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δ'''-shock wave solution to a nonstrictly hyperbolic system of conservation laws using weak asymptotic method

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Abstract. This article is concerned with the existence of a weak asymptotic solution for a 5×5 system of nonstrictly hyperbolic conservation laws. We provide additional weak asymptotic expansions within the framework of the weak asymptotic approach. Then, with the aid of these weak asymptotic expansions, we establish sufficient conditions for the existence of a weak asymptotic solution for the 5×5 system with initial data of Riemann type. Combining the Riemann problems allow us to form a weak asymptotic solution for a more general type of initial data.

Keywords. Hyperbolic conservation laws; δ'''-shock; weak asymptotic method; nonstrictly hyperbolic system.

1. Introduction

In the recent years, extensive study have been done for the solution to a system of conservation laws

$$U_t + (F(U))_x = 0, \quad x \in \mathbb{R},\ t \in (0, \infty),$$
$$U(x, 0) = U_0(x),$$

specifically for Riemann type initial data

$$U_0(x) = \begin{cases} U_-, & x<0, \\ U_+, & x > 0, \end{cases}$$

where U is the vector of conservative variables, F is the flux function with U_- and U_+ as constant states. Different methods like weak asymptotic method [1–3], vanishing viscosity method [4–6], flux approximation method [7–9], vanishing pressure limit method [10, 11], distributional product apporach [12, 13], etc. have been devised to study the hyperbolic system of conservation laws which admits δ-shock waves.

Analysis of solution to the Riemann problem for the system of hyperbolic conservation laws involving non-classical waves have been an intriguing yet a challenging journey for researchers so far. Many have been tried in the context of physical phenomena. First noted work on non-classical wave solution can be found in [14]. Later Tan and Zhang [15] observed δ-shock wave for two-dimensional Euler equations. To date many mathematicians and physicists have been focused on δ-shock wave solution, regarding these details we refer [16–19, 33] and the references cited therein.

Here with the help of weak asymptotic method we study the following system introduced by Joseph and Murthy [20]

$$(u_i)_t + \sum_{j=1}^{i} \left(\frac{u_j u_{i-j+1}}{2} \right)_x = 0, \quad i = 1, \ldots, n \tag{1.1}$$

for $n = 5$. For $n = 1$ the equation is the celebrated inviscid Burgers equation for which detailed analysis has been done. Hopf [21] and Cole [22] independently solved the initial value problem for viscous Burgers equation and by letting the viscosity term to zero they were able to find the weak entropy solution to the inviscid Burgers equation which lies in the space of BV functions when the initial data is considered to be of bounded measure. When $n = 2$, the system is a one-dimensional hyperbolic model for the large scale structure formation of universe [23] and also it is the well-known transport equations, where solution may contain nonclassical waves such as delta shock wave for one state variable. This 2×2 system has been studied in the context of vanishing viscosity method by Joseph [24]. With $n = 3$, the system is studied using weak asymptotic method in [2] where they have introduced a definition of δ'-shock wave type solution for the system as one of the state variable may contain higher order singularity than Dirac delta distribution for some initial data. Further, this 3×3 system has been discussed by Shelkovich in [25] by exploiting vanishing viscosity method where parabolic approximation is considered and with the help of Hopf–Cole transformation the parabolic system is reduced to the triple of linear heat equations. Then the weak limit is established and δ'-shock

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solution is constructed for some initial data. Also based on the distribution multiplication theory it has been analyzed in [26] where within an appropriate space of distributions they defined α-solution to the system. With the help of those they obtained all the α-independent exact solutions and observed the occurrence of δ'-wave in the solution. In [4], for $n = 4$, the system has been investigated in detail using vanishing viscosity method and it was found out that fourth component contains the measure δ and its derivative for specific Riemann data. In order to develop mathematical theory for $\delta^{(m)}$-shock ($m = 3$ in our case) now we turn our attention to the case $n = 5$. To the best of our knowledge, physical models which admit $\delta^{(m)}$-shocks ($m \geq 2$) are not reported yet in the literature. It is observed that the system (1.1) is not strictly hyperbolic as it admits repeated eigenvalues u. So we can not apply the classical theory of Glimm [27] and Lax [28] to the system (1.1). As for the case $n = 4$ we have noticed that u_3 and u_4 admit δ' and δ''-shock wave type solutions and the system has been analyzed by using weak asymptotic method in [29], so following the intuition for $n = 5$ we may expect u_5 to have higher order singularity. So measure solutions approach [6, 30] as well as well-known nonconservative product method [31, 32] is not applicable here. We consider the weak asymptotic method to study the system (1.1) for $n = 5$. With the transformation $(u_1, u_2, u_3, u_4, u_5) \to (2u_1, u_2, \frac{u_3}{4}, \frac{u_4}{24}, \frac{u_5}{96})$ we have the following reduced system

$$
\begin{aligned}
u_{1t} + (u_1^2)_x &= 0 \\
u_{2t} + (2u_1u_2)_x &= 0 \\
u_{3t} + 2(u_2^2 + u_1u_3)_x &= 0 \\
u_{4t} + 2(3u_2u_3 + u_1u_4)_x &= 0 \\
u_{5t} + (2u_1u_5 + 4u_2u_4 + 3u_3^2)_x &= 0.
\end{aligned} \tag{1.2}
$$

We consider the initial data in the following form

$$
\begin{aligned}
u_1(x,0) = u_1^0(x),\ u_2(x,0) = u_2^0(x),\ u_3(x,0) = u_3^0(x), \\
u_4(x,0) = u_4^0(x) \text{ and } u_5(x,0) = u_5^0(x).
\end{aligned} \tag{1.3}
$$

The order of the paper is as follows: in section 2, we revisit the preliminaries where the definition of weak asymptotic solution is recalled. In section 3, we provide some new weak asymptotic expansions and exploiting these we obtain a weak asymptotic solution for the initial data (1.3). Finally, concluding remarks are given in section 4.

2. Preliminaries

The framework of weak asymptotic method is to approximate the solution of the system of equations by smooth functions which are also dependent on arbitrary parameter, say, $\varepsilon > 0$ such that these approximate solutions converge to the original solution in the sense of distributions and also some product between these approximate solutions converge to some quantity whenever $\varepsilon \to 0$. With the help of [2], we define weak asymptotic solution. But first let us define the following:

$$
\begin{aligned}
\mathscr{L}_1[u_1] &= u_{1t} + (u_1^2)_x, \\
\mathscr{L}_2[u_1, u_2] &= u_{2t} + (2u_1u_2)_x, \\
\mathscr{L}_3[u_1, u_2, u_3] &= u_{3t} + 2(u_2^2 + u_1u_3)_x, \\
\mathscr{L}_4[u_1, u_2, u_3, u_4] &= u_{4t} + 2(3u_2u_3 + u_1u_4)_x, \\
\mathscr{L}_5[u_1, u_2, u_3, u_4, u_5] &= u_{5t} + (2u_1u_5 + 4u_2u_4 + 3u_3^2)_x.
\end{aligned}
$$

Definition 2.1 The functions $(u_1(x,t,\varepsilon), u_2(x,t,\varepsilon), u_3(x,t,\varepsilon), u_4(x,t,\varepsilon), u_5(x,t,\varepsilon))$ are called weak asymptotic solution to the system (1.2) for $\varepsilon > 0$ if

$$
\begin{aligned}
\int \mathscr{L}_1[u_1]\psi(x)dx &= o(1), \\
\int \mathscr{L}_2[u_1, u_2]\psi(x)dx &= o(1), \\
\int \mathscr{L}_3[u_1, u_2, u_3]\psi(x)dx &= o(1), \\
\int \mathscr{L}_4[u_1, u_2, u_3, u_4]\psi(x)dx &= o(1), \\
\int \mathscr{L}_5[u_1, u_2, u_3, u_4, u_5]\psi(x)dx &= o(1), \\
\int (u_1(x,0,\varepsilon) - u_1^0(x))\psi(x)dx &= o(1), \\
\int (u_2(x,0,\varepsilon) - u_2^0(x))\psi(x)dx &= o(1), \\
\int (u_3(x,0,\varepsilon) - u_3^0(x))\psi(x)dx &= o(1), \\
\int (u_4(x,0,\varepsilon) - u_4^0(x))\psi(x)dx &= o(1), \\
\int (u_5(x,0,\varepsilon) - u_5^0(x))\psi(x)dx &= o(1), \\
\varepsilon &\to 0,
\end{aligned} \tag{2.1}
$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$ with initial data (1.3).

For the system (1.2), we look for a weak asymptotic solution that takes the following form:

$$
\begin{aligned}
u_1(x,t,\varepsilon) =& u_{12}(x,t) + [u_1]\mathcal{H}_{u_1}(-x+\phi(t),\varepsilon) + \mathcal{R}_{u_1}(x,t,\varepsilon), \\
u_2(x,t,\varepsilon) =& u_{22}(x,t) + [u_2]\mathcal{H}_{u_2}(-x+\phi(t),\varepsilon) \\
&+ e(t)\delta_e(-x+\phi(t),\varepsilon) + \mathcal{R}_{u_2}(x,t,\varepsilon), \\
u_3(x,t,\varepsilon) =& u_{32}(x,t) + [u_3]\mathcal{H}_{u_3}(-x+\phi(t),\varepsilon) \\
&+ g(t)\delta_g(-x+\phi(t),\varepsilon) + h(t)\delta'_h(-x+\phi(t),\varepsilon) \\
&+ \mathcal{R}_{u_3}(x,t,\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
u_4(x,t,\varepsilon) =& u_{42}(x,t) + [u_4]\mathcal{H}_{u_4}(-x+\phi(t),\varepsilon)\\
&+ l(t)\delta_l(-x+\phi(t),\varepsilon) + m(t)\delta'_m(-x+\phi(t),\varepsilon)\\
&+ n(t)\delta''_n(-x+\phi(t),\varepsilon) + \mathcal{R}_{u_4}(x,t,\varepsilon),\\
u_5(x,t,\varepsilon) =& u_{52}(x,t) + [u_5]\mathcal{H}_{u_5}(-x+\phi(t),\varepsilon)\\
&+ o(t)\delta_o(-x+\phi(t),\varepsilon) + q(t)\delta'_q(-x+\phi(t),\varepsilon)\\
&+ r(t)\delta''_r(-x+\phi(t),\varepsilon) + s(t)\delta'''_s(-x+\phi(t),\varepsilon)\\
&+ \mathcal{R}_{u_5}(x,t,\varepsilon).
\end{aligned}
$$

It is obvious to consider the above ansatz with that specific form as we can expect the components to have respective order of singularities. We use the notation $[u_i] = u_{i1} - u_{i2}$, where u_{i1} and u_{i2} respectively are the values on the left and right of the discontinuity curve $\phi(t)$ of u_i for $i = 1, ..., 5$. The functions

$$
\mathcal{H}_{i'},\ \delta_{j'},\ \delta'_{k'},\ \delta''_{l'}, \delta'''_{m'},
$$

for

$$
i' = u_1, u_2, u_3, u_4, u_5,\ j' = e, g, l, o,\ k' = h, m, q,\ l' = n, r,\ m' = s,
$$

are regularization of the Heaviside function H, distributions δ, δ', δ'' and δ''' respectively. We represent $f(x,\varepsilon)$ as the following regularization for a distribution $f(x) \in \mathcal{D}'(\mathbb{R})$.

$$
f(x,\varepsilon) = f(x) * \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0,
$$

where the mollifier ω has the property (a)–(c) of Lemma 3.1 and $*$ denotes the convolution. Also $\mathcal{R}_i$, for $i = u_1, u_2, u_3, u_4$ and u_5, are correction terms and these satisfy

$$
\mathcal{R}_j(x,t,\varepsilon) = o_{\mathcal{D}'}(1), \quad \frac{\partial \mathcal{R}_j(x,t,\varepsilon)}{\partial t} = o_{\mathcal{D}'}(1),
$$

for

$$
i = u_1, u_2, u_3, u_4, u_5,\ \varepsilon \to 0.
$$

The weak asymptotic method considered in this manuscript has been applied to only specific systems but for more general system it would be difficult to construct singular superpositions (products) of distributions due to the dependence of the singular superpositions on the regularizations of the Heaviside function, delta function, its derivatives, and the correction functions.

3. Weak asymptotic solution

Let us begin with a Lemma where some asymptotic expansions, in the sense of distributions, are derived.

Lemma 3.1 *Let $\{\eta_k\}_{k\in I}$ be mollifiers satisfying*

(a) $\eta_k(x) = \eta_k(-x)$
(b) $\int \eta_k = 1$
(c) $\eta_k(x) \geq 0$.

Define

- $\mathcal{H}_k(x,\varepsilon) = \eta_{0k}(\frac{x}{\varepsilon}) = \int_{-\infty}^{\frac{x}{\varepsilon}} \eta_k(y)dy$
- $\delta_k(x,\varepsilon) = \frac{1}{\varepsilon}\eta_k(\frac{x}{\varepsilon})$
- $\delta^n_k(x,\varepsilon) = \frac{1}{\varepsilon^{n+1}}\eta^n_k(\frac{x}{\varepsilon})$

where $\mathcal{H}_k$, δ_k and δ^n_k are regularizations of Heaviside function $H(x)$ and the distributions $\delta(x)$, $\delta^k(x)$ respectively. Now we have the following weak asymptotic expansions:

$$
\left.
\begin{aligned}
(\mathcal{H}_k(x,\varepsilon))^n &= H(x) + O_{D'}(\varepsilon),\\
(\mathcal{H}_k(x,\varepsilon)\mathcal{H}_i(x,\varepsilon)) &= H(x) + O_{D'}(\varepsilon),\\
(\mathcal{H}_k(x,\varepsilon))^n\delta_i(x,\varepsilon) &= \delta(x)\int \eta^n_{0k}(\xi)\eta_i(\xi)d\xi + O_{D'}(\varepsilon),\\
(\delta_k(x,\varepsilon))^n &= \frac{1}{\varepsilon}\delta(x)\int \eta^2_k(\xi)d\xi + O_{D'}(\varepsilon),\\
\mathcal{H}_k(x,\varepsilon)\delta'_i(x,\varepsilon) &= -\frac{\delta(x)}{\varepsilon}\int \eta_k(\xi)\eta_i(\xi)d\xi + \delta'(x)\int \eta_{0k}(\xi)\eta_i(\xi)d\xi + O_{D'}(\varepsilon),\\
\mathcal{H}_k(x,\varepsilon)\varepsilon^2\delta'''_i(x,\varepsilon) &= \frac{1}{\varepsilon}\delta(x)\int \eta'_k(\xi)\eta'_i(\xi)d\xi + O_{D'}(\varepsilon),
\end{aligned}
\right\} \tag{3.1}
$$

$$
\left.
\begin{aligned}
\delta_k(x,\varepsilon)\delta_i(x,\varepsilon) &= \frac{\delta(x)}{\varepsilon}\int \eta_k(\xi)\eta_i(\xi)d\xi + O_{D'}(\varepsilon),\\
\delta_k(x,\varepsilon)\delta'_i(x,\varepsilon) &= -\frac{\delta'(x)}{\varepsilon}\int \xi\eta_k(\xi)\eta'_i(\xi)d\xi + O_{D'}(\varepsilon),\\
\delta_k(x,\varepsilon)\varepsilon^2\delta'''_i(x,\varepsilon) &= -\frac{\delta'(x)}{\varepsilon}\int \xi\eta_k(\xi)\eta'''_i(\xi)d\xi + O_{D'}(\varepsilon),\\
\mathcal{H}_k(x,\varepsilon)\varepsilon^2\delta''''_i(x,\varepsilon) &= -\frac{\delta'(x)}{\varepsilon}\int \xi\eta_{0k}(\xi)\eta''''_i(\xi)d\xi + O_{D'}(\varepsilon),\\
\mathcal{H}_k(x,\varepsilon)\delta''_i(x,\varepsilon) &= -\frac{\delta'(x)}{\varepsilon}\int \xi\eta_{0k}(\xi)\eta''_i(\xi)d\xi + \frac{\delta''(x)}{2}\int \xi^2\eta_{0k}(\xi)\eta''_i(\xi)d\xi + O_{D'}(\varepsilon),
\end{aligned}
\right\} \tag{3.2}
$$

$$\left.\begin{aligned}
\mathcal{H}_k(x,\varepsilon)\delta_i'''(x,\varepsilon) &= \frac{\delta(x)}{\varepsilon^3}\int \eta_{0k}(\xi)\eta_i'''(\xi)d\xi \\
&+\frac{\delta''(x)}{2\varepsilon}\int \xi^2\eta_{0k}(\xi)\eta_i'''(\xi)d\xi \\
&+\delta'''(x)\int \eta_{0k}(\xi)\eta_i(\xi)d\xi + O_{D'}(\varepsilon), \\
\mathcal{H}_k(x,\varepsilon)\varepsilon^2\delta_i^V(x,\varepsilon) &= \frac{\delta(x)}{\varepsilon^3}\int \eta_{0k}(\xi)\eta_i^V(\xi)d\xi \\
&+\frac{\delta''(x)}{2\varepsilon}\int \xi^2\eta_{0k}(\xi)\eta_i'''(\xi)d\xi + O_{D'}(\varepsilon), \\
\delta_k(x,\varepsilon)\delta_i''(x,\varepsilon) &= \frac{\delta(x)}{\varepsilon^3}\int \eta_k(\xi)\eta_i''(\xi)d\xi \\
&+\frac{\delta''(x)}{2\varepsilon}\int \xi^2\eta_k(\xi)\eta_i''(\xi)d\xi + O_{D'}(\varepsilon), \\
\delta_k(x,\varepsilon)\varepsilon^2\delta_i''''(x,\varepsilon) &= \frac{\delta(x)}{\varepsilon^3}\int \eta_k(\xi)\eta_i''''(\xi)d\xi \\
&+\frac{\delta''(x)}{2\varepsilon}\int \xi^2\eta_k(\xi)\eta_i''''(\xi)d\xi + O_{D'}(\varepsilon), \\
(\delta_k'(x,\varepsilon))^2 &= \frac{\delta(x)}{\varepsilon^3}\int (\eta_k'(\xi))^2 d\xi \\
&+\frac{\delta''(x)}{2\varepsilon}\int \xi^2(\eta_k'(\xi))^2 d\xi + O_{D'}(\varepsilon), \\
\delta_k'(x,\varepsilon)\varepsilon^2\delta_i'''(x,\varepsilon) &= \frac{\delta(x)}{\varepsilon^3}\int \eta_k'(\xi)\eta_i'''(\xi)d\xi \\
&+\frac{\delta''(x)}{2\varepsilon}\int \xi^2\eta_k'(\xi)\eta_i'''(\xi)d\xi + O_{D'}(\varepsilon), \\
(\varepsilon^2\delta_k'''(x,\varepsilon))^2 &= \frac{\delta(x)}{\varepsilon^3}\int (\eta_k'''(\xi))^2 d\xi \\
&+\frac{\delta''(x)}{2\varepsilon}\int \xi^2(\eta_k'''(\xi))^2 d\xi + O_{D'}(\varepsilon),
\end{aligned}\right\} \quad (3.3)$$

$\varepsilon \to 0^+$, $n = 1, 2.....$

Proof We can find the proof of (3.1) in [2] and (3.2) in [29] so we omit the details. Now we prove (3.3), considering ψ to be a test function which belongs to $D(\mathbb{R})$. In order to get the first relation of (3.3) consider the asymptotics of the product $\mathcal{H}_k(x,\varepsilon)\delta_i'''(x,\varepsilon)$. From the definition of $\mathcal{H}_k$ and δ_i''', using the change of variable $x = \varepsilon y$ and Taylor expansion, we have

$$\begin{aligned}
&\langle \mathcal{H}_k(x,\varepsilon)\delta_i'''(x,\varepsilon),\ \psi(x)\rangle \\
&= \int \eta_{0k}\left(\frac{x}{\varepsilon}\right)\frac{1}{\varepsilon^4}\eta_i'''\left(\frac{x}{\varepsilon}\right)\psi(x)dx \\
&= \int \eta_{0k}(y)\frac{1}{\varepsilon^3}\eta_i'''(y)\psi(\varepsilon y)dy \\
&= \int \eta_{0k}(y)\frac{1}{\varepsilon^3}\eta_i'''(y)(\psi(0)+\varepsilon y\psi'(0) \\
&+\frac{\varepsilon^2y^2}{2}\psi''(0)+\frac{\varepsilon^3y^3}{6}\psi'''(0))dy + O(\varepsilon)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\psi(0)}{\varepsilon^3}\int \eta_{0k}(y)\eta_i'''(y)dy + \frac{\psi'(0)}{\varepsilon^2}\int y\eta_{0k}(y)\eta_i'''(y)dy \\
&+\frac{\psi''(0)}{2\varepsilon}\int y^2\eta_{0k}(y)\eta_i'''(y)dy + \frac{\psi'''(0)}{6}\int y^3\eta_{0k}y)\eta_i'''(y)dy \\
&+O(\varepsilon).
\end{aligned} \quad (3.4)$$

Now from the property (a), we obtain

$$\int \eta_k(y)\eta_i'(y)dy = 0, \quad \int y\eta_k(y)\eta_i''(y)dy = 0,$$
$$\int y^3\eta_k(y)\eta_i''(y)dy = 0, \quad \int y^2\eta_k(y)\eta_i'(y)dy = 0,$$
$$\int y\eta_k(y)\eta_i(y)dy = 0.$$

Continuing from (3.4) and using the above relations we have the first relation of (3.3) as

$$\begin{aligned}
&\langle \mathcal{H}_k(x,\varepsilon)\delta_i'''(x,\varepsilon),\ \psi(x)\rangle \\
&= \frac{1}{\varepsilon^3}\delta(x)\int \eta_{0k}(y)\eta_i'''(y)dy + \frac{\delta''(x)}{2\varepsilon}\int y^2\eta_{0k}(y)\eta_i'''(y)dy \\
&+\delta'''(x)\int \eta_{0k}(y)\eta_i(y)dy + O(\varepsilon).
\end{aligned}$$

To establish the second relation of (3.3), we need to evaluate the weak asymptotic product $\mathcal{H}_k(x,\varepsilon)\varepsilon^2\delta_i^V(x,\varepsilon)$. By using the definition of $\mathcal{H}_k$ and δ_i^V, we have

$$\begin{aligned}
&\langle \mathcal{H}_k(x,\varepsilon)\varepsilon^2\delta_i^V(x,\varepsilon),\ \psi(x)\rangle \\
&= \int \eta_{0k}\left(\frac{x}{\varepsilon}\right)\varepsilon^2\frac{1}{\varepsilon^6}\eta_i^V\left(\frac{x}{\varepsilon}\right)\psi(x)dx \\
&= \int \eta_{0k}(y)\frac{1}{\varepsilon^3}\eta_i^V(y)\psi(\varepsilon y)dy \\
&= \int \eta_{0k}(y)\frac{1}{\varepsilon^3}\eta_i^V(y)(\psi(0)+\varepsilon y\psi'(0)+\frac{\varepsilon^2y^2}{2}\psi''(0) \\
&+\frac{\varepsilon^3y^3}{6}\psi'''(0))dy + O(\varepsilon) \\
&= \frac{1}{\varepsilon^3}\int \eta_{0k}(y)\eta_i^V(y)\psi(0)dy + \frac{1}{\varepsilon^2}\int y\eta_{0k}(y)\eta_i^V(y)\psi'(0)dy \\
&+\frac{1}{2\varepsilon}\int y^2\eta_{0k}(y)\eta_i^V(y)\psi''(0)dy \\
&+\frac{1}{6}\int y^3\eta_{0k}(y)\eta_i^V(y)\psi'''(0)dy + O(\varepsilon).
\end{aligned}$$

Using property (a) we obtain

$$\int y\eta_{0k}(y)\eta_i^V(y)dy = 0, \quad \int y^3\eta_{0k}(y)\eta_i^V(y)dy = 0.$$

So we can obtain the second relation of (3.3)

$$\langle \mathcal{H}_k(x,\varepsilon)\varepsilon^2\delta_i^V(x,\varepsilon),\ \psi(x)\rangle = \frac{\delta(x)}{\varepsilon^3}\int \eta_{0k}(y)\eta_i^V(y)dy + \frac{\delta''(x)}{2\varepsilon}\int y^2\eta_{0k}(y)\eta_i^V(y)dy + O(\varepsilon).$$

Similar procedure can be applied to prove the rest of the relations of (3.3) in the Lemma. Hence proved the Lemma. □

We have seen in [29] that the ansatz they have taken contains combination of $\delta,\ \delta'$ and δ'' with correction for a component. So here we assume that u_5 contains combination of $\delta,\ \delta',\ \delta''$ and δ'''. Hence in our ansatz we have taken the combinations of these singular waves with correction term. We use the Lemma 3.1 to obtain a weak asymptotic solution to the system (1.2).

Theorem 3.2 *The following ansatz :*

$$\left.\begin{aligned}
u_1(x,t,\varepsilon) &= u_{12}(x,t) + [u_1]\mathcal{H}_{u_1}(-x+\phi(t),\varepsilon),\\
u_2(x,t,\varepsilon) &= u_{22}(x,t) + [u_2]\mathcal{H}_{u_2}(-x+\phi(t),\varepsilon)\\
&\quad + e(t)\delta_e(-x+\phi(t),\varepsilon),\\
u_3(x,t,\varepsilon) &= u_{32}(x,t) + [u_3]\mathcal{H}_{u_3}(-x+\phi(t),\varepsilon)\\
&\quad + g(t)\delta_g(-x+\phi(t),\varepsilon)\\
&\quad + h(t)\delta_h'(-x+\phi(t),\varepsilon)\\
&\quad + \mathcal{R}_{u_3}(-x+\phi(t),\varepsilon),\\
u_4(x,t,\varepsilon) &= u_{42}(x,t) + [u_4]\mathcal{H}_{u_4}(-x+\phi(t),\varepsilon)\\
&\quad + l(t)\delta_l(-x+\phi(t),\varepsilon)\\
&\quad + m(t)\delta_m'(-x+\phi(t),\varepsilon)\\
&\quad + n(t)\delta_n''(-x+\phi(t),\varepsilon)\\
&\quad + \mathcal{R}_{u_4}(-x+\phi(t),\varepsilon),\\
u_5(x,t,\varepsilon) &= u_{52}(x,t) + [u_5]\mathcal{H}_{u_5}(-x+\phi(t),\varepsilon)\\
&\quad + o(t)\delta_o(-x+\phi(t),\varepsilon)\\
&\quad + q(t)\delta_q'(-x+\phi(t),\varepsilon)\\
&\quad + r(t)\delta_r''(-x+\phi(t),\varepsilon)\\
&\quad + s(t)\delta_s'''(-x+\phi(t),\varepsilon)\\
&\quad + \mathcal{R}_{u_5}(-x+\phi(t),\varepsilon),
\end{aligned}\right\} \tag{3.5}$$

where

$$\begin{aligned}
\mathcal{R}_{u_3}(x,t,\varepsilon) &= \varepsilon^2\mathcal{P}(t)\delta_{\mathcal{P}}'''(-x+\phi(t),\varepsilon),\\
\mathcal{R}_{u_4}(x,t,\varepsilon) &= \varepsilon^2(\mathcal{Q}(t)\delta_{\mathcal{Q}}'''(-x+\phi(t),\varepsilon)\\
&\quad + \mathcal{R}(t)\delta_{\mathcal{R}}''''(-x+\phi(t),\varepsilon)),\\
\mathcal{R}_{u_5}(x,t,\varepsilon) &= \varepsilon^2(\mathcal{S}(t)\delta_{\mathcal{S}}'''(-x+\phi(t),\varepsilon)\\
&\quad + \mathcal{T}(t)\delta_{\mathcal{T}}''''(-x+\phi(t),\varepsilon)\\
&\quad + \mathcal{U}(t)\delta_{\mathcal{U}}^V(-x+\phi(t),\varepsilon)),
\end{aligned} \tag{3.6}$$

is a weak asymptotic solution to the following problem

$$\left.\begin{aligned}
u_{1t} + (u_1^2)_x &= 0\\
u_{2t} + (2u_1u_2)_x &= 0\\
u_{3t} + 2(u_2^2 + u_1u_3)_x &= 0\\
u_{4t} + 2(3u_2u_3 + u_1u_4)_x &= 0\\
u_{5t} + (2u_1u_5 + 4u_2u_4 + 3u_3^2)_x &= 0
\end{aligned}\right\} \tag{3.7}$$

provided the subsequent relations hold (with initial data):

$$\begin{aligned}
&\mathscr{L}_1[u_1] = 0,\ \mathscr{L}_1[u_2] = 0,\quad \pm x > \phi(t),\\
&\mathscr{L}_2[u_1,v_1] = 0,\ \mathscr{L}_2[u_2,v_2] = 0,\quad \pm x > \phi(t),\\
&\mathscr{L}_3[u_1,v_1,w_1] = 0,\ \mathscr{L}_3[u_2,v_2,w_2] = 0,\quad \pm x > \phi(t),\\
&\dot\phi(t) = (u_{11}+u_{12})|_{x=\phi(t)},\ \dot e(t) = [u_1](u_{21}+u_{22})|_{x=\phi(t)},\\
&\dot g(t) = (2[u_1](u_{21}+u_{22}) + [u_1](u_{31}+u_{32}))|_{x=\phi(t)},\\
&\frac{d}{dt}(h(t)[u_1(\phi(t),t)]) = \frac{d(e^2(t))}{dt}\\
&\int \eta_{0u_1}(\xi)\eta_j(\xi)d\xi\\
&= \int \xi^2\eta_{0u_2}(\xi)\eta_e(\xi)d\xi = \frac{1}{2},\ j = e,g,h,\\
&\int \eta_{u_1}(\xi)\eta_h(\xi)d\xi = \int \eta_e^2(\xi)d\xi,\ \mathcal{P}(t) = \frac{B}{u_2(\phi(t),t)},
\end{aligned}$$

B is a constant,

$$\begin{aligned}
&\mathscr{L}_4[u_{11},u_{21},u_{31},u_{41}] = 0,\ \mathscr{L}_4[u_{12},u_{22},u_{32},u_{42}] = 0,\\
&\pm x > \phi(t),\\
&\dot l(t) = -[u_4]\dot\phi(t) + 2[3u_2u_3 + u_1u_4],\\
&\int \eta_{0u_1}(\xi)\eta_l(\xi)d\xi = \int \eta_{0u_1}(\xi)\eta_m(\xi)d\xi\\
&= \frac{1}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_n(\xi)d\xi = \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
\dot m(t) &= 2[3\{(u_{22} + [u_2]\int \eta_{0u_2}(\xi)\eta_g(\xi))g(t) + (u_{32}\\
&\quad + [u_3]\int \eta_{0u_3}(\xi)\eta_e(\xi)d\xi)e(t)\}\\
&\quad + 3\{(u_{22x} + [u_{2x}]\int \eta_{0u_2}(\xi)\eta_h(\xi)d\xi)h(t)\}\\
&\quad + (u_{12x} + [u_{1x}]\int \eta_{0u_1}(\xi)\eta_m(\xi)d\xi)m(t)\\
&\quad + (u_{12xx} + \frac{[u_{1xx}]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_n(\xi)d\xi)n(t)],\\
\dot n(t) &= 2[3\{(u_{22} + [u_1]\int \eta_{0u_2}(\xi)\eta_h(\xi)d\xi)h(t)\}\\
&\quad + 2(u_{22x} + [u_{1x}])n(t)],\\
\mathcal{R}(t) &= \frac{-1}{\{[u_1]\int \xi\eta_{0u_1}(\xi)\eta_{\mathcal{R}}''''(\xi)\}}[3e(t)h(t)\int \xi\eta_e(\xi)\eta_h'(\xi)d\xi\\
&\quad + 3e(t)\mathcal{P}(t)\int \xi\eta_e(\xi)\eta_{\mathcal{P}}'''(\xi)d\xi
\end{aligned}$$

$$+ n(t)[u_1]\int \xi\eta_{0u_1}(\xi)\eta''_n(\xi)d\xi],$$

$$\mathcal{Q}(t) = \frac{1}{\{[u_1]\int \eta'_u(\xi)\eta'_{\mathcal{Q}}(\xi)d\xi\}}[3h(t)[u_1]\int \eta_{u_2}(\xi)\eta_h(\xi)d\xi$$
$$- 3g(t)e(t)\int \eta_e(\xi)\eta_g(\xi)d\xi$$
$$- 3[u_1]\mathcal{P}(t)\int \eta'_{u_2}(\xi)\eta'_{\mathcal{P}}(\xi)dy + m(t)[u_1]\int \eta_{u_1}(\xi)\eta_m(\xi)d\xi$$
$$+ n(t)[u_{1x}]\int \xi\eta_{0u_1}(\xi)\eta''_n(\xi)d\xi$$
$$+ [u_{1x}]\mathcal{R}(t)\int \xi\eta_{0u_1}(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi],$$

$$\mathscr{L}_5[u_{11}, u_{21}, u_{31}, u_{41}, u_{51}] = 0,$$
$$\mathscr{L}_5[u_{12}, u_{22}, u_{32}, u_{42}, u_{52}] = 0,\ \pm x > \phi(t),$$
$$\int \eta_{0u_1}(\xi)\eta_o(\xi)d\xi = \frac{1}{2}, \int \eta_{0u_1}(\xi)\eta_q(\xi)d\xi = \frac{1}{2},$$
$$\int \xi^2\eta_{0u_1}(\xi)\eta_r(\xi)d\xi = 1, \int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)dy = 3,$$
$$\dot{o}(t) = [2u_1u_5 + 4u_2u_4 + 3u_3^2] - [u_5]\dot{\phi}(t),$$

$$\dot{q}(t) = 4\{(u_{22} + [u_1]\int \eta_{0u_2}(\xi)\eta_l(\xi)d\xi)l(t) + (u_{42}$$
$$+ [u_4]\int \eta_{0u_4}(\xi)\eta_e(\xi)d\xi)e(t)\}$$
$$+ 3\{2(u_{32} + [u_3]\int \eta_{0u_3}(\xi)\eta_g(\xi)d\xi)g(t)\}$$
$$+ 2(u_{22x} + [u_{1x}]\int \eta_{0u_1}(\xi)\eta_q(\xi)d\xi)q(t)$$
$$+ 4(u_{22x} + [u_{2x}]\int \eta_{0u_2}(\xi)\eta_m(\xi)d\xi)m(t)$$
$$+ 3\{2(u_{32x} + [u_{3x}]\int \eta_{0u_3}(\xi)\eta_h(\xi)d\xi)h(t)\}$$
$$+ 2(u_{12xx} + \frac{[u_{1xx}]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta''_r(\xi)d\xi)r(t)$$
$$+ 4(u_{22xx} + \frac{[u_{2xx}]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta''_n(\xi)d\xi)n(t)$$
$$+ 2(u_{1xxx} + \frac{[u_{1xxx}]}{6}\int \eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t),$$

$$\dot{r}(t) = 4(u_{22} + [u_1]\int \eta_{0u_2}(\xi)\eta_m(\xi)d\xi)m(t) + 3\{2(u_{22}$$
$$+ [u_3]\int \eta_{0u_3}(\xi)\eta_h(\xi)d\xi)h(t)\}$$
$$+ 2\{2(u_{22x} + \frac{[u_{1x}]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_r(\xi)d\xi)r(t)\}$$
$$+ 4\{2(u_{22x} + \frac{[u_{2x}]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta_n(\xi)d\xi)n(t)\}$$
$$+ 2\{3(u_{12xx} + \frac{[u_{1xx}]}{6}\int \eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)\},$$

$$\dot{s}(t) = 4(u_{22} + \frac{[u_1]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta''_n(\xi)d\xi)n(t)$$
$$+2\{3(u_{22x} + \frac{[u_{1x}]}{6}\int \eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)\}, \mathcal{U}(t)$$
$$= -\frac{1}{2[u_1]\int \eta_{0u_1}(\xi)\eta^V_{\mathcal{U}}d\xi}[2\{[u_1]s(t)\int \eta_{0u_1}(\xi)\eta'''_s(\xi)d\xi\}$$
$$+ 4e(t)n(t)\int \eta_e(\xi)\eta''_n(\xi)d\xi$$
$$+ e(t)\mathcal{R}(t)\int \eta_e(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi + 3\{h^2\int (\eta'_h(\xi))^2d\xi$$
$$+ 2h(t)\mathcal{P}(t)\int \eta'_h(\xi)\eta'''_{\mathcal{P}}(\xi)d\xi$$
$$+ \mathcal{P}^2(t)\int (\eta'''_{\mathcal{P}}(\xi))^2d\xi\}],$$

if we have

$$\int \eta_{0u_1}(\xi)\eta'''_s(\xi)d\xi = \frac{1}{2}\int \xi^2\eta_{0u_1}(\xi)\eta'''_s(\xi)d\xi,$$
$$\int \eta_{0u_1}(\xi)\eta^V_U(\xi)d\xi = \frac{1}{2}\int \xi^2\eta_{0u_1}(\xi)\eta^V_U(\xi)d\xi,$$
$$\int \eta_e(\xi)\eta''_n(\xi)d\xi = \frac{1}{2}\int \xi^2\eta_e(\xi)\eta''_n(\xi)d\xi,$$
$$\int \eta_e(\xi)\eta'''_R(\xi)d\xi = \frac{1}{2}\int \xi^2\eta_e(\xi)\eta'''_R(\xi)d\xi,$$
$$\int (\eta'_h(\xi))^2d\xi = \frac{1}{2}\int \xi^2(\eta'_h(\xi))^2d\xi,$$
$$\int \eta'_h(\xi)\eta'''_P(\xi)d\xi = \frac{1}{2}\int \xi^2\eta'_h(\xi)\eta'''_P(\xi)d\xi,$$
$$\int (\eta'''_P(\xi))^2d\xi = \frac{1}{2}\int \xi^2(\eta'''_P(\xi))^2d\xi,$$

else $\mathcal{U}(t) = 0$,

$$\mathcal{T}(t) = \frac{1}{2[u_1]\int \xi\eta_{0u_1}(\xi)\eta''''_{\mathcal{T}}(\xi)d\xi}[2\{2(\frac{[u_{1x}]s(t)}{2}$$
$$\int \xi^2\eta_{u_1}(\xi)\eta'''_s(\xi)d\xi + \frac{[u_{1x}]\mathcal{U}(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta^V_{\mathcal{U}}(\xi))\}$$
$$- 2\{[u_1]r(t)\int \xi\eta_{0u_1}(\xi)\eta''_r(\xi)d\xi\}$$
$$- 4\{n(t)[u_1]\int \xi\eta_{0u_1}(\xi)\eta''_n(\xi)d\xi$$
$$+ [u_1]\mathcal{R}(t)\int \xi\eta_{0u_2}(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi$$
$$+ e(t)m(t)\int \xi\eta_e(\xi)\eta''_m(\xi)d\xi$$
$$+ e(t)\mathcal{Q}(t)\int \xi\eta_e(\xi)\eta'''_{\mathcal{Q}}(\xi)d\xi\}$$
$$- 3\{2g(t)h(t)\int y\eta_g(\xi)\eta'_h(\xi)d\xi$$
$$+ 2g(t)\mathcal{P}(t)\int \xi\eta_g(\xi)\eta'''_{\mathcal{P}}(\xi)d\xi\}],$$

$$
\begin{aligned}
\mathcal{S}(t) =&\frac{1}{2[u_1]\int \eta'_u(\xi)\eta'_s(\xi)d\xi}[2[u_1]q(t)\int \xi\eta_{0u_1}(\xi)\eta''_q(\xi)d\xi\\
&-4\{m(t)[u_1]\int \eta_{u_2}(\xi)\eta_m(\xi)d\xi\\
&+[u_1]\mathcal{Q}(t)\int \eta'_{u_2}(\xi)\eta'_{\mathcal{Q}}(\xi)d\xi\\
&+e(t)l(t)\int \eta_e(\xi)\eta_l(\xi)d\xi\}\\
&-3\{2([u_3]\mathcal{P}(t)\int \eta'_w(\xi)\eta'_{\mathcal{P}}(\xi)d\xi\\
&-[u_3]h(t)\int \eta_w(\xi)\eta_h(\xi)d\xi\\
&+g^2(t)\int \eta_g^2(\xi)d\xi)\}\\
&+2\{[u_{1x}]r(t)\int \xi\eta_{0u_1}(\xi)\eta''_r(\xi)d\xi\\
&+[u_{1x}]\mathcal{T}(t)\int \xi\eta_{0u_1}(\xi)\eta''''_{\mathcal{T}}(\xi)d\xi\}\\
&+4\{n(t)[u_{2x}]\int \xi\eta_{0u_2}(\xi)\eta''_n(\xi)d\xi\\
&+[u_{2x}]\mathcal{R}(t)\int \xi\eta_{0u_2}(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi\}\\
&-2\{\frac{[u_1]_{xx}}{2}s(t)\int \xi^2\eta_{0u_1}(\xi)\eta'''_s(\xi)d\xi\\
&+\frac{[u_{1xxx}]}{2}\mathcal{U}(t)\int \xi^2\eta_{0u_1}(\xi)\eta^V_{\mathcal{U}}(\xi)d\xi\}].
\end{aligned}
$$

Proof In [29] it is proved that u_1, u_2, u_3 and u_4 is a weak asymptotic solution of (1.1) for $n=4$ if first twenty one relations hold. In order to prove $(u_1(x,t,\varepsilon),\ u_2(x,t,\varepsilon), u_3(x,t,\varepsilon),\ u_4(x,t,\varepsilon),\ u_5(x,t,\varepsilon))$ is a weak asymptotic solution to the system (1.2) we have to show that it satisfies (2.1). We only need to show that $(2.1)_e$ holds provided all the assumptions in the Theorem (3.2) are satisfied. So we need to consider the 5th equation of (1.2)

$$u_{5t} + (2u_1u_5 + 4u_2u_4 + 3u_3^2)_x = 0.$$

For $(2.1)_e$ we need to compute the weak asymptotic product u_1u_5, u_2u_4 and u_3^2 and the partial derivative of u_5 with respect to t to obtain a weak asymptotic solution. First we compute the asymptotic product u_1u_5. Using Lemma 3.1 we have

$$
\begin{aligned}
&u_1(x,t,\varepsilon)u_5(x,t,\varepsilon)\\
=&u_{12}u_{52}+[u_1u_5]H(-x+\phi(t))+(u_{12}\\
&+[u_1]\int \eta_{0u_1}(\xi)\eta_o(\xi)d\xi o(t)\delta(-x+\phi(t))\\
&+(u_{12}+[u_1]\int \eta_{0u_1}(\xi)\eta_q(\xi)d\xi q(t)\delta'(-x+\phi(t))+(u_{12}\\
&+\frac{[u_1]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta''_r(\xi)d\xi r(t)\delta''(-x+\phi(t))+(u_{12}\\
&+\frac{[u_1]}{6}\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi s(t)\delta'''(-x+\phi(t))\\
&+(-[u_1]q(t)\int \xi\eta_{0u_1}(\xi)\eta''_q(\xi)d\xi\\
&+[u_1]\mathcal{S}(t)\int \eta'_u(\xi)\eta'_{\mathcal{S}}(\xi))\frac{\delta(-x+\phi(t))}{\varepsilon}\\
&+([u_1]s(t)\int \eta_{0u_1}(\xi)\eta'''_s(\xi)d\xi\\
&+[u_1]\mathcal{U}(t)\int \eta_{0u_1}(\xi)\eta^V_{\mathcal{U}}(\xi)d\xi)\frac{\delta(-x+\phi(t))}{\varepsilon^3}\\
&+(-[u_1]r(t)\int \xi\eta_{0u_1}(\xi)\eta''_r(\xi)d\xi\\
&-[u_1]\mathcal{T}(t)\int \xi\eta_{0u_1}(\xi)\eta''''_{\mathcal{T}}(\xi)d\xi)\frac{\delta'(-x+\phi(t))}{\varepsilon}\\
&+(\frac{[u_1]}{2}s(t)\int \xi^2\eta_{0u_1}(\xi)\eta'''_s(\xi)d\xi\\
&+\frac{[u_1]}{2}\mathcal{U}(t)\int \xi^2\eta_{0u_1}(\xi)\eta^V_{\mathcal{U}}(\xi)d\xi)\frac{\delta''(-x+\phi(t))}{\varepsilon}\\
&+O_{D'}(\varepsilon).
\end{aligned}
\tag{3.8}
$$

Then we proceed to derive the asymptotic product u_2u_4. Using Lemma 3.1 we obtain

$$
\begin{aligned}
&u_2(x,t,\varepsilon)u_4(x,t,\varepsilon)\\
&=u_{22}u_{42}+[u_2u_4]H(-x+\phi(t))+\{(u_{22}\\
&+[u_2]\int \eta_{0u_2}(\xi)\eta_l(\xi)d\xi)l(t)\\
&+(u_{42}+[u_4]\int \eta_{0u_4}(\xi)\eta_e(\xi)dy)e(t)\}\delta(-x+\phi(t))\\
&+(u_{22}+[u_2]\int \eta_{0u_2}(\xi)\eta_m(\xi)dy)m(t)\delta'(-x+\phi(t))\\
&+(u_{22}+\frac{[u_2]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta''_n(\xi)d\xi)n(t)\delta''(-x+\phi(t))\\
&+\{-m(t)[u_2]\int \eta_{u_2}(\xi)\eta_m(\xi)d\xi+[u_1]\mathcal{Q}(t)\int \eta'_{u_2}(\xi)\eta'_{\mathcal{Q}}(\xi)d\xi\\
&+e(t)l(t)\int \eta_e(\xi)\eta_l(\xi)d\xi\}\frac{\delta(-x+\phi(t))}{\varepsilon}\\
&+\{e(t)n(t)\int \eta_e(\xi)\eta''_n(\xi)d\xi\\
&+e(t)\mathcal{R}(t)\int \eta_e(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi\}\frac{\delta(-x+\phi(t))}{\varepsilon^3}\\
&+\{-n(t)[u_2]\int \xi\eta_{0u_2}(\xi)\eta''_n(\xi)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&- [u_1]\mathcal{R}(t)\int \xi\eta_{0u_2}(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi \\
&- e(t)m(t)\int \xi\eta_e(\xi)\eta''_m(\xi)d\xi \\
&- e(t)\mathcal{Q}(t)\int \xi\eta_e(\xi)\eta'''_{\mathcal{Q}}(\xi)d\xi\}\frac{\delta'(-x+\phi(t))}{\varepsilon} \\
&\{\frac{e(t)n(t)}{2}\int \xi^2\eta_e(\xi)\eta''_n(\xi)d\xi \\
&+ \frac{e(t)\mathcal{R}(t)}{2}\int \xi^2\eta_e(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi\}\frac{\delta''(-x+\phi(t))}{\varepsilon} + O_{\mathcal{D}'}(\varepsilon).
\end{aligned} \tag{3.9}
$$

Lastly we estimate the asymptotic product for u_3^2 using Lemma 3.1. So we have the weak asymptotic expansion of u_3^2 as

$$
\begin{aligned}
&u_3^2(x,t,\varepsilon) \\
&= u_{32}^2 + [u_3^2]H(-x+\phi(t)) + 2(u_{32} \\
&+ [u_3]\int \eta_{0u_3}(\xi)\eta_g(\xi)d\xi)g(t)\delta(-x+\phi(t)) \\
&+ 2(u_{32} + [u_3]\int \eta_{0u_3}(\xi)\eta_h(\xi)d\xi)h(\xi)\delta'(-x+\phi(t)) \\
&+ 2\{[u_3]\mathcal{P}(t)\int \eta'_w(\xi)\eta'_{\mathcal{P}}(\xi)d\xi - [u_3]h(t)\int \eta_w(\xi)\eta_h(\xi)d\xi \\
&+ g^2(t)\int \eta_g^2(\xi)d\xi\}\frac{\delta(-x+\phi(t))}{\varepsilon} \\
&+ \{h^2(t)\int (\eta'_h(v))^2d\xi + 2h(t)\mathcal{P}(t)\int \eta'_h(\xi)\eta'''_{\mathcal{P}}(\xi)d\xi \\
&+ \mathcal{P}^2(t)\int (\eta'''_{\mathcal{P}}(\xi))^2d\xi\}\frac{\delta(-x+\phi(t))}{\varepsilon^3} \\
&+ \{-2g(t)h(t)\int \xi\eta_g(\xi)\eta'_h(\xi)d\xi \\
&- 2g(t)\mathcal{P}(t)\int \xi\eta_g(\xi)\eta'''_{\mathcal{P}}(\xi)d\xi\}\frac{\delta'(-x+\phi(t))}{\varepsilon} \\
&+ \{h^2(t)\int \xi^2(\eta'_h(\xi))^2d\xi + 2h(t)\mathcal{P}(t)\int \xi^2\eta'_h(\xi)\eta'''_{\mathcal{P}}(\xi)d\xi \\
&+ \mathcal{P}^2(t)\int \xi^2(\eta'''_{\mathcal{P}}(\xi))^2d\xi\}\frac{\delta''(-x+\phi(t))}{\varepsilon} + O_{\mathcal{D}'}(\varepsilon)
\end{aligned} \tag{3.10}
$$

Collecting the expansions of u_1u_5, u_2u_4 and u_3^2 from (3.8), (3.9) and (3.10) respectively and using the following identities

$$
\begin{aligned}
p(x,t)\delta'(-x+\phi(t)) &= p(\phi(t),t)\delta'(-x+\phi(t)) \\
&\quad + p_x(\phi(t),t)\delta(-x+\phi(t)), \\
p(x,t)\delta''(-x+\phi(t)) &= p(\phi(t),t)\delta''(-x+\phi(t)) \\
&\quad + 2p_x(\phi(t),t)\delta'(-x+\phi(t)) \\
&\quad + p_{xx}(\phi(t),t)\delta(-x+\phi(t)), \\
p(x,t)\delta'''(-x+\phi(t)) &= p(\phi(t),t)\delta'''(-x+\phi(t)) \\
&\quad + 3p_x(\phi(t),t)\delta''(-x+\phi(t)) \\
&\quad + 3p_{xx}(\phi(t),t)\delta'(-x+\phi(t)) \\
&\quad + p_{xxx}(\phi(t),t)\delta'''(-x+\phi(t)),
\end{aligned} \tag{3.11}
$$

where $p \in C^3$, we obtained the expansion of $2u_1u_5 + 4u_2u_4 + 3u_3^2$. After rearrangement of similar terms we have

$$
\begin{aligned}
&(2u_1(x,t,\varepsilon)u_5(x,t,\varepsilon) + 4u_2(x,t,\varepsilon)u_4(x,t,\varepsilon) + 3w^2(x,t,\varepsilon)) \\
&= (2u_{12}u_{52} + 4u_{22}u_{42} + 3u_{32}^2) + [2u_1u_5 + 4u_2u_4 \\
&+ 3u_3^2]H(-x+\phi(t)) + [2(u_{12} + [u_1]\int \eta_{0u_1}(\xi)\eta_o(\xi)d\xi)o(t) \\
&+ 4\{(u_{22} + [u_2]\int \eta_{0u_2}(\xi)\eta_l(\xi)d\xi)l(t) + (u_{42} \\
&+ [u_4]\int \eta_{0u_4}(\xi)\eta_e(\xi)d\xi)e(t)\} + 3\{2(u_{32} \\
&+ [u_3]\int \eta_{0u_3}(\xi)\eta_g(\xi)d\xi)g(t)\} + 2(u_{22x} \\
&+ [u_{1x}]\int \eta_{0u_1}(\xi)\eta_q(\xi)d\xi)q(t) \\
&+ 4(u_{22x} + [u_{2x}]\int \eta_{0u_2}(\xi)\eta_m(\xi)d\xi)m(t) + 3\{2(w_{2x} \\
&+ [u_{3x}]\int \eta_{0u_3}(\xi)\eta_h(\xi)d\xi)h(t)\} \\
&+ 2(u_{12xx} + \frac{[u_{1xx}]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta''_r(\xi)d\xi)r(t) \\
&+ 4(u_{22xx} + \frac{[u_{2xx}]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta''_n(\xi)d\xi)n(t) \\
&+ 2(u_{2xxx} \\
&+ \frac{[u_{1xxx}]}{6}\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)]|_{x=\phi(t)}\delta(-x+\phi(t)) \\
&+ [2(u_{12} + [u_1]\int \eta_{0u_1}(\xi)\eta_q(\xi)d\xi)q(t) + 4(u_{22} \\
&+ [u_2]\int \eta_{0u_2}(\xi)\eta_m(\xi)d\xi)m(t)
\end{aligned}
$$

$$
\begin{aligned}
&+3\{2(u_{32}+[u_3]\int \eta_{0u_3}(\xi)\eta_h(\xi)d\xi)h(t)\}2\{2(u_{1x}\\
&+\frac{[u_{1x}]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_r(\xi)d\xi)r(t)\} \ +4\{2(u_{22x}\\
&+\frac{[u_{2x}]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta_n(\xi)dy)n(t)\}+2\{3(u_{12xx}\\
&+\frac{[u_{1xx}]}{6}\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)\}]|_{x=\phi(t)} \quad \delta'(-x\\
&+\phi(t)) \ +[2(u_{12}+\frac{[u_1]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_r''(\xi)d\xi)r(t)\\
&+4(u_{22}+\frac{[u_2]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta_n''(\xi)d\xi)n(t) \ +2\{3(u_{22x}\\
&+\frac{[u_{1x}]}{6}\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)dy)s(t)\}]|_{x=\phi(t)} \quad \delta''(-x\\
&+\phi(t)) \ +[2\{(u_{12}\\
&+\frac{[u_1]}{6}\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)\}]|_{x=\phi(t)} \quad \delta'''(-x\\
&+\phi(t)) \ +[2\{-[u_1]q(t)\int \xi\eta_{0u_1}(\xi)\eta_q''(\xi)d\xi\\
&+[u_1]\mathcal{S}(t)\int \eta_u'(\xi)\eta_{\mathcal{S}}'(\xi)d\xi\}\\
&+4\{-m(t)[u_1]\int \eta_{u_2}(\xi)\eta_m(\xi)d\xi\\
&+[u_1]\mathcal{Q}(t)\int \eta_{u_2}'(\xi)\eta_{\mathcal{Q}}'(\xi)d\xi+e(t)l(t)\int \eta_e(\xi)\eta_l(\xi)d\xi\}\\
&+3\{2([u_3]\mathcal{P}(t)\int \eta_w'(\xi)\eta_{\mathcal{P}}'(\xi)d\xi\\
&-[u_3]h(t)\int \eta_w(\xi)\eta_h(\xi)d\xi+g^2(t)\int \eta_g^2(\xi)d\xi)\}\\
&+2\{-[u_{1x}]r(t)\int \xi\eta_{0u_1}(\xi)\eta_r''(\xi)d\xi\\
&-[u_{1x}]\mathcal{T}(t)\int y\eta_{0u_1}(\xi)\eta_{\mathcal{T}}''''(\xi)d\xi\}\\
&+4\{-n(t)[u_{2x}]\int \xi\eta_{0u_2}(\xi)\eta_n''(\xi)d\xi\\
&-[u_{2x}]\mathcal{R}(t)\int \xi\eta_{0u_2}(\xi)\eta_{\mathcal{R}}''''(\xi)d\xi\}\\
&+2\{\frac{[u_{1xx}]s(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_s'''(\xi)d\xi\\
&+\frac{[u_{1xx}]\mathcal{U}(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_{\mathcal{U}}^{V}(\xi)d\xi\}]|_{x=\phi(t)}\frac{\delta(-x+\phi(t))}{\varepsilon}\\
&+[2\{[u_1]s(t)\int \eta_{0u_1}(\xi)\eta_s'''(\xi)d\xi\\
&+[u_1]\mathcal{U}(t)\int \eta_{0u_1}(\xi)\eta_{\mathcal{U}}^{V}(\xi)d\xi\}\\
&+4\{e(t)n(t)\int \eta_e(\xi)\eta_n''(\xi)d\xi\\
&+e(t)\mathcal{R}(t)\int \eta_e(\xi)\eta_{\mathcal{R}}''''(\xi)d\xi\}+3\{h^2(t)\int (\eta_h'(\xi))^2d\xi\\
&+2h(t)\mathcal{P}(t)\int \eta_h'(\xi)\eta_{\mathcal{P}}'''(\xi)d\xi\\
&+\mathcal{P}^2(t)\int (\eta_{\mathcal{P}}'''(\xi))^2d\xi\}]|_{x=\phi(t)}\frac{\delta(-x+\phi(t))}{\varepsilon^3}\\
&+[2\{-[u_1]r(t)\int \xi\eta_{0u_1}(\xi)\eta_r''(\xi)d\xi\\
&-[u_1]\mathcal{T}(t)\int \xi\eta_{0u_1}(\xi)\eta_{\mathcal{T}}''''(\xi)d\xi\}\\
&+4\{-n(t)[u_1]\int \xi\eta_{0u_2}(\xi)\eta_n''(\xi)d\xi\\
&-[u_1]\mathcal{R}(t)\int \xi\eta_{0u_2}(\xi)\eta_{\mathcal{R}}''''(\xi)d\xi\\
&-e(t)m(t)\int \xi\eta_e(\xi)\eta_m''(\xi)d\xi\\
&-e(t)\mathcal{Q}(t)\int \xi\eta_e(\xi)\eta_{\mathcal{Q}}'''(\xi)d\xi\}\\
&+3\{-2g(t)h(t)\int \xi\eta_g(\xi)\eta_h'(\xi)d\xi\\
&-2g(t)\mathcal{P}(t)\int \xi\eta_g(\xi)\eta_{\mathcal{P}}'''(\xi)d\xi\}\\
&+2\{2(\frac{[u_{1x}]s(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_s'''(\xi)d\xi\\
&+\frac{[u_{1x}]\mathcal{U}(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_{\mathcal{U}}^{V}(\xi)d\xi)\}]|_{x=\phi(t)}\frac{\delta'(-x+\phi(t))}{\varepsilon}\\
&+[2\{\frac{[u_1]s(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_s'''(\xi)d\xi\\
&+\frac{[u_1]\mathcal{U}(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_{\mathcal{U}}^{V}(\xi)d\xi\}\\
&+4\{\frac{e(t)n(t)}{2}\int \xi^2\eta_e(\xi)\eta_n''(\xi)d\xi\\
&+\frac{e(t)\mathcal{R}(t)}{2}\int \xi^2\eta_e(\xi)\eta_{\mathcal{R}}''''(\xi)d\xi\}\\
&+3\{\frac{1}{2}h^2(t)\int \xi^2(\eta_h'(\xi))^2d\xi\\
&+2h(t)\mathcal{P}(t)\int \xi^2\eta_h'(\xi)\eta_{\mathcal{P}}'''(\xi)d\xi\\
&+\mathcal{P}^2(t)\int \xi^2(\eta_{\mathcal{P}}'''(\xi))^2d\xi\}]|_{x=\phi(t)}\frac{\delta''(-x+\phi(t))}{\varepsilon}+O_{D'}(\varepsilon).
\end{aligned} \tag{3.12}
$$

Then we need to calculate the expansion of the term u_{5t}. So after the differentiation of $u_5(x,t,\varepsilon)$ with respect to t we are left with the following expansion

$$
\begin{aligned}
u_{5t}(x,t,\varepsilon) = {} & u_{52_t}+[u_{5t}]H(-x+\phi(t))+(\dot{o}(t)\\
&+[u_5]\dot{\phi}(t))\delta(-x+\phi(t))+(o(t)\dot{\phi}(t)\\
&+\dot{q}(t))\delta'(-x+\phi(t))\\
&+(q(t)\dot{\phi}(t)+\dot{r}(t))\delta''(-x+\phi(t))+(r(t)\dot{\phi}(t)\\
&+\dot{s}(t))\delta'''(-x+\phi(t))+s(t)\dot{\phi}(t)\delta''''(-x+\phi(t))\\
&+O_{D'}(\varepsilon).
\end{aligned} \tag{3.13}
$$

By exploiting (3.12) and (3.13) we obtain

$$
\begin{aligned}
&u_{5t} + (2u_1(x,t,\varepsilon)u_5(x,t,\varepsilon) \\
&+ 4u_2(x,t,\varepsilon)u_4(x,t,\varepsilon) + 3u_3^2(x,t,\varepsilon))_x \\
&= u_{52t} + (2u_{12}u_{52} + 4u_{22}u_{42} + 3u_{32}^2)_x \\
&+ [p_t + (2u_1u_5 + 4u_2u_4 + 3u_3^2)_x]H(-x+\phi(t)) \\
&+ [\dot{o}(t) + [u_5]\dot{\phi}(t) - [2u_1u_5 + 4u_2u_4 + 3u_3^2]]\delta(-x+\phi(t)) \\
&+ [o(t)\dot{\phi}(t) + \dot{q}(t) - [2(u_{12} + [u_1]\int \eta_{0u_1}(\xi)\eta_o(\xi)d\xi)o(t) \\
&+ 4\{(u_{22} + [u_2]\int \eta_{0u_2}(\xi)\eta_l(\xi)d\xi)l(t) \\
&+ (u_{42} + [u_4]\int \eta_{0u_4}(\xi)\eta_e(\xi)d\xi)e(t)\} \\
&+ 3\{2(u_{32} + [u_3]\int \eta_{0u_3}(\xi)\eta_g(\xi)d\xi)g(t)\} \\
&+ 2(u_{22x} + [u_{1x}]\int \eta_{0u_1}(\xi)\eta_q(\xi)d\xi)q(t) + 4(u_{22x} \\
&+ [u_{2x}]\int \eta_{0u_2}(\xi)\eta_m(\xi)d\xi)m(t) \\
&+ 3\{2(w_{2x} + [u_{3x}]\int \eta_{0u_3}(\xi)\eta_h(\xi)d\xi)h(t)\} + 2(u_{12xx} \\
&+ \frac{[u_{1xx}]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_r''(\xi)d\xi)r(t) \\
&+ 4(u_{22xx} + \frac{[u_{2xx}]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta_n''(\xi)d\xi)n(t) + 2(u_{2xxx} \\
&+ \frac{[u_{1xxx}]}{6}\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)]]|_{x=\phi(t)}\delta'(-x+\phi(t)) \\
&+ [q(t)\dot{\phi}(t) + \dot{r}(t) - [2(u_{12} + [u_1]\int \eta_{0u_1}(\xi)\eta_q(\xi)dy)q(t) \\
&+ 4(u_{22} + [u_2]\int \eta_{0u_2}(\xi)\eta_m(\xi)d\xi)m(t) \\
&+ 3\{2(u_{32} + [u_3]\int \eta_{0u_3}(\xi)\eta_h(\xi)d\xi)h(t)\} + 2\{2(u_{1x} \\
&+ \frac{[u_{1x}]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_r(\xi)d\xi)r(t)\} \\
&+ 4\{2(u_{22x} + \frac{[u_{2x}]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta_n(\xi)d\xi)n(t)\} + 2\{3(u_{12xx} \\
&+ \frac{[u_{1xx}]}{6}\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)\}]]|_{x=\phi(t)}\delta''(-x+\phi(t)) \\
&+ [r(t)\dot{\phi}(t) + \dot{s}(t) - [2(u_{12} + \frac{[u_1]}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_r''(\xi)d\xi)r(t) \\
&+ 4(u_{22} + \frac{[u_2]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta_n''(\xi)d\xi)n(t) \\
&+ 2\{3(u_{22x} + \frac{[u_{1x}]}{6} \\
&\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)\}]]|_{x=\phi(t)}\delta'''(-x+\phi(t)) \\
&+ [s(t)\dot{\phi}(t) - [2\{(u_{12} \\
&+ \frac{[u_1]}{6}\int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi)s(t)\}]]|_{x=\phi(t)}\delta''''(-x+\phi(t)) \\
&- [2\{-[u_1]q(t)\int \xi\eta_{0u_1}(\xi)\eta_q''(\xi)d\xi \\
&+ [u_1]\mathcal{S}(t)\int \eta_u'(\xi)\eta_{\mathcal{S}}'(\xi)d\xi\} \\
&+ 4\{-m(t)[u_1]\int \eta_{u_2}(\xi)\eta_m(\xi)d\xi \\
&+ [u_1]\mathcal{Q}(t)\int \eta_{u_2}'(\xi)\eta_{\mathcal{Q}}'(\xi)d\xi + e(t)l(t)\int \eta_e(\xi)\eta_l(\xi)d\xi\} \\
&+ 3\{2([u_3]\mathcal{P}(t)\int \eta_w'(\xi)\eta_{\mathcal{P}}'(\xi)d\xi \\
&- [u_3]h(t)\int \eta_w(\xi)\eta_h(\xi)d\xi + g^2(t)\int \eta_g^2(\xi)d\xi)\} \\
&+ 2\{-[u_{1x}]r(t)\int \xi\eta_{0u_1}(\xi)\eta_r''(\xi)d\xi \\
&- [u_{1x}]\mathcal{T}(t)\int \xi\eta_{0u_1}(\xi)\eta_{\mathcal{T}}''''(\xi)d\xi\} \\
&+ 4\{-n(t)[u_{2x}]\int \xi\eta_{0u_2}(\xi)\eta_n''(\xi)d\xi \\
&- [u_{2x}]\mathcal{R}(t)\int \xi\eta_{0u_2}(\xi)\eta_{\mathcal{R}}''''(\xi)d\xi\} \\
&+ 2\{\frac{[u_{1xx}]s(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_s'''(\xi)d\xi \\
&+ \frac{[u_{1xx}]\mathcal{U}(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta_{\mathcal{U}}^{V}(\xi)d\xi\}]|_{x=\phi(t)}\frac{\delta'(-x+\phi(t))}{\varepsilon} \\
&- [2\{[u_1]s(t)\int \eta_{0u_1}(\xi)\eta_s'''(\xi)d\xi \\
&+ [u_1]\mathcal{U}(t)\int \eta_{0u_1}(\xi)\eta_{\mathcal{U}}^{V}(\xi)d\xi\} + 4\{e(t)n(t)\int \eta_e(\xi)\eta_n''(\xi)d\xi \\
&+ e(t)\mathcal{R}(t)\int \eta_e(\xi)\eta_{\mathcal{R}}''''(\xi)d\xi\} + 3\{h^2(t)\int (\eta_h'(\xi))^2d\xi \\
&+ 2h(t)\mathcal{P}(t)\int \eta_h'(\xi)\eta_{\mathcal{P}}'''(\xi)d\xi \\
&+ \mathcal{P}^2(t)\int (\eta_{\mathcal{P}}'''(\xi))^2d\xi\}]|_{x=\phi(t)}\frac{\delta'(-x+\phi(t))}{\varepsilon^3} \\
&- [2\{-[u_1]r(t)\int \xi\eta_{0u_1}(\xi)\eta_r''(\xi)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&-[u_1]\mathcal{T}(t)\int \xi\eta_{0u_1}(\xi)\eta''''_{\mathcal{T}}(\xi)d\xi\}\\
&+4\{-n(t)[u_1]\int \xi\eta_{0u_2}(\xi)\eta''_n(\xi)d\xi\\
&-[u_1]\mathcal{R}(t)\int \xi\eta_{0u_2}(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi\\
&-e(t)m(t)\int \xi\eta_e(\xi)\eta''_m(\xi)d\xi\\
&-e(t)\mathcal{Q}(t)\int \xi\eta_e(\xi)\eta'''_{\mathcal{Q}}(\xi)d\xi\}\\
&+3\{-2g(t)h(t)\int \xi\eta_g(\xi)\eta'_h(\xi)d\xi\\
&-2g(t)\mathcal{P}(t)\int \xi\eta_g(\xi)\eta'''_{\mathcal{P}}(\xi)d\xi\}\\
&+2\{2(\frac{[u_{1x}]s(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta'''_s(\xi)d\xi\\
&+\frac{[u_{1x}]\mathcal{U}(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta^V_{\mathcal{U}}(\xi)d\xi)\}]|_{x=\phi(t)}\frac{\delta''(-x+\phi(t))}{\varepsilon}\\
&-[2\{\frac{[u_1]s(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta'''_s(\xi)d\xi\\
&+\frac{[u_1]\mathcal{U}(t)}{2}\int \xi^2\eta_{0u_1}(\xi)\eta^V_{\mathcal{U}}(\xi)d\xi\}\\
&+4\{\frac{e(t)n(t)}{2}\int \xi^2\eta_e(\xi)\eta''_n(\xi)d\xi\\
&+\frac{e(t)\mathcal{R}(t)}{2}\int \xi^2\eta_e(\xi)\eta''''_{\mathcal{R}}(\xi)d\xi\}\\
&+3\{\frac{1}{2}h^2(t)\int \xi^2(\eta'_h(\xi))^2d\xi\\
&+2h(t)\mathcal{P}(t)\int \xi^2\eta'_h(\xi)\eta'''_{\mathcal{P}}(\xi)d\xi\\
&+\mathcal{P}^2(t)\int \xi^2(\eta'''_{\mathcal{P}}(\xi))^2d\xi\}]|_{x=\phi(t)}\frac{\delta'''(-x+\phi(t))}{\varepsilon}\\
&+O_{\mathcal{D}'}(\varepsilon).
\end{aligned}
\tag{3.14}
$$

We can obtain all the coefficient in (3.14) as zero except the error term by applying the conditions in the Theorem 3.2. So we have

$$\mathscr{L}_5[u_1,u_2,u_3,u_4,u_5]=o_{\mathcal{D}'}(1),\quad \varepsilon\to 0^+.$$

If the initial data also satisfy weak asymptotic relations then the ansatz that we have taken in the Theorem 3.2 is a weak asymptotic solution to the system (1.2). Hence proved. □

In the previous Theorem we have obtained a weak asymptotic solution where the initial data may not be constant. For that reason our conditions are more complicated compared to the constant initial data. But for the constant initial data case we have a simpler version of Theorem 3.2, which we have captured in the following Corollary. The following Corollary is very useful to construct weak asymptotic solution for more general initial data.

Corollary 3.3 *When $u_{1i},u_{2i},u_{3i},u_{4i}$ and u_{5i}, for $i=1,2$, are constants then the ansatz in the Theorem 3.2 is a weak asymptotic solution to (1.2) if the following relations hold.*

$$
\begin{aligned}
&\dot\phi(t)=(u_{11}+u_{12})|_{x=\phi(t)},\ \dot e(t)=[u_1](u_{21}+u_{22})|_{x=\phi(t)},\\
&\dot g(t)=(2[u_1](u_{21}+v_{22})+[u_1](u_{31}+u_{32}))|_{x=\phi(t)},\\
&\frac{d}{dt}(h(t)[u(\phi(t),t)])=\frac{d(e^2(t))}{dt}\\
&\int\eta_{0u_1}(\xi)\eta_j(\xi)d\xi=\int\xi^2\eta_{0u_2}(\xi)\eta_e(\xi)d\xi=\frac{1}{2},\ j=e,g,h,\\
&\int\eta_{u_1}(\xi)\eta_h(\xi)d\xi\\
&=\int\eta_e^2(\xi),\ P(t)=\frac{B}{u_2(\phi(t),t)},\ B\text{ is a constant},\\
&\dot l(t)=-[u_4]\dot\phi(t)+2[3u_2u_3+u_1u_4],\\
&\int\eta_{0u_1}(\xi)\eta_l(\xi)d\xi\\
&=\int\eta_{0u_1}(\xi)\eta_m(\xi)d\xi=\frac{1}{2}\int\xi^2\eta_{0u_1}(\xi)\eta_n(\xi)d\xi=\frac{1}{2},\\
&\dot m(t)=2[3\{(u_{22}+[u_1]\int\eta_{0u_2}(\xi)\eta_g(\xi))g(t)\\
&+(u_{32}+[u_3]\int\eta_{0u_3}(\xi)\eta_e(\xi)d\xi)e(t)\}],\\
&\dot n(t)=2[3\{(u_{22}+[u_2]\int\eta_{0u_2}(\xi)\eta_h(\xi)d\xi)h(t)\}],\\
&\mathcal{R}(t)=\frac{-1}{\{[u_1]\int\xi\eta_{0u_1}(\xi)\eta''''_{\mathcal{R}}(\xi)\}}[3e(t)h(t)\int\xi\eta_e(\xi)\eta'_h(\xi)d\xi\\
&+3e(t)\mathcal{P}(t)\int\xi\eta_e(\xi)\eta'''_{\mathcal{P}}(\xi)d\xi\\
&+n(t)[u_1]\int\xi\eta_{0u_1}(\xi)\eta''_n(\xi)d\xi],\\
&\mathcal{Q}(t)=\frac{1}{\{[u_1]\int\eta'_u(\xi)\eta'_{\mathcal{Q}}(\xi)d\xi\}}[3h(t)[u_1]\int\eta_{u_2}(\xi)\eta_h(\xi)d\xi\\
&-3g(t)e(t)\int\eta_e(\xi)\eta_g(\xi)d\xi\\
&-3[u_1]\mathcal{P}(t)\int\eta'_{u_2}(\xi)\eta'_{\mathcal{P}}(\xi)d\xi\\
&+m(t)[u_1]\int\eta_{u_1}(\xi)\eta_m(\xi)d\xi]\\
&\int\eta_{0u_1}(\xi)\eta_o(\xi)d\xi=\frac{1}{2},\int\eta_{0u_1}(\xi)\eta_q(\xi)d\xi=\frac{1}{2},
\end{aligned}
$$

$$\int \xi^2\eta_{0u_1}(\xi)\eta_r(\xi)d\xi = 1, \int \xi^3\eta_{0u_1}(\xi)\eta_s(\xi)d\xi = 3,$$

$$\dot{o}(t) = [2u_1u_5 + 4u_2u_4 + 3u_3^2] - [u_5]\dot{\phi}(t),$$

$$\dot{q}(t) = 4\{(u_{22} + [u_2]\int \eta_{0u_2}(\xi)\eta_l(\xi)d\xi)l(t) + (u_{42} + [u_4]\int \eta_{0u_4}(\xi)\eta_e(\xi)d\xi)e(t)\} + 3\{2(u_{32} + [u_3]\int \eta_{0u_3}(\xi)\eta_g(\xi)d\xi)g(t)\}$$

$$\dot{r}(t) = 4(u_{22} + [u_2]\int \eta_{0u_2}(\xi)\eta_m(\xi)d\xi)m(t) + 3\{2(u_{32} + [u_3]\int \eta_{0u_3}(\xi)\eta_h(\xi)d\xi)h(t)\},$$

$$\dot{s}(t) = 4(u_{22} + \frac{[u_2]}{2}\int \xi^2\eta_{0u_2}(\xi)\eta_n''(\xi)d\xi)n(t)$$

$$\mathcal{U}(t) = -\frac{1}{2[u_1]\int \eta_{0u_1}(\xi)\eta_{\mathcal{U}}^{V}d\xi}[2\{[u_1]s(t) \int \eta_{0u_1}(\xi)\eta_s'''(\xi)d\xi\} + 4e(t)n(t)\int \eta_e(\xi)\eta_n''(\xi)d\xi + e(t)\mathcal{R}(t)\int \eta_e(\xi)\eta_{\mathcal{R}}''''(\xi)d\xi + 3\{h^2\int (\eta_h'(\xi))^2d\xi + 2h(t)\mathcal{P}(t)\int \eta_h'(\xi)\eta_{\mathcal{P}}'''(\xi)d\xi + \mathcal{P}^2(t)\int (\eta_{\mathcal{P}}'''(\xi))^2d\xi\}],$$

$$\mathcal{T}(t) = \frac{1}{2[u_1]\int y\eta_{0u_1}(\xi)\eta_{\mathcal{T}}''''(\xi)d\xi}[-2\{[u_1]r(t) \int \xi\eta_{0u_1}(\xi)\eta_r''(\xi)d\xi\} - 4\{n(t)[u_1]\int \xi\eta_{0u_1}(\xi)\eta_n''(\xi)d\xi + [u_1]\mathcal{R}(t)\int \xi\eta_{0u_2}(\xi)\eta_{\mathcal{R}}''''(\xi)d\xi + e(t)m(t)\int \xi\eta_e(\xi)\eta_m''(\xi)d\xi + e(t)\mathcal{Q}(t)\int \xi\eta_e(\xi)\eta_{\mathcal{Q}}'''(\xi)d\xi\} - 3\{2g(t)h(t)\int \xi\eta_g(\xi)\eta_h'(\xi)d\xi + 2g(t)\mathcal{P}(t)\int \xi\eta_g(\xi)\eta_{\mathcal{P}}'''(\xi)d\xi\}],$$

$$\mathcal{S}(t) = \frac{1}{2[u_1]\int \eta_u'(\xi)\eta_s'(\xi)d\xi}[2[u_1]q(t)\int \xi\eta_{0u_1}(\xi)\eta_q''(\xi)d\xi - 4\{m(t)[u_1]\int \eta_{u_2}(\xi)\eta_m(\xi)d\xi + [u_1]\mathcal{Q}(t)\int \eta_{u_2}'(\xi)\eta_{\mathcal{Q}}'(\xi)d\xi + e(t)l(t) \int \eta_e(\xi)\eta_l(\xi)d\xi\} - 3\{2([u_3]\mathcal{P}(t)\int \eta_w'(\xi)\eta_{\mathcal{P}}'(\xi)d\xi - [u_3]h(t)\int \eta_w(\xi)\eta_h(\xi)d\xi + g^2(t)\int \eta_g^2(\xi)d\xi)\}]. \tag{3.15}$$

We know that piecewise constant functions can be used to approximate wide range of functions which are not necessarily constant. Now using the Corollary 3.3 we construct a weak asymptotic solution for the Riemann problem for certain initial data.

Theorem 3.4 *If $u_1^0,\ u_2^0,\ u_3^0,\ u_4^0$ and $u_5^0 \in L_\beta(\mathbb{R})$ for $\beta \in [1,\infty)$ then there exists a $T > 0$ such that (1.2) and (1.3) has a weak asymptotic solution for $t \in [0,T]$.*

Proof We know that when $u_1^0,\ u_2^0,\ u_3^0,\ u_4^0$ and $u_5^0 \in L_\beta(\mathbb{R})$ for $\beta \in [1,\infty)$, we can approximate these using simple functions. So for every $\varepsilon > 0$ we can find simple functions $u_{1\varepsilon}^0,\ u_{2\varepsilon}^0,\ u_{3\varepsilon}^0,\ u_{4\varepsilon}^0$ and $u_{5\varepsilon}^0$ such that

$$\left.\begin{array}{l} ||u_1^0 - u_{1\varepsilon}^0||_{L_\beta} < \varepsilon, \\ ||u_2^0 - u_{2\varepsilon}^0||_{L_\beta} < \varepsilon, \\ ||u_3^0 - u_{3\varepsilon}^0||_{L_\beta} < \varepsilon, \\ ||u_4^0 - u_{4\varepsilon}^0||_{L_\beta} < \varepsilon, \\ ||u_5^0 - u_{5\varepsilon}^0||_{L_\beta} < \varepsilon. \end{array}\right\} \tag{3.16}$$

Now let ψ be a test function having support in Ω such that these simple functions can be written as

$$\left.\begin{array}{l} u_{1\varepsilon}^0 = \sum_{i=1}^m u_{0i}(H(-x+a_i) - H(-x+a_{i-1})), \\ u_{2\varepsilon}^0 = \sum_{i=1}^m v_{0i}(H(-x+a_i) - H(-x+a_{i-1})), \\ u_{3\varepsilon}^0 = \sum_{i=1}^m w_{0i}(H(-x+a_i) - H(-x+a_{i-1})), \\ u_{4\varepsilon}^0 = \sum_{i=1}^m z_{0i}(H(-x+a_i) - H(-x+a_{i-1})), \\ u_{5\varepsilon}^0 = \sum_{i=1}^m p_{0i}(H(-x+a_i) - H(-x+a_{i-1})). \end{array}\right\} \tag{3.17}$$

Now the following ansatz

$$u_1(x,t,\varepsilon)=\sum_{i=1}^{m-1}(u_{1i}^0-u_{1i+1}^0)\mathcal{H}_{u_1}(-x+a_i+\phi_i(t),\varepsilon)+u_{1m}^0$$

$$u_2(x,t,\varepsilon)=\sum_{i=1}^{m-1}((u_{2i}^0-u_{2i+1}^0)\mathcal{H}_{u_2}(-x+a_i+\phi_i(t),\varepsilon)+u_{2m}^0 + \sum_{i=1}^{m-1}e_i(t)\delta_e(-x+a_i+\phi_i(t),\varepsilon)$$

$$u_3(x,t,\varepsilon)=\sum_{i=1}^{m-1}((u_{3i}^0-u_{3i+1}^0)\mathcal{H}_{u_3}(-x+a_i+\phi_i(t),\varepsilon)+u_{3m}^0 + \sum_{i=1}^{n-1}g_i(t)\delta_g(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{m-1}h_i\delta_h'(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{m-1}\mathcal{R}_{u_3 i}(-x+a_i+\phi_i(t),\varepsilon)$$

$$u_4(x,t,\varepsilon)=\sum_{i=1}^{m-1}((u_{4i}^0-u_{4i+1}^0)\mathcal{H}_{u_4}(-x+a_i+\phi_i(t),\varepsilon)+u_{4m}^0 + \sum_{i=1}^{m-1}l_i(t)\delta_l(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{m-1}\tilde{m}_i\delta_m'(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{m-1}n_i\delta_{\tilde{n}}''(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{n-1}\mathcal{R}_{u_4 i}(-x+a_i+\phi_i(t),\varepsilon)$$

$$u_5(x,t,\varepsilon)=\sum_{i=1}^{m-1}((u_{5i}^0-u_{5i+1}^0)\mathcal{H}_{u_5}(-x+a_i+\phi_i(t),\varepsilon)+u_{5m}^0 + \sum_{i=1}^{m-1}o_i(t)\delta_o(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{m-1}q_i\delta_q'(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{m-1}r_i\delta_r''(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{m-1}s_i\delta_s'''(-x+a_i+\phi_i(t),\varepsilon) + \sum_{i=1}^{m-1}\mathcal{R}_{u_5 i}(-x+a_i+\phi_i(t),\varepsilon),$$

is a weak asymptotic solution to (1.2) with initial data (1.3) for $t<T$ where T is the minimum of interaction time of the all different Riemann problems with

$$\phi_i(0)=e_i(0)=g_i(0)=h_i(0)=l_i(0)=\tilde{m}_i(0)=n_i(0)=0,\quad o_i(0)=q_i(0)=r_i(0)=s_i(0)=0,\quad i=1,...,m-1$$

where e_i, g_i, h_i, l_i, $\tilde{m}_i$, n_i, o_i, q_i, r_i, s_i, $\mathcal{R}_{u_3 i}$, $\mathcal{R}_{u_4 i}$ and $\mathcal{R}_{u_5 i}$ satisfy (3.15) with u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{51} and u_{52} replaced by u_{1i-1}^0, u_{1i}^0, u_{2i-1}^0, u_{2i}^0, u_{3i-1}^0, u_{3i}^0, u_{4i-1}^0, u_{4i}^0, u_{5i-1}^0 and u_{5i}^0 respectively and e, g, h, l, m, n, o, q, r and s are replaced by e_i, g_i, h_i, l_i, m_i, $\tilde{n}_i$, o_i, q_i, r_i and s_i respectively. So for every $\psi(x)\in\mathcal{D}(\mathbb{R})$ and $\varepsilon>0$, we have

$$\int \mathscr{L}_1[u_1]\psi(x)dx=O(\varepsilon)$$
$$\int \mathscr{L}_2[u_1,u_2]\psi(x)dx=O(\varepsilon)$$
$$\int \mathscr{L}_3[u_1,u_2,u_3]\psi(x)dx=O(\varepsilon)$$
$$\int \mathscr{L}_4[u_1,u_2,u_3,u_4]\psi(x)dx=O(\varepsilon)$$
$$\int \mathscr{L}_5[u_1,u_2,u_3,u_4,u_5]\psi(x)dx=O(\varepsilon)$$

and for initial conditions we have

$$\begin{aligned}&\int |u_1(x,0,\varepsilon)-u_1^0(x)|\psi(x)dx\\ &\le \int |u_1(x,0,\varepsilon)-u_{1\varepsilon}^0(x)|\psi(x)dx\\ &+\int |u_{1\varepsilon}^0(x)-u_1^0(x)|\psi(x)dx\\ &<O(\varepsilon)+C\varepsilon \quad \text{where C is max of } \psi\end{aligned}$$

and similar procedure can be applied for other initial functions. So we have weak asymptotic solution ($u_1(x,t,\varepsilon)$, $u_2(x,t,\varepsilon)$, $u_3(x,t,\varepsilon)$,$u_4(x,t,\varepsilon)$,$u_5(x,t,\varepsilon)$) for (2.1) with initial data u_1^0, u_2^0, u_3^0, u_4^0 and $u_5^0\in L_\beta(\mathbb{R})$ for $\beta\in[1,\infty)$. □

4. Conclusions

In this article, we considered a nonstrictly hyperbolic system of conservation laws and it is observed that one of the components of its solution contained a linear combination of Dirac measure and its first, second and third derivatives. Sufficient conditions are established in order to obtain a weak asymptotic solution. Then by exploiting Riemann type initial data, we constructed weak asymptotic solution for more general type initial data. In future, we would like to extend these type of solutions to different physical systems for more general initial data and also to work on the interaction of nonlinear waves using weak asymtotic method.

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