



Construction of interval-valued Pythagorean fuzzy Aczel-Alsina aggregation operators for decision making: a case study in emerging IT software company selection

TAPAN SENAPATI^{1,2}, ARUNODAYA RAJ MISHRA³, ABHIJIT SAHA⁴, VLADIMIR SIMIC^{5,*}, PRATIBHA RANI⁴ and RIFAQAT ALI⁶

¹Department of Mathematics, Padima Janakalyan Banipith, Kukrakhupi 721517, India

²School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

³Department of Mathematics, Government College Raigaon, Satna, MP 485441, India

⁴Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram, AP 522302, India

⁵Department of Transport and Traffic Engineering, University of Belgrade, Vojvode Stepe 305, 11010 Belgrade, Serbia

⁶Department of Mathematics, College of Science and Arts, Muhayil, King Khalid University, Abha 61413, Saudi Arabia

e-mail: vsima@sf.bg.ac.rs

MS received 10 February 2022; revised 20 July 2022; accepted 26 September 2022

Abstract. When there isn't enough evidence, it can be hard for decision-makers (DMs) to evaluate their evaluations of strategic decision issues in the right way. An expansion of Pythagorean fuzzy sets (PFSs), known as interval-valued PFSs (IVPFSs), can provide sufficient information space for DMs so that they can assess their evaluations using interval numbers. The purpose of this work is to examine the aggregation procedures used by IVPFSs using Aczel-Alsina operations. We first generalize the Aczel-Alsina t -norm and t -conorm to IVPF circumstances and introduce several unique IVPFS operations, such as Aczel-Alsina sum, Aczel-Alsina product, Aczel-Alsina scalar multiplication, and Aczel-Alsina exponentiation, that contribute to the emergence of many special IVPF aggregation operators, including the IVPF Aczel-Alsina weighted average (IVPFAAWA) operator, IVPF Aczel-Alsina order weighted average (IVPFAAOWA) operator, and IVPF Aczel-Alsina hybrid average (IVPFAAHA) operator. Furthermore, we define several features of the operators, illustrate them with a few specific examples, and investigate the relationships between the aforementioned operators. Additionally, we use these operators to develop a system for trying to manage multiple attribute decision-making (MADM) using IVPF data. Ultimately, a mathematical formulation involving the selection of an emerging IT software company is given to represent the decision steps of the recommended methodology. The outcome demonstrates the reasonableness and viability of the new methodology. We explore the effects of the parameter on decision-making results for a variety of values. A comparative study is also presented.

Keywords. MADM; IVPFEs; Aczel-Alsina operations; IVPF Aczel-Alsina average aggregation operators.

1. Introduction

Atanassov [1] developed the intuitionistic fuzzy set (IFS) by taking into consideration of the pairings of membership degree (MD) and non-membership degree (NMD) in such a way that their entire sum does not exceed one, which is a generalisation of the fuzzy set [2]. Following its launch, scientists have employed these speculations in various directions and realized that they are more productive in

dealing with the vulnerabilities throughout the investigation. Considering that the aforementioned hypotheses have already been effectively characterized, but sometimes, it struggles to deal with the circumstances by IFS. As an example, if a DM might take the MD and NMD of any component as 0.7 and 0.4, respectively, then obviously their total sum exceeds one, but the sum of their squares is less than one. Henceforth, in those circumstances, IFS has a type of inadequacy. To be able to address these challenges, the Pythagorean fuzzy set (PFS) [3, 4], an expansion of IFSs, has arisen as a powerful tool for portraying the

*For correspondence

vulnerability in the information. Henceforth, the PFS is more general compared to the IFS.

Leading to a shortage of existing knowledge, it could be challenging for DMs to characterize their feelings accurately with a crisp number in certain genuine decision-making situations, yet they can be addressed via an interval number within $[0,1]$. As a result, the concept of interval-valued PFSs (IVPFSs) is essential, as it allows MDs and NMDs to be authorized to a specific set with an interval value. Zhang [5] presented the notion of IVPFS. They should also satisfy the criterion that the squares of the upper limits of two intervals' are less than or equal to one [6]. Because of their amazing ability to deal with more faulty and ambiguous data and supervise sophisticated vulnerabilities in practice-oriented decision situations, IVPF sets have wider potential applications. According to this viewpoint, the utilization of the IVPFS hypothesis to MADM issues gets quite possibly the most encouraging directions in displaying unsure data for practical dynamic issues [7, 8]. IVPFSs allow the NMD and MD of a certain set to have an interval value within $[0, 1]$. Concurrently, IVPFSs are needed to fulfil the requirement that the sum of the square of both upper limits of the interval-valued MDs and NMDs be less than or equal to one.

A few valuable decision theories and strategies have been developed for managing MADM difficulties. For example, MADM with probabilities in an IVPF setting [9], IVPF Maclaurin symmetric mean operators in MADM difficulties [10], IVPF Frank power aggregation operators based on an isomorphic frank dual triple [11], methods for MADM with IVPF data [12], IVPF power average based MULTIMOORA technique for MADM issues [13], a novel outranking technique for MCDM with IVPF linguistic data [14], approaches to MAGDM based on induced IVPF Einstein hybrid aggregation operators [15], new operations and algorithms for IVPFSs [16, 17], a novel methodology for MADM concerns with IVPF linguistic data [18], IVPF dual Muirhead mean operators for MADM difficulties [19], some new generalized IVPF aggregation operators based on Einstein t -norms [20].

IVPF MADM has been widely employed in a variety of domains, for example, evaluation of solar photovoltaic technology development [21], bridge construction [22], green suppliers' selection [19], potential evaluation of emerging technology commercialization [23, 24], the configuration of a telecommunication network [25], sustainable supplier selection [26], car selection problem [27], risk assessment of technological innovation in high-tech companies [28, 29], providing financial support for the infrastructure development [30], treatment and management of health care waste [31].

We can see from these models and discussions that the majority of contemporary IVPF aggregation techniques are set up with the help of the algebraic product and algebraic sum of IVPFSs to express the aggregation technique. A

standard t -norm and t -conorm could be used to create a generalised union and intersection on IVPFSs. Aczel-Alsina [32] presented two new operations in 1982 named Aczel-Alsina t -norm (AA t -norm) and Aczel-Alsina t -conorm (AA t -conorm), which place a high premium on parameter adaptability. Recently, Senapati and his associates have opened new horizons in decision-making theories using the AA t -norms. They applied AA t -norms to decision-making difficulties under IFS [33], interval-valued IFS [34], hesitant fuzzy [35] and picture fuzzy [36] environments. To handle MADM issues including IVPF data, the motivation behind this paper is to produce some new aggregation operators dependent on AA t -norm and AA t -conorm. Besides, this research provides an adaptable and helpful way to manage different sorts of inclination data to adjust to the peculiarities of actual decision circumstances (figure 1).

We propose these sections in this paper: in section 2, we concisely discuss certain essential ideas of Aczel-Alsina triangular norms, IVPFSs, and a few working principles of IVPFEs. We suggest Aczel-Alsina operations for IVPFEs in section 3. In section 4, we propose certain IVPF aggregation operators by virtue of Aczel-Alsina operations, like the IVPFAAWA operator, IVPFAAOWA operator, and IVPFAAHA operator, and analyzed several attractive characteristics of the recommended operators. We determine the details of these new aggregation operators as well as specific situations. In section 5, we create a MADM strategy on the basis of the recommended operators under an induced IVPF situation. In section 6, a practical example is given to demonstrate the legitimacy and superiority of the recommended methodology. In section 7, we examine how a parameter influences decision outcomes. In section 8, then created a comparative study with the prevailing techniques. Finally, section 9 illustrates and discusses the conclusion and scope of future research.

2. Preliminaries

In the accompanying, we acquaint a few fundamental concepts closely tied with t -norm, t -conorm, AA t -norm, AA t -conorm and IVPFS.

2.1 t -norm, t -conorm, AA t -norm and AA t -conorm

Definition 1 [37] A t -norm is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1]^2 \rightarrow [0, 1]$ such that, for all $w, h, y \in [0, 1]$, the following axioms are satisfied:

- (i) $T(w, h) = T(h, w)$ (commutativity);
- (ii) $T(w, h) \leq T(w, y)$ if $h \leq y$ (monotonicity);
- (iii) $T(w, T(h, y)) = T(T(w, h), y)$ (associativity);
- (iv) $T(w, 1) = w$ (one identity).

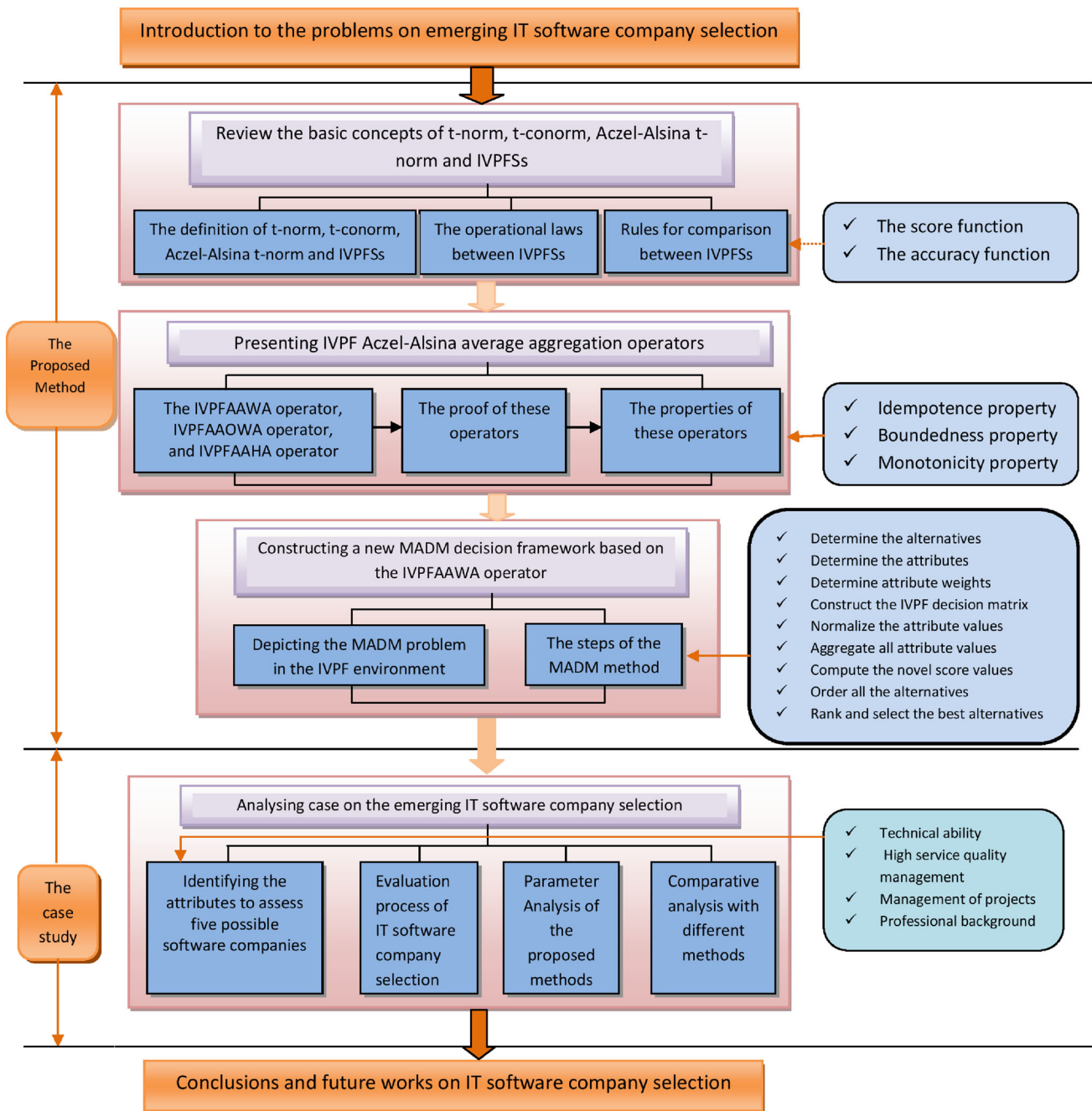


Figure 1. The framework of the study.

Example 1 Examples of popular t -norms:

- (i) $T_P(w, h) = w.h$ (product t -norm);
- (ii) $T_M(w, h) = \min(w, h)$ (minimum t -norm);
- (iii) $T_L(w, h) = \max(w + h - 1, 0)$ (Lukasiewicz t -norm);
- (iv) $T_D(w, h) = \begin{cases} w, & \text{if } h = 1 \\ h, & \text{if } w = 1 \\ 0, & \text{otherwise} \end{cases}$ (Drastic t -norm);

for all $w, h \in [0, 1]$.

Definition 2 [37] A t -conorm is a binary operation on the unit interval $[0, 1]$, i.e., a function $S : [0, 1]^2 \rightarrow [0, 1]$ such that, for all $w, h, y \in [0, 1]$, the following axioms are satisfied:

- (i) $S(w, h) = S(h, w)$ (commutativity);
- (ii) $S(w, h) \leq S(w, y)$ if $h \leq y$ (monotonicity);

- (iii) $S(w, S(h, y)) = S(S(w, h), y)$ (associativity);
- (iv) $S(w, 0) = w$ (one identity).

Example 2 Examples of popular t -conorms:

- (i) $S_P(w, h) = w + h - w.h$ (probabilistic sum);
- (ii) $S_M(w, h) = \max(w, h)$ (maximum t -conorm);
- (iii) $S_L(w, h) = \min(w + h, 1)$ (Lukasiewicz t -conorm);
- (iv) $S_D(w, h) = \begin{cases} w, & \text{if } h = 0 \\ h, & \text{if } w = 0 \\ 1, & \text{otherwise} \end{cases}$ (Drastic t -conorm);

for every $w, h \in [0, 1]$.

Additionally, it provided evidence that when T is a t -norm and S is a t -conorm, then $T(w, h) \leq \min\{w, h\}$ and $S(w, h) \geq \max\{w, h\}$ for all $w, h \in [0, 1]$ [37].

Definition 3 [32, 38](AA t -norm) In the early 1980s, Aczel-Alsina constructed the t -norm class within the context of analytic functions.

The class $(T_A^\zeta)_{\zeta \in [0, \infty]}$ of AA t -norms is portrayed through

$$T_A^\zeta(w, h) = \begin{cases} T_D(w, h), & \text{if } \zeta = 0 \\ \min(w, h), & \text{if } \zeta = \infty \\ e^{-((-\ln w)^\zeta + (-\ln h)^\zeta)^{1/\zeta}}, & \text{otherwise} \end{cases}$$

The class $(S_A^\zeta)_{\zeta \in [0, \infty]}$ of AA t -conorms is portrayed through

$$S_A^\zeta(w, h) = \begin{cases} S_D(w, h), & \text{if } \zeta = 0 \\ \max(w, h), & \text{if } \zeta = \infty \\ 1 - e^{-((-\ln(1-w))^\zeta + (-\ln(1-h))^\zeta)^{1/\zeta}}, & \text{otherwise} \end{cases}$$

Borderline Cases: $T_A^0 = T_D, T_A^1 = T_P, T_A^\infty = \min, S_A^0 = S_D, S_A^1 = S_P, S_A^\infty = \max$.

For each $\zeta \in [0, \infty]$, the t -norm T_A^ζ and t -conorm S_A^ζ are dual to each other. The class of AA t -norms increases strictly and the class of AA t -conorms decreases strictly.

2.2 IVPFSs

Zhang and Xu [4] presented the general mathematical form of PFS in the following way:

Definition 4 Let F be a fixed set, a PFS \tilde{Y} on F ascertained as

$$\tilde{Y} = \{ \langle \omega, \alpha_{\tilde{Y}}(\omega), \beta_{\tilde{Y}}(\omega) \rangle | \omega \in F \}$$

where MD $\alpha_{\tilde{Y}} : F \rightarrow [0, 1]$ and NMD $\beta_{\tilde{Y}} : F \rightarrow [0, 1]$ for all $\omega \in F$ fulfilling the condition $0 \leq \alpha_{\tilde{Y}}^2(\omega) + \beta_{\tilde{Y}}^2(\omega) \leq 1$, where the degree of indeterminacy

$$\pi_{\tilde{Y}}(\omega) = \sqrt{1 - \alpha_{\tilde{Y}}^2(\omega) - \beta_{\tilde{Y}}^2(\omega)}$$

Zhang [5] delivered the definition of IVPFSs by means of the following:

Definition 5 [5] Assume that $\Theta([0, 1])$ contains all closed subintervals of the unit interval $[0, 1]$. An IVPFS \tilde{Y} on F with MD $\tilde{\alpha}_{\tilde{Y}}(\omega) : F \rightarrow \Theta([0, 1])$ and NMD $\tilde{\beta}_{\tilde{Y}}(\omega) : F \rightarrow \Theta([0, 1])$ is assigned as

$$\tilde{Y} = \{ \langle \omega, \tilde{\alpha}_{\tilde{Y}}(\omega), \tilde{\beta}_{\tilde{Y}}(\omega) \rangle | \omega \in F \},$$

where $\tilde{\alpha}_{\tilde{Y}}(\omega) = [\alpha_{\tilde{Y}}^L(\omega), \alpha_{\tilde{Y}}^U(\omega)]$ and $\tilde{\beta}_{\tilde{Y}}(\omega) = [\beta_{\tilde{Y}}^L(\omega), \beta_{\tilde{Y}}^U(\omega)]$, for all $\omega \in F$, including the condition $0 \leq (\alpha_{\tilde{Y}}^U(\omega))^2 + (\beta_{\tilde{Y}}^U(\omega))^2 \leq 1$. $\pi_{\tilde{Y}}(\omega) = [\pi_{\tilde{Y}}^L(\omega), \pi_{\tilde{Y}}^U(\omega)]$ denotes the indeterminacy degree of element ω that belongs to \tilde{Y} , where

$$\pi_{\tilde{Y}}^L(\omega) = \sqrt{1 - (\alpha_{\tilde{Y}}^U(\omega))^2 - (\beta_{\tilde{Y}}^U(\omega))^2}$$

and

$$\pi_{\tilde{Y}}^U(\omega) = \sqrt{1 - (\alpha_{\tilde{Y}}^L(\omega))^2 - (\beta_{\tilde{Y}}^L(\omega))^2}$$

For benefit, we called $\tilde{Y} = \{ \langle \omega, [\alpha_{\tilde{Y}}^L(\omega), \alpha_{\tilde{Y}}^U(\omega)], [\beta_{\tilde{Y}}^L(\omega), \beta_{\tilde{Y}}^U(\omega)] \rangle | \omega \in F \}$ as IVPF element (IVPFE) defined by $\tilde{Y} = ([\alpha_{\tilde{Y}}^L, \alpha_{\tilde{Y}}^U], [\beta_{\tilde{Y}}^L, \beta_{\tilde{Y}}^U])$.

Liang *et al* [28] outlined score and accuracy function for contrasting two IVPFEs in this way:

Definition 6 [28] For an IVPFE $\tilde{Y} = ([\alpha_{\tilde{Y}}^L, \alpha_{\tilde{Y}}^U], [\beta_{\tilde{Y}}^L, \beta_{\tilde{Y}}^U])$, score function $\mathbb{Q}(\tilde{Y})$ and accuracy function $\mathbb{W}(\tilde{Y})$ can be computed as:

$$\mathbb{Q}(\tilde{Y}) = \frac{1}{2} [(\alpha_{\tilde{Y}}^L)^2 + (\alpha_{\tilde{Y}}^U)^2 - (\beta_{\tilde{Y}}^L)^2 - (\beta_{\tilde{Y}}^U)^2], \quad (1)$$

$$\mathbb{W}(\tilde{Y}) = \frac{1}{2} [(\alpha_{\tilde{Y}}^L)^2 + (\alpha_{\tilde{Y}}^U)^2 + (\beta_{\tilde{Y}}^L)^2 + (\beta_{\tilde{Y}}^U)^2], \quad (2)$$

where $\mathbb{Q}(\tilde{Y}) \in [-1, 1]$ and $\mathbb{W}(\tilde{Y}) \in [0, 1]$.

In this way, the ordering of two IVPFEs can be carried out dependent on score and accuracy function listed as follows.

Definition 7 [28] Assume that $\tilde{Y}_1 = ([\alpha_{\tilde{Y}_1}^L, \alpha_{\tilde{Y}_1}^U], [\beta_{\tilde{Y}_1}^L, \beta_{\tilde{Y}_1}^U])$ and $\tilde{Y}_2 = ([\alpha_{\tilde{Y}_2}^L, \alpha_{\tilde{Y}_2}^U], [\beta_{\tilde{Y}_2}^L, \beta_{\tilde{Y}_2}^U])$ are any two IVPFEs. Then:

- (1) if $\mathbb{Q}(\tilde{Y}_1) < \mathbb{Q}(\tilde{Y}_2)$ then $\tilde{Y}_1 \prec \tilde{Y}_2$,
- (2) if $\mathbb{Q}(\tilde{Y}_1) > \mathbb{Q}(\tilde{Y}_2)$ then $\tilde{Y}_1 \succ \tilde{Y}_2$,
- (3) if $\mathbb{Q}(\tilde{Y}_1) = \mathbb{Q}(\tilde{Y}_2)$ then

- (i) if $\mathbb{W}(\tilde{Y}_1) < \mathbb{W}(\tilde{Y}_2)$ then $\tilde{Y}_1 \prec \tilde{Y}_2$,
- (ii) if $\mathbb{W}(\tilde{Y}_1) > \mathbb{W}(\tilde{Y}_2)$ then $\tilde{Y}_1 \succ \tilde{Y}_2$,
- (iii) if $\mathbb{W}(\tilde{Y}_1) = \mathbb{W}(\tilde{Y}_2)$ then $\tilde{Y}_1 \sim \tilde{Y}_2$.

The fundamental operations of IVPFEs are constructed by Liang *et al* [28] and Zhang [5] in the following way.

Definition 8 Let $\tilde{Y} = ([\alpha_Y^L, \alpha_Y^U], [\beta_Y^L, \beta_Y^U])$, $\tilde{Y}_1 = ([\alpha_{Y_1}^L, \alpha_{Y_1}^U], [\beta_{Y_1}^L, \beta_{Y_1}^U])$ and $\tilde{Y}_2 = ([\alpha_{Y_2}^L, \alpha_{Y_2}^U], [\beta_{Y_2}^L, \beta_{Y_2}^U])$ be three IVPFEs, then:

(i) \oplus -union:

$$\tilde{Y}_1 \oplus \tilde{Y}_2 = \left(\left[\sqrt{(\alpha_{Y_1}^L)^2 + (\alpha_{Y_2}^L)^2 - (\alpha_{Y_1}^L)^2 (\alpha_{Y_2}^L)^2}, \sqrt{(\alpha_{Y_1}^U)^2 + (\alpha_{Y_2}^U)^2 - (\alpha_{Y_1}^U)^2 (\alpha_{Y_2}^U)^2} \right], \left[\beta_{Y_1}^L \beta_{Y_2}^L, \beta_{Y_1}^U \beta_{Y_2}^U \right] \right),$$

(ii) \otimes -intersection:

$$\tilde{Y}_1 \otimes \tilde{Y}_2 = \left(\left[\alpha_{Y_1}^L \alpha_{Y_2}^L, \alpha_{Y_1}^U \alpha_{Y_2}^U \right], \left[\sqrt{(\beta_{Y_1}^L)^2 + (\beta_{Y_2}^L)^2 - (\beta_{Y_1}^L)^2 (\beta_{Y_2}^L)^2}, \sqrt{(\beta_{Y_1}^U)^2 + (\beta_{Y_2}^U)^2 - (\beta_{Y_1}^U)^2 (\beta_{Y_2}^U)^2} \right] \right),$$

(iii) Multiplication:

$$\zeta \tilde{Y} = \left(\left[\sqrt{1 - (1 - (\alpha_Y^L)^2)^\zeta}, \sqrt{1 - (1 - (\alpha_Y^U)^2)^\zeta} \right], \left[(\beta_Y^L)^\zeta, (\beta_Y^U)^\zeta \right] \right), \zeta > 0, \tag{iii}$$

(iv) Exponentiation:

$$\tilde{Y}^\zeta = \left(\left[(\alpha_Y^L)^\zeta, (\alpha_Y^U)^\zeta \right], \left[\sqrt{1 - (1 - (\beta_Y^L)^2)^\zeta}, \sqrt{1 - (1 - (\beta_Y^U)^2)^\zeta} \right] \right), \zeta > 0,$$

(v) Complement:

$$\tilde{Y}^c = ([\beta_Y^L, \beta_Y^U], [\alpha_Y^L, \alpha_Y^U]).$$

3. Aczel-Alsina operations on IVPFEs

We explained Aczel-Alsina operations for IVPFEs in the context of AA t -norm and AA t -conorm.

Definition 9 Let $\tilde{Y} = ([\alpha_Y^L, \alpha_Y^U], [\beta_Y^L, \beta_Y^U])$, $\tilde{Y}_1 = ([\alpha_{Y_1}^L, \alpha_{Y_1}^U], [\beta_{Y_1}^L, \beta_{Y_1}^U])$ and $\tilde{Y}_2 = ([\alpha_{Y_2}^L, \alpha_{Y_2}^U], [\beta_{Y_2}^L, \beta_{Y_2}^U])$ be three IVPFEs, $\lambda \geq 1$ and $\zeta > 0$. Then, we describe the AA t -norm and t -conorm operations by means of the following:

(i)

$$\tilde{Y}_1 \oplus \tilde{Y}_2 = \left(\left[\sqrt{1 - e^{-((-\ln(1 - (\alpha_{Y_1}^L)^2))^\lambda + (-\ln(1 - (\alpha_{Y_2}^L)^2))^\lambda)^{1/\lambda}}, \sqrt{1 - e^{-((-\ln(1 - (\alpha_{Y_1}^U)^2))^\lambda + (-\ln(1 - (\alpha_{Y_2}^U)^2))^\lambda)^{1/\lambda}} \right], \left[\sqrt{e^{-((-\ln(\beta_{Y_1}^L)^2))^\lambda + (-\ln(\beta_{Y_2}^L)^2)^\lambda)^{1/\lambda}}, \sqrt{e^{-((-\ln(\beta_{Y_1}^U)^2))^\lambda + (-\ln(\beta_{Y_2}^U)^2)^\lambda)^{1/\lambda}} \right] \right),$$

(ii)

$$\tilde{Y}_1 \otimes \tilde{Y}_2 = \left(\left[\sqrt{e^{-((-\ln(\alpha_{Y_1}^L)^2))^\lambda + (-\ln(\alpha_{Y_2}^L)^2)^\lambda)^{1/\lambda}}, \sqrt{e^{-((-\ln(\alpha_{Y_1}^U)^2))^\lambda + (-\ln(\alpha_{Y_2}^U)^2)^\lambda)^{1/\lambda}} \right], \left[\sqrt{1 - e^{-((-\ln(1 - (\beta_{Y_1}^L)^2))^\lambda + (-\ln(1 - (\beta_{Y_2}^L)^2))^\lambda)^{1/\lambda}}, \sqrt{1 - e^{-((-\ln(1 - (\beta_{Y_1}^U)^2))^\lambda + (-\ln(1 - (\beta_{Y_2}^U)^2))^\lambda)^{1/\lambda}} \right] \right),$$

(iii)

$$\zeta \tilde{Y} = \left(\left[\sqrt{1 - e^{-((-\ln(1 - (\alpha_Y^L)^2))^\lambda)^{1/\lambda}}, \sqrt{1 - e^{-((-\ln(1 - (\alpha_Y^U)^2))^\lambda)^{1/\lambda}} \right], \left[\sqrt{e^{-((-\ln(\beta_Y^L)^2))^\lambda)^{1/\lambda}}, \sqrt{e^{-((-\ln(\beta_Y^U)^2))^\lambda)^{1/\lambda}} \right] \right),$$

(iv)

$$\tilde{Y}^\zeta = \left(\left[\sqrt{e^{-((-\ln(\alpha_Y^L)^2))^\lambda)^{1/\lambda}}, \sqrt{e^{-((-\ln(\alpha_Y^U)^2))^\lambda)^{1/\lambda}} \right], \left[\sqrt{1 - e^{-((-\ln(1 - (\beta_Y^L)^2))^\lambda)^{1/\lambda}}, \sqrt{1 - e^{-((-\ln(1 - (\beta_Y^U)^2))^\lambda)^{1/\lambda}} \right] \right),$$

Example 3 Let $\tilde{Y} = ([0.12, 0.19], [0.74, 0.84])$, $\tilde{Y}_1 = ([0.62, 0.67], [0.33, 0.40])$ and $\tilde{Y}_2 = ([0.51, 0.56], [0.42, 0.48])$ be three IVPFEs, then utilizing Aczel-Alsina operation in accordance with Definition 9 for $\lambda = 4$ and $\zeta = 7$, we get

(i)

$$\begin{aligned} \tilde{Y}_1 \oplus \tilde{Y}_2 &= \left(\left[\sqrt{1 - e^{-((-\ln(1-(0.62)^2))^4 + (-\ln(1-(0.51)^2))^4)^{1/4}}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-((-\ln(1-(0.67)^2))^4 + (-\ln(1-(0.56)^2))^4)^{1/4}}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{e^{-((-\ln(0.33)^2)^4 + (-\ln(0.42)^2)^4)^{1/4}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{e^{-((-\ln(0.40)^2)^4 + (-\ln(0.48)^2)^4)^{1/4}} \right] \right) \right) \\ &= ([0.628362, 0.679049], [0.301041, 0.368329]), \end{aligned}$$

(ii)

$$\begin{aligned} \tilde{Y}_1 \otimes \tilde{Y}_2 &= \left(\left[\sqrt{e^{-((-\ln(0.62)^2)^4 + (-\ln(0.51)^2)^4)^{1/4}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-((-\ln(0.67)^2)^4 + (-\ln(0.56)^2)^4)^{1/4}}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{1 - e^{-((-\ln(1-(0.33)^2))^4 + (-\ln(1-(0.42)^2))^4)^{1/4}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{1 - e^{-((-\ln(1-(0.40)^2))^4 + (-\ln(1-(0.48)^2))^4)^{1/4}} \right] \right) \right) \\ &= ([0.490393, 0.543179], [0.425614, 0.489478]), \end{aligned}$$

(iii)

$$\begin{aligned} 7\tilde{Y} &= \left(\left[\sqrt{1 - e^{-7(-\ln(1-(0.12)^2))^4}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-7(-\ln(1-(0.19)^2))^4}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{e^{-7(-\ln(0.74)^2)^4}}, \sqrt{e^{-7(-\ln(0.84)^2)^4}} \right] \right) \right) \\ &= ([0.152699, 0.240940], [0.612767, 0.753068]), \end{aligned}$$

(iv)

$$\begin{aligned} \tilde{Y}^7 &= \left(\left[\sqrt{e^{-7(-\ln(0.12)^2)^4}}, \sqrt{e^{-7(-\ln(0.19)^2)^4}} \right], \right. \\ &\quad \left[\sqrt{1 - e^{-7(-\ln(1-(0.74)^2))^4}}, \right. \\ &\quad \left. \left. \sqrt{1 - e^{-7(-\ln(1-(0.84)^2))^4}} \right] \right) \\ &= ([0.031785, 0.0671178], [0.851340, 0.929069]), \end{aligned}$$

Theorem 1 Let $\tilde{Y} = ([\alpha_Y^L, \alpha_Y^U], [\beta_Y^L, \beta_Y^U])$, $\tilde{Y}_1 = ([\alpha_{Y_1}^L, \alpha_{Y_1}^U], [\beta_{Y_1}^L, \beta_{Y_1}^U])$ and $\tilde{Y}_2 = ([\alpha_{Y_2}^L, \alpha_{Y_2}^U], [\beta_{Y_2}^L, \beta_{Y_2}^U])$ be three IVPFES, then we have

- (i) $\tilde{Y}_1 \oplus \tilde{Y}_2 = \tilde{Y}_2 \oplus \tilde{Y}_1$;
- (ii) $\tilde{Y}_1 \otimes \tilde{Y}_2 = \tilde{Y}_2 \otimes \tilde{Y}_1$;
- (iii) $\zeta(\tilde{Y}_1 \oplus \tilde{Y}_2) = \zeta\tilde{Y}_1 \oplus \zeta\tilde{Y}_2, \zeta > 0$;
- (iv) $(\zeta_1 + \zeta_2)\tilde{Y} = \zeta_1\tilde{Y} \oplus \zeta_2\tilde{Y}, \zeta_1, \zeta_2 > 0$;
- (v) $(\tilde{Y}_1 \otimes \tilde{Y}_2)^\zeta = \tilde{Y}_1^{\zeta} \otimes \tilde{Y}_2^{\zeta}, \zeta > 0$;
- (vi) $\tilde{Y}^{\zeta_1} \otimes \tilde{Y}^{\zeta_2} = \tilde{Y}^{\zeta_1 + \zeta_2}, \zeta_1, \zeta_2 > 0$.

Proof For the three PFEs \tilde{Y}, \tilde{Y}_1 and \tilde{Y}_2 , and $\zeta, \zeta_1, \zeta_2 > 0$, as stated in Definition 9, we can get

(i)

$$\begin{aligned} \tilde{Y}_1 \oplus \tilde{Y}_2 &= \left(\left[\sqrt{1 - e^{-((-\ln(1-(\alpha_{Y_1}^L)^2))^2 + (-\ln(1-(\alpha_{Y_2}^L)^2))^2)^{1/2}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-((-\ln(1-(\alpha_{Y_1}^U)^2))^2 + (-\ln(1-(\alpha_{Y_2}^U)^2))^2)^{1/2}}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{e^{-((-\ln(\beta_{Y_1}^L})^2)^2 + (-\ln(\beta_{Y_2}^L})^2)^2)^{1/2}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{e^{-((-\ln(\beta_{Y_1}^U})^2)^2 + (-\ln(\beta_{Y_2}^U})^2)^2)^{1/2}} \right] \right) \right) \\ &= \left(\left[\sqrt{1 - e^{-((-\ln(1-(\alpha_{Y_2}^L)^2))^2 + (-\ln(1-(\alpha_{Y_1}^L)^2))^2)^{1/2}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-((-\ln(1-(\alpha_{Y_2}^U)^2))^2 + (-\ln(1-(\alpha_{Y_1}^U)^2))^2)^{1/2}}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{e^{-((-\ln(\beta_{Y_2}^L})^2)^2 + (-\ln(\beta_{Y_1}^L})^2)^2)^{1/2}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{e^{-((-\ln(\beta_{Y_2}^U})^2)^2 + (-\ln(\beta_{Y_1}^U})^2)^2)^{1/2}} \right] \right) \right) \\ &= \tilde{Y}_2 \oplus \tilde{Y}_1 \end{aligned}$$

(ii) It is not complicated at all.

(iii) Let $t = \sqrt{1 - e^{-((-\ln(1-(\alpha_{Y_1}^L)^2))^2 + (-\ln(1-(\alpha_{Y_2}^L)^2))^2)^{1/\lambda}}$.
 Then $\ln(1 - t^2) = -((-\ln(1 - (\alpha_{Y_1}^L)^2))^2 + (-\ln(1 - (\alpha_{Y_2}^L)^2))^2)^{1/\lambda}$. Using this, we get

$$\begin{aligned} &\zeta(\tilde{Y}_1 \oplus \tilde{Y}_2) \\ &= \zeta\left(\left[\sqrt{1 - e^{-((-\ln(1-(\alpha_{Y_1}^L)^2))^2 + (-\ln(1-(\alpha_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-((-\ln(1-(\alpha_{Y_1}^U)^2))^2 + (-\ln(1-(\alpha_{Y_2}^U)^2))^2)^{1/\lambda}} \right], \right. \\ &\quad \left. \left[\sqrt{e^{-((-\ln(\beta_{Y_1}^L)^2))^2 + (-\ln(\beta_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-((-\ln(\beta_{Y_1}^U)^2))^2 + (-\ln(\beta_{Y_2}^U)^2))^2)^{1/\lambda}} \right] \right) \\ &= \left(\left[\sqrt{1 - e^{-\zeta((-\ln(1-(\alpha_{Y_1}^L)^2))^2 + (-\ln(1-(\alpha_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta((-\ln(1-(\alpha_{Y_1}^U)^2))^2 + (-\ln(1-(\alpha_{Y_2}^U)^2))^2)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{e^{-\zeta((-\ln(\beta_{Y_1}^L)^2))^2 + (-\ln(\beta_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \\ &\quad \left. \left. \sqrt{e^{-\zeta((-\ln(\beta_{Y_1}^U)^2))^2 + (-\ln(\beta_{Y_2}^U)^2))^2)^{1/\lambda}} \right] \right) \\ &= \left(\left[\sqrt{1 - e^{-\zeta(-\ln(1-(\alpha_{Y_1}^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta(-\ln(1-(\alpha_{Y_1}^U)^2))^2)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{e^{-\zeta(-\ln(\beta_{Y_1}^L)^2))^2)^{1/\lambda}}, \sqrt{e^{-\zeta(-\ln(\beta_{Y_1}^U)^2))^2)^{1/\lambda}} \right] \right) \\ &\oplus \left(\left[\sqrt{1 - e^{-\zeta(-\ln(1-(\alpha_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta(-\ln(1-(\alpha_{Y_2}^U)^2))^2)^{1/\lambda}} \right], \left[\sqrt{e^{-\zeta(-\ln(\beta_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-\zeta(-\ln(\beta_{Y_2}^U)^2))^2)^{1/\lambda}} \right] \right) = \zeta\tilde{Y}_1 \oplus \zeta\tilde{Y}_2 \end{aligned}$$

(iv)

$$\begin{aligned} \zeta_1\tilde{Y} \oplus \zeta_2\tilde{Y} &= \left(\left[\sqrt{1 - e^{-\zeta_1(-\ln(1-(\alpha_Y^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta_1(-\ln(1-(\alpha_Y^U)^2))^2)^{1/\lambda}} \right], \left[\sqrt{e^{-\zeta_1(-\ln(\beta_Y^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-\zeta_1(-\ln(\beta_Y^U)^2))^2)^{1/\lambda}} \right] \right) \oplus \left(\left[\sqrt{1 - e^{-\zeta_2(-\ln(1-(\alpha_Y^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta_2(-\ln(1-(\alpha_Y^U)^2))^2)^{1/\lambda}} \right], \right. \\ &\quad \left. \left[\sqrt{e^{-\zeta_2(-\ln(\beta_Y^L)^2))^2)^{1/\lambda}}, \sqrt{e^{-\zeta_2(-\ln(\beta_Y^U)^2))^2)^{1/\lambda}} \right] \right) \\ &= \left(\left[\sqrt{1 - e^{-((\zeta_1 + \zeta_2)(-\ln(1-(\alpha_Y^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-((\zeta_1 + \zeta_2)(-\ln(1-(\alpha_Y^U)^2))^2)^{1/\lambda}} \right], \right. \right. \\ &\quad \left. \left[\sqrt{e^{-((\zeta_1 + \zeta_2)(-\ln(\beta_Y^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-((\zeta_1 + \zeta_2)(-\ln(\beta_Y^U)^2))^2)^{1/\lambda}} \right] \right) = (\zeta_1 + \zeta_2)\tilde{Y}. \end{aligned}$$

(v)

$$\begin{aligned} (\tilde{Y}_1 \otimes \tilde{Y}_2)^\zeta &= \left(\left[\sqrt{e^{-\zeta((-\ln(\alpha_{Y_1}^L)^2))^2 + (-\ln(\alpha_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-\zeta((-\ln(\alpha_{Y_1}^U)^2))^2 + (-\ln(\alpha_{Y_2}^U)^2))^2)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{1 - e^{-\zeta((-\ln(1-(\beta_{Y_1}^L)^2))^2 + (-\ln(1-(\beta_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta((-\ln(1-(\beta_{Y_1}^U)^2))^2 + (-\ln(1-(\beta_{Y_2}^U)^2))^2)^{1/\lambda}} \right] \right)^\zeta \\ &= \left(\left[\sqrt{e^{-\zeta((-\ln(\alpha_{Y_1}^L)^2))^2 + (-\ln(\alpha_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-\zeta((-\ln(\alpha_{Y_1}^U)^2))^2 + (-\ln(\alpha_{Y_2}^U)^2))^2)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{1 - e^{-\zeta((-\ln(1-(\beta_{Y_1}^L)^2))^2 + (-\ln(1-(\beta_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta((-\ln(1-(\beta_{Y_1}^U)^2))^2 + (-\ln(1-(\beta_{Y_2}^U)^2))^2)^{1/\lambda}} \right] \right) \\ &= \left(\left[\sqrt{e^{-\zeta(-\ln(\alpha_{Y_1}^L)^2))^2)^{1/\lambda}}, \sqrt{e^{-\zeta(-\ln(\alpha_{Y_1}^U)^2))^2)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{1 - e^{-\zeta(-\ln(1-(\beta_{Y_1}^L)^2))^2)^{1/\lambda}}, \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta(-\ln(1-(\beta_{Y_1}^U)^2))^2)^{1/\lambda}} \right] \right) \\ &\oplus \left(\left[\sqrt{e^{-\zeta(-\ln(\alpha_{Y_2}^L)^2))^2)^{1/\lambda}}, \sqrt{e^{-\zeta(-\ln(\alpha_{Y_2}^U)^2))^2)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{1 - e^{-\zeta(-\ln(1-(\beta_{Y_2}^L)^2))^2)^{1/\lambda}}, \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\zeta(-\ln(1-(\beta_{Y_2}^U)^2))^2)^{1/\lambda}} \right] \right) \\ &= \tilde{Y}_1^\zeta \otimes \tilde{Y}_2^\zeta. \end{aligned}$$

(vi)

$$\begin{aligned} \tilde{Y}^{\zeta_1} \otimes \tilde{Y}^{\zeta_2} &= \left(\left[\sqrt{e^{-(\zeta_1(-\ln(\alpha_Y^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-(\zeta_1(-\ln(\alpha_Y^U)^2)^\lambda)^{1/\lambda}}}, \left[\sqrt{1 - e^{-(\zeta_1(-\ln(1-(\beta_Y^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{1 - e^{-(\zeta_1(-\ln(1-(\beta_Y^U)^2)^\lambda)^{1/\lambda}}}, \right] \right] \right) \\ &\otimes \left(\left[\sqrt{e^{-(\zeta_2(-\ln(\alpha_Y^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-(\zeta_2(-\ln(\alpha_Y^U)^2)^\lambda)^{1/\lambda}}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{1 - e^{-(\zeta_2(-\ln(1-(\beta_Y^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{1 - e^{-(\zeta_2(-\ln(1-(\beta_Y^U)^2)^\lambda)^{1/\lambda}}}, \right] \right] \right) \\ &= \left(\left[\sqrt{e^{-((\zeta_1+\zeta_2)(-\ln(\alpha_Y^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-((\zeta_1+\zeta_2)(-\ln(\alpha_Y^U)^2)^\lambda)^{1/\lambda}}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{1 - e^{-((\zeta_1+\zeta_2)(-\ln(1-(\beta_Y^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{1 - e^{-((\zeta_1+\zeta_2)(-\ln(1-(\beta_Y^U)^2)^\lambda)^{1/\lambda}}}, \right] \right] \right) \\ &= \tilde{Y}^{(\zeta_1+\zeta_2)}. \end{aligned}$$

4. IVPF Aczel-Alsina average aggregation operators

In this section, we define three IVPF average aggregation operators through using Aczel-Alsina operations.

Definition 10 Let $\tilde{Y}_\xi = ([\alpha_{Y_\xi}^L, \alpha_{Y_\xi}^U], [\beta_{Y_\xi}^L, \beta_{Y_\xi}^U])$ ($\xi = 1, 2, \dots, \psi$) be a collection of IVPFes. An IVPF Aczel-Alsina weighted average (IVPFAAWA) operator is a mapping $IVPFAAWA : IVPFE^\psi \rightarrow IVPFE$ such that

$$\begin{aligned} IVPFAAWA_\varsigma(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_\psi) &= \bigoplus_{\xi=1}^\psi (\varsigma_\xi \tilde{Y}_\xi) \\ &= \varsigma_1 \tilde{Y}_1 \oplus \varsigma_2 \tilde{Y}_2 \oplus \dots \oplus \varsigma_\psi \tilde{Y}_\psi \end{aligned}$$

where $\varsigma = (\varsigma_1, \varsigma_2, \dots, \varsigma_\psi)^T$ be the weight vector of \tilde{Y}_ξ ($\xi = 1, 2, \dots, \psi$) with $\varsigma_Y > 0$ and $\sum_{\xi=1}^\psi \varsigma_\xi = 1$.

In view of the Aczel-Alsina operation laws from Theorem 1, we can determine the accompanying Theorem 2.

Theorem 2 Let $\tilde{Y}_\xi = ([\alpha_{Y_\xi}^L, \alpha_{Y_\xi}^U], [\beta_{Y_\xi}^L, \beta_{Y_\xi}^U])$ ($\xi = 1, 2, \dots, \psi$) be a collection of IVPFes, then their aggregated value by employing the IVPFAAWA operator is also an IVPFE, and

$$\begin{aligned} IVPFAAWA_\varsigma(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_\psi) &= \bigoplus_{\xi=1}^\psi (\varsigma_\xi \tilde{Y}_\xi) \\ &= \left(\left[\sqrt{1 - e^{-\left(\sum_{\xi=1}^\psi \varsigma_\xi (-\ln(1-(\alpha_{Y_\xi}^L)^2)^\lambda)^{1/\lambda}\right)}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\left(\sum_{\xi=1}^\psi \varsigma_\xi (-\ln(1-(\alpha_{Y_\xi}^U)^2)^\lambda)^{1/\lambda}\right)}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{e^{-\left(\sum_{\xi=1}^\psi \varsigma_\xi (-\ln(\beta_{Y_\xi}^L)^2)^\lambda)^{1/\lambda}}}, \sqrt{e^{-\left(\sum_{\xi=1}^\psi \varsigma_\xi (-\ln(\beta_{Y_\xi}^U)^2)^\lambda)^{1/\lambda}}}, \right] \right] \right) \end{aligned} \tag{3}$$

where $\varsigma = (\varsigma_1, \varsigma_2, \dots, \varsigma_\psi)$ be the weight vector of \tilde{Y}_ξ ($\xi = 1, 2, \dots, \psi$), and $\varsigma_\xi > 0$, $\sum_{\xi=1}^\psi \varsigma_\xi = 1$, $\lambda > 0$.

Proof Using the process of mathematical induction, we can illustrate Theorem 2 as follows:

(i) When $\psi = 2$, on the basis of Aczel-Alsina operations of IVPFes, we get

$$\begin{aligned} \varsigma_1 \tilde{Y}_1 &= \left(\left[\sqrt{1 - e^{-(\varsigma_1(-\ln(1-(\alpha_{Y_1}^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-(\varsigma_1(-\ln(1-(\alpha_{Y_1}^U)^2)^\lambda)^{1/\lambda}}}, \left[\sqrt{e^{-(\varsigma_1(-\ln(\beta_{Y_1}^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{e^{-(\varsigma_1(-\ln(\beta_{Y_1}^U)^2)^\lambda)^{1/\lambda}}}, \right] \right] \right) \\ \varsigma_2 \tilde{Y}_2 &= \left(\left[\sqrt{1 - e^{-(\varsigma_2(-\ln(1-(\alpha_{Y_2}^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-(\varsigma_2(-\ln(1-(\alpha_{Y_2}^U)^2)^\lambda)^{1/\lambda}}}, \left[\sqrt{e^{-(\varsigma_2(-\ln(\beta_{Y_2}^L)^2)^\lambda)^{1/\lambda}}}, \right. \right. \right. \\ &\quad \left. \left. \left. \sqrt{e^{-(\varsigma_2(-\ln(\beta_{Y_2}^U)^2)^\lambda)^{1/\lambda}}}, \right] \right] \right). \end{aligned}$$

Based on Definition 9, we obtain

□

$$\begin{aligned}
 IVPFAAWA_{\zeta}(\tilde{Y}_1, \tilde{Y}_2) &= \zeta_1 \tilde{Y}_1 \oplus \zeta_2 \tilde{Y}_2 \\
 &= \left(\left[\sqrt{1 - e^{-\left(\zeta_1(-\ln(1-(\alpha_{\tilde{Y}_1}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{1 - e^{-\left(\zeta_1(-\ln(1-(\alpha_{\tilde{Y}_1}^U)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \left[\sqrt{e^{-\left(\zeta_1(-\ln(\beta_{\tilde{Y}_1}^L)^2)\right)^{1/2}}}, \sqrt{e^{-\left(\zeta_1(-\ln(\beta_{\tilde{Y}_1}^U)^2)\right)^{1/2}}} \right] \right] \right) \\
 &\oplus \left(\left[\sqrt{1 - e^{-\left(\zeta_2(-\ln(1-(\alpha_{\tilde{Y}_2}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{1 - e^{-\left(\zeta_2(-\ln(1-(\alpha_{\tilde{Y}_2}^U)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \left[\sqrt{e^{-\left(\zeta_2(-\ln(\beta_{\tilde{Y}_2}^L)^2)\right)^{1/2}}}, \sqrt{e^{-\left(\zeta_2(-\ln(\beta_{\tilde{Y}_2}^U)^2)\right)^{1/2}}} \right] \right] \right) \\
 &= \left(\left[\sqrt{1 - e^{-\left(\zeta_1(-\ln(1-(\alpha_{\tilde{Y}_1}^L)^2)\right)^{1/2} + \zeta_2(-\ln(1-(\alpha_{\tilde{Y}_2}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{1 - e^{-\left(\zeta_1(-\ln(1-(\alpha_{\tilde{Y}_1}^U)^2)\right)^{1/2} + \zeta_2(-\ln(1-(\alpha_{\tilde{Y}_2}^U)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \left[\sqrt{e^{-\left(\zeta_1(-\ln(\beta_{\tilde{Y}_1}^L)^2)\right)^{1/2} + \zeta_2(-\ln(\beta_{\tilde{Y}_2}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{e^{-\left(\zeta_1(-\ln(\beta_{\tilde{Y}_1}^U)^2)\right)^{1/2} + \zeta_2(-\ln(\beta_{\tilde{Y}_2}^U)^2)\right)^{1/2}} \right] \right) \\
 &= \left(\left[\sqrt{1 - e^{-\left(\sum_{\zeta=1}^2 \zeta_{\zeta}(-\ln(1-(\alpha_{\tilde{Y}_{\zeta}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{1 - e^{-\left(\sum_{\zeta=1}^2 \zeta_{\zeta}(-\ln(1-(\alpha_{\tilde{Y}_{\zeta}}^U)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \left[\sqrt{e^{-\left(\sum_{\zeta=1}^2 \zeta_{\zeta}(-\ln(\beta_{\tilde{Y}_{\zeta}}^L)^2)\right)^{1/2}}}, \sqrt{e^{-\left(\sum_{\zeta=1}^2 \zeta_{\zeta}(-\ln(\beta_{\tilde{Y}_{\zeta}}^U)^2)\right)^{1/2}}} \right] \right] \right).
 \end{aligned}$$

Hence, (3) is right for $\psi = 2$.

(ii) Suppose that (3) is valid for $\psi = k$, then we have

$$\begin{aligned}
 IVPFAAWA_{\zeta}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_k) &= \bigoplus_{\zeta=1}^k (\zeta_{\zeta} \tilde{Y}_{\zeta}) \\
 &= \left(\left[\sqrt{1 - e^{-\left(\sum_{\zeta=1}^k \zeta_{\zeta}(-\ln(1-(\alpha_{\tilde{Y}_{\zeta}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{1 - e^{-\left(\sum_{\zeta=1}^k \zeta_{\zeta}(-\ln(1-(\alpha_{\tilde{Y}_{\zeta}}^U)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \left[\sqrt{e^{-\left(\sum_{\zeta=1}^k \zeta_{\zeta}(-\ln(\beta_{\tilde{Y}_{\zeta}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{e^{-\left(\sum_{\zeta=1}^k \zeta_{\zeta}(-\ln(\beta_{\tilde{Y}_{\zeta}}^U)^2)\right)^{1/2}}} \right] \right] \right).
 \end{aligned}$$

Now for $\psi = k + 1$, then

$$\begin{aligned}
 IVPFAAWA_{\zeta}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_k, \tilde{Y}_{k+1}) &= \bigoplus_{\zeta=1}^k (\zeta_{\zeta} \tilde{Y}_{\zeta}) \oplus (\zeta_{k+1} \tilde{Y}_{k+1}) \\
 &= \left(\left[\sqrt{1 - e^{-\left(\sum_{\zeta=1}^k \zeta_{\zeta}(-\ln(1-(\alpha_{\tilde{Y}_{\zeta}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{1 - e^{-\left(\sum_{\zeta=1}^k \zeta_{\zeta}(-\ln(1-(\alpha_{\tilde{Y}_{\zeta}}^U)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \left[\sqrt{e^{-\left(\sum_{\zeta=1}^k \zeta_{\zeta}(-\ln(\beta_{\tilde{Y}_{\zeta}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{e^{-\left(\sum_{\zeta=1}^k \zeta_{\zeta}(-\ln(\beta_{\tilde{Y}_{\zeta}}^U)^2)\right)^{1/2}}} \right] \right] \right) \\
 &\oplus \left(\left[\sqrt{1 - e^{-\left(\zeta_{k+1}(-\ln(1-(\alpha_{\tilde{Y}_{k+1}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{1 - e^{-\left(\zeta_{k+1}(-\ln(1-(\alpha_{\tilde{Y}_{k+1}}^U)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \left[\sqrt{e^{-\left(\zeta_{k+1}(-\ln(\beta_{\tilde{Y}_{k+1}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{e^{-\left(\zeta_{k+1}(-\ln(\beta_{\tilde{Y}_{k+1}}^U)^2)\right)^{1/2}}} \right] \right] \right) \\
 &= \left(\left[\sqrt{1 - e^{-\left(\sum_{\zeta=1}^{k+1} \zeta_{\zeta}(-\ln(1-(\alpha_{\tilde{Y}_{\zeta}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{1 - e^{-\left(\sum_{\zeta=1}^{k+1} \zeta_{\zeta}(-\ln(1-(\alpha_{\tilde{Y}_{\zeta}}^U)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \left[\sqrt{e^{-\left(\sum_{\zeta=1}^{k+1} \zeta_{\zeta}(-\ln(\beta_{\tilde{Y}_{\zeta}}^L)^2)\right)^{1/2}}}, \right. \right. \\
 &\quad \left. \left. \sqrt{e^{-\left(\sum_{\zeta=1}^{k+1} \zeta_{\zeta}(-\ln(\beta_{\tilde{Y}_{\zeta}}^U)^2)\right)^{1/2}}} \right] \right] \right).
 \end{aligned}$$

Thus, (3) is true for $\psi = k + 1$.

Therefore, from (i) and (ii), it must be concluded that (3) are true for any ψ . \square

By employing the IVPFAAWA operator, it is simple to demonstrate the subsequent properties.

Theorem 3 (Idempotency) If $\tilde{Y}_{\zeta} = ([\alpha_{\tilde{Y}_{\zeta}}^L, \alpha_{\tilde{Y}_{\zeta}}^U], [\beta_{\tilde{Y}_{\zeta}}^L, \beta_{\tilde{Y}_{\zeta}}^U])$ ($\zeta = 1, 2, \dots, \psi$) is a collection of IVPFEs that are all identical, i.e., $\tilde{Y}_{\zeta} = \tilde{Y}$ for every ζ , then

$$IVPFAAWA_{\zeta}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) = \tilde{Y}.$$

Proof Since $\tilde{Y}_{\zeta} = ([\alpha_{\tilde{Y}_{\zeta}}^L, \alpha_{\tilde{Y}_{\zeta}}^U], [\beta_{\tilde{Y}_{\zeta}}^L, \beta_{\tilde{Y}_{\zeta}}^U]) = \tilde{Y}$ ($\zeta = 1, 2, \dots, \psi$), then we have by equation (3),

$$\begin{aligned} IVPFAAOWA_{\varpi}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) &= \bigoplus_{\xi=1}^{\psi} (\varpi_{\xi} \tilde{Y}_{\varrho(\xi)}) \\ &= \varpi_1 \tilde{Y}_{\varrho(1)} \oplus \varpi_2 \tilde{Y}_{\varrho(2)} \oplus \dots \oplus \varpi_{\psi} \tilde{Y}_{\varrho(\psi)} \end{aligned}$$

where $(\varrho(1), \varrho(2), \dots, \varrho(\psi))$ are the permutation of $(\xi = 1, 2, \dots, \psi)$, with the property that $\tilde{Y}_{\varrho(\xi-1)} \geq \tilde{Y}_{\varrho(\xi)}$ for all $\xi = 1, 2, \dots, \psi$.

The succeeding theorem is developed on the basis of the Aczel-Alsina product operation on IVPFES.

Theorem 6 Let $\tilde{Y}_{\xi} = ([\alpha_{\tilde{Y}_{\xi}}^L, \alpha_{\tilde{Y}_{\xi}}^U], [\beta_{\tilde{Y}_{\xi}}^L, \beta_{\tilde{Y}_{\xi}}^U])$ ($\xi = 1, 2, \dots, \psi$) denotes the set of IVPFES. A ψ -dimension IVPF Aczel-Alsina ordered weighted average (IVPFAAOWA) operator is a function $IVPFAAOWA : IVPFE^{\psi} \rightarrow IVPFE$ with related vector $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_{\psi})^T$ such that $\varpi_{\xi} > 0$, and $\sum_{\xi=1}^{\psi} \varpi_{\xi} = 1$. Then,

$$\begin{aligned} IVPFAAOWA_{\varpi}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) &= \bigoplus_{\xi=1}^{\psi} (\varpi_{\xi} \tilde{Y}_{\varrho(\xi)}) \\ &= \left(\left[\sqrt{1 - e^{-\left(\sum_{\xi=1}^{\psi} \varpi_{\xi} (-\ln(1 - (\alpha_{\tilde{Y}_{\varrho(\xi)}}^L)^2))\right)^{\lambda}}}, \right. \right. \\ &\quad \left. \left. \sqrt{1 - e^{-\left(\sum_{\xi=1}^{\psi} \varpi_{\xi} (-\ln(1 - (\alpha_{\tilde{Y}_{\varrho(\xi)}}^U)^2))\right)^{\lambda}}}, \right. \right. \\ &\quad \left. \left. \left[\sqrt{e^{-\left(\sum_{\xi=1}^{\psi} \varpi_{\xi} (-\ln(\beta_{\tilde{Y}_{\varrho(\xi)}}^L)^2)\right)^{\lambda}}}, \right. \right. \\ &\quad \left. \left. \sqrt{e^{-\left(\sum_{\xi=1}^{\psi} \varpi_{\xi} (-\ln(\beta_{\tilde{Y}_{\varrho(\xi)}}^U)^2)\right)^{\lambda}}} \right] \right)^{1/\lambda} \end{aligned}$$

where $(\varrho(1), \varrho(2), \dots, \varrho(\psi))$ are the permutation of $(\xi = 1, 2, \dots, \psi)$, for which $\tilde{Y}_{\varrho(\xi-1)} \geq \tilde{Y}_{\varrho(\xi)}$ for every $\xi = 1, 2, \dots, \psi$.

The following properties of the IVPFAAOWA operator can readily be demonstrated.

Theorem 7 (Idempotency) If all $\tilde{Y}_{\xi} = ([\alpha_{\tilde{Y}_{\xi}}^L, \alpha_{\tilde{Y}_{\xi}}^U], [\beta_{\tilde{Y}_{\xi}}^L, \beta_{\tilde{Y}_{\xi}}^U])$ ($\xi = 1, 2, \dots, \psi$) are identical, i.e. $\tilde{Y}_{\xi} = \tilde{Y}$ for every ξ , then $IVPFAAOWA_{\varpi}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) = \tilde{Y}$.

Theorem 8 (Boundedness) Let $\tilde{Y}_{\xi} = ([\alpha_{\tilde{Y}_{\xi}}^L, \alpha_{\tilde{Y}_{\xi}}^U], [\beta_{\tilde{Y}_{\xi}}^L, \beta_{\tilde{Y}_{\xi}}^U])$ ($\xi = 1, 2, \dots, \psi$) denotes the set of IVPFES. Assume $\tilde{Y}^- = \min_{\xi} \tilde{Y}_{\xi}$, and $\tilde{Y}^+ = \max_{\xi} \tilde{Y}_{\xi}$. Subsequently

$$\tilde{Y}^- \leq IVPFAAOWA_{\varpi}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) \leq \tilde{Y}^+$$

Theorem 9 (Monotonicity) Let \tilde{Y}_{ξ} and \tilde{Y}'_{ξ} ($\xi = 1, 2, \dots, \psi$) be two sets of IVPFES, if $\tilde{Y}_{\xi} \leq \tilde{Y}'_{\xi}$ for all ξ , then

$$\begin{aligned} IVPFAAOWA_{\varpi}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) \\ \leq IVPFAAOWA_{\varpi}(\tilde{Y}'_1, \tilde{Y}'_2, \dots, \tilde{Y}'_{\psi}). \end{aligned}$$

Theorem 10 (Commutativity) Let \tilde{Y}_{ξ} and \tilde{Y}'_{ξ} ($\xi = 1, 2, \dots, \psi$) be two IVPFES, then $IVPFAAOWA_{\varpi}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) = IVPFAAOWA_{\varpi}(\tilde{Y}'_1, \tilde{Y}'_2, \dots, \tilde{Y}'_{\psi})$ where \tilde{Y}'_{ξ} is any permutation of \tilde{Y}_{ξ} ($\xi = 1, 2, \dots, \psi$).

We can see in Definitions 10 and 11 that the IVPFAAWA operator only weights the IVPF values, whereas the IVPFAAOWA operator only weights the ordered positions of the IVPF values, not the weights of the IVPF values itself. Weights expressed in both the operators IVPFAAWA and IVPFAAOWA have been in various situations in this way. They are, however, regarded as simply one of them. We introduce the IVPF Aczel-Alsina hybrid averaging (IVPFAAHA) operator to avoid this flaw.

Definition 12 Assume that $\tilde{Y}_{\xi} = ([\alpha_{\tilde{Y}_{\xi}}^L, \alpha_{\tilde{Y}_{\xi}}^U], [\beta_{\tilde{Y}_{\xi}}^L, \beta_{\tilde{Y}_{\xi}}^U])$ ($\xi = 1, 2, \dots, \psi$) is a collection of IVPFES. A ψ -dimension IVPFAAHA operator is a mapping $IVPFAAHA : IVPFE^{\psi} \rightarrow IVPFE$, with related weight vector $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_{\psi})^T$ such that $\varpi_{\xi} > 0$, and $\sum_{\xi=1}^{\psi} \varpi_{\xi} = 1$. Therefore,

$$\begin{aligned} IVPFAAHA_{\varpi}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) &= \bigoplus_{\xi=1}^{\psi} (\varpi_{\xi} \dot{\tilde{Y}}_{\varrho(\xi)}) \\ &= \varpi_1 \dot{\tilde{Y}}_{\varrho(1)} \oplus \varpi_2 \dot{\tilde{Y}}_{\varrho(2)} \oplus \dots \oplus \varpi_{\psi} \dot{\tilde{Y}}_{\varrho(\psi)} \end{aligned}$$

where $\dot{\tilde{Y}}_{\varrho(\xi)}$ is the ξ -th largest weighted IVPF values $\dot{\tilde{Y}}_{\xi}$ ($\dot{\tilde{Y}}_{\xi} = \varpi_{\xi} \tilde{Y}_{\xi}$, $\xi = 1, 2, \dots, \psi$), and $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_{\psi})^T$ represents weight vector of \tilde{Y}_{ξ} with $\varpi_{\xi} > 0$ and $\sum_{\xi=1}^{\psi} \varpi_{\xi} = 1$, where ψ is balancing constant.

Specifically, if $\varpi = (1/\psi, 1/\psi, \dots, 1/\psi)^T$, the IVPFAAHA operator becomes an IVPFAAWA operator and if $\varpi = (1/\psi, 1/\psi, \dots, 1/\psi)^T$, the IVPFAAHA operator becomes an IVPFAAOWA operator.

We can prove the following Theorem 11 by using Aczel-Alsina sum operations of the IVPFES.

Theorem 11 Let $\tilde{Y}_{\xi} = ([\alpha_{\tilde{Y}_{\xi}}^L, \alpha_{\tilde{Y}_{\xi}}^U], [\beta_{\tilde{Y}_{\xi}}^L, \beta_{\tilde{Y}_{\xi}}^U])$ ($\xi = 1, 2, \dots, \psi$) be a collection of IVPFES. A ψ -dimension IVPFAAHA operator is a mapping $IVPFAAHA : IVPFE^{\psi} \rightarrow IVPFE$, with related weight vector $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_{\psi})^T$ such that $\varpi_{\xi} > 0$, and

$\sum_{\xi=1}^{\psi} \varpi_{\xi} = 1$. Therefore, IVPFAAHA operator can be evaluated as

$$\begin{aligned} \text{IVPFAAHA}_{\xi, \varpi}(\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\psi}) &= \bigoplus_{\xi=1}^{\psi} (\varpi_{\xi} \dot{Y}_{\varrho(\xi)}) \\ &= \left(\left[\sqrt{1 - e^{-\left(\sum_{\xi=1}^{\psi} \varpi_{\xi} (-\ln(1 - (\alpha_{Y_{\varrho(\xi)}}^L)^2)\right)^{\lambda}}}\right]^{1/\lambda}, \right. \\ &\quad \left. \sqrt{1 - e^{-\left(\sum_{\xi=1}^{\psi} \varpi_{\xi} (-\ln(1 - (\alpha_{Y_{\varrho(\xi)}}^U)^2)\right)^{\lambda}}}\right]^{1/\lambda}, \right. \\ &\quad \left[\sqrt{e^{-\left(\sum_{\xi=1}^{\psi} \varpi_{\xi} (-\ln(\beta_{Y_{\varrho(\xi)}}^L)^2)\right)^{\lambda}}}, \right. \\ &\quad \left. \sqrt{e^{-\left(\sum_{\xi=1}^{\psi} \varpi_{\xi} (-\ln(\beta_{Y_{\varrho(\xi)}}^U)^2)\right)^{\lambda}}}\right]^{1/\lambda} \bigg) \end{aligned}$$

Indicate a discrete set of alternatives by $\aleph = \{\aleph_1, \aleph_2, \dots, \aleph_u\}$ and the set of attributes by $\chi = \{\chi_1, \chi_2, \dots, \chi_{\psi}\}$. Let $\varsigma = (\varsigma_1, \varsigma_2, \dots, \varsigma_{\psi})^T$ be the weight vector of attributes, fulfilling $\varsigma_{\xi} > 0$ and $\sum_{\xi=1}^{\psi} \varsigma_{\xi} = 1$. We denote the preference values of every alternative \aleph_g with regard to the criterion χ_{ξ} by an IVPFE $\tilde{\varphi}_{\eta\xi} = ([\alpha_{Y_{\eta\xi}}^L, \alpha_{Y_{\eta\xi}}^U], [\beta_{Y_{\eta\xi}}^L, \beta_{Y_{\eta\xi}}^U])$, where $[\alpha_{Y_{\eta\xi}}^L, \alpha_{Y_{\eta\xi}}^U]$ states the positive MD that DM considers what the alternative \aleph_g should satisfy the criteria χ_{ξ} , and $[\beta_{Y_{\eta\xi}}^L, \beta_{Y_{\eta\xi}}^U]$ indicates the uncertain degree that DM considers what the alternative \aleph_g should not fulfill the criteria χ_{ξ} , where $[\alpha_{Y_{\eta\xi}}^L, \alpha_{Y_{\eta\xi}}^U] \subset D[0, 1]$, $[\beta_{Y_{\eta\xi}}^L, \beta_{Y_{\eta\xi}}^U] \subset D[0, 1]$ and $0 \leq (\alpha_{Y_{\eta\xi}}^U)^2 + (\beta_{Y_{\eta\xi}}^U)^2 \leq 1$, ($\eta = 1, 2, \dots, \phi$). Therefore we are able to elicit an IVPF decision matrix $R = (\tilde{\varphi}_{\eta\xi})_{\phi \times \psi}$ in the following form:

$$R = (\tilde{\varphi}_{\eta\xi})_{\phi \times \psi} \begin{matrix} & \chi_1 & \chi_2 & \dots & \chi_{\psi} \\ \aleph_1 & ([\alpha_{Y_{11}}^L, \alpha_{Y_{11}}^U], [\beta_{Y_{11}}^L, \beta_{Y_{11}}^U]) & ([\alpha_{Y_{12}}^L, \alpha_{Y_{12}}^U], [\beta_{Y_{12}}^L, \beta_{Y_{12}}^U]) & \dots & ([\alpha_{Y_{1\psi}}^L, \alpha_{Y_{1\psi}}^U], [\beta_{Y_{1\psi}}^L, \beta_{Y_{1\psi}}^U]) \\ \aleph_2 & ([\alpha_{Y_{21}}^L, \alpha_{Y_{21}}^U], [\beta_{Y_{21}}^L, \beta_{Y_{21}}^U]) & ([\alpha_{Y_{22}}^L, \alpha_{Y_{22}}^U], [\beta_{Y_{22}}^L, \beta_{Y_{22}}^U]) & \dots & ([\alpha_{Y_{2\psi}}^L, \alpha_{Y_{2\psi}}^U], [\beta_{Y_{2\psi}}^L, \beta_{Y_{2\psi}}^U]) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \aleph_u & ([\alpha_{Y_{\phi 1}}^L, \alpha_{Y_{\phi 1}}^U], [\beta_{Y_{\phi 1}}^L, \beta_{Y_{\phi 1}}^U]) & ([\alpha_{Y_{\phi 2}}^L, \alpha_{Y_{\phi 2}}^U], [\beta_{Y_{\phi 2}}^L, \beta_{Y_{\phi 2}}^U]) & \dots & ([\alpha_{Y_{\phi\psi}}^L, \alpha_{Y_{\phi\psi}}^U], [\beta_{Y_{\phi\psi}}^L, \beta_{Y_{\phi\psi}}^U]) \end{matrix}$$

where $\dot{Y}_{\varrho(\xi)}$ is the ξ -th biggest weighted IVPFEs \dot{Y}_{ξ} ($\dot{Y}_{\xi} = \psi \varsigma_{\xi} \tilde{Y}_{\xi}$, $\xi = 1, 2, \dots, \psi$), and $\varsigma = (\varsigma_1, \varsigma_2, \dots, \varsigma_{\psi})^T$ is weight vector of \tilde{Y}_{ξ} with $\varsigma_{\xi} > 0$ and $\sum_{\xi=1}^{\psi} \varsigma_{\xi} = 1$, where ψ is the balancing coefficient.

Proof Similarly to Theorem 2, Theorem 11 is simply obtained. \square

5. Recommended decision framework

In the following, we will use the suggested IVPFAAWA operator to create a technique to MADM utilizing IVPF data.

The following steps are included in the analysis focusing on the IVPFAAWA operator to evaluate the MADM concerns with IVIFEs:

Step 1. Convert decision matrix $R = (\tilde{\varphi}_{\eta\xi})_{\phi \times \psi}$ into the normalization matrix $\bar{R} = (\bar{\varphi}_{\eta\xi})_{\phi \times \psi}$.

$$\bar{\varphi}_{\eta\xi} = \begin{cases} \tilde{\varphi}_{\eta\xi} & \text{for benefit attribute } \chi_{\xi} \\ (\tilde{\varphi}_{\eta\xi})^c & \text{for cost attribute } \chi_{\xi} \end{cases} \quad (4)$$

where $(\tilde{\varphi}_{\eta\xi})^c$ is complement of $\tilde{\varphi}_{\eta\xi}$, so as $(\tilde{\varphi}_{\eta\xi})^c = ([\beta_{Y_{\eta\xi}}^L, \beta_{Y_{\eta\xi}}^U], [\alpha_{Y_{\eta\xi}}^L, \alpha_{Y_{\eta\xi}}^U])$.

If all of the attributes χ_{ξ} ($\xi = 1, 2, \dots, \psi$) are of an identical type, there is no need to normalize the attribute values. However, if a MADM issue has both benefit and cost attributes, we are able to transform cost type rating

Table 1. IVPF decision matrix.

	χ_1	χ_2	χ_3	χ_4
\aleph_1	([0.26,0.32],[0.58,0.68])	([0.55,0.65],[0.20,0.30])	([0.62,0.72],[0.16,0.26])	([0.65,0.70],[0.25,0.35])
\aleph_2	([0.47,0.51],[0.40,0.47])	([0.37,0.41],[0.47,0.57])	([0.44,0.50],[0.38,0.48])	([0.59,0.69],[0.26,0.31])
\aleph_3	([0.21,0.31],[0.52,0.57])	([0.74,0.77],[0.19,0.24])	([0.66,0.71],[0.26,0.33])	([0.74,0.79],[0.16,0.21])
\aleph_4	([0.66,0.71],[0.23,0.29])	([0.21,0.28],[0.57,0.69])	([0.56,0.63],[0.30,0.37])	([0.57,0.63],[0.32,0.38])
\aleph_5	([0.57,0.62],[0.35,0.43])	([0.26,0.31],[0.47,0.53])	([0.65,0.75],[0.20,0.25])	([0.50,0.60],[0.35,0.40])

values into the benefit type rating values. As a consequence $R = (\tilde{\varphi}_{\eta\xi})_{\phi \times \psi}$ change into IVPF decision matrix $\bar{R} = (\tilde{\varphi}_{\eta\xi})_{\phi \times \psi}$.

Step 2. Use the IVPFAAWA operator with the decision information expressed in matrix R to acquire the overall preference values $\tilde{\varphi}_\eta$ ($\eta = 1, 2, \dots, \phi$) of the alternative \aleph_η i.e.,

$$\tilde{\varphi}_\eta = IVPFAAWA_\zeta(\tilde{\varphi}_{\eta 1}, \tilde{\varphi}_{\eta 2}, \dots, \tilde{\varphi}_{\eta \psi}) = \bigoplus_{\xi=1}^{\psi} (\zeta_\xi \tilde{\varphi}_{\eta \xi})$$

$$= \left(\left[\sqrt{1 - e^{-\left(\sum_{\xi=1}^{\psi} \zeta_\xi (-\ln(1 - (\alpha_{Y_\xi}^L)^2)^{1/\lambda}\right)^{1/\lambda}}}\right)}, \sqrt{1 - e^{-\left(\sum_{\xi=1}^{\psi} \zeta_\xi (-\ln(1 - (\beta_{Y_\xi}^U)^2)^{1/\lambda}\right)^{1/\lambda}}}\right)}, \left[\sqrt{e^{-\left(\sum_{\xi=1}^{\psi} \zeta_\xi (-\ln(\beta_{Y_\xi}^L)^2)^{1/\lambda}\right)^{1/\lambda}}}, \sqrt{e^{-\left(\sum_{\xi=1}^{\psi} \zeta_\xi (-\ln(\alpha_{Y_\xi}^U)^2)^{1/\lambda}\right)^{1/\lambda}}} \right] \right)$$

Step 3. Determine the scores $\mathbb{Q}(\tilde{\varphi}_\eta)$ ($\eta = 1, 2, \dots, \phi$) of the ultimate IVPFEs $\tilde{\varphi}_\eta$ ($\eta = 1, 2, \dots, \phi$) to evaluate all of the alternatives \aleph_η ($\eta = 1, 2, \dots, \phi$) and then to choose the most effective one(s) (if there is no difference between two scores $\mathbb{Q}(\tilde{\varphi}_\eta)$ and $\mathbb{Q}(\tilde{\varphi}_\xi)$, then we must evaluate the accuracy degrees $\mathbb{W}(\tilde{\varphi}_\eta)$ and $\mathbb{W}(\tilde{\varphi}_\xi)$ of the collective overall preference values $\tilde{\varphi}_\eta$ and $\tilde{\varphi}_\xi$, respectively, and then rank the alternatives \aleph_η and \aleph_ξ by means of accuracy degrees $\mathbb{W}(\tilde{\varphi}_\eta)$ and $\mathbb{W}(\tilde{\varphi}_\xi)$.

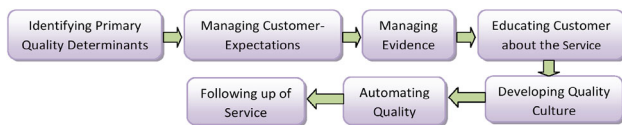


Figure 2. High service quality management.

Step 4. Rank all the possibilities \aleph_η ($\eta = 1, 2, \dots, \phi$) and pick the optimal one(s) according to $\mathbb{Q}(\tilde{\varphi}_\eta)$ ($\eta = 1, 2, \dots, \phi$).

Step 5. End.

6. Numerical analysis

MADM’s technique can be applied to a broader range of human choices and decisions, ranging from commercial to governmental to socioeconomically frameworks. Let’s look at an example of a professional decision-making difficulty.

6.1 Problem description

A travel agency named Chongqing China International Travel Service has dominated in giving travel-related services to international tourists from different countries. The agency’s consumers should be offered additional services, such as comprehensive briefings, online reservation capabilities, the ability to reserve and sell airline tickets, and certain other transport services. As a result, the corporation intends to locate an appropriate information technology (IT) software development company capable of delivering affordable arrangements via software development. To accomplish the above rationale, the organisation establishes a group of five alternatives (companies), to be specific, Chongqing Temiluo Technology Co Ltd (\aleph_1), Chongqing Zhuangwang Technology Co Ltd (\aleph_2), Chongqing Zhangzhiwo Technology Co Ltd (\aleph_3), Chongqing Siyuan Software Company (\aleph_4), and Wujue Software Co Ltd (\aleph_5). They use the following criteria to evaluate five possible software companies:

χ_1 : Technical ability (Technical abilities are the specific skills and knowledge necessary to successfully accomplish complex actions, activities, and processes in the fields of physical and computational technology, including a wide range of other businesses. The key challenges in technical abilities include standard operating systems, programming languages, software proficiency, graphic designing, and analytical techniques),

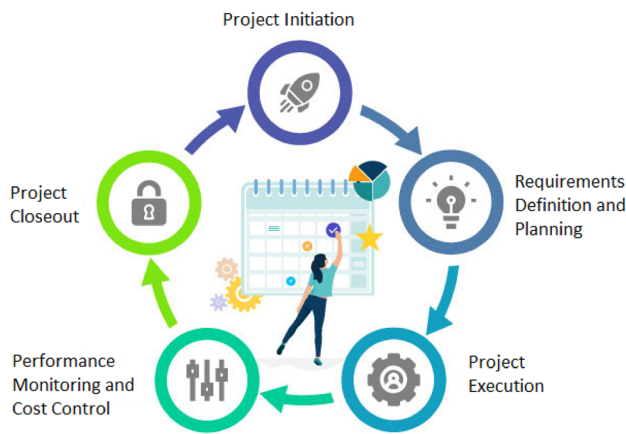


Figure 3. Project management.

- χ_2 : High-quality service management (Generally speaking, service quality refers to a customer’s evaluation of service results in terms of the organization’s business. An organisation with superior service quality is more likely to successfully fulfil customers’ needs while being globally competitive in its sector), (figure 2).
- χ_3 : Management of projects (Project management is the utilization of explicit information, abilities, apparatuses and strategies to provide something of value to individuals. A project management life cycle comprises five particular stages including initiation, planning, execution, monitoring, and closure that consolidate to transform a project idea into a working product), (figure 3).
- χ_4 : Professional background (A professional experience is a summary of prior work experience and performance. It’s most commonly used throughout the application process for a job. This is more than a list of previous roles; it should showcase your most important and relevant accomplishments).

The DM gives weight of the attribute as $\varsigma = (0.30, 0.30, 0.25, 0.15)^T$. The DM utilizes the IVPFEs to give assessment data of five alternatives \aleph_η ($\eta = 1, 2, \dots, 5$) in relation to aforesaid four attributes χ_ξ ($\xi = 1, 2, 3, 4$), as indexed in table 1.

As every one of the criteria values are of the same kind, the initial decision matrix does not need to be normalized. With the aim of designating proper IT companies \aleph_η ($\eta = 1, 2, \dots, 5$), we utilize the IVPFAAWA operator. The IVPFEs presented in table 1 are assessed accordance with the following arrangements:

- **Step 1.** Expect to be that $\lambda = 5$. Then, at that point, by employing the IVPFAAWA operator to assess the overall decision values $\tilde{\varphi}_\eta$ of software systems \aleph_η are as $\tilde{\varphi}_1 = ([0.59526, 0.67766], [0.21049, 0.31484])$,

$$\begin{aligned} \tilde{\varphi}_2 &= ([0.51168, 0.60141], [0.35793, 0.42258]), \\ \tilde{\varphi}_3 &= ([0.70605, 0.74565], [0.21936, 0.27472]), \\ \tilde{\varphi}_4 &= ([0.60963, 0.66228], [0.28787, 0.35181]), \\ \tilde{\varphi}_5 &= ([0.59392, 0.68748], [0.27990, 0.33633]). \end{aligned}$$

- **Step 2.** Calculate the score values $\mathbb{Q}(\tilde{\varphi}_\eta)$ ($\eta = 1, 2, \dots, 5$) of the entire IVPFEs $\tilde{\varphi}_\eta$ ($\eta = 1, 2, \dots, 5$) by Definition 6, and get $\mathbb{Q}(\tilde{\varphi}_1) = 0.33506$, $\mathbb{Q}(\tilde{\varphi}_2) = 0.15841$, $\mathbb{Q}(\tilde{\varphi}_3) = 0.46545$, $\mathbb{Q}(\tilde{\varphi}_4) = 0.30181$, $\mathbb{Q}(\tilde{\varphi}_5) = 0.31695$.
- **Step 3.** Subsequently $\mathbb{Q}(\tilde{\varphi}_3) > \mathbb{Q}(\tilde{\varphi}_1) > \mathbb{Q}(\tilde{\varphi}_5) > \mathbb{Q}(\tilde{\varphi}_4) > \mathbb{Q}(\tilde{\varphi}_2)$, then $\tilde{\varphi}_3 > \tilde{\varphi}_1 > \tilde{\varphi}_5 > \tilde{\varphi}_4 > \tilde{\varphi}_2$, and thus we have $\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$, where “ \succ ” represents “be superior to”.
- **Step 4.** According to the priority ranking of alternatives, \aleph_3 is selected as the supreme IT software company.

7. The effects of parameter λ on the ranking results

It is noted that we accept $\lambda = 10$ in the aforementioned analysis. In general, DMs can assign different values to the parameter with respect to their preferences. To be able to demonstrate the effects of the parameter λ on the decision-making outcome of this example, we use the different values λ to rank the alternatives. We set the values within 1 and 100 and generate the score values of five IT software companies \aleph_η ($\eta = 1, 2, \dots, 5$) to evaluate the variation in the ranking of these companies concerning the values of the parameter. The differences in the rankings regarding the parameter values λ can be seen explicitly in table 2, and graphically in figure 4. Table 2 portrays the score values of alternatives produced by employing the IVPFAAWA operator, where we have the ability to recognize that score values of each alternative grow significantly whilst the values of λ change from 1 to 100. When $1 \leq \lambda \leq 8$, the order of preference is $\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$ and when $9 \leq \lambda \leq 100$, the order of preference is $\aleph_3 \succ \aleph_5 \succ \aleph_1 \succ \aleph_4 \succ \aleph_2$. It is observed that there are two different orders of preference but the corresponding best choice is always \aleph_3 .

8. Comparative analysis

We compare our proposed method to a variety of other classical techniques namely the IVPF weighted averaging (IVPFWA) operator [39], IVPF weighted geometric (IVPFWG) operator [39], IVPF Einstein weighted averaging (IVPFEWA) operator [40] and IVPF Einstein weighted geometric (IVPFEWG) operator [41] in this section. Table 3 summarises the comparison outcomes, which are represented graphically in figure 5. Tables 2 and 3 show

Table 2. Preference order of alternatives by IVPFAAWA operator with different parameter values λ .

λ	$\mathbb{Q}(\tilde{\varphi}_1)$	$\mathbb{Q}(\tilde{\varphi}_2)$	$\mathbb{Q}(\tilde{\varphi}_3)$	$\mathbb{Q}(\tilde{\varphi}_4)$	$\mathbb{Q}(\tilde{\varphi}_5)$	Order of priority
1	0.224445	0.054338	0.341155	0.176379	0.184222	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
2	0.276603	0.080621	0.403154	0.233986	0.235558	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
3	0.305275	0.108259	0.434056	0.266154	0.270585	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
4	0.322926	0.13474	0.452704	0.286904	0.296637	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
5	0.335063	0.158409	0.465455	0.301811	0.316954	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
6	0.344117	0.178732	0.474882	0.313278	0.333124	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
7	0.351256	0.195849	0.482222	0.322482	0.346138	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
8	0.357104	0.21017	0.488144	0.330074	0.356715	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
9	0.36202	0.222155	0.493046	0.336454	0.365399	$\aleph_3 \succ \aleph_5 \succ \aleph_1 \succ \aleph_4 \succ \aleph_2$
10	0.366234	0.232228	0.497184	0.341887	0.372605	$\aleph_3 \succ \aleph_5 \succ \aleph_1 \succ \aleph_4 \succ \aleph_2$
50	0.40915	0.31069	0.53673	0.388844	0.427579	$\aleph_3 \succ \aleph_5 \succ \aleph_1 \succ \aleph_4 \succ \aleph_2$
100	0.416482	0.320499	0.543837	0.39511	0.434433	$\aleph_3 \succ \aleph_5 \succ \aleph_1 \succ \aleph_4 \succ \aleph_2$

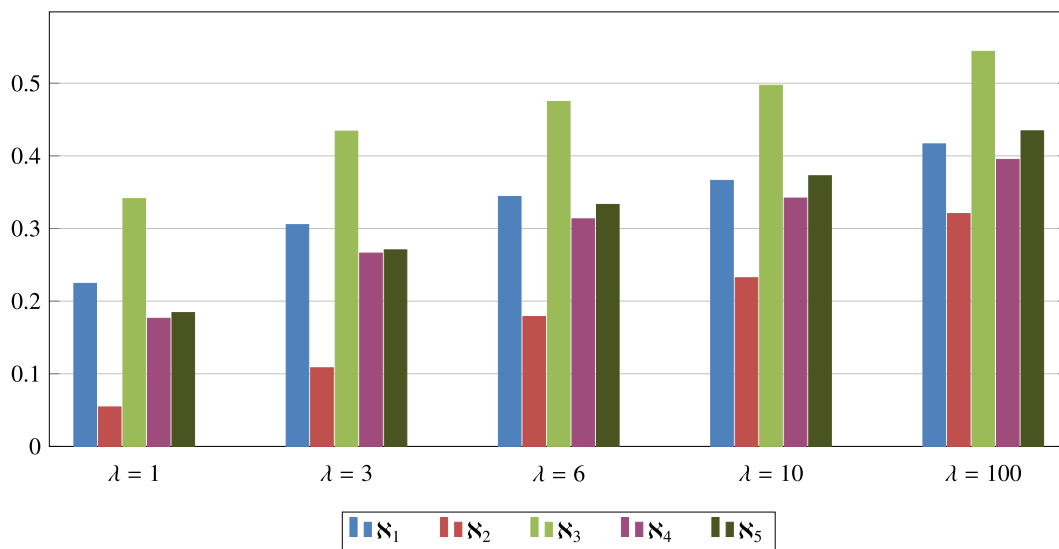


Figure 4. Score values of the alternatives for different values λ by IVPFAAWA operator.

Table 3. Comparison analysis with several commonly used methodologies.

Techniques	$\mathbb{Q}(\tilde{\varphi}_1)$	$\mathbb{Q}(\tilde{\varphi}_2)$	$\mathbb{Q}(\tilde{\varphi}_3)$	$\mathbb{Q}(\tilde{\varphi}_4)$	$\mathbb{Q}(\tilde{\varphi}_5)$	Rank order
IVPFWA operator (Garg [39])	0.224444856	0.054338	0.341155042	0.176379043	0.184222486	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
IVPFWG operator (Garg [39])	0.074822264	0.022226497	0.149304748	0.025519903	0.083556542	$\aleph_3 \succ \aleph_5 \succ \aleph_1 \succ \aleph_4 \succ \aleph_2$
IVPFEWA operator (Rahman <i>et al</i> [40])	0.211944885	0.050065164	0.323531609	0.162529203	0.173194524	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
IVPFEWG operator (Rahman <i>et al</i> [41])	0.094489798	0.025502385	0.173496455	0.044204094	0.094200715	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$
Recommended IVPFAAWA operator	0.335063134	0.158409254	0.465454586	0.301811332	0.316954187	$\aleph_3 \succ \aleph_1 \succ \aleph_5 \succ \aleph_4 \succ \aleph_2$

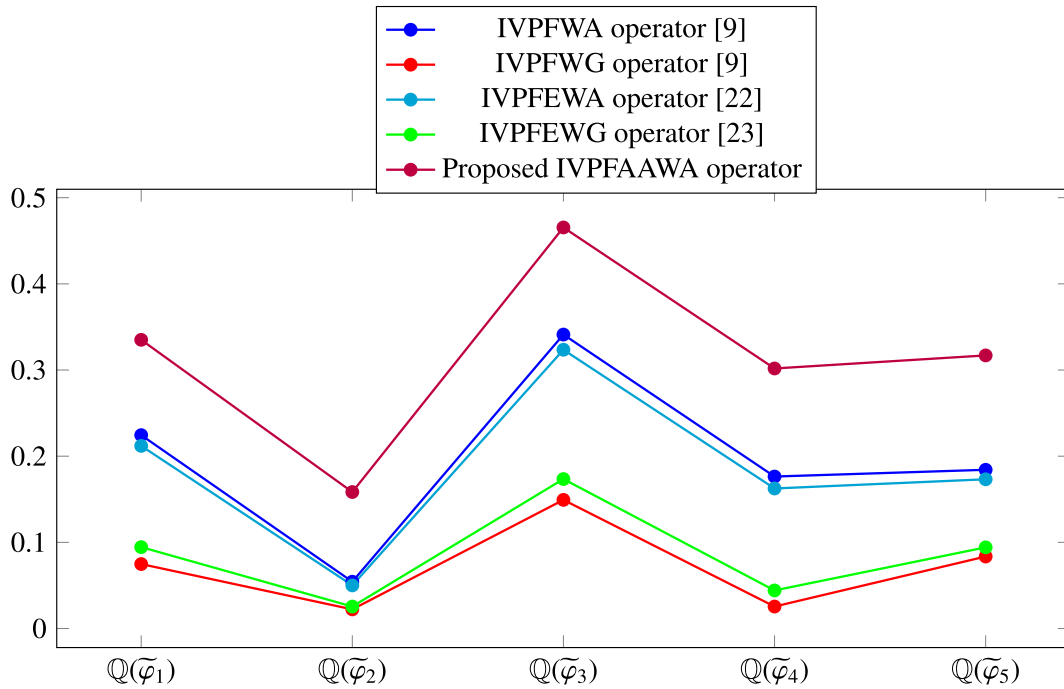


Figure 5. Comparative analysis employing a few prevalent approaches.

Table 4. Comparisons with several existing techniques in terms of their characteristics.

Techniques	Whether this is easier to assess fuzzy data	Is it possible to use a parameter to facilitate information aggregation simpler
IVPFWA operator (Garg [39])	Yes	No
IVPFWG operator (Garg [39])	Yes	No
IVPFEWA operator (Rahman <i>et al.</i> [40])	Yes	No
IVPFEWG operator (Rahman <i>et al.</i> [41])	Yes	No
Recommended IVPFAAWA operator	Yes	Yes

that the IVPFWA operator is a particular case of our recommended IVPFAAWA operator, and that it occurs when $\lambda = 1$.

As a result, our suggested techniques are much more comprehensive and diverse than many conventional techniques for dealing with IVPF MADM difficulties.

9. Conclusions

In this paper, bearing in mind AA *t*-norm and AA *t*-conorm, we have presented different operational rules for IVPFEs. Subsequently, we have proposed the IVPFAAWA operator, IVPFAAOWA operator, and IVPFAAHA operator based on the proposed Aczel-Alsina operations and analyzed a lot of important characteristics of these operators. Based on IVPFAAWA operator, we developed a methodology to manage conventional MADM issue, delivered a representative example to adequately justify the methodology, and

examined the impacts of the adaptable parameter on the ultimate aggregation results. The relative investigation further exhibited and found that the outcomes coincide with the current procedures which confirms the constancy of the methodology.

When it comes to future research, we will additionally sum up these operators by the utilization of the power operator and Bonferroni mean operator or expand the uses of the aggregation operators to different areas, as for example multi-objective optimization problem, clustering, supply chain management, and pattern recognition. Moreover, we will stretch out the suggested methodology to tackle the MADM issues under a dual probabilistic linguistic environment [42], etc.

Acknowledgements

The author, Rifaqat Ali, extends his appreciation to Deanship of Scientific Research at King Khalid University,

for funding this work through General Research Project under grant number (GRP/93/43).

References

- [1] Attanassov K T 1986 Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* 20: 87–96
- [2] Zadeh L A 1965 Fuzzy sets. *Inform Control.* 8: 338–353
- [3] Yager R R and Abbasov A M 2013 Pythagorean membership grades, complex numbers and decision making. *Int. J. Intell Syst.* 28: 436–452
- [4] Zhang X and Xu Z 2014 Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *Int. J. Intell Syst.* 29: 1061–1078
- [5] Zhang X 2016a Multicriteria Pythagorean fuzzy decision analysis: A hierarchical QUALIFLEX approach with the closeness index-based ranking methods. *Inf. Sci.* 330: 104–124
- [6] Peng X and Yang Y 2016 Fundamental properties of interval-valued pythagorean fuzzy aggregation operators. *Int. J. Intell Syst.* 31(5): 444–487
- [7] Chen T Y 2018 An outranking approach using a risk attitudinal assignment model involving Pythagorean fuzzy information and its application to financial decision making. *Appl. Soft Comput.* 71: 460–487
- [8] Liang D, Darko A P and Xu Z 2018, Interval-valued Pythagorean fuzzy extended Bonferroni mean for dealing with heterogenous relationship among attributes. *Int. J. Intell Syst.* 33(7): 1381–1411
- [9] Liu Y, Qin Y and Han Y 2018 Multiple criteria decision making with probabilities in interval-valued Pythagorean fuzzy setting. *Int. J. Fuzzy Syst.* 20(2): 558–571
- [10] Wei G, Garg H, Gao H and Wei C 2018 Interval-valued pythagorean fuzzy Maclaurin symmetric mean operators in multiple attribute decision making. *IEEE Access.* 6: 67866–67884
- [11] Yang Y, Chen Z S, Chen Y H and Chin K S 2018 Interval-valued Pythagorean fuzzy Frank power aggregation operators based on an isomorphic Frank dual triple. *Int. J. Comput. Int. Sys.* 11: 1091–1110
- [12] Li Z, Wei G and Gao H 2018 Methods for multiple attribute decision making with interval-valued Pythagorean fuzzy information. *Mathematics.* 6: 228; <https://doi.org/10.3390/math6110228>
- [13] Liang D, Darko A P and Zeng J 2019 Interval-valued pythagorean fuzzy power average-based MULTIMOORA method for multi-criteria decision-making. *J. Exp. Theor. Artif. In.* 32(5): 845–874
- [14] Xian S, Yu D X, Sun Y and Liu Z 2020 A novel outranking method for multiple criteria decision making with interval-valued Pythagorean fuzzy linguistic information. *Comput. Appl. Math.* 39: 58; <https://doi.org/10.1007/s40314-020-1064-5>.
- [15] Rahman K, Abdullah S, Ali A and Amin F 2019 Approaches to multi-attribute group decision making based on induced interval-valued Pythagorean fuzzy Einstein hybrid aggregation operators. *Bull. Braz. Math. Soc. New Series.* 50: 845–869
- [16] Peng X 2019 New operations for interval-valued Pythagorean fuzzy set. *Scientia Iranica E.* 26(2): 1049–1076
- [17] Peng X and Li W 2019 Algorithms for interval-valued Pythagorean fuzzy sets in emergency decision making based on multiparametric similarity measures and WDBA. *IEEE Access.* 7: 7419–7441
- [18] Du Y, Hou F, Zafar W, Yu Q and Zhai Y 2017 A novel method for multiattribute decision making with interval-valued Pythagorean fuzzy linguistic information. *Int. J. Intell Syst.* 32(10): 1085–1112
- [19] Tang X Y, Wei G W and Gao H 2019 Models for multiple attribute decision making with interval-valued pythagorean fuzzy Muirhead mean operators and their application to green suppliers selection. *Informatica.* 30(1): 153–186
- [20] Rahman K and Abdullah S 2019 Some new generalized interval-valued Pythagorean fuzzy aggregation operators using einstein t -norm and t -conorm. *J. Intell. Fuzzy Syst.* 37(3): 3721–3742
- [21] Haktanir E and Kahraman C 2019 A novel interval-valued Pythagorean fuzzy QFD method and its application to solar photovoltaic technology development. *Comput. Ind. Eng.* 132: 361–372
- [22] Chen T Y 2018 An interval-valued Pythagorean fuzzy compromise approach with correlation-based closeness indices for multiple-criteria decision analysis of bridge construction methods. *Complexity.* Article ID 6463039, 29 pages
- [23] Huang Y H and Wei G W 2018 TODIM method for interval-valued Pythagorean fuzzy multiple attribute decision making. *Int. J. Knowl-based Intell Eng. Syst.* 22: 249–259
- [24] Sajjad Ali Khan M and Abdullah S 2018 Interval-valued Pythagorean fuzzy GRA method for multiple-attribute decision making with incomplete weight information. *Int. J. Intell Syst.* 33(8): 1689–1716
- [25] Enayattabar M, Ebrahimnejad A and Motameni H 2019 Dijkstra algorithm for shortest path problem under interval-valued Pythagorean fuzzy environment. *Complex Intell Syst.* 5: 93–100
- [26] Yu C, Shao Y, Wang K and Zhang L 2019 A group decision making sustainable supplier selection approach using extended TOPSIS under interval-valued Pythagorean fuzzy environment. *Expert Syst. Appl.* 121: 1–17
- [27] Yanmaz O, Turgut Y, Can E N and Kahraman C 2020 Interval-valued Pythagorean fuzzy EDAS method: An application to car selection problem. *J. Intell. Fuzzy Syst.* 38: 4061–4077
- [28] Liang W, Zhang X and Liu M 2015 The maximizing deviation method based on interval-valued Pythagorean fuzzy weighted aggregating operator for multiple criteria group decision analysis. *Discrete Dyn. Nat. Soc.* Article ID 746572, 15 pages; <https://doi.org/10.1155/2015/746572>
- [29] Wang L and Li N 2019 Continuous interval-valued Pythagorean fuzzy aggregation operators for multiple attribute group decision making. *J. Intell. Fuzzy Syst.* 36: 6245–6263
- [30] Biswas A and Sarkar B 2019 Interval-valued Pythagorean fuzzy TODIM approach through point operator-based similarity measures for multicriteria group decision making. *Kybernetes.* 48(3): 496–519
- [31] Liang D, Darko A P, Xu Z and Quan W 2018 The linear assignment method for multicriteria group decision making based on interval-valued Pythagorean fuzzy Bonferroni mean. *Int. J. Intell Syst.* 33(11): 2101–2138

- [32] Aczel J and Alsina C 1982 Characterization of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements. *Aequationes Math.* 25(1): 313-315
- [33] Senapati T, Chen G, and Yager R R 2021 Aczel-Alsina aggregation operators and their application to intuitionistic fuzzy multiple attribute decision making. *Int. J. Intell Syst.* 37(2): 1529–1551
- [34] Senapati T, Chen G, Mesiar R and Yager R R 2021 Novel Aczel-Alsina operations-based interval-valued intuitionistic fuzzy aggregation operators and its applications in multiple attribute decision-making process. *Int. J. Intell Syst.* 37(8): 5059–5081
- [35] Senapati T, Chen G, Mesiar R, Yager R R and Saha A 2022 Novel Aczel-Alsina operations-based hesitant fuzzy aggregation operators and their applications in cyclone disaster assessment. *Int. J. Gen. Syst.* 51(5): 511–546
- [36] Senapati T, Approaches to multi-attribute decision-making based on picture fuzzy Aczel-Alsina average aggregation operators. *Comp. Appl. Math.* 2022; 41: 40, <https://doi.org/10.1007/s40314-021-01742-w>
- [37] Klement E P, Mesiar R and Pap E 2000 *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.
- [38] Alsina C, Frank M J and Schweizer B 2006 *Associative Functions-Triangular Norms and Copulas*, World Scientific Publishing, Danvers, MA.
- [39] Garg H 2016 A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi-criteria decision making problem. *J. Intell Fuzzy Syst.* 31(1): 529-540
- [40] Rahman K, Ali A, Abdullah S and Amin F 2018 Approaches to multi-attribute group decision making based on induced interval-valued Pythagorean fuzzy Einstein aggregation operator. *New Math. Natural Comput.* 14(3): 343–361
- [41] Rahman K and Abdullah S 2019 Some induced generalized interval-valued Pythagorean fuzzy Einstein geometric aggregation operators and their application to group decision-making. *Comput. Appl. Math.* 38(3): Art. no. 139
- [42] Saha A, Senapati T and Yager R R 2021 Hybridizations of generalized Dombi operators and Bonferroni mean operators under dual probabilistic linguistic environment for group decision-making. *Int. J. Intell Syst.* 36(11): 6645–6679