3x_0+2}{5}$, $\frac{\partial f}{\partial x_1} > 0$ when $x_1 = x_1^R$ and $f < 0$ when both $x_1 = \frac{3x_0+2}{5}$ and $x_1 = x_1^R$.

Similarly, $h(x_0, x_1) < 0$ because $\frac{\partial^2 h}{\partial x_1^2} > 0$, $\frac{\partial h}{\partial x_1} > 0$ when $x_1 = \frac{3x_0+2}{5}$ and $h = 0$ when $x_1 = x_1^R$.

Proof of socially optimal locations in Proposition 2

Since demand is inelastic, the socially optimal locations are those that minimize transportation costs as given in (5). To find them, remember first that, as shown above, $q_0 = \frac{x_0+x_1}{2}$ minimizes transportation costs for any locations x_0, x_1. Replacing this value back in (5) yields TC as a function of x_0 and x_1 as given in (10). The first-order conditions to minimize TC in (10) with respect to both x_0 and x_1 are:

$$\frac{\partial TC}{\partial x_0} = x_0^3 - \left(\frac{x_1 - x_0}{2}\right)^3 = 0$$

$$\frac{\partial TC}{\partial x_1} = \left(\frac{x_1 - x_0}{2}\right)^3 - (1 - x_1)^3 = 0$$

from where we can obtain, respectively, $x_0 = \frac{x_1}{3}$ and $x_0 = 3x_1 - 2$ and, therefore, $x_0 = \frac{1}{4}$, $x_1 = \frac{3}{4}$. The second order conditions are also satisfied.

Firm 0's demand q_0 written as an explicit function of p_0 and p_1

If $-\left(x_1^3 - x_0^3\right) \le p_1 - p_0 \le -\left(x_1 - x_0\right)^3$ then:

$$q_0 = \frac{x_0 + x_1}{2} - \sqrt{\frac{4\left(p_0 - p_1\right) - \left(x_1 - x_0\right)^3}{12\left(x_1 - x_0\right)}}$$

If $-\left(x_1 - x_0\right)^3 \le p_1 - p_0 \le \left(x_1 - x_0\right)^3$ then:

$$q_0 = \frac{\left(x_0 + x_1\right)}{2} + \left(\frac{\sqrt{4\left(p_1 - p_0\right)^2 + \left(x_1 - x_0\right)^6}}{8} - \frac{p_0 - p_1}{4}\right)^{1/3} - \frac{\left(x_1 - x_0\right)^2}{4\left(\frac{\sqrt{4\left(p_1 - p_0\right)^2 + \left(x_1 - x_0\right)^6}}{8} - \frac{p_0 - p_1}{4}\right)^{1/3}}$$

If $\left(x_1 - x_0\right)^3 \le p_1 - p_0 \le \left(1 - x_0\right)^3 - \left(1 - x_1\right)^3$ then:

$$q_0 = \frac{x_0 + x_1}{2} + \sqrt{\frac{4\left(p_1 - p_0\right) - \left(x_1 - x_0\right)^3}{12\left(x_1 - x_0\right)}}$$

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