Sādhanā (2022)47:52
https://doi.org/10.1007/s12046-022-01820-7

Collision of nonlinear waves in logotropic system with a Coulomb-type friction

MINHAJUL[1] and T. RAJA SEKHAR[2,*]

[1]Department of Mathematics, Birla Institute of Techonology and Science Pilani, K K Birla Goa Campus, Goa 403726, India
[2]Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India
e-mail: minhajul@goa.bits-pilani.ac.in; trajasekhar@maths.iitkgp.ac.in

MS received 11 September 2021; revised 7 December 2021; accepted 12 January 2022

Abstract. In this article, we consider the logotropic system of gasdynamics with a Coulomb-type friction to explore all possible collisions of elementary waves. We discuss the elementary waves and their properties in the phase plane to describe the exact Riemann solution. Further, we analyze all possible cases of the elementary wave interactions between same and different families of waves in the phase plane employing the solution of the Riemann problem.

Keywords. Riemann problem; elementary waves; wave interactions; logotropic system.

1. Introduction

The theory of hyperbolic quasilinear conservation laws has variety applications in the field of gas dynamics, oil industries, applied science, engineering physics, multiphase flows and etc. It can be noted that the conservation of mass, momentum and energy form a common starting point of this study. However, in terms of field variables these reduce to partial differential equations (PDEs) under quite natural assumptions. In most of the cases, these PDEs are either quasilinear or nonlinear with source terms. Therefore, it is not easy to understand the solution and their properties as in the context of linear theory. Conservation laws are system of PDEs which can be written in the form of divergence. There are many physical phenomena which are modeled by the systems of conservation laws those are hyperbolic in nature. Hence, it is very interesting and challenging task to study the mathematical theory of hyperbolic conservation laws. Nowadays, the study of Riemann problem and wave interactions becomes very much popular in the theory of hyperbolic quasilinear PDEs.

The Riemann problem is a particular case of the Cauchy problem where the initial data is piecewise constant and having a single jump discontinuity. In order to construct the solution to the general initial value problem by exploiting the random choice method [1], the solution of local Riemann problem plays a very important role. In general, the solution of Riemann problem consists of shock wave, rarefaction wave and contact discontinuity which are called as elementary waves. The study of Riemann problem exhibits some fundamental properties of the elementary waves and detailed picture of solution. Therefore, this study has its own significance due to its wide practical applications. So, the researchers are attracted towards this topic and analyze the solution to the Riemann problem for the system of conservation laws and it becomes a very important topic in the context of quasilinear hyperbolic conservation laws.

In the last few decades, many researchers have been attracted towards the study of wave interactions for hyperbolic system of conservation laws due to its wide practical applications. Minhajul *et al* [2] studied the Riemann problem and collision between weak shocks in two-phase flows which describes isentropic drift-flux model. Kuila and Raja Sekhar [3] established Von Neumann's result related to overtaking of two weak shocks belong to same family in the context of drift-flux isothermal multiphase flows. Raja Sekhar and Sharma [4] discussed the existence of vacuum state and wave interactions briefly in isentropic magnetogasdynamics. By exploiting characteristic analysis methodology, elementary wave interactions in ideal magnetogasdynamics have been analyzed by Liu and Sun [5]. Sen *et al* [6] discussed stability of Riemann solutions and their asymptotic behaviour for system of strictly hyperbolic conservation laws. Collision of weak discontinuity with contact discontinuity and shock in isothermal drift-flux model have been analyzed in [7].

There are several interesting ways to study the problem of wave interactions among which phase plane analysis and characteristic analysis are widely adopted by the researchers. For example, elementary wave interactions for diverse practical problems [8, 9] have been analyzed using the phase plane analysis. The collision of elementary classical

*For correspondence
Published online: 15 March 2022

waves and stability analysis of Riemann solution for various physical problems have been investigated in [10, 11] using the method of characteristic analysis. For basic concepts on analysis of nonlinear waves in the context of hyperbolic quasilinear systems we refer [12–14]. Sun [15] constructed solutions for one-dimensional traffic flow problem when initial data consist of three constant states. Singh and Jena [16] computed transmitted and reflected wave amplitudes after the interaction between strong shock and acceleration wave in reacting polytropic gases. Classes of exact solutions to generalized Riemann problem for traffic flow model have been developed in [17] by using differential constraint method. A class of double wave solutions to nonlinear model of extended thermodynamics with six fields is determined and nonlinear wave interactions have been illustrated by Curro and Manganaro [18]. Chaudhary and Singh [19] established existence and uniqueness of Riemann solution for isentropic dusty gas flows and wave interactions are discussed. Stability of Riemann solutions are proved in [20] for the chromatography system under the small perturbations of local Riemann data. Sen *et al* [21] studied stability of Riemann solution, which consists of classical waves and delta shock, for strictly hyperbolic conservation laws system. Nonlinear wave interactions for a Temple-class hyperbolic system of conservation laws consisting of three scalar equations have been analyzed by Wei and Sun [22]. Zhang and Zhang [23] constructed the global structure of solutions through the nonlinear wave interactions investigation.

In the current study, we are concerned with the elementary wave interactions for Euler system with logarithmic equation of state and Coulomb type friction terms. The system of PDEs can be expressed by balance laws of the following form [24]

$$\begin{aligned} &\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0,\\ &\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho v^2+A\ln\rho)=\eta\rho, \end{aligned} \tag{1}$$

where the independent variables x and t denote the space and time, respectively, while the dependent variables ρ and v, respectively, denote the density and velocity of the gas. Here, A and η are positive constant parameters. When $\eta=0$, the system of PDEs has been introduced in the area of astrophysics to study various properties of molecular clouds which may not be well understood in general for the case of isothermal distribution.

The logarithmic equation of state is used to investigate the logotropic dark fluid as a unification of dark matter and dark energy [25–27]. The system (1) can be reduced to homogeneous conservative form through a new state variable $w(x,t)=v(x,t)-\eta t$ as done in [28] and the corresponding conservative form is given by

$$\begin{aligned} &\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho(w+\eta t))=0,\\ &\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho w(w+\eta t)+A\ln\rho)=0. \end{aligned} \tag{2}$$

The Riemann initial data for the system (2) is given by

$$(\rho,w)(x,0)=\begin{cases}(\rho_l,w_l), & \text{if } x<0,\\ (\rho_r,w_r), & \text{if } x>0,\end{cases} \tag{3}$$

where ρ_l, w_l, ρ_r and w_r are constants. For the system (2)-(3), an exact Riemann solver has been developed in [24]. Recently, the authors in [29] discussed the limiting behaviour of the solution to the Riemann problem (1) and (3). The solution of the original Riemann problem (1) and (3) can be determined from the solution of (2) and (3) by substituting the value of new variable $(\rho,v)(x,t)=(\rho,w+\eta t)(x,t)$. In the present work, our main objective is to study various possible wave interactions of the elementary waves of (2) in the phase plane. The motivation of this study is to analyse the wave interactions problems to the nonhomogeneous hyperbolic system with the logotropic equation of state because of its various practical applications in the field of aerodynamics, cosmology, engineering physics and astrophysics. To the best of our knowledge, no one has attempted this type of wave interactions for the system (1) till now.

Organization of the rest of this paper is as follows. In section 2, we recall the elementary waves of (2) and their properties very briefly. In section 3, we discuss in detail about the interactions between elementary waves for all possible cases in the phase plane. Finally, a brief conclusions are drawn in section 4.

2. Preliminaries

The quasilinear form of the system (2) can be written as

$$\frac{\partial V}{\partial t}+A(V)\frac{\partial V}{\partial x}=0. \tag{4}$$

Here, the primitive variable V and the Jacobian matrix $A(V)$ are, respectively, given by

$$V=\begin{pmatrix}\rho\\ w\end{pmatrix},\quad A(V)=\begin{pmatrix} w+\eta t & \rho\\ \dfrac{A}{\rho^2} & w+\eta t\end{pmatrix}.$$

The eigenvalues of the Jacobian matrix $A(V)$ are given by

$$\lambda_1=w+\eta t-\sqrt{\frac{A}{\rho}}\quad\text{and}\quad \lambda_2=w+\eta t+\sqrt{\frac{A}{\rho}}. \tag{5}$$

The respective eigenvectors corresponding to the eigenvalues are

$$r^1 = \left(-\rho, \sqrt{\frac{A}{\rho}}\right)^{Tr} \text{ and } r^2 = \left(\rho, \sqrt{\frac{A}{\rho}}\right)^{Tr} \quad (6)$$

where Tr denotes the transposition. It can be noted that both the characteristic fields are genuinely nonlinear as $\nabla\lambda_i \cdot r^i \neq 0\,(i = 1, 2)$. Hence, the solution of the Riemann problem consists of either shock wave (bounded discontinuous solution) or rarefaction wave (continuous solution). The Riemann invariants corresponding to these characteristic fields are, respectively, given by

$$1 - \textit{Riemann invariant}\, \Pi_1 = w - 2\sqrt{\frac{A}{\rho}}, \quad (7)$$

$$2 - \textit{Riemann invariant}\, \Pi_2 = w + 2\sqrt{\frac{A}{\rho}}. \quad (8)$$

We have already seen from the characteristic analysis that the solution of the Riemann problem consists of shock and rarefaction waves. Now, we discuss some properties of these two elementary waves very briefly. For more details about the elementary waves corresponding to the system (2), one can refer [24].

2.1 *Shock waves*

Shock wave is a discontinuous solution of the system (2) satisfying the Rankine-Hugoniot jump conditions and Lax entropy conditions. Suppose σ denotes the speed of shock then the Rankine-Hugoniot jump conditions across the shock wave are given by

$$\begin{aligned} \sigma[\rho] &= [(\rho(w + \eta t))], \\ \sigma[\rho w] &= [\rho w(w + \eta t) + A \ln \rho], \end{aligned} \quad (9)$$

where $[V] = V_l - V_r$, $V_l = V(x(t) - 0, t)$ and $V_r = V(x(t) + 0, t)$, denotes the jump of V across the shock. Suppose $V = (\rho, w)$ and $V_l = (\rho_l, w_l)$ indicate the right and left-hand states respectively. If $\sigma = 0$, then we get the trivial solution $V = V_l$. So, we assume $\sigma \neq 0$ and the 1-shock curve passing through V_l is denoted by $S_1(V_l)$ which satisfies the following

$$S_1(V_l) := \begin{cases} w = w_l - \sqrt{A(\frac{1}{\rho_l} - \frac{1}{\rho})(\ln \rho - \ln \rho_l)}, \\ \sigma = \dfrac{\rho w - \rho_l w_l}{\rho - \rho_l} + \eta t, \\ w < w_l, \rho_l < \rho. \end{cases} \quad (10)$$

Similarly, 2-shock curve passing through V_l is represented by $S_2(V_l)$ which is given by

$$S_2(V_l) := \begin{cases} w = w_l - \sqrt{A(\frac{1}{\rho_l} - \frac{1}{\rho})(\ln \rho - \ln \rho_l)}, \\ \sigma = \dfrac{\rho w - \rho_l w_l}{\rho - \rho_l} + \eta t, \\ w < w_l, \rho_l > \rho. \end{cases} \quad (11)$$

One can easily prove the following properties of the shock curve and we explore these properties in the succeeding sections.

Lemma 1 *The 1-shock curve, S_1, is monotonically decreasing and convex whilst the 2-shock curve, S_2, is monotonically increasing and concave.*

2.2 *Rarefaction waves*

Using the property of Riemann invariants (7)-(8) across the rarefaction wave region, the 1-rarefaction wave curve through V_l is denoted by $R_1(V_l)$ and represented by

$$R_1(V_l) := \begin{cases} w = w_l - 2\sqrt{\dfrac{A}{\rho_l}} + 2\sqrt{\dfrac{A}{\rho}}, \\ \dfrac{dx}{dt} = \lambda_1 = w + \eta t - \sqrt{\dfrac{A}{\rho}}, \\ w \geq w_l, \rho_l \geq \rho. \end{cases} \quad (12)$$

In the same manner, the 2-rarefaction wave curve through V_l is represented by $R_2(V_l)$ and expressed as

$$R_2(V_l) := \begin{cases} w = w_l + 2\sqrt{\dfrac{A}{\rho_l}} - 2\sqrt{\dfrac{A}{\rho}}, \\ \dfrac{dx}{dt} = \lambda_2 = w + \eta t + \sqrt{\dfrac{A}{\rho}}, \\ w \geq w_l, \rho_l \leq \rho. \end{cases} \quad (13)$$

One can easily prove the following properties of the rarefaction wave curve and these can be exploited in the succeeding sections.

Lemma 2 *The 1-rarefaction curve, R_1, is convex and monotonically decreasing whilst the 2-rarefaction curve, R_2, is concave and monotonically increasing.*

Using the properties of elementary waves we can easily prove the following theorem.

Theorem 1 *The curves of shock and rarefaction waves of same family, say, S_1 and R_1 (respectively, S_2 and R_2) have the second order contact at V_l.*

2.3 *Solution structure of Riemann problem*

Let us consider the Riemann problem for (2) with initial data given by

$$V(x,0) = \begin{cases} V_l, & \text{if } x<0, \\ V_r, & \text{if } x>0. \end{cases} \quad (14)$$

It is observed from figure 1 that the elementary waves divide the phase plane into four disjoint regions namely, I, II, III and IV. Depending on the position of V_r, the solution of Riemann problem can be constructed for a given V_l. For example, if V_r lies in region I, then V_l can be connected to V_r by a 1-shock $S_1(V_l)$ followed by a 2-rarefaction wave $R_2(V_l)$, i.e., the Riemann solution involves 1-shock and 2-rarefaction wave. Similarly, if $V_r \in II$, solution of Riemann problem contains a 1-shock followed by a 2-shock. Consequently, if $V_r \in III$ then the solution of Riemann problem consists of a 1-rarefaction followed by a 2-shock wave. Finally, if $V_r \in IV$ then the Riemann solution possess of a 1-rarefaction followed by a 2-rarefaction wave. Therefore, we can state the following theorem without proof.

Theorem 2 *If $V_l, V_r \in \mathbb{R}^+ \times \mathbb{R}$ with V_l fixed and V_r is allowed to vary then the Riemann problem is solvable if and only if V_r lies on any one of the four regions I, II, III and IV.*

3. Elementary wave interactions

In order to determine all possible cases of wave interactions for the system (2), we consider the following initial data with three piecewise constant states

$$V(x,0) = \begin{cases} V_l, & -\infty<x<x_-, \\ V_m, & x_-<x<x_+, \\ V_r, & x>x_+, \end{cases} \quad (15)$$

where x_-, x_+ are arbitrary real numbers and we choose V_m and V_r with reference to V_l. Therefore, with this choice of data (15) with system (2) leads to two Riemann problems locally at x_- and x_+. An elementary wave of the first Riemann problem may interact with an elementary wave of the second Riemann problem and at the time of interaction a new Riemann problem is formed. In this article, the symbol $S_2R_1 \Rightarrow R_1S_2$ indicate that a 2-shock, S_2, associated with the first Riemann problem which connects V_l to V_m, collide with 1-rarefaction wave, R_1, corresponding to the second Riemann problem which connects V_m to V_r and the collision generates a new Riemann problem with a left hand state V_l and a right hand state V_r and the solution of this new Riemann problem consists of a 1-rarefaction wave, R_1, and a 2-shock wave S_2 (i.e., R_1S_2). There are four possible cases of collisions of elementary waves corresponding to various families which are R_2R_1, R_2S_1, S_2R_1, and S_2S_1 whilst there are six possible cases of collisions of elementary waves corresponding to the same family which are R_1S_1, S_1S_1, S_1R_1, S_2S_2 R_2S_2 and S_2R_2.

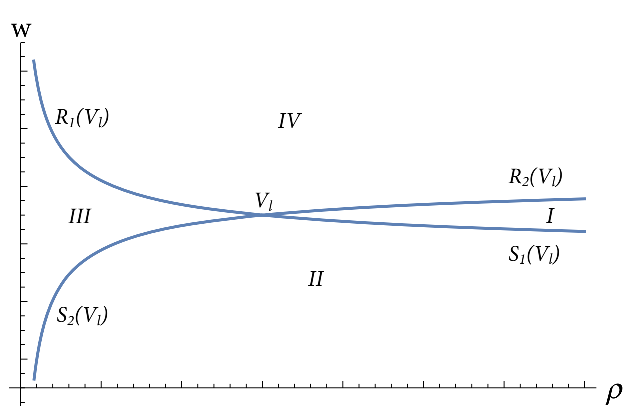

Figure 1. Elementary waves in (ρ, w)-plane.

3.1 *Wave interactions between different families of elementary waves*

(i) *Collision of two shocks* (S_2S_1):
Let us assume that V_l and V_m are connected by S_2 associated with the Riemann first problem and V_m and V_r are connected by S_1 corresponding to the second Riemann problem. Therefore, for a fixed given V_l, we choose V_m and V_r in such a way that $\rho_m<\rho_l$, $w_m = w_l - F(\rho, \rho_l)$ and $\rho_m<\rho_r, w_r = w_m - F(\rho_r, \rho_m)$ where $F(a,b) = \sqrt{A\left(\frac{1}{a} - \frac{1}{b}\right)(\ln b - \ln a)}$. One can verify from the Lax entropy condition that

$$\sigma_2(V_l, V_m) > \lambda_2(V_m) > \lambda_1(V_m) > \sigma_1(V_m, V_r). \quad (16)$$

Therefore, 2-shock corresponding to first Riemann problem moves faster than the 1-shock of the second Riemann problem. Hence, interaction will take place and S_2 overtakes S_1 after a finite time. In order to solve this problem, we need to determine the region in which V_r lies with respect to V_l. Now, we show that for any given arbitrary state V_l, the curve $S_1(V_l)$ lies above the curve $S_1(V_m)$ which implies that the state $V_r \in II$. In order to prove this, it is enough to show that $F(\rho_m, \rho_l) + F(\rho_m, \rho) - F(\rho_l, \rho) > 0$ for $\rho<\rho_l$ and $\rho_m<\rho$.
We assume that $F(\rho_m, \rho_l) + F(\rho_m, \rho) - F(\rho_l, \rho) \leq 0$ and we prove the inequality by method of contradiction. Therefore, we have

$$F^2(\rho_m, \rho_l) + F^2(\rho_m, \rho) + 2F(\rho_m, \rho_l)F(\rho_m, \rho) \leq F^2(\rho_l, \rho)$$

which implies that

$$A\left(\frac{1}{\rho_l}-\frac{1}{\rho_m}\right)\left(\ln\rho_m-\ln\rho\right)+A\left(\frac{1}{\rho}-\frac{1}{\rho_m}\right)\left(\ln\rho_m-\ln\rho_l\right)+2F(\rho_m,\rho_l)F(\rho_m,\rho)\le 0. \tag{17}$$

For $\rho > \rho_l > \rho_m$, the left hand side of (17) is positive, which is a contradiction and therefore $F(\rho_m,\rho_l)+F(\rho_m,\rho)-F(\rho_l,\rho)>0$. Further, if $\rho_l > \rho > \rho_m$ then also the left hand side of inequality (17) is positive as $A\left(\frac{1}{\rho_l}-\frac{1}{\rho_m}\right)\left(\ln\rho_m-\ln\rho\right)>0$ and $A\left(\frac{1}{\rho}-\frac{1}{\rho_m}\right)\left(\ln\rho_m-\ln\rho_l\right)>0$. Therefore, V_r lies in the region II and hence the interaction result is $S_2S_1 \Rightarrow S_1S_2$. The graphical representation for this case is drawn in figure 2.

(ii) *Interaction of 2-shock and 1-rarefaction* (S_2R_1): Here, we consider V_m and V_r with reference to a given V_l in such a way that $V_m \in S_2(V_l)$ and $V_r \in R_1(V_m)$. It follows that, $\rho_m < \rho_l$, $w_m = w_l - F(\rho,\rho_l)$ and $\rho_r \le \rho_m, w_r = w_m - 2\sqrt{\frac{A}{\rho_m}}+2\sqrt{\frac{A}{\rho_r}}$. One can evaluate that $\sigma_2(V_l,V_m)-\lambda_1(V_m)=\frac{\rho_l(w_l-w_m)}{\rho_l-\rho_m}+\frac{A}{\sqrt{\rho_m}}>0$. Therefore, 1-rarefaction has less speed compare to 2-shock which leads S_2 to overtake R_1 after a finite time. In order to prove that V_r lies in the region III, it is enough to show that $F(\rho,\rho_l)-2\sqrt{\frac{A}{\rho_l}}+2\sqrt{\frac{A}{\rho_m}}>0$ for $\rho<\rho_m\le\rho_l$ which is in fact true as left hand side is always positive for $\rho<\rho_m\le\rho_l$. Therefore, $R_1(V_m)$ lies below the curve $R_1(V_l)$ and hence V_r lies in region III. So, the result of interaction is $S_2R_1 \Rightarrow R_1S_2$ as depicted in figure 3.

(iii) *Interaction between two rarefaction waves* (R_2R_1): In this case, $V_m \in R_2(V_l)$ and $V_r \in R_1(V_m)$. Therefore, for a given fixed V_l, we choose V_m and V_r in such a way that $\rho \ge \rho_l$, $w_m = w_l + 2\sqrt{\frac{A}{\rho_l}}-2\sqrt{\frac{A}{\rho_m}}$ and $\rho_r \le \rho_m$, $w_r = w_m - 2\sqrt{\frac{A}{\rho_m}}+2\sqrt{\frac{A}{\rho_r}}$. Since, $\lambda_2(V_m)>\lambda_1(V_m)$, the tail of 2-rarefaction wave of first Riemann problem has greater speed than the head of 1-rarefaction wave of second Riemann problem. Hence, interaction will take place after a finite time. Here, we prove that curve $R_1(V_l)$ lies below $R_1(V_m)$. As, $4\left(\sqrt{\frac{A}{\rho_l}}-\sqrt{\frac{A}{\rho_m}}\right)>0$ whenever $\rho_m \ge \rho_l$, hence $R_1(V_l)$ lies below the curve $R_1(V_m)$ for any ρ_r satisfying $\rho_m \ge \rho_l \ge \rho_r$ or $\rho_l \le \rho_r \le \rho_m$. Therefore, we can conclude that the result of interaction is $R_2R_1 \Rightarrow R_1R_2$ and the computed result is illustrated in figure 4.

(iv) *Collision of 2-rarefaction and 1-shock* (R_2S_1): Here, $V_m \in R_2(V_l)$ and $V_r \in S_1(V_m)$. Hence, we set V_m and V_r with reference to a given V_l in such a manner that $\rho_m \ge \rho_l$, $w_m = w_l + 2\sqrt{\frac{A}{\rho_l}}-2\sqrt{\frac{A}{\rho_m}}$ and $\rho_m<\rho_r, w_r = w_m - F(\rho_r,\rho_m)$. From Lax entropy condition we have, $\lambda_2(V_m)>\sigma_1(V_m,V_r)$, which follows that the trailing end of R_2 corresponding

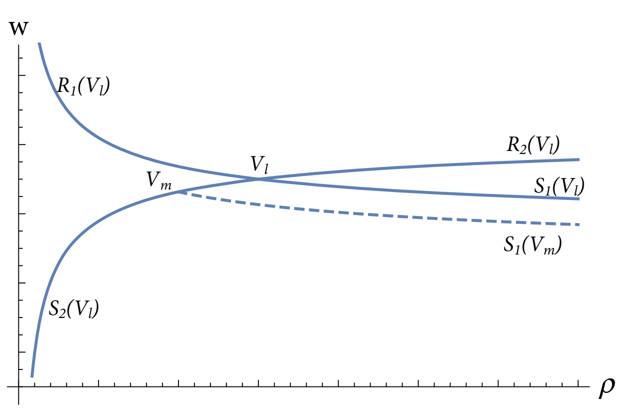

Figure 2. Collision of S_2S_1.

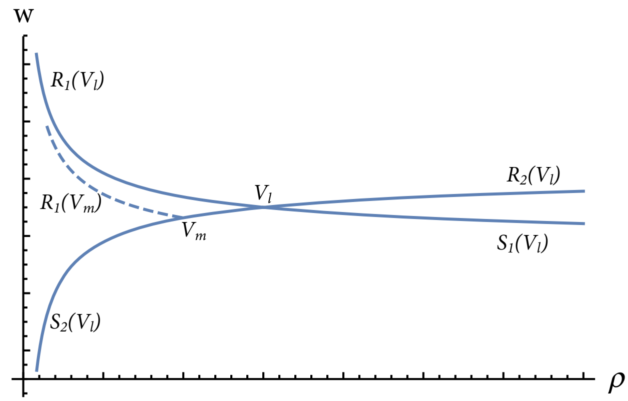

Figure 3. Collision of S_2R_1.

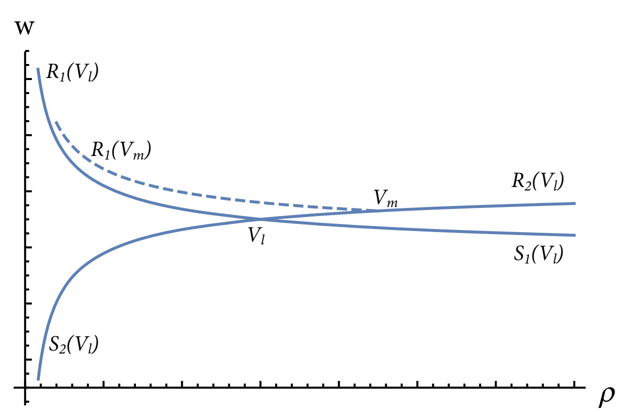

Figure 4. Collision of R_2R_1.

to the Riemann first problem has more speed compare to S_1 associated with Riemann second problem. Thus, R_2 overtakes S_1 and interaction will take place after a finite time. Now for a given V_l, we show that $V_r \in I$. In order to prove this, it is sufficient to show that $2\sqrt{\frac{A}{\rho_l}} - 2\sqrt{\frac{A}{\rho_m}} + F(\rho,\rho_l) - F(\rho,\rho_m) > 0$. Clearly, $2\sqrt{\frac{A}{\rho_l}} - 2\sqrt{\frac{A}{\rho_m}} > 0$ for $\rho_m \geq \rho_l$ and an easy computation yields $F(\rho,\rho_l) - F(\rho,\rho_m) > 0$. Therefore, $S_1(V_m)$ lies above the curve $S_1(V_l)$ and hence the result of interaction is $R_2S_1 \Rightarrow S_1R_2$ and the computed result is exhibited in figure 5.

3.2 *Interaction of elementary waves of same family*

(i) *1-rarefaction overtakes 1-shock* (R_1S_1):
In this case, V_l and V_m are connected by a 1-rarefaction of first Riemann problem while the states V_m and V_r are connected by 1-shock of second Riemann problem. In other words, for a given V_l, we choose V_m and V_r in such a way that $\rho_m \leq \rho_l$, $w_m = w_l - 2\sqrt{\frac{A}{\rho_l}} + 2\sqrt{\frac{A}{\rho_m}}$ and $\rho_m < \rho_r$, $w_r = w_m - F(\rho_m, \rho_r)$. From Lax entropy inequality, we have $\lambda_1(V_m) > \sigma_1(V_l, V_m)$ which indicates that the trailing end of R_1 corresponding to Riemann first problem has more speed compare to S_1 associated with Riemann second problem. So, R_1 collides with S_1 and interaction occurs after a finite time. Now, we show that the curve $S_1(V_m)$ lies below the curve $R_1(V_l)$ for $\rho_m < \rho \leq \rho_l$. In order to complete the argument, it is sufficient to establish that

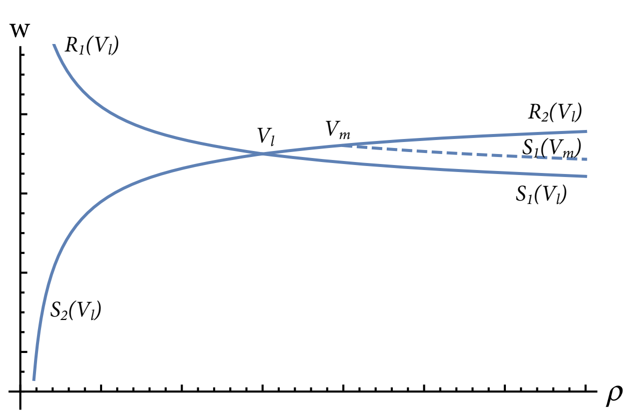

Figure 5. Collision of R_2S_1.

$$2\left(\sqrt{\frac{A}{\rho_m}} - \sqrt{\frac{A}{\rho}}\right) - F(\rho,\rho_m) < 0, \qquad (18)$$

for $\rho_m < \rho \leq \rho_l$ which is equivalent to the inequality

$$2\left(\sqrt{\frac{\rho}{\rho_m}} - 1\right) - \sqrt{\left(1 - \frac{\rho}{\rho_m}\right)\ln\frac{\rho_m}{\rho}} < 0. \qquad (19)$$

In order to prove (19), let $y = \frac{\rho}{\rho_m}$ then $\rho_m < \rho \leq \rho_l$ implies $1 < y \leq \frac{\rho_l}{\rho_m}$. Therefore, we have to prove that $2(\sqrt{y} - 1) - \sqrt{(y-1)\ln y} < 0$ for $y > 1$. One can show that $\ln y < y - 1$ for $y > 1$ using the monotonicity of the function $g_1(y) = y - \ln y$. Further, by applying Lagrange mean value theorem on the interval $[1, y]$ to the function $g_2(y) = \sqrt{y}$, we obtain $\frac{\sqrt{y}-1}{y-1} < \frac{1}{2}$ and which implies the inequality (19) and hence (18) for $\rho_m < \rho \leq \rho_l$.
Next, we prove that the curve $S_1(V_l)$ lies above the curve $S_1(V_m)$ whenever $\rho_m < \rho_l \leq \rho$. In order to prove this it is sufficient to show that

$$2\left(\sqrt{\frac{A}{\rho_m}} - \sqrt{\frac{A}{\rho_l}}\right) - F(\rho,\rho_m) + F(\rho,\rho_l) < 0. \qquad (20)$$

In order to prove (20), let $g_3(\rho) = 2\left(\sqrt{\frac{A}{\rho_m}} - \sqrt{\frac{A}{\rho_l}}\right) - F(\rho,\rho_m) + F(\rho,\rho_l)$. Then from (18), it is clear that $g_3(\rho_l) < 0$. Moreover, $\frac{dg_3}{d\rho} = \frac{dF}{d\rho}(\rho,\rho_l) - \frac{dF}{d\rho}(\rho,\rho_m) < 0$ as $F(\rho,\rho_m)$ is a decreasing function of ρ_m for $\rho_m < \rho$. Therefore, $g_3(\rho) < g_3(\rho_l) < 0$ which proves the inequality (20) and hence $S_1(V_m)$ lies below the curve $S_1(V_l)$ for $\rho_m < \rho_l \leq \rho$.
Lastly, we prove that $S_1(V_m)$ intersects $S_2(V_l)$ at some point, say, $V_1 = (\rho_1, w_1)$ with $\rho_m < \rho_1 \leq \rho_l$. To prove this, let us consider the function $g_4(\rho) = 2\left(\sqrt{\frac{A}{\rho_m}} - \sqrt{\frac{A}{\rho_l}}\right) - F(\rho,\rho_m) + F(\rho,\rho_l)$ with $\rho_l \geq \rho \geq \rho_m$. Therefore, $g_4(\rho_l) < 0$ and $g_4(\rho_m) > 0$ and hence by applying monotonicity and the intermediate value theorem of the function we can find a unique ρ_1 such that $g_4(\rho_1) = 0$ satisfying $\rho_m < \rho_1 \leq \rho_l$. Therefore, the point of intersection between $S_1(V_m)$ and $S_2(V_l)$ is uniquely obtained. Now, based upon the choice of ρ_r, there will be three possible cases which are

(a) When $\rho_r < \rho_1$ then $V_r \in III$ and the result of interaction is $R_1S_1 \Rightarrow R_1S_2$.
(b) When $\rho_r = \rho_1$ then $V_r \in S_2(V_l)$ and the interaction results as $R_1S_1 \Rightarrow S_2$, i.e., the wave of second family occurs after the collision of two 1-family waves.

(c) When $\rho_r > \rho_1$ then $V_r \in II$ and consequently the interaction leads to $R_1S_1 \Rightarrow S_1S_2$. The corresponding configuration of wave interactions are illustrated in figure 6.

(ii) *1-shock overtakes 1-rarefaction* (S_1R_1):
In this case, V_l is connected to V_m by 1-shock of first Riemann problem and V_m is connected to V_r by a 1-rarefaction wave of second Riemann problem. Therefore, for a given V_l, we choose V_m and V_r are such that $\rho_l < \rho_m$, $w_m = w_l - F(\rho_l, \rho_m)$ and $\rho_r \leq \rho_m$, $w_r = w_m - 2\sqrt{\frac{A}{\rho_l}} + 2\sqrt{\frac{A}{\rho}}$. Lax entropy condition for 1-shock of first Riemann problem is given by $\lambda_1(V_m) < \sigma_1(V_l, V_m) < \lambda_2(V_m)$ which follows that the trailing end of R_1 associated with Riemann second problem has less velocity compare to the velocity of S_1 corresponding to the Riemann first problem. Consequently, 1-shock collides with 1-rarefaction wave and interaction occurs after a certain time. Here, we prove that $S_1(V_l)$ lies above the curve $R_1(V_m)$ whenever $\rho_l < \rho < \rho_m$. To establish this, it is enough to show that

$$F(\rho_m, \rho_l) - F(\rho, \rho_l) + 2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho}} > 0. \quad (21)$$

To prove (21), let us consider the function $g_5(\rho) = F(\rho_m, \rho_l) - F(\rho, \rho_l) + 2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho}}$ such that $g_5(\rho_m) = 0$. An easy computation yields

$$\frac{dg_5}{d\rho} = \frac{\sqrt{A}}{\rho^2}\left(\sqrt{\rho} + \frac{\ln\frac{\rho_l}{\rho} + (1 - \frac{\rho}{\rho_l})}{2\sqrt{(\frac{1}{\rho} - \frac{1}{\rho_l})\ln\frac{\rho_l}{\rho}}}\right). \quad (22)$$

Now, we claim that $\frac{dg_5}{d\rho} < 0$ for $\rho_l < \rho < \rho_m$. In order to prove our claim, let us take $\xi = \frac{\rho}{\rho_l} > 1$ and it is sufficient to prove that

$$2\sqrt{(\xi - 1)\ln\xi} - \ln\xi + (1 - \xi) < 0. \quad (23)$$

Applying the property of arithmetic mean (A. M.) and geometric mean (G. M.) between the two positive real numbers $\xi - 1$ and $\ln\xi$, we get $(\xi - 1) + \ln\xi > 2\sqrt{(\xi - 1)\ln\xi}$ (as $A.M. > G.M.$) which leads to the inequality (23). Therefore, $g_5(\rho)$ is decreasing function of ρ and hence $g_5(\rho) > g_5(\rho_m) = 0$ which implies that $R_1(V_m)$ lies below the curve $S_1(V_l)$ for $\rho_l < \rho < \rho_m$.
Now, we show that $R_1(V_m)$ lies below the curve $R_1(V_l)$ for $\rho < \rho_l < \rho_m$. To prove this it is sufficient to establish that $0 < F(\rho_m, \rho_l) + 2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho_l}}$ for $\rho < \rho_l < \rho_m$. The right hand side of this inequality is same as $g_5(\rho_m)$ which is established already as positive quantity.
Finally, we prove that the curve $R_1(V_m)$ intersects with $S_2(V_l)$ at some unique point say (ρ_2, w_2). To establish this, we have to show that $g_5(\rho) = 0$ has a unique root at $\rho = \rho_2$ satisfying $\rho_2 < \rho_l$. Clearly, $g_5(\rho_l) > 0$ and $g_5(\rho)$ is negative when ρ is near zero. Hence by applying the monotonicity and intermediate value theorem to the shock and rarefaction wave curve there exist a unique root of $\rho = \rho_2$ with $\rho_2 < \rho_l$. Now, depending upon the choice of ρ_r, there will be three possibilities which are

(a) When $\rho_r < \rho_2$ then $V_r \in III$ and the result of interaction is $S_1R_1 \Rightarrow R_1S_2$.
(b) When $\rho_r = \rho_2$ then $V_r \in S_2(V_l)$ and the interaction results as $R_1S_1 \Rightarrow S_2$, which means that after collision of two waves of 1-family provides a new wave of the other family.
(c) When $\rho_r > \rho_2$ then $V_r \in II$ and consequently the interaction leads to $S_1R_1 \Rightarrow S_1S_2$. The configuration of wave interactions are depicted in figure 7.

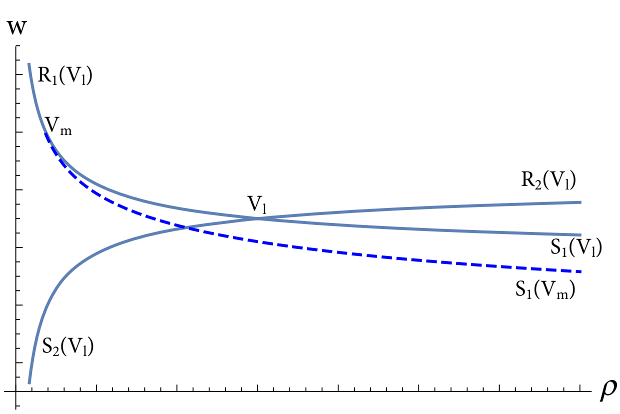

Figure 6. R_1 overtakes S_1.

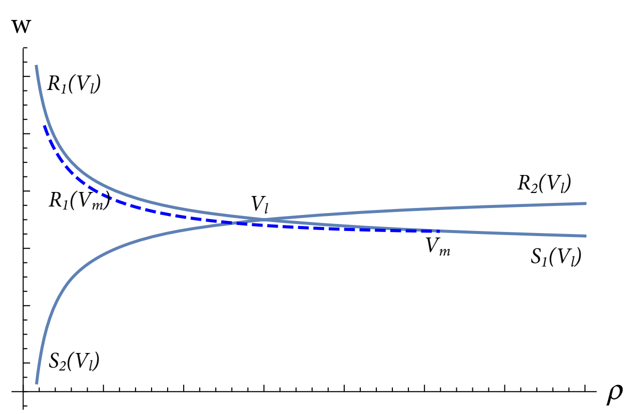

Figure 7. S_1 overtakes R_1.

(iii) *1-shock overtakes 1-shock* (S_1S_1):
In this case, for a given V_l, we choose V_m and V_r in such a way that V_l is connected to V_m by 1-shock of first Riemann problem and V_m to V_r are connected by 1-shock of second Riemann problem. In other words, $\rho_l < \rho_m$, $w_m = w_l - F(\rho_l, \rho_m)$ and $\rho_m < \rho_r$, $w_r = w_m - F(\rho_m, \rho_r)$. From Lax entropy condition one can easily verify that $\sigma_1(V_l, V_m) > \lambda_1(V_m) > \sigma_1(V_m, V_r)$ and hence 1-shock of first Riemann problem overtakes 1-shock of second Riemann problem and interaction occurs after a finite time. Now, we show that the curve $S_1(V_m)$ lies above the curve $S_1(V_l)$ for $\rho_l < \rho_m < \rho$. In order to prove this, it is sufficient to show that

$$F(\rho, \rho_l) - F(\rho_m, \rho_l) - F(\rho, \rho_m) > 0, \qquad (24)$$

for $\rho_l < \rho_m < \rho$. In order to prove (24), let us consider the function $g_7(\rho) = F(\rho, \rho_l) - F(\rho_m, \rho_l) - F(\rho, \rho_m)$ such that $g_7(\rho_m) = 0$. Differentiating $g_7(\rho)$ with respect to ρ gives

$$\frac{dg_7}{d\rho}(\rho) = \frac{dF}{d\rho}(\rho, \rho_l) - \frac{dF}{d\rho}(\rho, \rho_m). \qquad (25)$$

Since, $\frac{dF}{d\rho}$ is a decreasing function of ρ for $\rho > \rho_l$ which implies that $\frac{dg_7}{d\rho}(\rho) > 0$ and hence $g_7(\rho) > g_7(\rho_m) > 0$ which proves the inequality (24) for $\rho_l < \rho_m < \rho$. Therefore, $V_r \in I$ and the result of interaction is $S_1S_1 \Rightarrow S_1R_2$. The computed result is established in figure 8.

(iv) *2-shock overtakes 2-shock* (S_2S_2):
Here, for a given V_l, we choose V_m and V_r in such a way that V_l and V_m are connected by a 2-shock of first Riemann problem and V_m and V_r are connected by 2-shock of second Riemann problem. It can be proved in a same manner as in previous case that $V_r \in III$ and the result of interaction is $S_2S_2 \Rightarrow R_1S_2$. The computed result is characterized in figure 9.

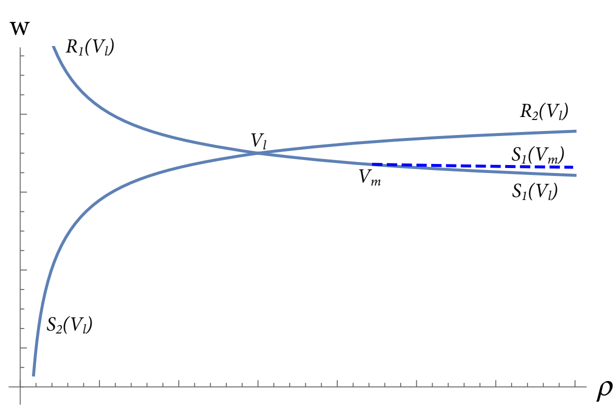

Figure 8. S_1 overtakes S_1.

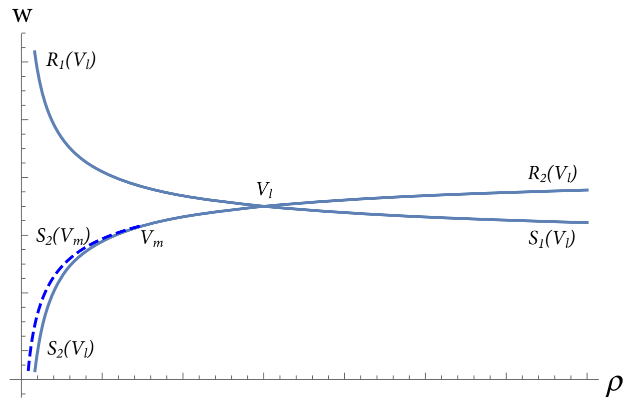

Figure 9. S_2 overtakes S_2.

(v) *2-rarefaction overtakes 2-shock* (R_2S_2):
In this case, we choose V_m and V_r with respect to V_l in such a way that $V_m \in R_2(V_l)$ and $V_r \in S_2(V_m)$. In other words, we have $\rho_m \geq \rho_l$, $w_m = w_l + 2\sqrt{\frac{A}{\rho_l}} - 2\sqrt{\frac{A}{\rho_m}}$ and $\rho_r < \rho_m$, $w_r = w_m - F(\rho_r, \rho_m)$. From Lax entropy condition, we have $\lambda_2(V_m) > \sigma_2(V_l, V_m)$ which implies that R_2 overtakes S_2 after a finite time. First we prove that $R_2(V_l)$ lies above $S_2(V_m)$ for $\rho_l \leq \rho < \rho_m$. In order to show this, it is sufficient to prove

$$2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho}} + F(\rho, \rho_m) > 0 \qquad (26)$$

for $\rho_l \leq \rho < \rho_m$. The inequality (26) can be written as

$$\sqrt{\frac{A}{\rho}}\left[2\left(\sqrt{\frac{\rho}{\rho_m}} - 1\right) + \sqrt{\left(1 - \frac{\rho}{\rho_m}\right)\ln\frac{\rho_m}{\rho}}\right] > 0. \qquad (27)$$

Now, let $\tau = \frac{\rho}{\rho_m} < 1$. To prove (27), it is sufficient to show that $g_8(\tau) = 2(\sqrt{\tau} - 1) + \sqrt{(\tau - 1)\ln\tau} > 0$ for $\tau < 1$. Now, one can easily prove that $g_8(\tau)$ is a decreasing function of τ on the interval $[\tau, 1]$. Hence, we obtain that $g_8(\tau) > g_8(1) = 0$ which implies our required inequality (27).
Next, we show that $S_2(V_m)$ lies below the curve $S_2(V_l)$ for $\rho \leq \rho_l < \rho_m$. In order to prove this it is sufficient to show that

$$2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho}} + F(\rho, \rho_m) - F(\rho, \rho_l) > 0 \qquad (28)$$

for $\rho \leq \rho_l < \rho_m$. To prove (28), let $g_9(\rho) = 2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho}} + F(\rho, \rho_m) - F(\rho, \rho_l)$ such that $g_9(\rho_l)$ is nothing but the left hand side of the inequality (26) which is already shown to be positive, i.e., $g_9(\rho_l) > 0$. Now,

$$\frac{dg_9}{d\rho}(\rho) = \frac{dF}{d\rho}(\rho, \rho_m) - \frac{dF}{d\rho}(\rho, \rho_l). \qquad (29)$$

Since, $\frac{dF}{d\rho}$ is a decreasing function of ρ_m for $\rho_m > \rho_l$ which implies that $\frac{dg_9}{d\rho}(\rho) < 0$. Therefore, $g_9(\rho) > g_9(\rho_l) > 0$ which implies the inequality.
Finally, we prove that $S_2(V_m)$ intersects with $S_1(V_l)$ uniquely at some point, say, (ρ_3, w_3) with $\rho_l < \rho_3 < \rho_m$. This can be proved using the same argument as done in preceding cases. Here also, depending upon the choice of ρ_r, there will be three cases which are

(a) When $\rho_r < \rho_3$ then $V_r \in II$ and the result of interaction is $R_2S_2 \Rightarrow S_1S_2$.
(b) When $\rho_r = \rho_3$ then $V_r \in S_1(V_l)$ and the interaction results as $R_2S_2 \Rightarrow S_1$, i.e., after interaction of two waves of 2-family gives rise to a wave of the 1-family.
(c) When $\rho_r > \rho_3$ then $V_r \in I$ and the result of interaction is $R_2S_2 \Rightarrow S_1R_2$. The computed results are interpreted in figure 10.

(vi) *2-shock overtakes 2-rarefaction* (S_2R_2):
Here, V_l is connected to V_m by a 2-shock of the first Riemann problem and V_m is connected to V_r by a 2-rarefaction wave of the second Riemann problem, i.e., $V_m \in S_2(V_l)$ and $V_r \in R_2(V_m)$. It follows that, for $\rho_m < \rho_l$, $w_m = w_l - F(\rho_l, \rho_m)$ and for $\rho_r \geq \rho_m$, $w_r = w_m + 2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho_r}}$. Since $\sigma_2(V_l, V_m) > \lambda_2(V_m)$, 2-shock overtakes 2-rarefaction after a finite time. First we show that the curve $S_2(V_l)$ lies above the curve $R_2(V_m)$ for $\rho_m < \rho < \rho_l$. In order to prove this, it is enough to show that

$$2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho}} + F(\rho, \rho_l) - F(\rho_m, \rho_l) > 0, \qquad (30)$$

for $\rho_m < \rho < \rho_l$. Now, we set a new function $g_{10}(\rho) = 2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho}} + F(\rho, \rho_l) - F(\rho_m, \rho_l)$ such a way that $g_{10}(\rho_m) = 0$. Moreover, one can prove that $\frac{dg_{10}}{d\rho}(\rho) > 0$ which implies that $g_{10}(\rho) > g_{10}(\rho_m) = 0$ and hence the inequality (30). Thus, $S_2(V_l)$ lies above the curve $R_2(V_m)$ for $\rho_m < \rho < \rho_l$. Now, we prove that $R_2(V_m)$ lies below the curve $R_2(V_l)$ for $\rho > \rho_l > \rho_m$. To show this it is sufficient to prove that $0 < 2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho_l}} - F(\rho_m, \rho_l)$ for $\rho > \rho_l \leq \rho_m$. But, the left hand side of the inequality is nothing but $g_9(\rho_m)$ which is already shown to be positive. Thus, $R_2(V_l)$ lies above the curve $R_2(V_m)$ for $\rho_m < \rho_l < \rho$.
Finally, we prove that $R_2(V_m)$ intersects uniquely with $S_1(V_l)$ at some point, say, (ρ_4, w_4) with $\rho_m < \rho_l < \rho_4$. In order to show this, it is sufficient to prove that $2\sqrt{\frac{A}{\rho_m}} - 2\sqrt{\frac{A}{\rho}} + F(\rho, \rho_l) - F(\rho_m, \rho_l) = 0$ has a root for $\rho_m < \rho_l < \rho$ which can be proved similarly as done in preceding cases. Again, there will be three possible cases which are given by

(a) When $\rho_r < \rho_4$ then $V_r \in II$ and the result of interaction is $S_2R_2 \Rightarrow S_1R_2$.
(b) When $\rho_r = \rho_4$ then $V_r \in S_1(V_l)$ and the interaction results as $S_2R_2 \Rightarrow S_1$, i.e., after interaction of two waves of 2-family gives rise to a wave of the 1-family.
(c) When $\rho_r > \rho_4$ then $V_r \in I$ and the result of interaction is $S_2R_2 \Rightarrow S_1R_2$. The computed results are represented in figure 11.

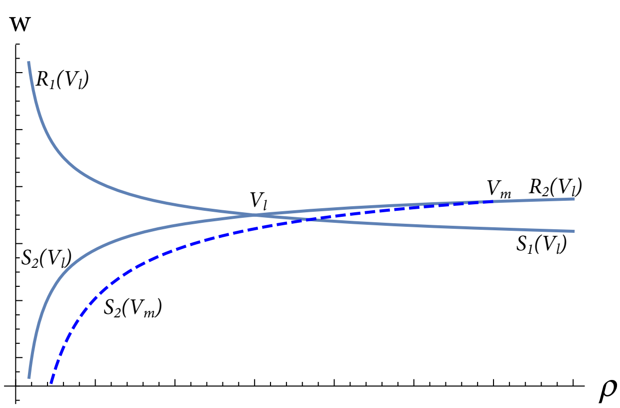

Figure 10. R_2 overtakes S_2.

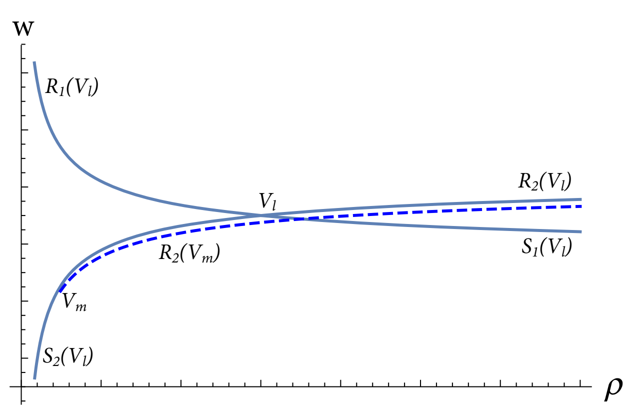

Figure 11. S_2 overtakes R_2.

4. Conclusion

We studied characteristic analysis of logotropic system of gasdynamics with a Coulomb-type friction and established the properties of shocks and rarefaction curves of one-parameter family. For a given arbitrary state, we constructed the Riemann solution in phase plane. Further, we developed locally two Riemann problems by considering appropriate initial data and analyzed all possible interactions between shocks and rarefactions in the phase plane which provide the basic features of solution with rich geometric structure. It is observed that Riemann problem is uniquely solvable and no vacuum occurs in the solution.

Acknowledgements

We would like to thank the anonymous referees for their fruitful comments and valuable suggestions. The second author (TRS) would like to thank SERB, DST, India (Ref. No. MTR/2019/001210) for its financial support through MATRICS grant.

References

[1] Glimm J 1965 Solutions in the large for nonlinear hyperbolic systems of equations. *Commun. Pure Appl. Math.* 18(4): 697–715
[2] Minhajul, Zeidan D and Raja Sekhar T 2018 On the wave interactions in the drift-flux equations of two-phase flows. *Appl. Math. Comput.* 327: 117–131
[3] Kuila S and Raja Sekhar T 2018 Interaction of weak shocks in drift-flux model of compressible two-phase flows. *Chaos, Solitons & Fractals* 107: 222–227
[4] Raja Sekhar T and Sharma V D 2010 Riemann problem and elementary wave interactions in isentropic magnetogasdynamics. *Nonlinear Anal. Real World Appl.* 11(2): 619–636
[5] Liu Y and Sun W 2016 Elementary wave interactions in magnetogasdynamics. *Indian J. Pure Appl. Math.* 47(1): 33–57
[6] Sen A, Raja Sekhar T and Sharma V D 2017 Wave interactions and stability of the Riemann solution for a strictly hyperbolic system of conservation laws. *Quart. Appl. Math.* 75(3): 539–554
[7] Minhajul and Raja Sekhar T 2019 Interaction of elementary waves with a weak discontinuity in an isothermal drift-flux model of compressible two-phase flows. *Quart. Appl. Math.* 77(3): 671–688
[8] Raja Sekhar T and Sharma V D 2010 Wave interactions for the pressure gradient equations. *Methods and Applications of Analysis* 17(2): 165–178
[9] Minhajul and Raja Sekhar T 2021 Nonlinear wave interactions in a macroscopic production model. *Acta Math. Sci. Ser. B Engl. Ed.* 41(3): 764-780
[10] Minhajul, Raja Sekhar T and Raja Sekhar G P 2019 Stability of solutions to the Riemann problem for a thin film model of a perfectly soluble anti-surfactant solution. *Commun. Pure Appl. Anal.* 18(6): 3367–3386
[11] Liu Y and Sun W 2018 Wave interactions and stability of Riemann solutions of the Aw–Rascle model for generalized chaplygin gas. *Acta Appl. Math.* 154(1): 95–109
[12] Sharma V D 2010 *Quasilinear hyperbolic systems, compressible flows, and waves* CRC Press
[13] Smoller J 2012 *Shock waves and reaction diffusion equations*, volume 258. Springer Science & Business Media
[14] Godlewski E and Raviart P A 2013 *Numerical approximation of hyperbolic systems of conservation laws*, volume 118, Springer Science & Business Medi
[15] Sun M 2009 Interactions of elementary waves for the Aw–Rascle model. *SIAM J. Appl. Math.* 69(6): 1542–1558
[16] Singh R and Jena J 2013 Interaction of an acceleration wave with a strong shock in reacting polytropic gases. *Appl. Math. and Comput.* 225: 638–644
[17] Curro C and Manganaro N 2013 Riemann problems and exact solutions to a traffic flow model. *J. Math. Phys.* 54 (7): 071503
[18] Curro C and Manganaro N 2019 Nonlinear wave interactions for a model of extended thermodynamics with six fields. *Ric. di Mat.* 68(1): 131–143
[19] Chaudhary J P and Singh L P 2019 Riemann problem and elementary wave interactions in dusty gas. *Appl. Math. Comput.* 342: 147–165
[20] Shen C 2010 Wave interactions and stability of the Riemann solutions for the chromatography equations. *J. Math. Anal. Appl.* 365(2): 609–618
[21] Sen A, Raja Sekhar T and Zeidan D 2019 Stability of the Riemann solution for a 2 × 2 strictly hyperbolic system of conservation laws. *Sadhana* 44: 1–8
[22] Wei Z and Sun M 2021 Riemann problem and wave interactions for a Temple-class hyperbolic system of conservation laws. *Bull. Malays. Math. Sci. Soc.* 44(6): 4195–4221
[23] Zhang Y and Zhang Y 2020 Riemann problem and wave Interactions for a Class of strictly hyperbolic systems of conservation laws *Bull. Braz. Math. Soc.* 51: 1017–1040
[24] R K Chaturvedi and L P Singh 2020 Riemann solutions to the logotropic system with a Coulomb-type friction, *Ricerche di Matematica*, 1–14
[25] P H Chavanis and C Sire 2007 Logotropic distributions, Physica A Stat. Mech. Appl., 375(1), 140-158
[26] P H Chavanis 2016 The logotropic dark fluid as a unification of dark matter and dark energy. *Phys. Lett. B*, 758, 59-66
[27] Odintsov S D, Oikonomou V, Timoshkin A, ESaridakis E N and Myrzakulov R 2018 Cosmological fluids with logarithmic equation of state. *Ann. Phys.* 398: 238–253
[28] Faccanoni G and Mangeney A 2013 Exact solution for granular flows. *Int. J. Numer. Anal. Methods Geomech.* 37(10): 1408–1433
[29] Sen A and Raja Sekhar T 2021 The limiting behavior of the Riemann solution to the isentropic Euler system for logarithmic equation of state with a source term. *Math. Methods Appl. Sci.* 44(8): 7207–7227