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Urban chaos and perplexing dynamics of urbanization

Yanguang Chen

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Abstract A city with fractal structure used to be thought of as a kind of spatial chaotic attractor. Several chaotic attractors indeed can be found by simulating urbanization dynamics through numerical iterations. However, the results lend little support to the suggestion that real cities are chaotic systems. The rural-urban population interaction model does not display chaotic behavior in normal state, but chaos will happen only if the parameter values of the model deviate from the reality. Accordingly, whether or not complex urban systems are chaotic is posed as a pending question. Varied simulation experiments based on the urbanization dynamics imply that the complex patterns of cities occur on the edge of chaos rather than in chaotic state. This result presents an angle of view for us to understand Holland's question, i.e., why the interactions that form a city are typically stable in the real world.

Keywords Chaos · Fractals · Chaotic attractor · Complexity · Urbanization · Urban dynamics

JEL Classification C61

After chaos theory came into being, the concepts from chaos are always employed to explain complex human socioeconomic phenomena, including the emergence of urban and regional patterns (e.g. Dendrinos and Sonis 1990; Kiel and Elliott 1996; Nijkamp 1990; Nijkamp and Reggiani 1998; Parker and Stacey 1994; Portugali 2000; Puu 2003; Van der Leeuw and McGlade 1997). Nowadays, chaos has even changed from a kind of science to a cultural metaphor. In terms of geography, four representative types of models were usually applied to chaotic spatial dynamics.

Y. Chen (✉)
Department of Geography, College of Urban and Environmental Sciences, Peking University, Beijing 100871, People's Republic of China
e-mail: chenyg@pku.edu.cn

The first is the Verhulst-Pearl model and the generalized logit models (Nijkamp and Reggiani 1991, 1998), the second is the log-linear model from the Cobb-Douglas production function (Dendrinos and Sonis 1990), the third is the predator-prey relation based on the Lotka-Volterra dynamics (Dendrinos and Mullally 1985; Nijkamp and Reggiani 1998), and the fourth, the gravity model associated with spatial interaction principle (Nijkamp and Reggiani 1991; Zhang and Jarrett 1998). No matter what kind of models is adopted, the contents are mainly related to the interaction between different locations/regions, or to origin-destination (O-D) network. Little research has been reported on chaos of urbanization so far. Modeling urbanization dynamics provides a new way of looking at chaos because it engages a number of mathematical models related to chaos such as the logistic model, the predator-prey model, and the gravity model in geography (Batty and Karmeshu 1983; Chen 2008).

In fact, both cities as systems and systems of cities are complex systems, and urban form and structure are demonstrated to be fractals (e.g. Batty 2005; Batty and Longley 1994; Chen and Zhou 2006; Dendrinos and El Naschie 1994). It is inevitable for us to speculate that urban spatial interaction and urbanization process are likely to be a kind of chaotic process. The students who have studied cities know that urban form, urban system, and urbanization process have intrinsic relations in evolutional mechanism. The indications of urban complexity, including fractal structure and rank-size distribution, suggest that urbanization process should take on chaotic behavior (Chen and Zhou 2008). However, lots of simulation experiments of urbanization show that it is too early to interpret urban phenomenon using the ideas from chaos. In this letter, chaotic attractors are created with the models of urbanization dynamics. The results are unexpected or even perplexing, but revealing for us to understand both cities and chaos in its right perspective.

1 Chaotic behaviors of urbanization dynamics

Recently, the author advanced several possible models on urbanization dynamics based on the census data or statistic data sets of America, China and Indian. One form of the rural-urban interaction models of population is as follows

$$\begin{cases} \frac{\mathrm{d}r(t)}{\mathrm{d}t} = ar(t) + bu(t) - er(t)u(t), \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} = cu(t) + dr(t) + fr(t)u(t), \end{cases} \tag{1}$$

where $r(t)$ and $u(t)$ respectively represent the rural population and urban population in a region at time t ($r(t) > 0$, $u(t) > 0$), and a, b, c, d, e, f are all parameters. For a close regional system with no population exchange with outside, we have $e = f$. Moreover, we will get $b = 0$ if the counter-urbanization process does not occur, i.e., no urban population flows into the country.

This urbanization model is similar to some extent to the predator-prey interaction model in ecology. In a sense the latter can be regarded as a special case of the former where the mathematical expression is concerned. Researching (1) is helpful for us to answer the Holland's question. After discussing the Lotka-Volterra model, Holland

(1995, p. 18) once observed: "In the long run, extensions of such models should help us understand why predator-prey interaction exhibit strong oscillations, whereas the interactions that form a city are typically more stable."

For simplicity, suppose there exists an open region ($e \neq f$) with no counter-urbanization ($b = 0$). The iteration method can be employed to find the numerical solution to the model. First of all, (1) should be discretized as a 2-dimension map

$$\begin{cases} r(t+1) = (1+a)r(t) - er(t)u(t), \\ u(t+1) = (1+c)u(t) + dr(t) + fr(t)u(t). \end{cases} \quad (2)$$

The numerical simulation based on (2) with normal parameter values suggests that the urban total population $P(t) = r(t) + u(t)$ and the level of urbanization $L(t) = u(t)/[r(t) + u(t)]$ just increase in the form of S-curve, which look like the stable logistic process in diagram.

Equation (2) can present very complex periodic oscillation and chaotic behavior. For example, the chaotic attractor in Fig. 1 emerges when the original values are set as $r(0) = 1$, $u(0) = 0.2$, and the parameter values are taken as $a = 1.05$, $c = -2$, $d = 0.9$, $e = 0.8$, and $f = 0.2$. Clearly, the trajectory in Fig. 1 is infinitely enlaced in the limited phase space, but never repeats itself. This kind of strange attractor can be named rural-urban interaction attractor, whose box-counting dimension is about 1.5, and the correlation dimension is around 0.75. However, as will be illuminated later, it can only appear in an imaginary world instead of the real world.

The values of $u(t)$ and $r(t)$ oscillate randomly as time t increases, tracing two irregular curves which look like white noises (Fig. 2). This suggests that the system

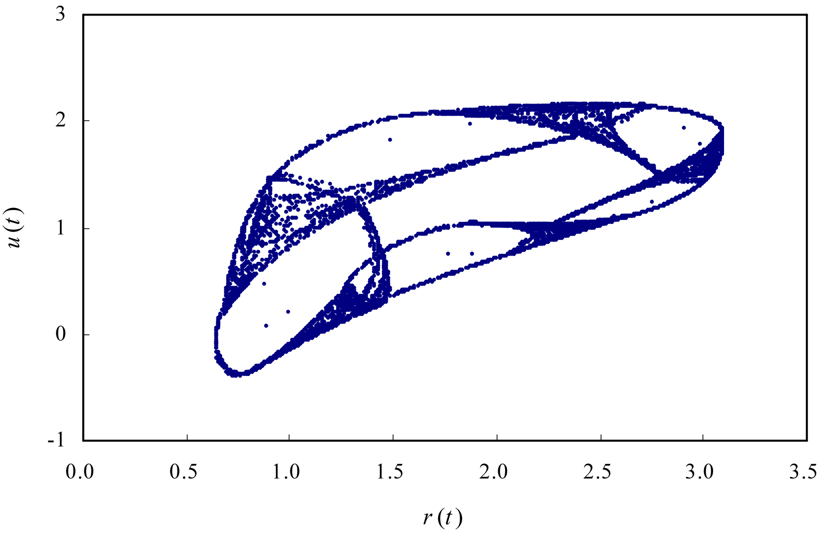


Fig. 1 The chaotic attractor produced by the rural-urban interaction model of population. *Note*: The time of iterations is 10,000. The axes are 'time series' of rural and urban population created by the 2D map of urbanization. Given the numerical values of the initial conditions, the shape of the attractor depends on the parameter values. Things are similar in Figs. 3 and 4

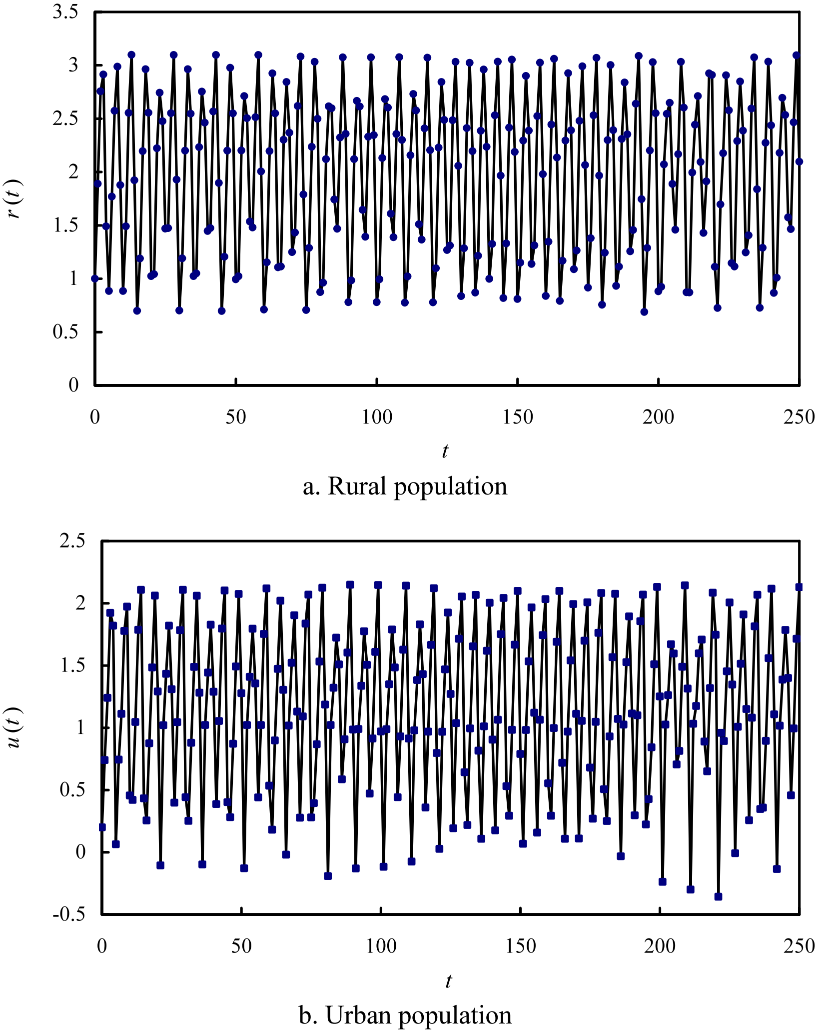
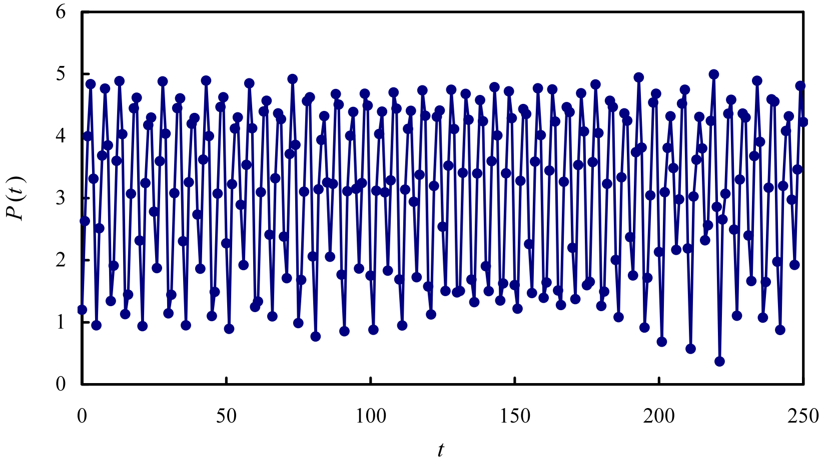


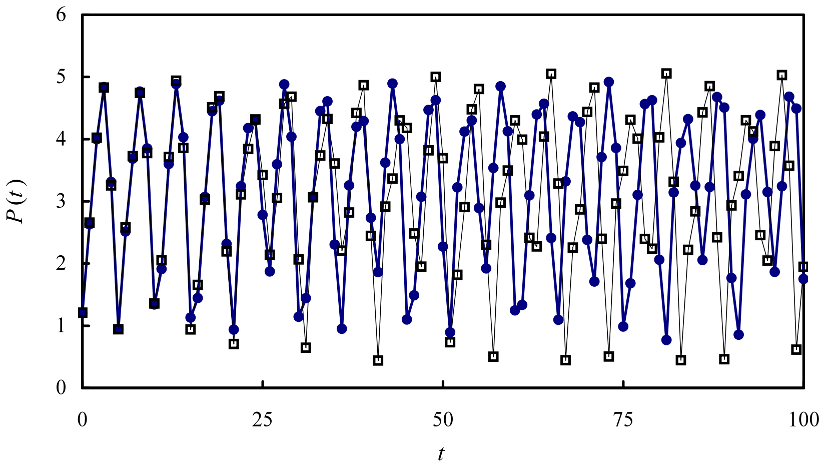
Fig. 2 Chaotic change curves of rural population and urban population based on the two-population interaction model

is unpredictable. Of course, they are not real white noises because that the phase space supported by $r(t)$ and $u(t)$ has fixed geometric structure. If they were Gaussian white noises, the phase portrait should be of sphericity instead of fractal patterns. Both $u(t)$ and $r(t)$ are not stable, and never converge. But the trajectories never go beyond certain spatial scope (Fig. 1). This indicates that population changes are globally stable but locally unstable. There exists an order underlying the apparent chaos, and the local separation of trajectories coexists with the global attraction. These are just the basic properties of stranger attractors.

The information of urban-rural population $u(t)$ and $r(t)$ can be reflected by total population $P(t)$. The curve of $P(t)$ with respect to iteration times t displays no reg-



a. Chaotic curve of the total population



b. Population curves based on different initial values

Fig. 3 Chaotic change curves of total population based on the two-population interaction model. *Note*: The *solid dots* and *real line* represent the curve based on the unchanged initial values, while the *hollow dots* and *broken line* represent the curve based on the changed original value

ularity. That is to say, the time series of $P(t)$ values is a deterministic noise (Fig. 3a). Further, if the original values is altered, for example, the initial value of rural population can be changed as $r(0) = 1.01$, then the difference between the new curve and the primary curve becomes larger and larger with the iteration times increasing. Eventually the two curves are completely different from one another (Fig. 3b), while the shape of the attractor only varies a little. In one word, the growth curve of total population is sensitively dependent on the initial conditions, but the shape of the chaotic attractor will keep stable when the initial values, $r(0)$ and $u(0)$, are made different. However, if the initial values come beyond certain range, the attractor will come to disaggregation suddenly.

2 More complex behaviors

Changing the parameter values of (2) can yield various complex behaviors, such as periodic oscillation, quasi-periodic fluctuation and chaos. Of course, the patterns are different to a degree from the doubling bifurcation and chaos of the logistic map. Several results are displayed in Fig. 4. As mentioned above, both the total population $P(t)$ and the level of urbanization $L(t)$ grow in the form of S-curve on condition that the parameter values of the model are reasonable. If and only if the parameter values become beyond the understandable range, the periodic oscillation and chaos emerge from the rural-urban interaction dynamics. In short, only when the parameter values are irrational, or have no physical meaning, the urbanization process comes

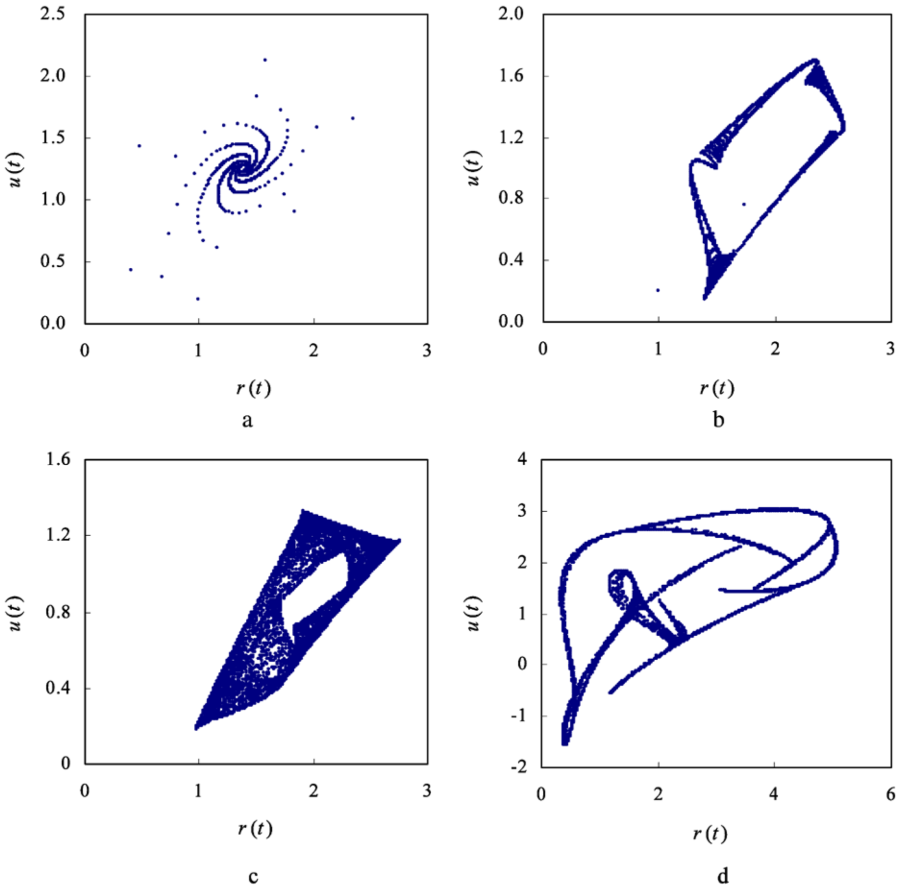


Fig. 4 Patterns of the rural-urban population interaction dynamics in phase space (examples). *Note*: The axes are 'actual' populations, and the times of iterations are 10,000

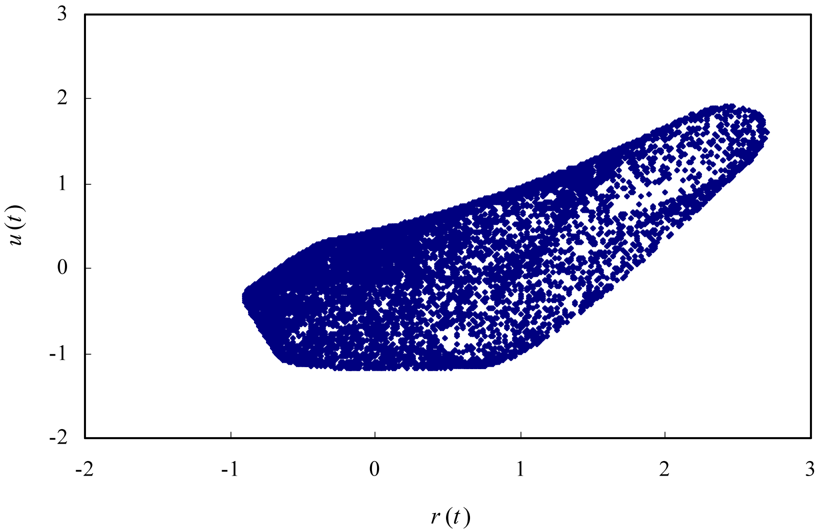


Fig. 5 Another chaotic attractor—a strange urbanization footprint. *Note*: The times of iterations are 10,000

into the chaotic state (Figs. 4b, c). What is more, sometimes the physical meaning of the simulation results cannot be interpreted reasonably (Figs. 1, 4d).

Chaotic behavior also exhibits when the urbanization model becomes abnormal and thus impenetrable. In this case, the structure of the model cannot be explained rationally. Suppose that there exists open region ($e \neq f$) with a counter-urbanization process ($e = 0$, $c = -1$). The model can be restated as

$$\begin{cases} \frac{\mathrm{d}r(t)}{\mathrm{d}t} = ar(t) + bu(t), \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} = -u(t) + dr(t) + fr(t)u(t). \end{cases} \tag{3}$$

The first formula in (3) does not comprise a crossing item indicating interaction, and the second one cannot be normally understood. However, it is worth noting that this model can also exhibit periodic oscillation and chaotic behavior. Let the original values be $r(0) = 1$ and $u(0) = 0.2$, and the parameter values be taken as $a = 0.2$, $b = -1.5$, $d = 1$, $f = -0.7$. Then after discretizing (3) as a 2-dimensional map, we can produce a chaotic attractor as illustrated in Fig. 5. Maybe this attractor can be termed "urban footprint" in light of shape. It is actually a sick footprint because that the physical meaning of parameters $b < 0$ and $f < 0$ in the model can not be explained. As a matter of fact, even if $f > 0$, the map from (3) can also display chaotic behavior as long as $b < 0$, although the range of parameter values guaranteeing chaotic behavior is very narrow.

3 Conditions of urban chaos

There are lots of evidences showing that cities and systems of cities possess the features of complex system (Batty 2005; Chen 2008). It has been demonstrated that the city size distribution on large scale follows the negative power law implying complexity (e.g. Albeverio et al. 2008; Chen and Zhou 2008; Zanette and Manrubia 1997), and urban systems have hierarchical structure similar to the cascade of the period-doubling bifurcation (Chen and Zhou 2003). We can conclude that urbanization is a dynamical process of complexity in view of the intrinsic relation between systems of cities and urbanization. However, urban complexity seems not to indicate periodic oscillation and chaotic behavior of urbanization course. The fractal structure of urban systems is not necessarily associated with chaotic behavior of urbanization. The numerical simulation of urbanization dynamics in this study lends further support to the suggestion that complexity always exists on the edge of chaos (Kauffman 1993). It also gives further weight to the viewpoint that the simultaneous appearance of chaos and fractals does not imply necessary relation between the two phenomena (Bak 1996).

In order to reveal the conditions of chaotic urbanization, it is necessary to show the actual parameter values of the urban model in terms of the observed data. Let's take the urbanization process of China as a natural example. We can fit the Chinese census and statistical data of urban and rural population from 1949 to 2000[1] (the data after 2000 fail to be calibrated with the newest census) to the 2-dimension map comes from discretizing (1). For simplicity, we should use the initial value of rural population to divide all the numbers to "normalize" the variables. Thus we get $r(0) = 1$. If time interval $\Delta t = 1$ as given, then we have $\mathrm{d}x/\mathrm{d}t \propto \Delta x/\Delta t = \Delta x$. Taking $r(t)$, $u(t)$, and $r(t)^{*}u(t)$ as independent variables, while taking $\Delta u(t)/\Delta t$ and $\Delta r(t)/\Delta t$ as dependent variables respectively, we can estimate the model parameter values through the multivariate stepwise regression. A least squares calculation yields a rural-urban interaction model such as

$$\begin{cases} \frac{\Delta r(t)}{\Delta t} = r(t)[0.0252 - 0.0366u(t)], \\ \frac{\Delta u(t)}{\Delta t} = 0.0245r(t)u(t). \end{cases} \tag{4}$$

The model can be successfully employed to explain and predict the urbanization course and trend in China. This suggests that, in the real world, the natural parameter values are as follows: $a > 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $e > 0$, $f > 0$ (Table 1). If so, no chaotic behavior appears. Otherwise, the parameter values are abnormal and cannot be understood reasonably. Only when the parameter $c < 0$, we can find irregular oscillation or even chaos.

Through a similar approach, we can make a two-population interaction model for Indian urbanization, as a contrast, based on the decennial census data from 1901 to 2001[2] ($\Delta t = 10$). One of the alternate models for India is as follows

$$\begin{cases} \frac{\Delta r(t)}{\Delta t} = 0.2821u(t) - 0.0212r(t) - 0.0576r(t)u(t), \\ \frac{\Delta u(t)}{\Delta t} = 0.0343u(t). \end{cases} \tag{5}$$

[1]The data is available from http://www.stats.gov.cn/tjsj/ndsj/.

[2]The data is available from http://www.censusindia.net/results/eci14_page2.html.

Table 1 Comparison between the parameter values for artificial systems and those for real systems

Scene	Model	Parameter					
		a	*b*	*c*	*d*	*e*	*f*
Artificial systems with chaotic behavior	Case 1	**1.05**	0	**−2**	0.9	0.8	0.2
	Case 2	0.2	**−1.5**	**−1**	**1**	**0**	**−0.7**
Real systems without chaotic behavior	China	0.0252	0	0	0	0.0366	0.0245
	India	**−0.0212**	0.2821	0.0343	0	0.0576	**0**
Real systems with reasonable parameters	Map based on (1)	$1 > a > 0$	$1 > b \geq 0$	$1 > c \geq 0$	$1 > d \geq 0$	$1 > e > 0$	$1 > e > 0$

Note: The numbers with shading are abnormal by reasonable parameter values. The models with reasonable parameter values can give logical, rational, and sound explanation and prediction for urbanization process

This model is abnormal since some parameter values go outside the bounds of the reasonable scale, but it cannot exhibit any chaotic behavior. Now, based on the 2D map from (1), a comparison can be drawn between the parameter values for the artificial systems with chaotic behavior (Figs. 1, 5) and those for the real systems (China, India) with no chaotic behavior (Table 1).

The basic condition of urban chaos is nonlinearity indicated by a crossing item, i.e., the rural-urban coupling item $r(t) * u(t)$. Without nonlinearity no chaos would appear. More results of numerical simulation based on variety kinds of nonlinear model of urbanization dynamics show that chaotic behavior takes place only if, at least, one of the following phenomena occurs.

First, the structure of model is eccentric or cannot be reasonably understood. Taking (3) as an example, the change of rural population is supposed to relate to a crossing item. However, there is no crossing item in the first formula. The growth rate of urban population should be in positive proportion to urban population, but the second formula shows a negative proportion of urban growth rate to the urban population size.

Second, the parameters or predicted values of population have no physical meaning. For example, the value of urban population $u(t)$, or rural population $r(t)$, or total population $P(t)$, is a negative, the percentage of urban population, $L(t)$, is less than 0 or greater than 1, and so on. For Figs. 3a and c, the predicted results are acceptable, but the parameter values of the model are unexplainable; As for Figs. 1 and 3d, both the predicted results and parameter values of the model are uninterpretable.

The source of urban chaos lies with nonlinearity and model parameters. One of necessary conditions of chaotic urbanization is that the parameter values of the nonlinear models are of no physical meaning. For instance, the growth rate of urban population is in inverse proportion to city size. This violates the reality and our intuition. However, this is just the precondition of the chaotic attractors displayed in Figs. 1 and 3.

4 Conclusions

By the numerical simulation, empirical studies, and parameter analysis of urbanization dynamics, the main conclusions can be reached as follows.

Firstly, urban chaos principally emerges from the artificial systems, or the possible world, rather than the real systems. Chaotic behavior does not arise in an urbanization model when the parameter values are of physical meaning. In other words, in case periodic oscillation or chaotic behaviors appear, the parameter values or even the structure of the urban model defy our understanding.

Secondly, spatial complexity of self-organized network of cities comes on the edge of chaos instead of chaos itself. Urban evolution seems to struggle between order and chaos, but it hardly falls into chaotic state in reality. Maybe it is just the trend to chaos rather than chaos itself of urban evolvement that gives rise to the emergence of complex geographical patterns.

Thirdly, the study of chaos in urbanization is revealing for us to answer the puzzling but attractive question that urban evolution is typically stable. Both urban systems and ecosystems can be described by similar models such as logistic model and predator-prey interaction model. However, the interactions forming a city are more stable than that forming an ecosystem. The key may rest with that the probability of urban chaos is too small in the real world.

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